

Working Paper

Integrated Optimization of Procurement, Processing and Trade of Commodities

Sripad K. Devalkar

Stephen M. Ross School of Business
at the University of Michigan

Ravi Anupindi

Stephen M. Ross School of Business
at the University of Michigan

Amitabh Sinha

Stephen M. Ross School of Business
at the University of Michigan

Ross School of Business Working Paper Series
Working Paper No. 1095
August 2007

This paper can be downloaded without charge from the
Social Sciences Research Network Electronic Paper Collection:
<http://ssrn.com/abstract=1004742>

UNIVERSITY OF MICHIGAN

Integrated Optimization of Procurement, Processing and Trade of Commodities

Sripad K Devalkar, Ravi Anupindi, Amitabh Sinha

Stephen M. Ross School of Business, University of Michigan, Ann Arbor, MI 48109
devalkar@umich.edu, anupindi@umich.edu, amitabh@umich.edu

August 2007

Abstract

We consider an integrated optimization problem for a firm involved in procurement, processing and trading of commodities. We first derive optimal policies for a risk-neutral firm, when the processed commodity(ies) are sold using futures instruments. We find that the optimal procurement quantity is governed by a threshold policy, where the threshold is independent of the starting inventory level, and it is optimal to postpone all processing till the last possible period. We extend the model to include risk-averse firms, using a Value-at-Risk constraint on the total expected profits. We show that the optimal procurement quantity for a risk-averse firm is never greater than that for a risk-neutral firm and a risk-averse firm may find it optimal to process and sell the output commodity in earlier periods. We conduct numerical studies to quantify the benefit from integrated decision making and the impact of risk-aversion on expected profits.

Keywords: Integrated Optimization; Commodities; Risk; Value-at-Risk

1 Introduction

Consider a firm that procures an input commodity, processes it into one or more output commodities, and trades both input and output commodities for profit. For such firms, there are three critical decision-making stages: the procurement of the input commodity, the commitment of the input commodity to processing (an irreversible transformation of the input commodity into output commodities), and the trading of the input and output commodity(ies). Examples of such input-output commodity sets include corn and ethanol; soybean and soymeal/oil; oranges and orange juice; crude oil and refined petroleum products; etc. The firm's objective of profit maximization is affected by the interplay of decisions in all three stages: procurement, processing and trading. In the literature (reviewed in §2), however, typically these stages are analyzed independently of one another, leading to possibly sub-optimal strategies for the overall integrated problem. While the processing costs may be somewhat well-predictable or deterministic, the procurement costs and revenues from trading are driven by spot and futures prices of the commodities in international exchanges, as well as local prices (trading with small-scale farmers or independent users of the commodities), which are stochastic and not predictable with certainty.

An additional level of complexity is added when firms' risk profile is considered. Commodity prices are time-varying and stochastic, and the correlation between prices of the input and output commodities are not perfect. The stochastic prices result in the potential for huge downside losses if, for example, commodity prices fall after the input commodity has already been procured and held in inventory for processing and /or trading at a later point in time. Naturally, firms wish to guard against such downside risk by adopting risk-averse behavior strategies, which further modify the optimal policies for the three stages of procurement, processing and trading.

Traditional research in operations management has addressed the problem in each of the decision stages independently, usually under the assumption of risk neutrality. Resultantly, the overall integrated optimization problem presents both an interesting challenge and an opportunity to fill a substantial gap in the literature. This paper seeks to fill some of the gap by deriving integrated optimal policies across the three decision stages under different scenarios of the general problem.

1.1 Our Contributions

We consider a firm that earns revenues by procuring and processing an input commodity, and committing to sell the output of the processing in a futures contract and/or salvaging the input

inventory in a spot market at the end of the horizon. We begin with the study of a risk-neutral firm and find that the optimal procurement policy is a threshold policy, where the threshold is *independent* of the starting inventory level. We also find that it is optimal for a risk-neutral firm to postpone the processing and sale of the output using a futures contract until the last period before the maturity of the futures contract.

When risk-averse behavior is considered, however, the firm may in fact find it optimal to commit to sell the output in periods other than the last. However, these commitments are solely to manage its risk and result in lower expected profits. The procurement policy is still a threshold policy, but the threshold may depend on the starting inventory level.

We also conduct a numerical study, highlighting the impact of risk-averse behavior as well as the benefits of integrated decision making. While firms not practicing integrated decision-making can and do follow a variety of different operational strategies, we compare the benefits with respect to a specific policy¹ we term the ‘full-commitment’ policy, described in §3.1.1. We find that there is a significant difference in expected profits between the optimal and full-commitment policies, and risk-aversion plays a significant role in optimal policies and expected profits, confirming the theoretical results.

Finally, we also consider the case of a firm that has access to multiple futures contracts for the output. For a risk-neutral firm, the structure of the procurement policy is unchanged. However, it may be optimal to commit to process before the end of the horizon. If this is done, the commitment is always in a period just before the expiration of a futures contract and only if the margin from the expiring contract exceeds the maximum expected margin of retaining unprocessed inventory. Furthermore, if such an option is exercised, all available inventory is committed.

1.2 Motivation

The original motivation for this work comes from the innovative practices of one of India’s largest private sector companies, The ITC Group (www.itcportal.com). While ITC is a diversified company, the International Business Division (IBD) of ITC exports agricultural commodities such as soybean meal, rice, wheat and wheat products, etc. As a buyer of these agricultural commodities, ITC-IBD faced the consequences of an inefficient farm-to-market supply chain amidst increasing competition in a liberalized economy. In response, in the year 2000 ITC-IBD (hereafter referred to as ITC) embarked on an initiative to deploy information and communication technology (ICT) to

¹This policy is used by The ITC Group, the firm that motivated this research, as described in §1.2.

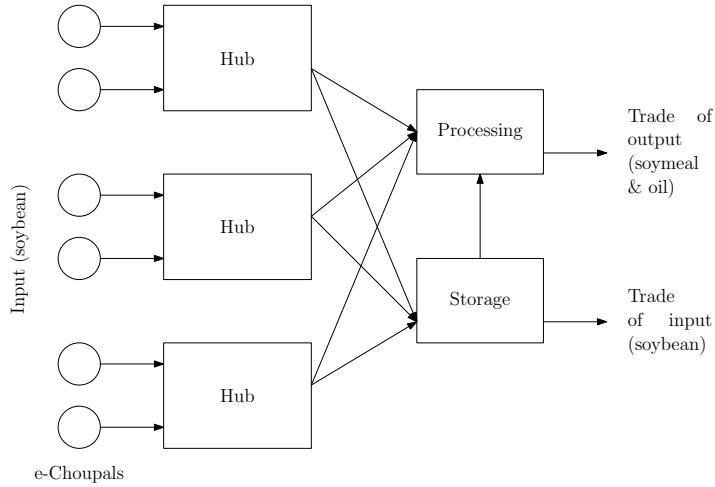


Figure 1: ITC e-Choupal Network.

re-engineer the procurement of soybeans from rural India. ICT kiosks (called *e-Choupals*) consisting of a personal computer with internet access were setup at the villages. Soybean farmers could access this kiosk for information on prices, but had a choice to sell their produce either at the local spot market (called a *mandi*) or directly to ITC at their hub locations. A hub location would service a cluster of e-Choupals. By purchasing directly from the farmers, ITC significantly improved the efficiency of the channel and created value for both the farmer and itself. The e-Choupal experiment for soybeans procurement has been well documented by Anupindi and Sivakumar (2006) and the experiment has been extremely successful for ITC.

The procured soybean is processed to produce soybean oil and soymeal, which are sold using futures instruments traded on global commodity exchanges such as the Chicago Board of Trade (CBOT). The network of procurement hubs gives ITC a cost advantage in procuring soybean along with an ability to store the excess soybean that is not immediately required for processing. Therefore, in addition to processing and selling the soymeal, ITC also sells the soybean to other processors, primarily in the off-season, if it is profitable to do so. A schematic of the network is shown in Figure 1.

Managing this network requires decisions regarding procurement, trading, and demand management to maximize profits. Procurement decisions include price and quantity decisions for each hub. Since the farmers have a choice of whether or not to sell to ITC directly, these decisions are important and form the supply curve. For the soybean procured, ITC needs to make decisions regarding

whether to trade the bean (typically at the end of the planning horizon, which is the off-season for procurement but may still have processing activity arising from other firms) or process it and trade the oil and meal. Finally, the procurement decision needs to be integrated with the decision to manage the demand in terms of the form of output commodity and channels to trade in. Based on our extensive discussions with ITC, we observe that the decisions of procurement, allocation, and sale are not coordinated. This disconnect is also seen in the literature, with relatively little academic work on the integrated optimization problem.

While the ITC e-choupal network was our introduction to the area and our initial motivation, the model we analyze is quite generic and applies to other contexts as well. Any firm engaged in the procurement of an input commodity with a choice of whether, and when, to irreversibly process it into an output commodity faces such a decision-making problem. For instance, the increasing use of ethanol as an alternative to fossil fuels presents a similar optimization problem for corn producers and procurers. The model can also be extended in a variety of other directions, some of which are discussed in the conclusion of the paper.

1.3 Outline

In §2 we review the literature related to commodity procurement and processing, and joint operational and financial hedging. §3 describes the mathematical model for the integrated optimization problem. We derive optimal policies for a risk-neutral and risk-averse firm when there is a single futures contract available for trading the output in §3.1 and §3.2 respectively. Numerical calculations for the single futures case are presented in §3.3. We also consider the optimal policy for a risk-neutral firm in the presence of multiple futures contracts with different maturities in §4. Conclusions and open research questions are discussed in §5.

2 Literature Review

The trading of commodities is a fairly old economic activity, and a steady stream of literature has developed on the modeling of commodity prices and derivatives and their trading. Working (1949) is one of the earliest to study the relation between storage decisions and commodity prices and introduced the idea of *convenience yield*². Geman (2005) is a recent and comprehensive book on

²The return on storage, or the convenience yield, is the benefit of avoiding frequent deliveries and frequent production schedule changes to meet demand, when one has stock of the commodity available.

commodity prices, including agricultural commodities. Other recent papers on pricing commodities in the spot and futures markets include Gibson and Schwartz (1990), Pindyck (2001), Routledge et al. (2000) and Routledge et al. (2001). The work of Gibson and Schwartz (1990) was generalized by Schwartz and Smith (2000), who use a general two-factor model comprised of a long-run equilibrium as well as short-term mean-reverting fluctuations. We use a variation of this model in our preliminary numerical analysis to validate our findings. We observe here that all these papers (with the exception of Routledge et al. (2001)) focus on single commodities, and do not model the relationship between the prices of two commodities, one of which is an input to and the other the output of some process.

The widespread use of futures markets to trade and hedge risk has led to a substantial body of associated literature as well. Working (1953) is among the earliest papers to study the use of futures markets for trading and hedging. Risk management in agriculture was studied by Goy (1999), who explores various hedging strategies available to farmers in the U.S. Anderson and Danthine (1995), Tsang and Leuthold (1990) and Dahlgran (2002), among others, study a single period problem of hedging positions in multiple commodities using futures instruments while Myers and Hanson (1996) consider the problem of dynamical hedging the risk from a single commodity over multiple periods. It has been observed that commodity processing decisions in the aggregate are correlated with output commodity prices; an exploration of this phenomenon in the soybean crushing industry by Plato (2001) finds some empirical evidence that firms strategically use the commodity markets to optimally time their operational (processing) decisions. Most of the papers mentioned above consider either a single commodity or single period in their analysis, but not both. In contrast, we study the dynamic hedging and optimization of multiple commodities over a horizon.

In the past few years, a series of papers in the operations literature have begun to focus on using financial hedging strategies to mitigate inventory and other operational risks. These include Caldentey and Haugh (2006), who view the operations of the firm as an asset for investment and use portfolio analysis techniques; Gaur and Seshadri (2005), who study the hedging of inventory risk in a newsvendor setting; and Zhu and Kapuscinski (2006) and Chowdhry and Howe (1999), who consider operational and financial hedging for multinational firms facing exchange rate risk in addition to uncertain demand. Perhaps most closely related to our work is the recent work of Goel and Gutierrez (2006), who analyze the integration of spot and futures markets for optimal procurement strategies of commodities in a multi-period setting. All of these papers, however, continue to focus on trade on only one side: either the input (procurement) or the output commodity, without analyzing the

integrated decision of optimizing strategies over both commodities.

As can be seen by the literature survey above, there is substantial academic work on the individual pieces of the decision making involved in the type of firm we study, such as procurement over spot and futures markets, hedging inventory with markets, etc. However, there are no studies that we are aware of, which look at the integrated problem of procuring, processing and trading of commodities. It is this gap in the literature that we hope to address in the current paper.

3 Model Description

We consider a finite horizon model for the integrated procurement and processing decisions of a firm that maximizes discounted expected profit over the horizon. The firm may be risk-neutral or risk-averse and the two cases are analyzed in §3.1 and §3.2 respectively.

The time periods are indexed by $n = 1, 2, \dots, N - 1, N$, with $n = 1$ being the first decision period. In any period n , let S_n denote the spot market price for the input commodity. The firm sells all the processed product (output) using futures contracts that are traded on an exchange.

A *futures contract* is an agreement between two parties to buy or sell a certain quantity of a commodity at a certain time in the future for a certain price (Hull 1997). Futures contracts are normally traded on an exchange, with the exchange specifying certain standardized features of the contract, such as the quality and delivery location. The price specified on the futures contract at which the commodity (the processed product or output in the current model) can be sold or bought is known as the futures price and this price changes over time. Let F_n^l denote the futures price on a futures contract l for the output, with maturity $N_l > n$. We assume that there are L futures contracts, with maturities N_l , $l = \{1, 2, \dots, K\}$ with $N_i < N_j$ for $i < j$.

Any leftover inventory of the input at the end of the horizon is salvaged at the prevailing spot price in the last period, S_N . In the ITC context, the planning horizon can be considered as the procurement season, when bulk of the procurement happens. The end of the horizon can be thought of as the off-season, when most of the trading of the input (soybean) occurs. Therefore, S_N models the off-season trading price for the input and it may be substantially different from the spot prices during the procurement season. Under certain conditions, the margins from just holding the input inventory and trading it at the end of the horizon might result in significantly higher profits. Considering this potential for trade is critical in our integrated decision-making.

Let \mathcal{I}_n denote all the relevant information regarding the spot market prices, futures prices and

the end of the horizon salvage value available to the firm in period n . Thus, \mathcal{I}_n includes the realized spot market price, futures prices and could include other information like aggregate inventory levels of the commodities, inventory levels with other processors, etc. A definition of \mathcal{I}_n at this general level is sufficient for the purposes of model being considered.

The availability of labor, handling equipment and other operational constraints at the procurement hub impose a restriction on the amount of the input commodity that can be procured in any given period. For simplicity, we assume that the procurement capacity in every period is the same and let $K > 0$ denote the maximum quantity of the input that can be procured in any given period at the hub; i.e., $x_n \leq K$ for all $n \leq N - 1$. We later show, in §3.1.3, that relaxing this assumption does not alter the structural results obtained.

Thus, in each period n , based on \mathcal{I}_n , the firm decides the quantity of input, x_n , to be procured and the quantity of the processed product, q_n^l , to be committed for sale using a futures contract l , with the revenues being realized in period N_l . Any leftover inventory of the input at the end of the horizon is sold to other firms, at a per-unit salvage value of S_N .

Naturally, a commitment to sell the output can be made only using a futures contract that matures later in the horizon; i.e., q_n^l makes sense only when $n < N_l$. Furthermore, since we consider a situation where sales of a futures contract are settled by actual delivery of the processed product, it is costly to reverse a commitment. Therefore, we require that at the time of expiration of a futures contract, the total amount committed does not exceed the total amount procured:

$$\sum_{j=1}^l \sum_{i=1}^{N_j-1} q_i^j \leq \sum_{i=1}^{N_l-1} x_i \quad \forall l \leq L \quad (1)$$

However, temporary over-commitment is allowed: the firm at an intermediate point of time may have more commitments than the available inventory, as long as the shortfall is made up before expiration of the futures contract.

We assume that there are no processing capacity restrictions and the quantity committed is limited only by the total amount procured. This assumption is made for analytical tractability and to focus attention on the value of integrated decision making. Observe that infinite processing capacity implies that the firm would never process the input without a commitment to sell the output.

The firm also has an endogenous procurement cost function $C(S_n, x_n)$, which is the total cost incurred to procure x_n units if the spot price is S_n . In the simplest case (that of constant marginal costs), $C(S_n, x_n)$ is simply $S_n \times x_n$, but in general, the cost of procurement may be increasing and

convex due to market factors. An alternative view of this cost function is in the context of ITC; ITC announces a one-day forward price for input procurement directly at its hubs. The resulting supply is a function of the price announced by ITC as well as the prevailing spot price. Inverting this supply function results in the cost function $C(S_n, x_n)$.

For ease of exposition and without loss of generality, we assume there is no discounting and that the physical costs of holding inventory are negligible³. The firm, however, incurs a processing cost of p per unit of input that is processed.

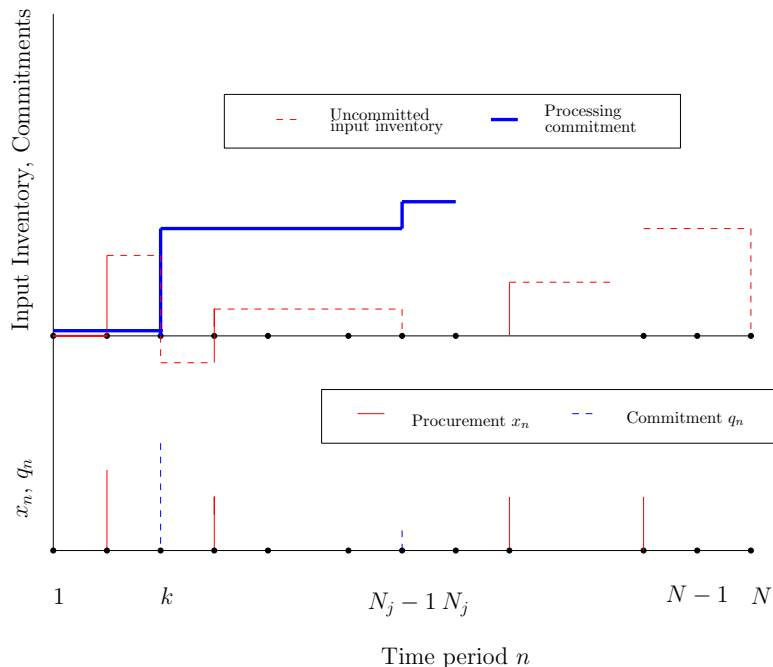


Figure 2: Sample Path for Inventory and Processing Commitment.

A theoretical sample path for the input inventory and processing commitments is shown in Figure 2. The top portion shows the net input inventory after commitments and the cumulative commitments against contract j , expiring in period N_j . Period N is the end of the horizon (beginning of the off-season) and period $N - 1$ is the last period in which any procurement or commitments can be made. The bottom portion of the figure shows the procurement and commitment (against futures contract j), x_n and q_n^j , in each period n . Above, the firm has an over-commitment at the end of period k . However, because of (1), the firm can additionally commit only a small quantity

³From the analysis that follows, assuming a discount factor $\alpha < 1$ and imposing a positive holding cost on inventory does not alter the structure of the optimal policy discussed below and hence these assumptions are not restrictive.

in the last period, $N_j - 1$, before the contract expires.

3.1 Single Futures Contract: Risk-neutral Firm

We begin by considering the case when the firm is risk-neutral and a single futures contract is available for selling the processed product; later we extend the analysis to include risk-aversion and multiple futures contracts.

Since there is only one futures contract, we drop the superscript l in the notation for the rest of this section. W.l.o.g., we assume the futures contract expires in the last period, N , and F_n denotes the futures price on the contract in period n . If the firm decides to commit q_n to be sold against the futures contract in period n , the revenue realized (at the end of the horizon) is given by $(F_n - p)q_n$.

Let e_n denote the cumulative excess (or shortfall) of the input commodity over commitments already made at the beginning of period n . That is, $e_n = e_1 + \sum_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} q_i$, where e_1 is the quantity of the input available at beginning of period 1. Only the uncommitted inventory at the beginning of period n is relevant for the procurement and processing decisions in period n . Therefore, the pair (e_n, \mathcal{I}_n) is sufficient to describe the state at period n .

We consider an efficient market for all commodities, i.e., a market without arbitrage opportunities. A well known result in the financial literature is that in a risk-neutral world, the futures price in any period is equal to the expected spot price of the commodity at maturity (see Hull (1997) sec. 3.9, Bjork (2004) sec. 7.6). That is, $F_n = E[Y_N | \mathcal{I}_n]$, where $E[\cdot]$ denotes the expectation operator, Y_N is the spot price of the commodity underlying the futures contract at maturity. It follows that $F_{n+1} = E[Y_N | \mathcal{I}_{n+1}]$, and $E[F_{n+1} | \mathcal{I}_n] = E[E[Y_N | \mathcal{I}_{n+1}] | \mathcal{I}_n] = E[Y_N | \mathcal{I}_n] = F_n$. Therefore, the following assumption holds for the remaining analysis in this section.

Assumption 1 *The markets for the input and output commodities are efficient and the futures prices for the output satisfy the following property: $E[F_{n+1} | \mathcal{I}_n] = F_n$ for $n = 1, 2, \dots, N - 1$.*

Let $V_n(e_n, \mathcal{I}_n)$ denote the optimal expected profit starting from period n ; i.e., if (e_n, \mathcal{I}_n) is the state at the start of period n , then $V_n(e_n, \mathcal{I}_n)$ denotes the additional maximal expected profits that the firm can earn if optimal decisions are made in period n and all subsequent periods.

For $e_n \geq 0$, $n = 1, 2, \dots, N - 1$, define $J_n(e_n, q_n, x_n, \mathcal{I}_n)$ as follows:

$$J_n(e_n, q_n, x_n, \mathcal{I}_n) = (F_n - p)q_n - C(S_n, x_n) + E_{\mathcal{I}_n}[V_{n+1}(e_n + x_n - q_n, \mathcal{I}_{n+1})] \quad (2)$$

$V_n(e_n, \mathcal{I}_n)$ then satisfies the following dynamic programming equation:

$$V_n(e_n, \mathcal{I}_n) = \max_{q_n \geq 0, 0 \leq x_n \leq K} \{J_n(e_n, q_n, x_n, \mathcal{I}_n)\} \quad (3)$$

$$\text{and } V_N(e_N, \mathcal{I}_N) = \begin{cases} S_N e_N & \text{if } e_N \geq 0 \\ -\infty & \text{if } e_N < 0 \end{cases}$$

The definition of $V_N(e_N, \mathcal{I}_N)$ implies that we do not allow the total commitment to exceed the total available inventory of the input when the futures contract expires, as required by (1).

Consider the period $n = N - 1$. When $e_{N-1} \geq 0$, the marginal revenue from committing to sell a unit of the input as processed product against the futures contract is $F_{N-1} - p$. The marginal revenue from holding unprocessed inventory and salvaging it at the end of the horizon is $E_{\mathcal{I}_{N-1}}[S_N]$. We define *processing margin* as the expected margin from selling the output using the futures contract and *trade margin* as the margin from holding unprocessed inventory and salvaging it at the end of the horizon. We see that it is optimal to commit to sell the output only if the *processing margin* is at least as much as the expected *trade margin*; i.e., $F_{N-1} - p \geq E_{\mathcal{I}_{N-1}}[S_N]$. Because of the convex cost of procurement, the total quantity to procure is given by the standard first order condition where marginal cost of procurement is equal to the marginal revenue.

The following theorem formalizes this intuition and describes the optimal policy for $n = N - 1$. Proofs for all the theorems are given in §A in the e-companion.

Theorem 1 *In period $N - 1$, for $e_{N-1} \geq 0$, the optimal policy is as follows.*

1. *Procurement: The procurement decision is characterized by two critical values, \hat{x}_{N-1} and \tilde{x}_{N-1} which satisfy the following first order conditions.*

$$\frac{\partial C(S_{N-1}, \hat{x}_{N-1})}{\partial x_{N-1}} = F_{N-1} - p \qquad \frac{\partial C(S_{N-1}, \tilde{x}_{N-1})}{\partial x_{N-1}} = E[S_N | \mathcal{I}_{N-1}]$$

The optimal quantity to procure, x_{N-1}^ , is then given by*

$$x_{N-1}^* = \begin{cases} \min\{\hat{x}_{N-1}, K\} & \text{if } F_{N-1} - p \geq E[S_N | \mathcal{I}_{N-1}] \\ \min\{\tilde{x}_{N-1}, K\} & \text{if } F_{N-1} - p < E[S_N | \mathcal{I}_{N-1}] \end{cases}$$

2. *Processing: It is optimal to commit to sell the processed product against the futures contract if and only if the processing margin is greater than the trade margin; i.e.,*

$$q_{N-1}^* = \begin{cases} e_{N-1} + x_{N-1}^* & \text{if } F_{N-1} - p \geq E_{\mathcal{I}_{N-1}}[S_N] \\ 0 & \text{if } F_{N-1} - p < E_{\mathcal{I}_{N-1}}[S_N] \end{cases}$$

Furthermore, $V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})$ can be expressed as

$$V_{N-1}(e_{N-1}, \mathcal{I}_{N-1}) = \max\{F_{N-1} - p, E[S_N | \mathcal{I}_{N-1}]\} \cdot e_{N-1} + B_{N-1}$$

$$\text{where } B_{N-1} = \begin{cases} (F_{N-1} - p)x_{N-1}^* - C(S_{N-1}, x_{N-1}^*) & \text{if } F_{N-1} - p \geq E[S_N | \mathcal{I}_{N-1}] \\ E[S_N | \mathcal{I}_{N-1}]x_{N-1}^* - C(S_{N-1}, x_{N-1}^*) & \text{if } F_{N-1} - p < E[S_N | \mathcal{I}_{N-1}] \end{cases}$$

From the above theorem, we see that $V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})$ is linear in e_{N-1} and the marginal revenue of a unit of inventory is $\max\{F_{N-1} - p, E_{\mathcal{I}_{N-1}}[S_N]\}$. Notice that the marginal benefit of a unit of inventory is always at least $F_{N-1} - p$ for all realizations of \mathcal{I}_{N-1} , for all $e_{N-1} \geq 0$. Thus in period $n = N - 2$, the marginal benefit of postponing the sale of the processed product against the futures contract and carrying the inventory to period $N - 1$ is at least $E_{\mathcal{I}_{N-2}}[F_{N-1} - p]$. By Assumption 1, we have $E_{\mathcal{I}_{N-2}}[F_{N-1} - p] = F_{N-2} - p$. But $F_{N-2} - p$ is the marginal revenue of committing to sell the output against the futures contract in period $N - 2$. Therefore, the marginal benefit of postponing the sale is at least as much as the marginal benefit from committing to the sale.

In fact, this property extends to all $n < N - 1$ and the marginal benefit of carrying an additional unit of inventory of the input is always greater than or equal to $F_n - p$ in any period $n \leq N - 1$. The next theorem states this result formally for a general period n .

Theorem 2 *The value function $V_n(e_n, \mathcal{I}_n)$ for $n \leq N - 1$ is linear for all $e_n \geq 0$. Moreover, the marginal benefit of an additional unit of inventory is at least $F_n - p$ for all $e_n \geq 0$.*

In any period $n < N$, the optimal procurement and processing decisions are as described below:

1. *Procurement Policy: The optimal procurement policy is characterized by a critical value \hat{x}_n which satisfies the following first order condition:*

$$\frac{\partial C(S_n, \hat{x}_n)}{\partial x_n} = E[\max\{F_{N-1} - p, E[S_N | \mathcal{I}_{N-1}]\} | \mathcal{I}_n]$$

The optimal procurement quantity, x_n^ , in period n is given by $x_n^* = \min\{\hat{x}_n, K\}$.*

2. *Processing Policy: It is optimal to not commit for processing any of the available input inventory in any period n such that $n < N - 1$. In period $N - 1$, it is optimal to commit all the available inventory for sale as processed product against the futures contract only if $F_{N-1} - p \geq E[S_N | \mathcal{I}_{N-1}]$ and not to commit anything otherwise. That is, the optimal policy for processing is given by*

$$q_n^* = \begin{cases} 0 & \text{if } n < N - 1 \\ 0 & \text{if } F_{N-1} - p < E_{\mathcal{I}_{N-1}}[S_N] \\ e_{N-1} + x_{N-1}^* & \text{if } F_{N-1} - p \geq E_{\mathcal{I}_{N-1}}[S_N] \end{cases}$$

Thus it is optimal to carry any available inventory of the input and postpone all processed product sale commitments until the last possible period when commitments can be made, i.e., period $N - 1$. This result may seem counter-intuitive and puzzling at first sight. However, maintaining the inventory as input until the last possible instance allows the firm to retain the option of trading it as either input or output. Also, since the futures prices satisfy Assumption 1, there is no decrease in the expected revenue by postponing the processing decision. As described in Plato (2001), we can consider any available inventory of the input commodity as a call option that pays the higher of the margin from processing, $F_n - p$ and the expected margin from salvaging, $E_{\mathcal{I}_n}[S_N]$. The results obtained here agree with what is known in the financial literature (see Hull (1997), Bjork (2004), for instance) - that it is optimal to postpone the exercise of a call option on a non-dividend paying stock until the last possible instance. Here, the option to process expires after period $N - 1$ and hence it is optimal to delay exercising the processing option until then.

From the analysis above, we see that the optimal policy has the following characteristics:

1. *Threshold policy in procurement:* The procurement quantity in any period is governed by a critical value determined by the convex cost of procurement. However, it is important to note that this threshold is *independent* of the current inventory level e_n .
2. *No ‘early commitment’ for processing:* Any commitment to process the input and sell the processed product is made in the last possible period to do so.
3. *‘All or nothing’ commitment:* If it is optimal to commit in the last possible period, all available inventory is committed to processing, and nothing otherwise.

Figure 3 illustrates a sample path of the input inventory and commitment profile over the horizon. The top portion shows the uncommitted input inventory and cumulative commitment for each period. The bottom portion of the figure shows the optimal procurement and commitment quantities, x_n^* and q_n^* , in every period. For instance, in period 2, procurement is up to capacity K , because the marginal benefit of an additional inventory is very high, possibly because of a high futures price realized. In the penultimate period, $N - 1$, the realized futures price is such that the margin from processing and selling the output is higher than the expected margin from selling the input itself at the end of the horizon. Therefore, all the available inventory is committed to processing and there is no inventory to trade at the end of the horizon.

Notice that if the system starts with non-negative inventory of the input, i.e. $e_1 \geq 0$, by following an optimal policy it will never reach a state where there will be a shortfall in the inventory. That

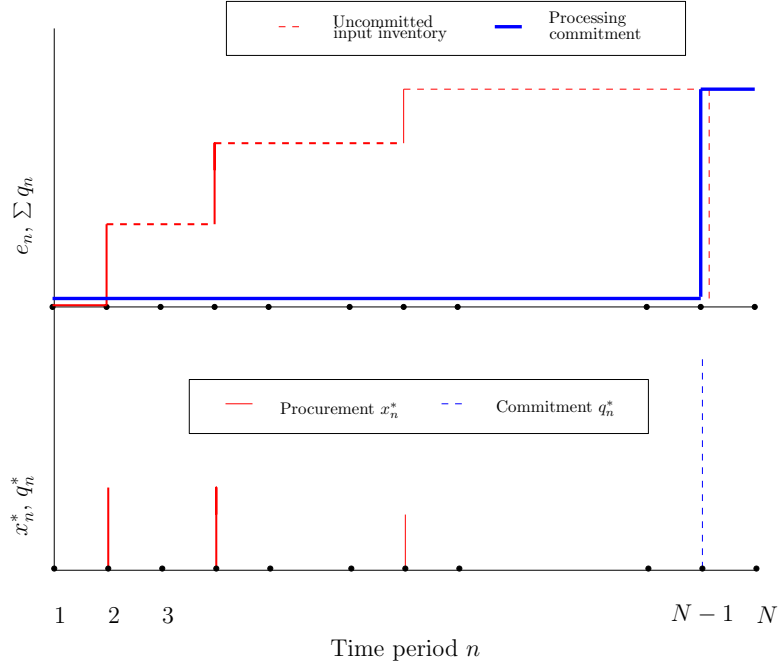


Figure 3: Sample Inventory and Commitment Profile for a Risk-neutral Firm.

is, $e_n \geq 0$ for all $n \leq N$, if $e_1 \geq 0$ under the optimal policy. (Hence, following an optimal policy, a sample path such as the one originally shown in Figure 2 would not be realized.)

3.1.1 Comparison with Full-Commitment Policy.

While firms not practicing integrated decision making can and do follow a wide variety of different operational strategies, we choose to compare against a version of the policy followed by ITC. Here, managers procure up to an optimal threshold, based on the revenues from immediate commitment, i.e., $F_n - p$ and commit all available inventory for processing immediately. We label this the ‘full-commitment’ policy.

In a full-commitment policy, in every-period, we have $q_n^{fc} = e_n + x_n^{fc}$ for all $n < N$, where q_n^{fc} and x_n^{fc} are the commitment and procurement quantities under the full-commitment policy. The problem de-couples into $N - 1$ single period problems and the procurement quantity in each period, x_n^{fc} is given by $x_n^{fc} = \min\{\tilde{x}_n, K\}$ where \tilde{x}_n is given by $\frac{\partial C(S_n, \tilde{x}_n)}{\partial x_n} = F_n - p$. Since $C(S_n, x_n)$ is convex in x_n and $\tilde{x}_n < \hat{x}_n$ where \hat{x}_n is as defined in Theorem 2, we have $x_n^{fc} \leq x_n^*$.

The marginal benefit of a unit of inventory under the full-commitment policy is equal to $F_n - p$, while it is equal to $E_{\mathcal{I}_n}[\max\{F_{N-1} - p, E_{\mathcal{I}_{N-1}}[S_N]\}]$ under the optimal policy. Thus, the benefits

from the optimal policy over the full-commitment policy accrue from (a) higher marginal benefit for every unit of inventory and (b) higher procurement quantity in every period.

3.1.2 Special Case: Constant Marginal Costs.

As a further illustration of our findings, consider the special case of constant marginal costs of procurement, i.e., $C(S_n, x_n) = S_n x_n$. Since the marginal benefit of a unit of inventory is not dependent on the procurement cost structure (when we start from non-negative inventory of the input), the marginal benefit of inventory in any period n is still given by $E_{\mathcal{I}_n}[\max\{F_{N-1} - p, E_{\mathcal{I}_{N-1}}[S_N]\}]$. The optimal processing policy remains the same as the one described in Theorem 2. The procurement policy is much simpler and is given by an ‘all or nothing’ policy; that is, if $E_{\mathcal{I}_n}[\max\{F_{N-1} - p, E_{\mathcal{I}_{N-1}}[S_N]\}] \geq S_n$, then it is optimal to procure up to the procurement capacity K in period n and 0 otherwise.

3.1.3 General Procurement Capacities.

While we have assumed that the procurement capacity per period is a constant K , we show here that relaxing this assumption is fairly straightforward. From the analysis above, the marginal benefit of inventory is not dependent on the level of inventory. Therefore, even if the procurement capacity is not the same in every period, the optimal processing policy still remains the same. The only modification will be that the optimal procurement quantity would be given by $x_n^* = \min\{\hat{x}_n, K_n\}$ where K_n is the procurement capacity in period n and \hat{x}_n is as described in Theorem 2.

3.2 Single Futures Contract: Risk-averse Firm

Notice that the optimal policy in §3.1 requires the firm to keep the entire input inventory uncommitted till the last possible period. Thus, there is significant uncertainty in the profits realized and the firm is exposed to substantial down-side risk if prices fall. Typically, firms in the commodities business have limited appetite for such risk. In this section, we explore how the optimal policy changes when risk-aversion is incorporated as a constraint.

Risk aversion has been modeled in many different ways in the financial and agricultural economics literature; Goy (1999) provides a good discussion on different approaches to modeling risk and risk management tools that have been developed in the context of agricultural producers. There are two major approaches to modeling risk attitude: (a) Value-at-Risk (VaR), and (b) various forms of utility functions. VaR is defined as the maximum loss of value that a firm can incur

for a given confidence level and a time interval. Linsmeier and Pearson (2000) provide a discussion on the concept of VaR and describe various methods used for computing it. VaR is widely used in practice; for instance, Manfredo and Leuthold (1999) provide an analysis of VaR and its potential applications for firms involved in the procurement and processing of agricultural commodities. We also found that ITC uses a VaR measure to manage risk in their agribusiness. Based on all these factors, we choose to model risk-aversion by using a VaR constraint.

The VaR constraint is characterized by a critical level of wealth, VaR, and a probability α . The VaR constraint requires that the probability of wealth at the end of the time interval being below the critical value VaR is no more than α . In our multi-period problem, in each period optimal x_n and q_n values need to be computed which account for this critical level, given profits already accumulated from past actions (which are deterministic and known). Therefore, we have a period-specific value for the critical level, VaR_n , (which incorporates past actions and revenues) which only constrains actions taken in present and future periods.

In general, under a risk-averse probability measure, the futures price might not necessarily be equal to the expectation of the future spot price. However, Assumption 1 ($E[F_{n+1}|\mathcal{I}_n] = F_n$) is often made in the literature (see Myers and Hanson (1996), Dahlgran (2002), e.g.). Therefore, Assumption 1 continues to hold in our model as well.

Recall that $V_n(e_n, \mathcal{I}_n)$ represents the total expected profits to go from period n until the end of the horizon. If we define wealth at the end of period n as the sum of the immediate profits from actions in period n plus the total expected profits from period $n + 1$ onwards, the VaR constraint for period n can then be expressed as $\mathbf{P}\{(F_n - p)q_n - C(S_n, x_n) + V_{n+1}(e_n + x_n - q_n, \mathcal{I}_{n+1}) \leq VaR_n | \mathcal{I}_n\} \leq \alpha$, where α is the maximum allowable probability that the total wealth at the end of the period will be less than the critical level, and the probability measure is over all future realizations of spot and futures prices of the two commodities.

The dynamic programming formulation for a risk-averse firm becomes

$$V_n(e_n, \mathcal{I}_n) = \max_{(q_n \geq 0, 0 \leq x_n \leq K)} \{J_n(e_n, q_n, x_n, \mathcal{I}_n)\} \quad (4)$$

$$\text{s.t. } \mathbf{P}\{(F_n - p)q_n - C(S_n, x_n) + V_{n+1}(e_n + x_n - q_n, \mathcal{I}_{n+1}) \leq VaR_n | \mathcal{I}_n\} \leq \alpha \quad (5)$$

In the last period, since there is no uncertainty in profits, the VaR constraint is irrelevant. Thus the profit function is given by $V_N(e_N, \mathcal{I}_N) = S_N e_N$ if $e_N \geq 0$ and $-\infty$ otherwise.

A commitment to process the input and sell the output in period n gives a risk-free marginal revenue of $F_n - p$ per unit. Carrying uncommitted inventory to the end of the horizon gives an

expected marginal revenue of $E[S_N|\mathcal{I}_n]$ per unit. However, the realized marginal revenue from uncommitted inventory is uncertain and hence risky. Thus, we can consider a commitment to process as an investment in a risk-free asset while carrying uncommitted inventory of the input is analogous to investing in a risky asset.

In a financial portfolio investment problem, for a risk-averse investor, the parameters of the VaR constraint are such that investing all the available wealth into the risk-free asset will satisfy the VaR constraint (see Arzac and Bawa (1977) e.g.). In the our model, this means that committing to process all available inventory in any given period should satisfy the VaR constraint (5). Indeed, the problem would be meaningless if this were not the case, because no combination of procurement and processing quantities would meet the VaR constraint. We therefore assume that the following holds throughout this section.

Assumption 2 *The VaR constraint, equation (5), is always satisfied by committing to process all the available inventory. That is, for all $n < N$, we have $\mathbf{P}\{(F_n - p)(e_n + x_n) - C(S_n, x_n) + V_{n+1}(0, \mathcal{I}_{n+1}) \leq VaR_n|\mathcal{I}_n\} \leq \alpha$ for all $x_n \geq 0$ and all realizations of \mathcal{I}_n .*

At $n = N - 1$, the firm's problem can be formulated as

$$V_{N-1}(e_{N-1}, \mathcal{I}_{N-1}) = \max_{q_{N-1} \geq 0, 0 \leq x_{N-1} \leq K} \{J_{N-1}(e_{N-1}, q_{N-1}, x_{N-1}, \mathcal{I}_{N-1})\} \quad (6)$$

$$\text{s.t. } \mathbf{P}\{(F_{N-1} - p)q_{N-1} - C(S_{N-1}, x_{N-1}) + S_N(e_{N-1} + x_{N-1} - q_{N-1}) \leq VaR_{N-1}|\mathcal{I}_{N-1}\} \leq \alpha \quad (7)$$

Define S_N^α as follows: $\mathbf{P}\{S_N \leq S_N^\alpha|\mathcal{I}_{N-1}\} = \alpha$. That is, S_N^α is the value of S_N corresponding to the critical fractile, α . (Since the distribution of S_N is dependent on \mathcal{I}_{N-1} , the value S_N^α is a function of \mathcal{I}_{N-1} . We suppress this dependence for notational convenience.) The following theorem characterizes $V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})$ and gives the optimal policy for $n = N - 1$.

Theorem 3 *$V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})$ is concave in e_{N-1} and such that $\frac{\partial V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})}{\partial e_{N-1}} \geq F_{N-1} - p$.*

The optimal processing and procurement policy is as described below:

1. *Procurement Policy: The optimal procurement quantity is characterized by two critical values, \hat{x}_{N-1} and \tilde{x}_{N-1} , which satisfy the following first order conditions*

$$\frac{\partial C(S_{N-1}, \hat{x}_{N-1})}{\partial x_{N-1}} = F_{N-1} - p \qquad \frac{\partial C(S_{N-1}, \tilde{x}_{N-1})}{\partial x_{N-1}} = E[S_N|\mathcal{I}_{N-1}]$$

respectively, and the optimal procurement quantity is given by

$$x_{N-1}^* = \begin{cases} \min\{\hat{x}_{N-1}, K\} & \text{if } F_{N-1} - p \geq E[S_N | \mathcal{I}_{N-1}] \\ \min\{\hat{x}_{N-1}, K\} & \text{if } F_{N-1} - p < E[S_N | \mathcal{I}_{N-1}], \\ & \text{and } S_N^\alpha(e_{N-1} + x_{N-1}^*) - C(S_{N-1}, x_{N-1}^*) < VaR_{N-1} \\ \min\{\tilde{x}_{N-1}, K\} & \text{if } F_{N-1} - p < E[S_N | \mathcal{I}_{N-1}], \\ & \text{and } S_N^\alpha(e_{N-1} + x_{N-1}^*) - C(S_{N-1}, x_{N-1}^*) \geq VaR_{N-1} \end{cases}$$

2. *Processing Policy: The optimal quantity to commit for processing is given by*

$$q_{N-1}^* = \begin{cases} e_{N-1} + x_{N-1}^* & \text{if } F_{N-1} - p \geq E[S_N | \mathcal{I}_{N-1}] \\ \frac{VaR_{N-1} - [S_N^\alpha(e_{N-1} + x_{N-1}^*) - C(S_{N-1}, x_{N-1}^*)]}{(F_{N-1} - p) - S_N^\alpha} & \text{if } F_{N-1} - p < E[S_N | \mathcal{I}_{N-1}], \\ & \text{and } S_N^\alpha(e_{N-1} + x_{N-1}^*) \\ & - C(S_{N-1}, x_{N-1}^*) < VaR_{N-1} \\ 0 & \text{if } F_{N-1} - p < E[S_N | \mathcal{I}_{N-1}], \\ & \text{and } S_N^\alpha(e_{N-1} + x_{N-1}^*) \\ & - C(S_{N-1}, x_{N-1}^*) \geq VaR_{N-1} \end{cases}$$

From the above theorem, we find that in period $N - 2$, the marginal revenue from committing to process a unit of input inventory, $F_{N-2} - p$, is less than the expected marginal revenue from carrying the inventory into period $N - 1$, since $E_{\mathcal{I}_{N-2}}[\frac{\partial V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})}{\partial e_{N-1}}] \geq F_{N-2} - p$. Therefore, any commitment to process in period $N - 2$ reduces the expected profits, while increasing the certainty of the profits (committing to process results in a known revenue of $F_{N-2} - p$ for every unit committed to be sold as the output). Thus, it is optimal to commit to process only the minimum quantity required to meet the VaR constraint and carry the rest as uncommitted input inventory.

From the definition of S_N^α , we have that $\mathbf{P}\{V_N(e_N, \mathcal{I}_N) \leq S_N^\alpha e_N | \mathcal{I}_{N-1}\} = \alpha$. Therefore, $S_N^\alpha e_N$ can be interpreted as the value of $V_N(e_N, \mathcal{I}_N)$ corresponding to the α fractile. For any general period n , we define $V_n^\alpha(e_n)$ as the value of $V_n(e_n, \mathcal{I}_n)$ corresponding to the α fractile. That is, $\mathbf{P}\{V_n(e_n, \mathcal{I}_n) \leq V_n^\alpha(e_n) | \mathcal{I}_{n-1}\} = \alpha$. (As in the case of S_N^α , we suppress the dependence of $V_n^\alpha(e_n)$

on \mathcal{I}_{n-1} for notational convenience.) The next theorem describes the optimal policy for a risk-averse firm in any period $n < N - 1$.

Theorem 4 *For all $n < N - 1$ and $e_n \geq 0$, the value function $V_n(e_n, \mathcal{I}_n)$ is concave and increasing in e_n , for all realizations of \mathcal{I}_n and the marginal benefit of an unit of uncommitted inventory is always greater than or equal to the processing margin; i.e., $\frac{\partial V_n(e_n, \mathcal{I}_n)}{\partial e_n} \geq F_n - p$. The optimal procurement and processing policy is as given below*

1. *Procurement Policy: The optimal procurement quantity is characterized by two critical values, \hat{x}_n and \tilde{x}_n , which satisfy the following first order conditions*

$$\frac{\partial C(S_n, \hat{x}_n)}{\partial x_n} = F_n - p \quad \frac{\partial C(S_n, \tilde{x}_n)}{\partial x_n} = E \left[\frac{\partial V_{n+1}(e_n + \tilde{x}_n)}{\partial e_{n+1}} | \mathcal{I}_n \right]$$

respectively, and the optimal procurement quantity is given by

$$x_n^* = \begin{cases} \min\{\hat{x}_n, K\} & \text{if } V_{n+1}^\alpha(e_n + x_n^*) - C(S_n, x_n^*) < VaR_n \\ \min\{\tilde{x}_n, K\} & \text{if } V_{n+1}^\alpha(e_n + x_n^*) - C(S_n, x_n^*) \geq VaR_n \end{cases}$$

2. *Processing Policy: The optimal quantity to commit for processing is given by*

$$q_n^* = \begin{cases} \frac{VaR_n - [V_{n+1}^\alpha(e_n + \tilde{x}_n - q_n^*) - C(S_n, x_n^*)]}{(F_n - p)} & \text{if } V_{n+1}^\alpha(e_n + x_n^*) - C(S_n, x_n^*) < VaR_n \\ 0 & \text{if } V_{n+1}^\alpha(e_n + x_n^*) - C(S_n, x_n^*) \geq VaR_n \end{cases}$$

Figure 4 shows a sample path for the commitment and uncommitted input inventory profile. From the figure, in period 3 the firm needs to commit a portion of the available input inventory for processing to reduce the uncertainty in profits. Also, in the penultimate period, $N - 1$, the expected margin from trading is higher than the margin from processing. Hence the firm finds it optimal to commit only a portion of the available inventory to meet its VaR constraint and trade the rest as input at the end of the horizon. Similarly, the marginal cost of procurement is high enough in intermediate periods that there is almost zero procurement in those periods.

3.2.1 Comparison with Risk-neutral Case.

We note the following observations in comparing the findings of the risk-neutral and risk-averse cases:

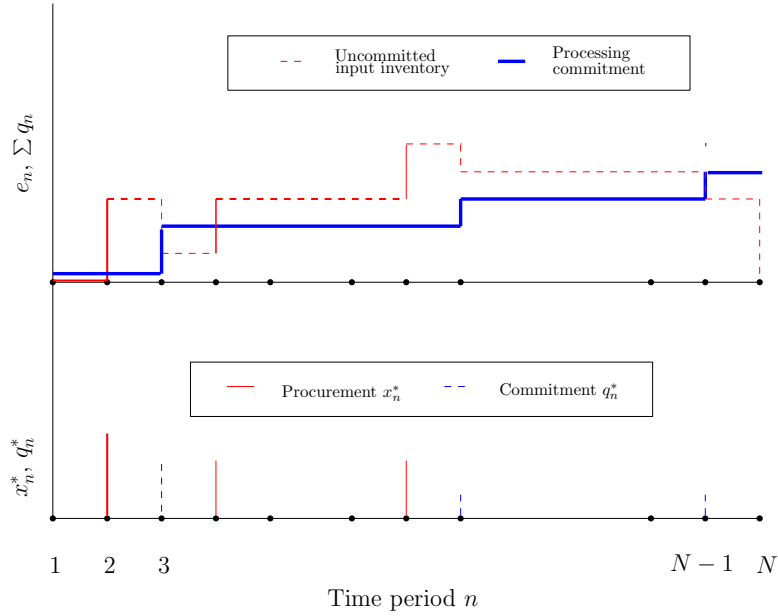


Figure 4: Sample Inventory and Commitment Profile for a Risk-averse Firm.

- The marginal benefit of carrying uncommitted inventory is always higher than the marginal benefit from committing to process in all periods for both risk-neutral and risk-averse firms. This is because of the fact that the firm retains an option to process or trade by keeping the inventory uncommitted to processing. Therefore, any commitment to process in earlier periods is purely for hedging.
- The quantity procured by a risk-neutral firm is at least as much as the quantity procured by a risk-averse firm, in every period. This result is similar to the result in inventory theory that the quantity procured by a risk-averse newsvendor with a concave utility function is less than the quantity procured by a risk-neutral newsvendor (Eeckhoudt et al. 1995).
- The quantity committed to processing by a risk-averse firm is at least as much as the quantity committed to processing by a risk-neutral firm, in every period. This is because the risk-averse firm sacrifices some of the expected profits for reducing the uncertainty in the expected profits.

3.3 Single Futures Contract: Numerical Study

In this section, we illustrate the findings and implications of the analytical models described in §3.1 and §3.2 by a numerical simulation study for a specific set of parameters. The aim of this numerical

study is two-fold:

1. *Quantify benefit from integrated decision making:* As mentioned in §1, the decision-making of the three stages of procurement, processing and trading are often done in isolation, in the literature and in practice. One of the aims of this paper is to develop an integrated decision-making policy, raising the question of how much better (w.r.t. expected profits) the integrated policy is compared to the full-commitment policy (one of the many possible non-integrated policies).
2. *Quantify the impact of the VaR constraint:* Imposing a VaR constraint on the expected profits limits the probability of making severe losses. However, this reduction in the downside comes at the cost of sacrificing some of the expected profits from waiting to commit until the end, which is the optimal risk-neutral policy. The numerical study will help quantify the impact of the VaR constraint on the expected profits and the distribution of profits by comparing the optimal policies for a risk-neutral and risk-averse firm.

3.3.1 Implementation.

The implementation study was conducted on a specific chosen set of parameters, described below. The optimal policy was calculated for each period for each combination of (e_n, S_n, F_n) over a range of values of these three quantities, for every $n = 1, 2, \dots, N$. (For the purposes of this study, \mathcal{I}_n consists of just the spot and futures prices realized in the current period.) The distribution of (S_{n+1}, F_{n+1}) , given (S_n, F_n) for every pair (S_{n+1}, F_{n+1}) was estimated using the price process described in §B in the electronic companion. The distribution thus generated was then used to estimate $E_{\mathcal{I}_n}[V_{n+1}(e_n, \mathcal{I}_{n+1})]$ and $V_{n+1}^\alpha(e_{n+1})$, given $V_{n+1}(e_{n+1}, \mathcal{I}_{n+1})$ for each combination of $(e_{n+1}, \mathcal{I}_{n+1})$, in the range. Once $E_{\mathcal{I}_n}[V_{n+1}(e_n, \mathcal{I}_{n+1})]$ and $V_{n+1}^\alpha(e_{n+1})$ are known, $V_n(e_n, \mathcal{I}_n)$, $x_n^*(e_n, \mathcal{I}_n)$ and $q_n^*(e_n, \mathcal{I}_n)$ can be calculated for each combination of (e_n, \mathcal{I}_n) using the optimality equations (4) and (5). Thus, starting with $V_N(e_N, \mathcal{I}_N) = S_N e_N$, the quantities $V_n(e_n, \mathcal{I}_n)$, $x_n^*(e_n, \mathcal{I}_n)$ and $q_n^*(e_n, \mathcal{I}_n)$ were estimated for each value of (e_n, \mathcal{I}_n) in the range.

Once the policy parameters $x_n^*(e_n, \mathcal{I}_n)$ and $q_n^*(e_n, \mathcal{I}_n)$ were calculated, forward simulation runs were implemented. Let $\Pi(e_1, \mathcal{I}_1)$ denote the profit over the entire horizon, for one sample path, starting from an initial state of (e_1, \mathcal{I}_1) . The expectation of $\Pi(e_1, \mathcal{I}_1)$ over multiple sample paths gives $V_1(e_1, \mathcal{I}_1)$. The forward simulation runs were conducted in the following manner.

1. Set $\Pi(e_1, \mathcal{I}_1) = 0$.

2. For $n \neq N$, for a starting value of (e_n, \mathcal{I}_n) , choose x_n^* and q_n^* .
3. Update $e_{n+1} = e_n + x_n^* - q_n^*$ and $n = n + 1$ and $\Pi(e_1, \mathcal{I}_1) = \Pi(e_1, \mathcal{I}_1) + (F_n - p)q_n^* - C(S_n, x_n^*)$.
4. For the given values of (S_n, F_n) , generate the next period prices (S_{n+1}, F_{n+1}) .
5. Repeat steps 2 to 4 until $n = N$.
6. For $n = N$, set $\Pi(e_1, \mathcal{I}_1) = \Pi(e_1, \mathcal{I}_1) + S_N e_N$ and stop.

The optimal policies were computed for a horizon with $N = 5$ periods. With each period in the model corresponding to 15 real days, a horizon of $N = 5$ models a significant portion of the procurement season. The procurement capacity in each period, K , was normalized to 1 unit. The uncommitted inventory levels, e , ranged from 0 to $N * K = 5$, in steps of 0.1. The processing cost was set to $p = 5$ per-unit. A procurement cost function, $C(S_n, x_n) = S_n x_n^{1.5}$ was used.

For scaling purposes, in the numerical study, the long term equilibrium value of the input spot price was set to 25. Correspondingly, the long run equilibrium of the output was scaled to 31. With these values, policies were computed over a input spot price range of $[10, 40]$ and output futures price range of $[11, 51]$, both in increments of 0.25. These limits were chosen such that the realized spot and futures prices over the horizon would fall within the range 95% of the time. Observe also that a processing cost of $p = 5$ corresponds to an expected processing margin of 1 (approximately 3.3%) which allows us to model situations when the actual realized processing margin may be large, marginal or negative.

While this numerical study is based on specific values, we note that it is for illustration only. If different parameters are chosen, the analysis in §3.1 and §3.2 shows that the broad conclusions (greater profit with integrated decision-making, and risk-reward tradeoff with incorporation of risk-aversion) will continue to hold; only their magnitudes will change.

3.3.2 Benefit from Integrated Optimization.

Optimal policies were calculated for the risk-neutral case and a total of 10,000 simulation runs were conducted using the optimal policy parameters generated. To exclude boundary effects, only those simulation runs where the maximum futures price realized across the horizon is less than or equal to 42 were considered from these simulation runs (At the boundary of the range of futures price, Assumption 1 is violated and hence the optimal policies and the simulation run results corresponding to these boundary values were not considered.). Similar, independent simulation

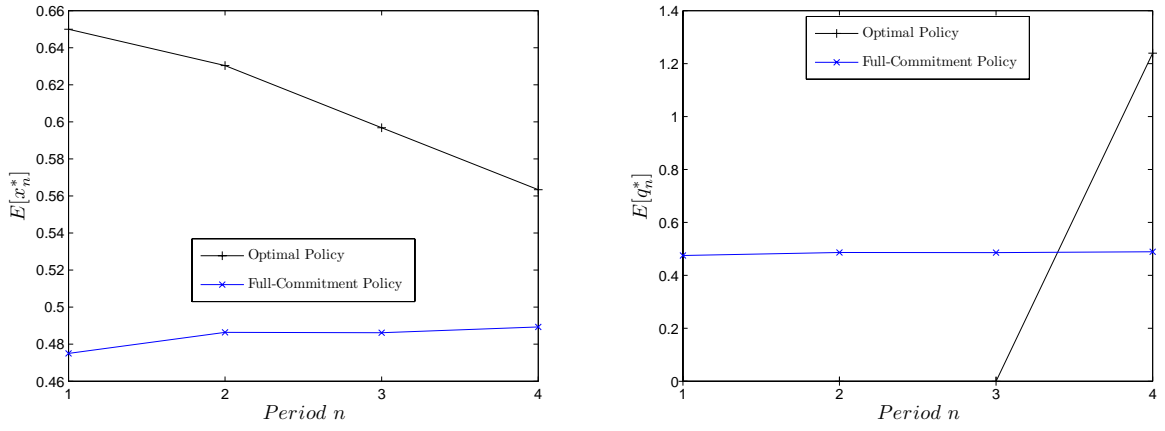


Figure 5: Comparison of expected optimal procurement (left figure) and expected optimal commitment (right figure) under optimal and full-commitment policies respectively.

Case	Expected Profits	Std Dev. of Profits
Optimal Policy	23.5094	12.0700
Full-Commitment	18.0173	7.4411

Table 1: Benefits from Optimal Policy.

runs were conducted for the ‘full-commitment’ policy. The results from the simulation runs for the two policies are as given in Table 1.

We see that the benefits from following the optimal policy is close to 30% in terms of expected profits for a risk-neutral firm. However, this does not mean that the optimal policy performs better than the ‘full-commitment’ policy along every sample path; for example, if the futures and spot prices in period $N - 1$ fall significantly, the inventory procured till $N - 1$ is sunk and the total profits in this case are lower for the optimal policy. Because of the large number of simulation runs ($\sim 10,000$), the difference in the expected profits over the horizon is statistically significant (p-value ~ 0), in spite of the high standard deviation.

This improvement in expected profits is accompanied by an increased uncertainty, as is evident from the standard deviation of profits for the two policies. The full-commitment policy is an extremely risk-averse policy, since all the procured stock is processed immediately and no open inventory is carried. Thus, the difference between the expected profits is the risk-premium associated

VaR	Expected Profits	Std Dev. of Profits
Risk-Neutral	23.5094	12.0700
15	23.4109	12.2971
20	18.1625	7.7952
25	18.0478	7.3994
Full-Commitment	18.0173	7.4411

Table 2: Impact of the VaR constraint.

with the optimal policy. The expected procurement and commitment quantities for each period under the two policies are shown in Figure 5.

In period $N - 1 = 4$, which is the penultimate period, the procurement under the optimal and full-commitment policy would be the same for all realizations of \mathcal{I}_{N-1} such that $F_{N-1} - p \geq E_{\mathcal{I}_{N-1}}[S_N]$. Therefore, the difference in the expected procurement quantities is minimum in this period. Also, both policies would commit all available inventory for processing under these conditions. However, because of higher procurement in earlier periods and the availability of additional inventory, the expected commitments in period $N - 1$ are higher under the optimal policy.

3.3.3 Impact of the VaR Constraint.

Based on the expected value and standard deviation of the total profits for a risk-neutral firm, different values of the critical profit level, VaR , were set, along with $\alpha = 0.05$, and optimal policies generated for those cases.⁴ Simulation runs were then conducted in a manner similar to that described in §3.3.2. The results from the simulation runs for the various cases are summarized in Table 2.

Since the the coefficient of variation of profits is quite high in the risk-neutral case, ($\sigma/\mu = 0.513$), imposing a VaR constraint with a low critical value ($VaR = 15$), does not seem to affect the distribution of profits significantly. As expected, as the VaR value is increased, the expected profits, along with the standard deviation, also decrease.

The expected procurement and commitment quantities for the different cases are shown in Figure 6 below. As described in §3.2, the expected procurement decreases as VaR increases.

⁴For each of the cases, the critical level, VaR_n , was set to a common value in each period, denoted by VaR .

Conversely, the expected commitment to processing increases as VaR increases.

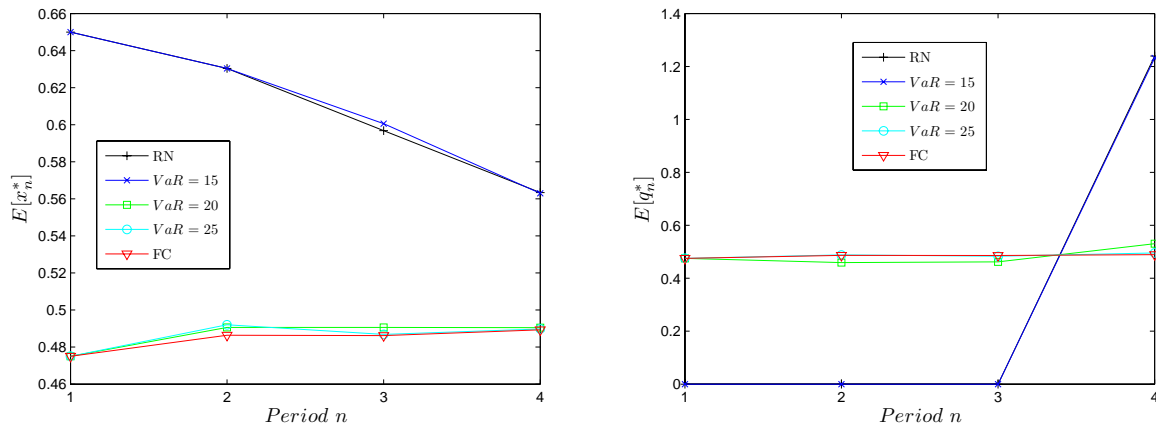


Figure 6: Comparison of expected optimal procurement (left figure) and expected optimal commitment quantities (right figure) for different values of VaR .

4 Multiple Futures Contracts: Risk-neutral Firm

While we have focused so far on a single futures contract expiring at the end of the horizon, in reality, there are futures contracts with different maturities that are traded at the same time and futures contracts that are traded in different markets/exchanges. Therefore, in any period, the firm could choose any one of the many futures contracts available to sell the output. At the outset, it is not clear how this affects the firm's optimal trading strategy. In this section, we explore how the availability of multiple futures contracts affects the optimal policies of a risk-neutral firm.

Let there be L futures contracts available for the processed product, with contract i expiring in period N_i . Without loss of generality, we assume that the contracts have different maturities and index the contracts such that $N = N_L > N_{L-1} > \dots > N_2 > N_1 > 1$. Let F_n^i denote the period n futures price on contract i , for all $n < N_i$. We assume that the futures price on each contract satisfies the conditions of Assumption 1.

Consider any period n such that $N_{L-1} \leq n < N$. At this time, only the futures contract with maturity in period N is traded. From the discussion in §3.1 we know that it is optimal not to make any output sale commitments against futures contract L for any $n < N_L - 1$. In period $n = N_{L-1} - 1$, it would still be optimal not to make any output sale commitments against the

futures contract L , which matures in period N . However, there is also another futures contract that expires in period N_{L-1} that is available, against which potential commitments could be made.

The marginal benefit of committing a unit of available inventory of the input against the futures contract maturing at N_{L-1} is $F_{N_{L-1}-1}^{L-1} - p$. The marginal benefit of carrying this inventory to the next period is given by $E_{\mathcal{I}_{N_{L-1}-1}}[\max\{F_{N-1}^L - p, E_{\mathcal{I}_{N-1}}[S_N]\}]$. Therefore, it is optimal to commit to sell the processed product against the futures contract maturing at N_{L-1} only when $F_{N_{L-1}-1}^{L-1} - p \geq E_{\mathcal{I}_{N_{L-1}-1}}[\max\{F_{N-1}^L - p, E_{\mathcal{I}_{N-1}}[S_N]\}]$. Unlike the single-contract case, it is possible that $F_{N_{L-1}-1}^{L-1} - p \geq E_{\mathcal{I}_{N_{L-1}-1}}[\max\{F_{N-1}^L - p, E_{\mathcal{I}_{N-1}}[S_N]\}]$, as we are comparing the futures price on contracts with different maturities, possibly traded on different exchanges. Thus, it may be optimal to commit against a futures contract for $n < N - 1$.

The presence of multiple futures contracts thus affects the marginal benefit of a unit of inventory. However, the optimal procurement quantity would still be governed by a threshold such that the marginal cost of procurement at the threshold is equal to the marginal benefit from an additional unit of inventory. As in the single futures case, this threshold would not depend on the current level of inventory and the optimal procurement quantity would be the lesser of this threshold value, \hat{x}_n , and the procurement capacity, K , in any period n .

The next theorem formalizes the intuition described in the previous paragraphs and shows that it is never optimal to commit against a futures contract i in a period n for which $n \neq N_i - 1$. We assume that $e_1 \geq 0$; that is, the initial uncommitted inventory of the input commodity is non-negative. Define

$$M_n = E_{\mathcal{I}_n}[\max\{F_{N_{i+1}-1}^{i+1} - p, E_{\mathcal{I}_{N_{i+1}-1}}[\max\{\dots \max\{F_{N-1}^L - p, E_{\mathcal{I}_{N-1}}[S_N]\} \dots\}]\}]$$

Theorem 5 *When multiple futures contracts of the output commodity are available, the marginal benefit of inventory is given by $\max\{F_{N_i-1}^i - p, E_{\mathcal{I}_{N_i-1}}[M_{N_i}]\}$ when $n = N_i - 1$ for some i and M_n otherwise. The optimal policies are as described below:*

1. *Procurement Policy: The optimal procurement quantity is characterized by a critical value \hat{x}_n which satisfies the following first order condition*

$$\frac{\partial C(S_n, \hat{x}_n)}{\partial x_n} = \begin{cases} M_n & \text{if } N_i \leq n < N_{i+1} - 1 \\ \max\{F_{N_i-1}^i - p, E_{\mathcal{I}_{N_i-1}}[M_{N_i}]\} & \text{if } n = N_i - 1 \end{cases}$$

and the optimal procurement quantity x_n^ is given by $x_n^* = \min\{\hat{x}_n, K\}$.*

2. *Processing Policy:* The optimal quantity of output to process and sell against any futures contract i is given by

$$q_n^{i*} = \begin{cases} 0 & \text{if } n \neq N_i - 1 \\ 0 & \text{if } n = N_i - 1 \text{ and } F_{N_i-1}^i - p < E_{\mathcal{I}_{N_i-1}}[M_{N_i}] \\ e_n + x_n^* & \text{if } n = N_i - 1 \text{ and } F_{N_i-1}^i - p \geq E_{\mathcal{I}_{N_i-1}}[M_{N_i}] \end{cases}$$

Notice that $E_{\mathcal{I}_{N_i-1}}[M_{N_i}] \geq E_{\mathcal{I}_{N_i-1}}[F_{N_j-1}^j - p]$ for all $j = i + 1, i + 2, \dots, L$. Therefore, the condition in the above theorem includes the intuitive condition that it is never optimal to commit to a futures contract i when there exists a contract j , with $N_j > N_i$ and $F_{N_i-1}^j > F_{N_i-1}^i$.

M_n , the expected marginal benefit of inventory in period n , where $N_i \leq n < N_{i+1} - 1$, is analogous to the marginal benefit of inventory in the case of a single futures contract given by $E_{\mathcal{I}_n}[\max\{F_{N-1} - p, E_{\mathcal{I}_{N-1}}[S_N]\}]$. This marginal benefit of inventory accounts for the fact that there are multiple *processing margins* (through the multiple futures contracts) available. As in the case with a single futures contract, the expected marginal benefit of a unit of inventory is at least as much as the marginal benefit from committing to sell the output using *any* of the remaining futures contracts.

4.1 Comparison with Single Futures Case

The multiple futures contracts case can be treated as one where each unit of inventory is a call option on *all* the processing margins from futures contracts that are yet to expire and the salvage margin. Looking at it from this perspective, the optimal procurement and processing policy is a direct extension of the result in §3.1 for a single futures contract and exhibit similar properties as in the single futures case.

It is optimal to make a commitment against a specific futures contract only if the processing margin from the contract is at least as much as the maximal expected benefit from all the futures contracts that are yet to expire and the salvage value; i.e., the value from exercising the option is at least as much as the value from waiting. The value from waiting is nothing but the marginal benefit of inventory, $E_{\mathcal{I}_{N_i-1}}[M_{N_i}]$, which is at least as much as $\max\{F_{N_i-1}^j - p\}$ for all $j = i + 1, i + 2, \dots, L$. Therefore we see that the simple condition, $F_{N_i-1}^i \geq F_{N_i-1}^j$, is not enough to commit against futures contract i in period $N_i - 1$.

5 Conclusions

In this paper we study the integrated procurement, processing and trading decisions for a firm dealing in commodities. We first analyzed the case when a risk-neutral firm has a single futures contract available for trading the processed product. We find that the available inventory of the input commodity can be considered as a call option on the processing margin and the optimal policy is to postpone the decision to process and sell the output until the last possible period in which the output can be sold. When the procurement costs are convex, the optimal procurement policy is governed by a threshold that is independent of the current inventory level and it is optimal to procure up to the threshold quantity.

We then considered a risk-averse firm which has a Value-at-Risk (VaR) constraint. In this case, we find that the risk-averse firm finds it optimal to trade some portion of the available inventory as processed product in every period. The quantity to process is dependent on the starting inventory level and the processing decision is purely for managing risk and satisfying the VaR constraint. The procurement policy is again governed by a threshold value, but the threshold is not necessarily independent of the starting inventory levels. Moreover, the quantity procured in any period is no greater than that procured by a risk-neutral firm.

Finally, we look at optimal policies for a risk-neutral firm when there are multiple futures contract with different maturities that are traded. The optimal policies in this case are very similar to those in the single futures case. We find it is optimal to postpone selling the output against any futures contract until the last period in which a sale can be made against that contract. The optimal procurement policy is again a threshold policy, where the procurement threshold is independent of the starting inventory level.

In summary, we find that incorporating ideas from the financial world into operational problems provides significant insights in the analysis of integrated problems such as the one we consider. Furthermore, this yields significant managerial insights and decision support tools to improve performance in a variety of contexts. For example, our finding that it is optimal for a risk-neutral firm to postpone the processing and sale of the output until the last possible opportunity provides a practical guideline for managerial decision making. Similarly, managers can adopt operational strategies to manage risk, supplementing financial risk management.

5.1 Future Research

While this paper provides a start towards analyzing integrated decision making for firms involved in the commodities business, there is much work that remains. We highlight some of the open questions and future research directions in which the results in the paper can be extended. In particular, the cases of a risk-averse firm trading in multiple futures contracts as well incorporating finite processing capacities is work in progress.

Agricultural input commodities like soybean and corn are grown in farms spread over large geographic areas, the firm's processing capabilities may be in factories in fixed locations, and the output commodity(ies) may need to be delivered to specific locations such as ports of trade. The opportunity to maximize profit is affected by the network characteristics such as distances, capacities and transportation times and the firm needs to consider the network effects while deciding on its procurement and processing decision.

This paper only considers a single input commodity being processed into a single output commodity. In some industries, the firm can choose what output to process the input commodity into. For example, corn can be processed into ethanol or cornmeal. In a similar manner, the firm might have a choice in terms of the input commodity. We believe our research has the potential for spurring further research into these and other related problems.

6 Acknowledgment

The authors gratefully acknowledge the collaboration and support of The ITC Group. Their generous hospitality and access to key personnel and data greatly enhanced this research.

References

- Anderson, R.W., J. Danthine. 1995. Hedging and joint production: Theory and illustrations. *The Journal of Finance* **35**(2) 487–498.
- Anupindi, R., S. Sivakumar. 2006. Supply chain re-engineering in agri-business: A case study of ITC's e-choupal. Hau L. Lee, Chung-Yee Lee, eds., *Supply Chain Issues in Emerging Economies*. Elsevier-Springer.
- Arzac, E.R., V.S. Bawa. 1977. Portfolio choice and equilibrium in capital markets with safety-first investors. *Journal of Financial Economics* **4** 277–288.
- Bjork, T. 2004. *Arbitrage Theory in Continuous Time*. Oxford University Press, New York.

- Caldentey, R., M. Haugh. 2006. Optimal control and hedging of operations in the presence of financial markets. Working Paper, Stern School of Business, New York University.
- Chowdhry, B., J.T.B. Howe. 1999. Corporate risk management for multinational corporations: Financial and operational hedging policies. *European Finance Review* **2** 229–246.
- Dahlgran, R.A. 2002. Inventory and transformation risks in soybean processing. Paper presented at the NCR-134 Conference on Applied Commodity Price Analysis, Forecasting, and Market Risk Management.
- Eeckhoudt, L., C. Gollier, H. Schlesinger. 1995. The risk-averse (and prudent) newsboy. *Management Science* **41**(5) 786–794.
- Gaur, V., S. Seshadri. 2005. Hedging inventory risk through market instruments. *Manufacturing and Service Operations Management* **7**(2) 103–120.
- Geman, H. 2005. *Commodities and Commodity Derivatives: Modeling and Pricing for Agriculturals, Metals and Energy*. Wiley.
- Gibson, R., E.S. Schwartz. 1990. Stochastic convenience yield and the pricing of oil contingent claims. *Journal of Finance* **45**(3) 959–976.
- Goel, A., G.J. Gutierrez. 2006. Integrating commodity markets in the optimal procurement policies of a stochastic inventory system. Working Paper, Management Department, University of Texas at Austin.
- Goy, B.A. 1999. Essays on agricultural risk management and agribusiness marketing. Ph.D. thesis. Purdue University.
- Hull, J.C. 1997. *Options, Futures and Other Derivatives*. Prentice Hall, Upper Saddle River, New Jersey.
- Linsmeier, T.J., N.D. Pearson. 2000. Value at risk. *Financial Analysts Journal* **56**(2) 47–67.
- Manfredo, M.R., R.M. Leuthold. 1999. Value-at-risk analysis: A review and the potential for agricultural applications. *Review of Agricultural Economics* **21**(1) 99–111.
- Myers, R.J., S.D. Hanson. 1996. Optimal dynamic hedging in unbiased futures markets. *American Journal of Agricultural Economics* **78**(1) 13–20.
- Pindyck, R.S. 2001. The dynamics of commodity spot and futures markets: A primer. *Energy Journal* **22**(3) 1–29.
- Plato, G. 2001. The soybean processing decision: Exercising a real option on processing margins. Electronic Report from the Economic Research Service, Technical Bulletin Number 1897, United States Department of Agriculture.
- Routledge, B.R., D.J. Seppi, C.S. Spatt. 2000. Equilibrium forward curves for commodities. *Journal of Finance* **55**(3) 1297–1338.
- Routledge, B.R., D.J. Seppi, C.S. Spatt. 2001. The spark spread: An equilibrium model of cross-commodity price relationships in electricity. Working Paper, Tepper School of Business, Carnegie Mellon University.

- Schwartz, E., J.E. Smith. 2000. Short-term variations and long-term dynamics in commodity prices. *Management Science* **46**(7) 893–911.
- Tsang, D., R.M. Leuthold. 1990. Hedge ratios under inherent risk reduction in a commodity complex. *The Journal of Futures Markets* **10**(5) 497–504.
- Working, H. 1949. The theory of price of storage. *American Economic Review* **39**(6) 1254–1262.
- Working, H. 1953. Futures trading and hedging. *American Economic Review* **43**(3) 314–343.
- Zhu, W., R. Kapuscinski. 2006. Optimal operational versus financial hedging for a risk-averse firm. Working Paper, Ross School of Business, University of Michigan.

A Proofs of Theorems

Theorem 1 *In period $N - 1$, for $e_{N-1} \geq 0$, the optimal policy is as follows.*

1. *Procurement: The procurement decision is characterized by two critical values, \hat{x}_{N-1} and \tilde{x}_{N-1} which satisfy the following first order conditions.*

$$\frac{\partial C(S_{N-1}, \hat{x}_{N-1})}{\partial x_{N-1}} = F_{N-1} - p \qquad \frac{\partial C(S_{N-1}, \tilde{x}_{N-1})}{\partial x_{N-1}} = E[S_N | \mathcal{I}_{N-1}]$$

The optimal quantity to procure, x_{N-1}^ , is then given by*

$$x_{N-1}^* = \begin{cases} \min\{\hat{x}_{N-1}, K\} & \text{if } F_{N-1} - p \geq E[S_N | \mathcal{I}_{N-1}] \\ \min\{\tilde{x}_{N-1}, K\} & \text{if } F_{N-1} - p < E[S_N | \mathcal{I}_{N-1}] \end{cases}$$

2. *Processing: It is optimal to commit to sell the processed product against the futures contract if and only if the processing margin is greater than the trade margin; i.e.,*

$$q_{N-1}^* = \begin{cases} e_{N-1} + x_{N-1}^* & \text{if } F_{N-1} - p \geq E_{\mathcal{I}_{N-1}}[S_N] \\ 0 & \text{if } F_{N-1} - p < E_{\mathcal{I}_{N-1}}[S_N] \end{cases}$$

Furthermore, $V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})$ can be expressed as

$$V_{N-1}(e_{N-1}, \mathcal{I}_{N-1}) = \max\{F_{N-1} - p, E[S_N | \mathcal{I}_{N-1}]\} \cdot e_{N-1} + B_{N-1}$$

$$\text{where } B_{N-1} = \begin{cases} (F_{N-1} - p)x_{N-1}^* - C(S_{N-1}, x_{N-1}^*) & \text{if } F_{N-1} - p \geq E[S_N | \mathcal{I}_{N-1}] \\ E[S_N | \mathcal{I}_{N-1}]x_{N-1}^* - C(S_{N-1}, x_{N-1}^*) & \text{if } F_{N-1} - p < E[S_N | \mathcal{I}_{N-1}] \end{cases}$$

Proof: By substituting for $V_N(e_N, \mathcal{I}_N)$, we have

$$V_{N-1}(e_{N-1}, \mathcal{I}_{N-1}) = \max_{q_{N-1} \geq 0, 0 \leq x_{N-1} \leq K} \{(F_{N-1} - p)q_{N-1} - C(S_{N-1}, x_{N-1}) + E_{\mathcal{I}_{N-1}}[S_N(e_{N-1} + x_{N-1} - q_{N-1})]\} \quad (8)$$

Observe that the expression to be optimized above is linear in q_{N-1} . Therefore, for a given x_{N-1} , the contribution of q_{N-1} is maximized at the boundary; that is, the optimal processing decision is given by

$$q_{N-1}^* = \begin{cases} e_{N-1} + x_{N-1} & \text{if } F_{N-1} - p \geq E_{\mathcal{I}_{N-1}}[S_N] \\ 0 & \text{if } F_{N-1} - p < E_{\mathcal{I}_{N-1}}[S_N] \end{cases} \quad (9)$$

Thus, (8) can be written as

$$V_{N-1}(e_{N-1}, \mathcal{I}_{N-1}) = \begin{cases} \max_{x_{N-1}} \{(F_{N-1} - p)e_{N-1} + (F_{N-1} - p)x_{N-1} - C(S_{N-1}, x_{N-1})\} & \text{if } F_{N-1} - p \geq E_{\mathcal{I}_{N-1}}[S_N] \\ \max_{x_{N-1}} \{E_{\mathcal{I}_{N-1}}[S_N]e_{N-1} + E_{\mathcal{I}_{N-1}}[S_N]x_{N-1} - C(S_{N-1}, x_{N-1})\} & \text{if } F_{N-1} - p < E_{\mathcal{I}_{N-1}}[S_N] \end{cases}$$

Because of the convexity of $C(S_{N-1}, x_{N-1})$, both the functions to be maximized above are concave in x_{N-1} and have unique maximizers, \hat{x}_{N-1} and \tilde{x}_{N-1} , satisfying the respective first order conditions.

The rest of the theorem follows from the above results. \square

Theorem 2 *The value function $V_n(e_n, \mathcal{I}_n)$ for $n \leq N - 1$ is linear for all $e_n \geq 0$. Moreover, the marginal benefit of an additional unit of inventory is at least $F_n - p$ for all $e_n \geq 0$.*

In any period $n < N$, the optimal procurement and processing decisions are as described below:

1. *Procurement Policy: The optimal procurement policy is characterized by a critical value \hat{x}_n which satisfies the following first order condition:*

$$\frac{\partial C(S_n, \hat{x}_n)}{\partial x_n} = E[\max\{F_{N-1} - p, E[S_N | \mathcal{I}_{N-1}]\} | \mathcal{I}_n]$$

The optimal procurement quantity, x_n^ , in period n is given by $x_n^* = \min\{\hat{x}_n, K\}$.*

2. *Processing Policy: It is optimal to not commit for processing any of the available input inventory in any period n such that $n < N - 1$. In period $N - 1$, it is optimal to commit*

all the available inventory for sale as processed product against the futures contract only if $F_{N-1} - p \geq E[S_N | \mathcal{I}_{N-1}]$ and not to commit anything otherwise. That is, the optimal policy for processing is given by

$$q_n^* = \begin{cases} 0 & \text{if } n < N - 1 \\ 0 & \text{if } F_{N-1} - p < E_{\mathcal{I}_{N-1}}[S_N] \\ e_{N-1} + x_{N-1}^* & \text{if } F_{N-1} - p \geq E_{\mathcal{I}_{N-1}}[S_N] \end{cases}$$

Proof: We prove the theorem by induction. As the basis for induction, we know from Theorem 1 that the above holds for $n = N - 1$. Suppose it is true for period $n + 1 \leq N - 1$. Then, V_{n+1} can be expressed in linear form as follows:

$$V_{n+1}(e_{n+1}, \mathcal{I}_{n+1}) = A_{n+1}e_{n+1} + B_{n+1} \quad (10)$$

where A_{n+1} and B_{n+1} are functions of \mathcal{I}_{n+1} alone and $A_{n+1} \geq F_{n+1} - p$.

From (3), we have

$$V_n(e_n, \mathcal{I}_n) = \max_{q_n \geq 0, 0 \leq x_n \leq K} \{L_n(e_n, q_n, x_n, \mathcal{I}_n)\}$$

where $L_n(e_n, q_n, x_n, \mathcal{I}_n) = (F_n - p)q_n - C(S_n, x_n) + E_{\mathcal{I}_n}[A_{n+1}(e_n + x_n - q_n) + B_{n+1}]$.

Consider the coefficient of q_n in $L_n(\cdot)$. We have

$$\begin{aligned} F_n - p - E_{\mathcal{I}_n}[A_{n+1}] &\leq F_n - p - E_{\mathcal{I}_n}[F_{n+1} - p] \\ &= F_n - p - F_n - p \\ &= 0 \end{aligned}$$

The inequality above is by the induction hypothesis and the first equality due to Assumption 1. Therefore, it is optimal to have $q_n = 0$. We can then write the maximization as

$$V_n(e_n, \mathcal{I}_n) = \max_{x_n} \{E_{\mathcal{I}_n}[A_{n+1}](e_n + x_n) + E_{\mathcal{I}_n}[B_{n+1}] - C(S_n, x_n)\}$$

The function to be maximized is concave in x_n and has a unique maximizer, \hat{x}_n which satisfies the first order condition,

$$\frac{\partial C(S_n, \hat{x}_n)}{\partial x_n} = E_{\mathcal{I}_n}[A_{n+1}] \quad (11)$$

The optimal procurement quantity is therefore given by $x_n^* = \min\{\hat{x}_n, K\}$. Substituting, we get

$$V_n(e_n, \mathcal{I}_n) = E_{\mathcal{I}_n}[A_{n+1}]e_n + E_{\mathcal{I}_n}[A_{n+1}]x_n^* + E_{\mathcal{I}_n}[B_{n+1}] - C(S_n, x_n^*) \quad (12)$$

We see that $V_n(e_n, \mathcal{I}_n)$ is linear with $A_n = E_{\mathcal{I}_n}[A_{n+1}]$ and $B_n = E_{\mathcal{I}_n}[A_{n+1}]x_n^* + E_{\mathcal{I}_n}[B_{n+1}] - C(S_n, x_n^*)$. Observe also that $A_n = E_{\mathcal{I}_n}[A_{n+1}] \geq E_{\mathcal{I}_n}[F_{n+1} - p] = F_n - p$. Hence by induction, the theorem is proved. \square

Theorem 3 $V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})$ is concave in e_{N-1} and such that $\frac{\partial V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})}{\partial e_{N-1}} \geq F_{N-1} - p$.

The optimal processing and procurement policy is as described below:

1. *Procurement Policy: The optimal procurement quantity is characterized by two critical values, \hat{x}_{N-1} and \tilde{x}_{N-1} , which satisfy the following first order conditions*

$$\frac{\partial C(S_{N-1}, \hat{x}_{N-1})}{\partial x_{N-1}} = F_{N-1} - p \quad \frac{\partial C(S_{N-1}, \tilde{x}_{N-1})}{\partial x_{N-1}} = E[S_N | \mathcal{I}_{N-1}]$$

respectively, and the optimal procurement quantity is given by

$$x_{N-1}^* = \begin{cases} \min\{\hat{x}_{N-1}, K\} & \text{if } F_{N-1} - p \geq E[S_N | \mathcal{I}_{N-1}] \\ \min\{\hat{x}_{N-1}, K\} & \text{if } F_{N-1} - p < E[S_N | \mathcal{I}_{N-1}], \\ & \text{and } S_N^\alpha(e_{N-1} + x_{N-1}^*) - C(S_{N-1}, x_{N-1}^*) < VaR_{N-1} \\ \min\{\tilde{x}_{N-1}, K\} & \text{if } F_{N-1} - p < E[S_N | \mathcal{I}_{N-1}], \\ & \text{and } S_N^\alpha(e_{N-1} + x_{N-1}^*) - C(S_{N-1}, x_{N-1}^*) \geq VaR_{N-1} \end{cases}$$

2. *Processing Policy: The optimal quantity to commit for processing is given by*

$$q_{N-1}^* = \begin{cases} e_{N-1} + x_{N-1}^* & \text{if } F_{N-1} - p \geq E[S_N | \mathcal{I}_{N-1}] \\ \frac{VaR_{N-1} - [S_N^\alpha(e_{N-1} + x_{N-1}^*) - C(S_{N-1}, x_{N-1}^*)]}{(F_{N-1} - p) - S_N^\alpha} & \text{if } F_{N-1} - p < E[S_N | \mathcal{I}_{N-1}], \\ & \text{and } S_N^\alpha(e_{N-1} + x_{N-1}^*) \\ & - C(S_{N-1}, x_{N-1}^*) < VaR_{N-1} \\ 0 & \text{if } F_{N-1} - p < E[S_N | \mathcal{I}_{N-1}], \\ & \text{and } S_N^\alpha(e_{N-1} + x_{N-1}^*) \\ & - C(S_{N-1}, x_{N-1}^*) \geq VaR_{N-1} \end{cases}$$

Proof: We examine two different cases, according to the trade and processing margins:

Case (i): $F_{N-1} - p \geq E[S_N | \mathcal{I}_{N-1}]$

In this case, the margin from processing and selling the output is higher than the expected margin

from salvaging the input. Also, the margin from committing to process has no uncertainty and hence eliminates all risk. Therefore, it is optimal for the firm to commit to process all available inventory, i.e.,

$$q_{N-1}^* = e_{N-1} + x_{N-1}^* \quad (13)$$

where x_{N-1}^* is the optimal quantity of the input to procure. Substituting in (6) we get

$$V_{N-1}(e_{N-1}, \mathcal{I}_{N-1}) = \max_{0 \leq x_{N-1} \leq K} \{(F_{N-1} - p)(e_{N-1} + x_{N-1}) - C(S_{N-1}, x_{N-1})\}$$

which gives $x_{N-1}^* = \min\{\hat{x}_{N-1}, K\}$, where \hat{x}_{N-1} satisfies

$$\frac{\partial C(S_{N-1}, \hat{x}_{N-1})}{\partial x_{N-1}} = F_{N-1} - p$$

We therefore have

$$V_{N-1}(e_{N-1}, \mathcal{I}_{N-1}) = (F_{N-1} - p)(e_{N-1} + x_{N-1}^*) - C(S_{N-1}, x_{N-1}^*) \quad (14)$$

Concavity of V_{N-1} now follows from the functional form above. Differentiating the above with respect to e_{N-1} gives $\partial V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})/\partial e_{N-1} = F_{N-1} - p$.

Case (ii): $F_{N-1} - p < E[S_N|\mathcal{I}_{N-1}]$

For the case of a risk-neutral firm, the optimal solution in this case would be to have $q_{N-1} = 0$ and $x_{N-1}^{rn} = \min\{\tilde{x}_{N-1}, K\}$ where \tilde{x}_{N-1} satisfies

$$\frac{\partial C(S_{N-1}, \tilde{x}_{N-1})}{\partial x_{N-1}} = E[S_N|\mathcal{I}_{N-1}]$$

Recall that S_N^α is the value of S_N corresponding to the α fractile. Using S_N^α , we can modify (7) as

$$VaR_{N-1} + C(S_{N-1}, x_{N-1}) \leq S_N^\alpha(e_{N-1} + x_{N-1} - q_{N-1}) + (F_{N-1} - p)q_{N-1} \quad (15)$$

Therefore, if we have $VaR_{N-1} \leq S_N^\alpha(e_{N-1} + x_{N-1}^{rn}) - C(S_{N-1}, x_{N-1}^{rn})$, then $(0, x_{N-1}^{rn})$ satisfied the VaR constraint and would be an optimal solution for the risk-averse firm as well. We then have

$$V_{N-1}(e_{N-1}, \mathcal{I}_{N-1}) = E[S_N|\mathcal{I}_{N-1}](e_{N-1} + x_{N-1}^{rn}) - C(S_{N-1}, x_{N-1}^{rn}) \quad (16)$$

Taking derivatives, we get $\partial V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})/\partial e_{N-1} = E[S_N|\mathcal{I}_{N-1}] > F_{N-1} - p$.

Suppose we have

$$S_N^\alpha(e_{N-1} + x_{N-1}) - C(S_{N-1}, x_{N-1}) < VaR_{N-1} \quad (17)$$

for all $x_n \geq 0$. By Assumption 2, full-commitment satisfies the VaR constraint; i.e.,

$$(F_{N-1} - p)(e_{N-1} + x_{N-1}) - C(S_{N-1}, x_{N-1}) \geq VaR_{N-1}$$

Therefore, there exists a \tilde{q}_{N-1} such that

$$(F_{N-1} - p)\tilde{q}_{N-1} + S_N^\alpha(e_{N-1} + x_{N-1} - \tilde{q}_{N-1}) - C(S_{N-1}, x_{N-1}) = VaR_{N-1} \quad (18)$$

That is, if committing nothing does not meet the VaR constraint but committing everything does, then there exists an intermediate value of the commitment amount that satisfies the VaR constraint exactly. (Notice that assumption 2 is satisfied only if $S_N^\alpha < F_{N-1} - p$.)

The quantity to process, \tilde{q}_{N-1} , is therefore given by

$$\tilde{q}_{N-1} = \frac{VaR_{N-1} + C(S_{N-1}, x_{N-1}) - S_N^\alpha(e_{N-1} + x_{N-1})}{(F_{N-1} - p) - S_N^\alpha} \quad (19)$$

Notice that the quantity to commit is a function of the quantity procured, x_{N-1} . Substituting this in the maximization problem, (6), we have

$$V_{N-1}(e_{N-1}, \mathcal{I}_{N-1}) = \max_{0 \leq x_{N-1} \leq K} \{ (F_{N-1} - p)(\tilde{q}_{N-1}) - C(S_{N-1}, x_{N-1}) \\ + E[S_N | \mathcal{I}_{N-1}](e_{N-1} + x_{N-1} - \tilde{q}_{N-1}) \}$$

The function to be maximized is concave and has a unique maximum. The maximizer, \hat{x}_{N-1} satisfies the following first order condition

$$(F_{N-1} - p) \frac{\partial \tilde{q}_{N-1}}{\partial x_{N-1}} - \frac{\partial C(S_{N-1}, x_{N-1})}{\partial x_{N-1}} + E[S_N | \mathcal{I}_{N-1}] \left(1 - \frac{\partial \tilde{q}_{N-1}}{\partial x_{N-1}} \right) \Big|_{x_{N-1} = \hat{x}_{N-1}} = 0 \quad (20)$$

Substituting $\partial \tilde{q}_{N-1} / \partial x_{N-1}$ from (19) and simplifying the above equation gives

$$\frac{\partial C(S_{N-1}, x_{N-1})}{\partial x_{N-1}} \Big|_{x_{N-1} = \hat{x}_{N-1}} = F_{N-1} - p \quad (21)$$

Thus we have $x_{N-1}^* = \min\{\hat{x}_{N-1}, K\}$ and $q_{N-1}^* = \max\{\tilde{q}_{N-1}, 0\}$. Thus, we have

$$V_{N-1}(e_{N-1}, \mathcal{I}_{N-1}) = (F_{N-1} - p)(\tilde{q}_{N-1}) - C(S_{N-1}, x_{N-1}^*) \\ + E[S_N | \mathcal{I}_{N-1}](e_{N-1} + x_{N-1}^* - \tilde{q}_{N-1}) \quad (22)$$

Taking partial derivatives, we have

$$\frac{\partial V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})}{\partial e_{N-1}} = (F_{N-1} - p) \frac{E[S_N | \mathcal{I}_{N-1}] - S_N^\alpha}{(F_{N-1} - p) - S_N^\alpha}$$

We find that $V_{N-1}(e_{N-1}, \mathcal{I}_{N-1})$ is concave $\partial V_{N-1}(e_{N-1}, \mathcal{I}_{N-1}) / \partial e_{N-1} \geq F_{N-1} - p$ for all cases considered. \square

Theorem 4 For all $n < N - 1$ and $e_n \geq 0$, the value function $V_n(e_n, \mathcal{I}_n)$ is concave and increasing in e_n , for all realizations of \mathcal{I}_n and the marginal benefit of an unit of uncommitted inventory is always greater than or equal to the processing margin; i.e., $\frac{\partial V_n(e_n, \mathcal{I}_n)}{\partial e_n} \geq F_n - p$. The optimal procurement and processing policy is as given below

1. *Procurement Policy: The optimal procurement quantity is characterized by two critical values, \hat{x}_n and \tilde{x}_n , which satisfy the following first order conditions*

$$\frac{\partial C(S_n, \hat{x}_n)}{\partial x_n} = F_n - p \quad \frac{\partial C(S_n, \tilde{x}_n)}{\partial x_n} = E \left[\frac{\partial V_{n+1}(e_n + \tilde{x}_n)}{\partial e_{n+1}} \middle| \mathcal{I}_n \right]$$

respectively, and the optimal procurement quantity is given by

$$x_n^* = \begin{cases} \min\{\hat{x}_n, K\} & \text{if } V_{n+1}^\alpha(e_n + x_n^*) - C(S_n, x_n^*) < VaR_n \\ \min\{\tilde{x}_n, K\} & \text{if } V_{n+1}^\alpha(e_n + x_n^*) - C(S_n, x_n^*) \geq VaR_n \end{cases}$$

2. *Processing Policy: The optimal quantity to commit for processing is given by*

$$q_n^* = \begin{cases} \frac{VaR_n - [V_{n+1}^\alpha(e_n + \tilde{x}_n - q_n^*) - C(S_n, x_n^*)]}{(F_n - p)} & \text{if } V_{n+1}^\alpha(e_n + x_n^*) - C(S_n, x_n^*) < VaR_n \\ 0 & \text{if } V_{n+1}^\alpha(e_n + x_n^*) - C(S_n, x_n^*) \geq VaR_n \end{cases}$$

Proof: We prove the theorem by induction. Suppose the claim in the theorem is true for periods $n + 1, n + 2, \dots, N - 1$.

The VaR constraint, (5), can be written as

$$VaR_n + C(S_n, x_n) \leq V_{n+1}^\alpha(e_n + x_n - q_n) + (F_n - p)q_n \quad (23)$$

For a risk-neutral firm, by the induction hypothesis, it would never be optimal to commit any quantity for processing in period n . Also, in the risk-neutral case, it would be optimal to procure $x_n^{rn} = \min\{\tilde{x}_n, K\}$ where \tilde{x}_n satisfies

$$\frac{\partial C(S_n, \tilde{x}_n)}{\partial x_n} = E \left[\frac{\partial V_{n+1}(e_n + \tilde{x}_n, \mathcal{I}_{n+1})}{\partial e_{n+1}} \middle| \mathcal{I}_n \right] \quad (24)$$

So, if $V_{n+1}^\alpha(e_n + x_n^{rn}) - C(S_n, x_n^{rn}) \geq VaR_n$, then the risk-neutral optimal solution would also be optimal for the risk-averse firm and we would have

$$V_n(e_n, \mathcal{I}_n) = E[V_{n+1}(e_n + x_n^{rn}, \mathcal{I}_{n+1}) | \mathcal{I}_n] - C(S_n, x_n^{rn}) \quad (25)$$

Concavity and continuity of V_n follows from the induction hypothesis. Taking partial derivatives, we have

$$\begin{aligned}\frac{\partial V_n(e_n, \mathcal{I}_n)}{\partial e_n} &= E\left[\frac{\partial V_{n+1}(e_n + x_n^{rn}, \mathcal{I}_{n+1})}{\partial e_{n+1}} \middle| \mathcal{I}_n\right] \\ &\geq E[F_{n+1} - p | \mathcal{I}_n] \\ &= F_n - p\end{aligned}$$

where the inequality follows from the induction hypothesis and the last equality from the assumption about the futures prices. The interchange of the derivative and expectation in the above equation is justified by the Lebesgue Dominated Convergence Theorem.

Now suppose

$$V_{n+1}^\alpha(e_n + x_n) - C(S_n, x_n) < VaR_n \quad (26)$$

for any $x_n \geq 0$. By Assumption 2, we have

$$V_{n+1}^\alpha(0) - C(S_n, x_n) + (F_n - p)(e_n + x_n) \geq VaR_n \quad (27)$$

By continuity of all the terms in the above equation, there exists at least one $q_n \leq e_n + x_n$ such that

$$V_{n+1}^\alpha(e_n + x_n - q_n) - C(S_n, x_n) + (F_n - p)q_n = VaR_n \quad (28)$$

Let \tilde{q}_n be the smallest value satisfying the following equation

$$\tilde{q}_n = \frac{VaR_n + C(S_n, x_n) - V_{n+1}^\alpha(e_n + \tilde{x}_n - q_n)}{(F_n - p)} \quad (29)$$

Consider $\epsilon_1 > 0$, where $q_n = e_n + x_n - \epsilon_1$ such that

$$V_{n+1}^\alpha(\epsilon_1) - C(S_n, x_n) + (F_n - p)(e_n + x_n - \epsilon_1) \geq VaR_n$$

Consider the derivative of the left hand side with respect to ϵ_1 . If this derivative is non-negative, i.e., if $\partial V_{n+1}^\alpha(\epsilon_1)/\partial e_{n+1} \geq (F_n - p)$, then by increasing ϵ_1 (reducing the quantity committed for processing), the VaR constraint can still be satisfied. However, since \tilde{q}_n satisfies (28), there must be a $\epsilon_2 < e_n + x_n - \tilde{q}_n$ such that

$$\frac{\partial V_{n+1}^\alpha(\epsilon_2)}{\partial e_{n+1}} < F_n - p$$

(Otherwise, we can find a $q_n < \tilde{q}_n$ for which the VaR constraint is satisfied and the firm will find it optimal to reduce the quantity committed for processing.) Therefore, $\partial V_{n+1}^\alpha(e_n + x_n - \tilde{q}_n)/\partial e_{n+1} < F_n - p$.

Substituting (29) into the optimality equation, we have

$$V_n(e_n, \mathcal{I}_n) = \max_{0 \leq x_n \leq K} \{(F_n - p)\tilde{q}_n - C(S_n, x_n) + E[V_{n+1}(e_n + x_n - \tilde{q}_n, \mathcal{I}_{n+1})|\mathcal{I}_n]\} \quad (30)$$

The optimal procurement quantity is given by $x_n^* = \min\{\hat{x}_n, K\}$ where \hat{x}_n satisfies the following first order condition

$$(F_n - p)\frac{\partial \tilde{q}_n}{\partial x_n} - \frac{\partial C(S_n, x_n)}{\partial x_n} + E\left[\frac{\partial V_{n+1}(e_n + x_n - \tilde{q}_n, \mathcal{I}_{n+1})}{\partial e_{n+1}}\Big|\mathcal{I}_n\right]\left(1 - \frac{\partial \tilde{q}_n}{\partial x_n}\right)\Big|_{x_n=\hat{x}_n} = 0 \quad (31)$$

Taking partial derivative with respect to x_n for equation (29) we have

$$\frac{\partial \tilde{q}_n}{\partial x_n} = \frac{\frac{\partial C(S_n, x_n)}{\partial x_n} - \frac{\partial V_{n+1}^\alpha(e_n + x_n - \tilde{q}_n)}{\partial e_{n+1}}}{(F_n - p) - \frac{\partial V_{n+1}^\alpha(e_n + x_n - \tilde{q}_n)}{\partial e_{n+1}}} \quad (32)$$

Substituting into (31) and simplifying, we get

$$\left[E[V_{(n+1,e)}|\mathcal{I}_n] - V_{(n+1,e)}^\alpha\right]\left[(F_n - p) - \frac{\partial C(S_n, x_n)}{\partial x_n}\right]\Big|_{x_n=\hat{x}_n} = 0 \quad (33)$$

where $V_{(n+1,e)} = \partial V_{n+1}(e_n + x_n - \tilde{q}_n, \mathcal{I}_{n+1})/\partial e_{n+1}$ and $V_{(n+1,e)}^\alpha = \partial V_{n+1}^\alpha(e_n + x_n^* - \tilde{q}_n)/\partial e_{n+1}$.

Because we have $\partial V_{n+1}^\alpha(e_n + x_n - \tilde{q}_n)/\partial e_{n+1} < F_n - p \leq E[\partial V_{n+1}(e_n + x_n - \tilde{q}_n, \mathcal{I}_{n+1})/\partial e_{n+1}|\mathcal{I}_n]$, it follows that \hat{x}_n satisfies

$$\frac{\partial C(S_n, \hat{x}_n)}{\partial x_n} = F_n - p \quad (34)$$

Therefore, we have

$$V_n(e_n, \mathcal{I}_n) = (F_n - p)\tilde{q}_n - C(S_n, x_n^*) + E[V_{n+1}(e_n + x_n^* - \tilde{q}_n, \mathcal{I}_{n+1})|\mathcal{I}_n]$$

Taking partial derivatives, we have

$$\frac{\partial V_n(e_n, \mathcal{I}_n)}{\partial e_n} = (F_n - p)\frac{E[V_{(n+1,e)}|\mathcal{I}_n] - V_{(n+1,e)}^\alpha}{(F_n - p) - V_{(n+1,e)}^\alpha}$$

From the above equation, we see that $\partial V_n(e_n, \mathcal{I}_n)/\partial e_n \geq F_n - p$. Taking the second order partial derivative and simplifying, we have

$$\begin{aligned} \frac{\partial^2 V_n(e_n, \mathcal{I}_n)}{\partial e_n^2} &= (F_n - p)\left[\frac{E[V_{(n+1,ee)}](F_n - p - V_{(n+1,e)}^\alpha)}{((F_n - p) - V_{(n+1,e)}^\alpha)^2} \right. \\ &\quad \left. + \frac{V_{(n+1,ee)}^\alpha(E[V_{(n+1,e)}|\mathcal{I}_n] - F_n - p)}{((F_n - p) - V_{(n+1,e)}^\alpha)^2}\right] \\ &\leq 0 \end{aligned}$$

where $E[V_{(n+1,ee)}] = E[\partial^2 V_{n+1}(e_n + x_n - \tilde{q}_n, \mathcal{I}_{n+1})/\partial e_{n+1}^2 | \mathcal{I}_n]$ and $V_{(n+1,ee)}^\alpha = \partial^2 V_{n+1}^\alpha(e_n + x_n^* - \tilde{q}_n)/\partial e_{n+1}^2$. \square

Theorem 5 *When multiple futures contracts of the output commodity are available, the marginal benefit of inventory is given by $\max\{F_{N_i-1}^i - p, E_{\mathcal{I}_{N_i-1}}[M_{N_i}]\}$ when $n = N_i - 1$ for some i and M_n otherwise. The optimal policies are as described below:*

1. *Procurement Policy: The optimal procurement quantity is characterized by a critical value \hat{x}_n which satisfies the following first order condition*

$$\frac{\partial C(S_n, \hat{x}_n)}{\partial x_n} = \begin{cases} M_n & \text{if } N_i \leq n < N_{i+1} - 1 \\ \max\{F_{N_i-1}^i - p, E_{\mathcal{I}_{N_i-1}}[M_{N_i}]\} & \text{if } n = N_i - 1 \end{cases}$$

and the optimal procurement quantity x_n^* is given by $x_n^* = \min\{\hat{x}_n, K\}$.

2. *Processing Policy: The optimal quantity of output to process and sell against any futures contract i is given by*

$$q_n^{i*} = \begin{cases} 0 & \text{if } n \neq N_i - 1 \\ 0 & \text{if } n = N_i - 1 \text{ and } F_{N_i-1}^i - p < E_{\mathcal{I}_{N_i-1}}[M_{N_i}] \\ e_n + x_n^* & \text{if } n = N_i - 1 \text{ and } F_{N_i-1}^i - p \geq E_{\mathcal{I}_{N_i-1}}[M_{N_i}] \end{cases}$$

Proof: From Theorem 2 when $e_n \geq 0$, we have from equation (12)

$$\begin{aligned} A_n &= E_{\mathcal{I}_n}[A_{n+1}] \\ &= E_{\mathcal{I}_n}[E_{\mathcal{I}_{n+1}}[A_{n+2}]] \\ &\vdots \\ &= E_{\mathcal{I}_n}[\dots E_{\mathcal{I}_{N-2}}[A_{N-1}]] \\ &= E_{\mathcal{I}_n}[\max\{F_{N-1} - p, E_{\mathcal{I}_{N-1}}[S_N]\}] \end{aligned} \tag{35}$$

for the case when only a single futures contract is available.

For any n such that $N_{L-1} \leq n < N - 1$, where N_{L-1} is the period in which the futures contract with second longest maturity expires, the situation is the same as if there was only a single futures contract. Therefore, the above equation holds for all n such that $N_{L-1} \leq n < N - 1$.

In period $n = N_{L-1} - 1$, we know from Theorem 2 that it is not optimal to commit against the futures contract maturing in period N . However, committing against the futures contract maturing at N_{L-1} gives a marginal benefit of $F_{N_{L-1}-1}^{L-1} - p$ per unit of soybean inventory, while not

committing and carrying inventory into the future gives a marginal benefit of $E_{\mathcal{I}_{N_{L-1}-1}}[A_{N-1}]$ per unit. Therefore, at $n = N_{L-1} - 1$, the marginal benefit of a unit of soybean inventory is given by

$$\max\{F_{N_{L-1}-1}^{L-1} - p, E_{\mathcal{I}_{N_{L-1}-1}}[\max\{F_{N-1}^L - p, E_{\mathcal{I}_{N-1}}[S_N]\}]\} \quad (36)$$

Denote this marginal benefit by $M_{N_{L-1}-1}$. Notice that $M_{N_{L-1}-1} \geq F_{N_{L-1}-1}^{L-1} - p$ and also $M_{N_{L-1}-1} \geq F_{N_{L-1}-1}^L - p$.

Now, for any n such that $N_{L-2} \leq n < N_{L-1} - 1$, the marginal benefit from committing against futures contract L or $L - 1$ is less than the marginal benefit of carrying uncommitted inventory into the next period. For $n = N_{L-2} - 1$, we have, the marginal benefit of a unit of soybean inventory is

$$\begin{aligned} M_{N_{L-2}-1} &= \max\{F_{N_{L-2}-1}^{L-2} - p, E_{\mathcal{I}_{N_{L-2}-1}}[M_{N_{L-1}-1}]\} \\ &= \max\{F_{N_{L-2}-1}^{L-2} - p, E_{\mathcal{I}_{N_{L-2}-1}}[\max\{F_{N_{L-1}-1}^{L-1} - p, \\ &\quad E_{\mathcal{I}_{N_{L-1}-1}}[\max\{F_{N-1}^L - p, E_{\mathcal{I}_{N-1}}[S_N]\}]\}]\} \end{aligned}$$

Extending this argument, for any period n such that $N_i \leq n < N_{i+1} - 1$, it is not optimal to commit against any futures contract $i + 1, i + 2, \dots, L$. For any such period n , the marginal benefit of a unit of inventory is given by

$$M_n = E_{\mathcal{I}_n}[\max\{F_{N_{i+1}-1}^{i+1} - p, E_{\mathcal{I}_{N_{i+1}-1}}[\max\{\dots \max\{F_{N-1}^L - p, E_{\mathcal{I}_{N-1}}[S_N]\} \dots\}]\}]$$

Therefore, in any period n such that $n = N_i - 1$, it is optimal to commit against futures contract i only if $F_{N_i-1}^i - p \geq E_{\mathcal{I}_{N_i-1}}[M_{N_i}]$.

The optimal procurement policy described in the theorem follows as \hat{x}_n and \hat{x}_{N_i-1} are the threshold values at which the marginal cost of procurement is equal to the marginal revenue in the respective periods. \square

B Price Model for Numerical Study

B.1 Input Spot Price

We use a two-factor model for the input spot price in the spirit of Schwartz and Smith (2000). The spot price S_t is modeled as $\ln S_t = X_t + \mu_t$, where X_t represents the short-term deviation in prices and μ_t the equilibrium price level. Since the period of interest in our model is a single season, we assume μ_t to be fixed for all t to a constant value, μ . (This is unlike the model in Schwartz and Smith (2000) where μ_t follows a Brownian motion process). X_t follows a mean-reverting process

given by $dX_t = -\kappa X_t dt + \sigma_x dW_t^x$, where dW_t^x is the increment of a standard Brownian motion and κ is the mean-reversion coefficient.

Using Ito's lemma and integrating the above equation gives:

$$X_T = e^{-\kappa(T-t)} X_t + \sigma_x \frac{\sqrt{1 - e^{-2\kappa(T-t)}}}{\sqrt{2\kappa(T-t)}} (W_T^x - W_t^x)$$

Discretizing this equation, we get $X_{t+1} = e^{-\kappa} X_t + \sigma_x \frac{\sqrt{1 - e^{-2\kappa}}}{\sqrt{2\kappa}} \zeta_x$, where ζ_x is a random variable with a standard normal distribution.

Finally, substituting in the two-factor spot price model, we get $S_{t+1} = \exp\{X_{t+1} + \mu\} = \exp\left\{e^{-\kappa} X_t + \sigma_x \frac{\sqrt{1 - e^{-2\kappa}}}{\sqrt{2\kappa}} \zeta_x + \mu\right\} = \exp\left\{(1 - e^{-\kappa})\mu + e^{-\kappa} \ln S_t + \sigma_x \frac{\sqrt{1 - e^{-2\kappa}}}{\sqrt{2\kappa}} \zeta_x\right\}$.

B.2 Estimating the parameters

From the expression for S_{t+1} derived above, we have $\ln S_{t+1} = (1 - e^{-\kappa})\mu + e^{-\kappa} \ln S_t + \sigma_x \frac{\sqrt{1 - e^{-2\kappa}}}{\sqrt{2\kappa}} \zeta_x$. Thus, by fitting a linear regression model on $\ln S_t$ and $\ln S_{t+1}$, one can estimate the value of the parameters μ , κ and σ_x . (Note that S_{t+1} is the next period price, where a period can be appropriately chosen and not necessarily the immediate next day price.)

Based on the National Commodities and Derivatives Exchange Ltd. (NCDEX)⁵ data on soybean spot prices for the period of 1st August, 2006 to 30th December, 2006, prices with a 15 day lag (i.e., $S_{t+1} = S_{t+15}$ regressed with S_t) show a fit with a $R^2 = 0.424$ and an adjusted $R^2 = 0.419$. By considering a period in our model to correspond to 15 actual days (i.e., procurement and processing decisions are made once every two weeks), we can use the results from this data for modeling the spot price process.

B.3 Output Futures Price

For the model studied in this paper, the processed product is sold entirely using futures contracts. However, there was no reliable data available on the futures prices of soy meal and soy oil on the NCDEX. (the futures markets are illiquid and there are very few trades that happened on these markets.)

In order to overcome this deficiency and to serve the purposes of this numerical study, we assume that the futures prices are equal to the expected spot prices at maturity. We consider a

⁵A commodities exchange in India, http://www.ncdex.com/Market_Data/hist_spot_price_2006.aspx, retrieved July 2007.

model for the spot prices of soymeal and soyoil and then derive the futures price from those spot prices. We consider a simple case of a single output commodity in this paper. Hence, we model the total output, soymeal and soyoil, as a single output commodity and model the price of a composite output.

The output spot price is modeled as $\lambda * M_t + (1 - \lambda) * O_t$ where λ is the amount of soymeal produced from a unit of soybean (and the remaining is assumed to be soyoil), M_t is the spot price of soymeal and O_t is the spot price of soyoil. Considering this composite output as a commodity, we use the same model as for the input for these prices. Therefore we have $\ln S_t = Y_t + \mu_c$, where S_t is the spot price of the composite, Y_t is the short term deviation in the price of the composite and μ_c is the long run equilibrium price level of the composite product. Y_t has the following dynamics: $dY_t = -\kappa_y Y_t dt + \sigma_y dW_t^y$.

Along similar lines as before, we have, upon integration:

$$Y_T = e^{-\kappa_y(T-t)} Y_t + \sigma_y \frac{\sqrt{1 - e^{-2\kappa_y(T-t)}}}{\sqrt{2\kappa_y(T-t)}} (W_T^y - W_t^y)$$

Discretizing as before, we have $Y_{t+1} = e^{-\kappa_y} Y_t + \sigma_y \frac{\sqrt{1 - e^{-2\kappa_y}}}{\sqrt{2\kappa_y}} \zeta_y$, where ζ_y is a standard normal random variable. Moreover, ζ_x and ζ_y are correlated with a correlation coefficient ρ .

From the equations above we have $S_T = \exp \left\{ (1 - e^{-\kappa_y(T-t)}) \mu_y + e^{-\kappa_y(T-t)} \ln S_t + \sigma_y \frac{\sqrt{1 - e^{-2\kappa_y(T-t)}}}{\sqrt{2\kappa_y}} \zeta_y \right\}$ and $S_{t+1} = \exp \left\{ (1 - e^{-\kappa_y}) \mu_y + e^{-\kappa_y} \ln S_t + \sigma_y \frac{\sqrt{1 - e^{-2\kappa_y}}}{\sqrt{2\kappa_y}} \zeta_y \right\}$. The parameters μ_y , κ_y , σ_y can again be estimated from a linear regression model with S_{t+1} against S_t as described in the section on input spot prices.

Based on the NCDEX data on soymeal and soyoil spot prices for the period of 1st August, 2006 to 30th December, 2006, prices with a 15 day lag (i.e., $S_{t+1} = S_{t+15}$ regressed with S_t) show a fit with a $R^2 = 0.484$ and an adjusted $R^2 = 0.479$. (As in the case of the input spot prices, we can consider a single period in the model to correspond to 15 real days and hence these results can be used for the purposes of this paper.)

To derive the futures price process for the output, we assume that the rational expectation hypothesis holds and the futures price F_t^T for a contract maturing in period T , is an unbiased estimator for the future spot price S_T in period T . Therefore, we have $F_t^T = E[S_T | S_t] = \exp \left\{ (1 - e^{-\kappa_y(T-t)}) \mu_y + e^{-\kappa_y(T-t)} \ln S_t + \sigma_y^2 \frac{(1 - e^{-2\kappa_y(T-t)})}{4\kappa_y} \right\}$.

We need to be able to express F_{t+1}^T in terms of F_t^T to generate the probability distribution of the next period futures price, given the current period futures price. Through algebraic manipulation,

Parameter	Bean	Oil + Meal
κ	0.332	0.321
σ	0.050	0.029
μ	7.187	7.410
ρ	0.945	

Table 3: Parameters for Spot and Futures Price Processes.

we obtain $F_{t+1}^T = F_t^T \exp\left\{-\sigma_y^2 \frac{(e^{2\kappa_y} - 1)}{4\kappa_y} e^{-2\kappa_y(T-t)} + e^{-\kappa_y(T-t)} \sigma_y \frac{\sqrt{e^{2\kappa_y} - 1}}{\sqrt{2\kappa_y}} \zeta_y\right\}$.

B.4 Implementation

The various parameters were estimated using linear regression models on the spot prices of the input and output. The correlation coefficient between ζ_x and ζ_y was estimated using the correlation coefficient of the residuals of the regressions. The values of the parameters are summarized in Table 3.

For scaling purposes, in the numerical study, the long term equilibrium value of the input spot price was set to 25. Correspondingly, the long run equilibrium of the output was scaled to 31. With these long run equilibrium values, policies were computed over a input spot price range of $[10, 40]$ and output futures price range of $[11, 51]$, both in increments of 0.25. These limits were chosen such that the realized spot and futures prices over the horizon would fall within the range 95% of the time. Observe also that a processing cost of $p = 5$ corresponds to an expected processing margin of 1 (approximately 3.3%) which allows us to model situations when the actual realized processing margin may be large, marginal or negative.