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Mapping the *G*-structures and supersymmetric vacua of five-dimensional $\mathcal{N} = 4$ supergravity

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Abstract
We classify the supersymmetric vacua of $\mathcal{N} = 4$, $d = 5$ supergravity in terms of G-structures. We identify three classes of solutions: with $\mathbb{R}^3$, $SU(2)$ and Id structure. Using the Killing spinor equations, we fully characterize the first two classes and partially solve the latter. With the $\mathcal{N} = 4$ graviton multiplet decomposed in terms of $\mathcal{N} = 2$ multiplets: the graviton, vector and gravitino multiplets, we obtain new supersymmetric solutions corresponding to turning on fields in the gravitino multiplet. These vacua are described in terms of an $SO(5)$ vector sigma model coupled with gravity, in three or four dimensions. A new feature of these $\mathcal{N} = 4$ vacua, which is not seen from an $\mathcal{N} = 2$ point of view, is the possibility for preserving more exotic fractions of supersymmetry. We give a few concrete examples of these new supersymmetric (albeit singular) solutions. Additionally, we show how by truncating the $\mathcal{N} = 4$, $d = 5$ set of fields to minimal supergravity coupled with one vector multiplet we recover the known two-charge solutions.

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1. Introduction and summary

One of the most important principles underlying much of physics is the use of symmetries as a means of classifying and understanding physical phenomena. This is especially true in the theoretical realm, where the use of standard symmetries such as Lorentz and gauge invariance has played a key rôle in the development of quantum field theories of the Standard Model and beyond. Along these lines, the use of supersymmetry has been at the forefront of many recent explorations into both formal string theory as well as string and particle phenomenology. After all, supersymmetry is a natural extension of the Poincaré symmetry of spacetime, and furthermore may be argued to be a natural ingredient of any realistic theory of quantum gravity.

Given an underlying supersymmetric theory, it is of course expected that many interesting vacua or configurations may break some or all of the supersymmetries. In fact, it is precisely the BPS states, namely the configurations with partially broken symmetry, that are of much interest in the field. This is because potential corrections to these objects are much better controlled, whether through multiplet shortening or related non-renormalization theorems. As a result, BPS states are an important tool in the study of strong/weak coupling dualities, where otherwise one would naturally expect large corrections to appear.

Through the use of duality, BPS objects such as black holes and branes often have multiple descriptions. On one side of a duality, they may be constructed as exact solutions within a particular supergravity framework, while on the other side they may be fundamental strings, D-branes or other such objects. From this point of view, the construction and classification of exact BPS solutions has certainly given rise to many important advances. This is especially true in the development of our understanding of D-branes and the counting of black hole microstates, both of which depended greatly on the existence of corresponding supergravity solutions.

In addition, the classification of supersymmetric vacua is of current interest in the program of mapping out the string landscape. Ideally one would like to be able to answer the question of what string, brane or flux compactifications are possible that lead to realistic $\mathcal{N} = 1$ models in four dimensions. While this has been answered in the conventional perturbative heterotic picture by $SU(3)$ holonomy (i.e. Calabi-Yau) manifolds, less is known in the presence of fluxes and branes. Nevertheless, progress is being made in this direction, and much of that has been due to better understanding of fluxes and G-structures.

Much of the recent work on classifying and constructing supersymmetric configurations involves the invariant tensor analysis originally developed in [1, 2] and further developed in [3–6]. In this analysis, one first postulates the existence of a Killing spinor ϵ. Given such a Killing spinor, one is then able to construct a set of invariant tensors formed out of spinor bilinears. The existence of such invariant tensors ensures the existence of a preferred G-structure. This G-structure, along with its intrinsic torsion classes then provides a framework for the classification of all supersymmetric solutions. To proceed to an actual construction, one must examine the 'differential relations' which follow from the actual Killing spinor equations. Here we note that solving these relations to arrive at an explicit field configuration is often the most challenging step in the construction. Finally, as partially broken supersymmetry does not necessarily imply the full set of equations of motion, one may have to examine an appropriate subset of them to complete the construction. This is generally the origin of the resulting second-order 'harmonic function' equations.

The invariant tensor analysis has been particularly fruitful in theories with eight supercharges. This includes four-dimensional $\mathcal{N} = 2$ ungauged [1, 2] and gauged [7, 8] supergravity, five-dimensional $\mathcal{N} = 2$ (minimal) ungauged [4] and gauged [9–11] supergravity, and six-dimensional $\mathcal{N} = (1, 0)$ ungauged [12] and gauged [13] supergravity. The classification in terms of G-structures is also reasonably well developed for eleven-dimensional supergravity, with 32 supercharges. However, the actual construction of all possible solutions is a great technical challenge in the models with more supersymmetries, as there are many more bosonic degrees of freedom that must be pinned down. One way to overcome these difficulties is to impose additional isometries on the BPS solutions. This method has been used to great success in the bubbling AdS work of [14]. However, this still leaves open the question of what is the full class of solutions without any restriction on the isometries.

Given a well-developed set of techniques applied to theories with eight supercharges, it is then natural to explore the construction of all supersymmetric solutions in theories with 16 supercharges as an intermediate step on the way to theories with 32 supercharges. Proceeding towards this goal, a G-structure classification for seven-dimensional supergravity was given in [15, 16], and a construction of all supersymmetric configurations of $\mathcal{N} = 4$ ungauged supergravity in four dimensions was recently given in [17].

In this paper, we continue the study of theories with 16 supercharges by constructing all supersymmetric solutions to $\mathcal{N} = 4$ ungauged supergravity in five dimensions. This theory is in fact closely related by dimensional reduction to $\mathcal{N} = 4$ supergravity in four dimensions, which was investigated in [17]. However, the present five-dimensional case is somewhat more general, and we find that solutions break up into three classes, namely those with either: (i) a timelike Killing vector and $SU(2)$ structure, (ii) a timelike Killing vector and Id structure,

$$\mathrm{d}s^2 = -f^2(\mathrm{d}t + \omega)^2 + f^{-1}h_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n,$$

or (iii) a null Killing vector and $\mathbb{R}^3$ structure

$$\mathrm{d}s^2 = H^{-1}(\mathcal{F}\,\mathrm{d}u^2 + 2\,\mathrm{d}u\,\mathrm{d}v) + H^2h_{mn}(\mathrm{d}y^m + a^m\,\mathrm{d}u)(\mathrm{d}y^n + a^n\,\mathrm{d}u).$$

In the cases (i) and (iii), the metric h_{mn} is obtained by solving a gravitating $SO(5)$ vector sigma model corresponding to a vector u^a $(a = 1, \ldots, 5)$ with unit norm

$$\hat{R}_{mn} = -\partial_m u^a \partial_n u^a, \qquad \Box u^a = u^a u^b \Box u^b.$$

More precisely, we find that the timelike with $SU(2)$ structure supersymmetric backgrounds are specified by the set of functions $u^a, h_{mn}, \mathcal{H}_{1,2}, G^+_{1,2}$:

$$\begin{aligned}
&\mathrm{d}s^2 = -\big(\mathcal{H}_1\mathcal{H}_2^2\big)^{-2/3}(\mathrm{d}t + \omega)^2 + \big(\mathcal{H}_1\mathcal{H}_2^2\big)^{1/3}h_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n,\\
&G = -d\big[\mathcal{H}_1^{-1}(\mathrm{d}t + \omega)\big] - G_1^+, \qquad F^a = d\big[u^a\mathcal{H}_2^{-1}(\mathrm{d}t + \omega)\big] + u^a G_2^+,\\
&\mathrm{e}^{\frac{3}{\sqrt{6}}\phi} = \mathcal{H}_2/\mathcal{H}_1,
\end{aligned}$$

where

$$\mathrm{d}G_1^+ = 0, \qquad \mathrm{d}\big(u^a G_2^+\big) = 0, \qquad \Box_4\mathcal{H}_1 = \tfrac{1}{2}(G_2^+)^2, \qquad (\Box_4 - \hat{R})\mathcal{H}_2 = \tfrac{1}{2}G_1^+G_2^+.$$

All supersymetric solutions with a null Killing vector are of the form

$$\begin{aligned}
&\mathrm{d}s^2 = \big(\mathcal{H}_1\mathcal{H}_2^2\big)^{-1/3}(\mathcal{F}\,\mathrm{d}u^2 + 2\mathrm{d}u\,\mathrm{d}v) + \big(\mathcal{H}_1\mathcal{H}_2^2\big)^{2/3}h_{mn}(\mathrm{d}y^m + a^m\,\mathrm{d}u)(\mathrm{d}y^n + a^n\,\mathrm{d}u)\\
&G = G_{+\bar{m}}e^+ \wedge \mathrm{e}^{\bar{m}} - \tfrac{1}{2}\epsilon_{mn}{}^p\partial_p\mathcal{H}_1(\mathrm{d}y^m + a^m\,\mathrm{d}u) \wedge (\mathrm{d}y^n + a^n\,\mathrm{d}u)\\
&F^a = F_{+\bar{m}}e^+ \wedge \mathrm{e}^{\bar{m}} + \tfrac{1}{2}\epsilon_{mn}{}^p(u^a\partial_p\mathcal{H}_2 - \mathcal{H}_2\partial_p u^a)(\mathrm{d}y^m + a^m\,\mathrm{d}u) \wedge (\mathrm{d}y^n + a^n\,\mathrm{d}u)\\
&\mathrm{e}^{\frac{3}{\sqrt{6}}\phi} = \mathcal{H}_1/\mathcal{H}_2,
\end{aligned}$$

where $\Box_3\mathcal{H}_1 = 0$, $(\Box_3 - \hat{R})\mathcal{H}_2 = 0$, and where the fluxes along the null direction as well as the dependence of the scalar functions $\mathcal{H}_{1,2}$, u^a on the null u coordinate are further constrained by differential equations (see section 4).

Lastly, for supersymmetric solutions with a timelike Killing vector and identity structure, the fünfbein are completely determined in terms of the Killing spinor.

In the rigid (constant u^a) case, the generic solution preserves 1/4 of the supersymmetries, although special configurations preserve either 1/2 or all of the supersymmetries. This rigid case admits a natural $\mathcal{N} = 2$ interpretation in terms of supergravity coupled to a single vector multiplet.

The non-rigid cases (whether for timelike or null Killing isometries) are rather more unusual, as they have no direct correspondence in the $\mathcal{N} = 2$ theory. From an $\mathcal{N} = 2$ perspective, these cases correspond to exciting the gauge fields in the gravitino multiplet. As a result, they give rise to true $\mathcal{N} = 4$ configurations. Furthermore, it appears that these non-rigid solutions may preserve any of 0, 1, 2, 3, 4, 6, 8 or 16 of the $\mathcal{N} = 4$ supersymmetries. We present some examples, although we have yet to find a completely regular solution in this class which is free of all singularities.

In the following section, we review the $\mathcal{N}=4$, $d=5$ ungauged supergravity theory and proceed to construct the spinor bilinears. Use of the Fierz identities allows us to deduce the G-structure classification indicated above. In section 3, we specialize to the timelike Killing vector case and present a complete investigation of the solutions preserving an $SU(2)$ structure and say a few words about the Id structure case. Following this, we take up the null case in section 4. Finally, we conclude in section 5 with a few concrete examples of non-rigid solutions. Some of the technical details are relegated to the appendices. In particular, appendix A contains a set of important Fierz identities, and appendix B tabulates the differential identities following from the Killing spinor equations.

2. $\mathcal{N}$ = 4 supergravity and G-structures

Five-dimensional $\mathcal{N}=4$ supergravity was first constructed in [18], and is formulated in terms of a five-dimensional $USp(4)$ symplectic Majorana spinor ϵ^i. In the minimal ungauged case, the bosonic fields consist of a metric, a scalar ϕ, and six abelian gauge fields $A_\mu^{[ij]|}$ and B_μ (with field strengths $F_{\mu\nu}^{[ij]|}$ and $G_{\mu\nu}$), transforming under $USp(4)$ as the **5** and **1**, respectively. The fermionic fields are comprised by the four gravitini ψ_μ^i and the four dilatini χ^i, both of which transform as the **4** of $USp(4)$.

Up to terms quartic in fermions, the Lagrangian is

$$\begin{aligned}
\mathrm{e}^{-1}\mathcal{L} = R &- \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{8}\,\mathrm{e}^{\frac{2}{\sqrt{6}}\phi}\big(F_{\mu\nu}^{ij}\big)^2 - \frac{1}{4}\,\mathrm{e}^{-\frac{4}{\sqrt{6}}\phi}(G_{\mu\nu})^2 - \frac{1}{2}\bar\psi_\mu^i\Gamma^{\mu\nu\rho}\nabla_\nu\psi_i - \frac{1}{2}\bar\chi^i\Gamma^\mu\nabla_\mu\chi_i\\
&+ \frac{1}{16}\epsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}^{ij}F_{\rho\sigma ij}B_\lambda - \frac{\mathrm{i}}{2\sqrt{2}}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}\bar\chi^i\Gamma^\mu\Gamma^\nu\psi_{\mu i}\partial_\nu\phi\\
&+ \frac{1}{4\sqrt{3}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\bar\chi^i\Gamma^\mu\Gamma^{\rho\sigma}\psi_\mu^jF_{\rho\sigma ij} - \frac{1}{4\sqrt{3}}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}\bar\chi^i\Gamma^\mu\Gamma^{\rho\sigma}\psi_{\mu i}G_{\rho\sigma}\\
&- \frac{\mathrm{i}}{24}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\bar\chi^i\Gamma^{\rho\sigma}\chi^jF_{\rho\sigma ij} + \frac{5}{48}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}\bar\chi^i\Gamma^{\rho\sigma}\chi_iG_{\rho\sigma}\\
&- \frac{\mathrm{i}}{8}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\big[\bar\psi_\mu^i\Gamma^{\mu\nu\rho\sigma}\psi_\nu^j + 2\bar\psi^{\rho i}\psi^{\sigma j}\big]F_{\rho\sigma ij}\\
&- \frac{\mathrm{i}}{16}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}\big[\bar\psi_\mu^i\Gamma^{\mu\nu\rho\sigma}\psi_{\nu i} + 2\bar\psi^{\rho i}\psi_i^\sigma\big]G_{\rho\sigma},
\end{aligned}\tag{2.1}$$

where we have rescaled some of the fields of [18]. Note that we work with signature $(-,+,+,+,+)$.

The supersymmetry transformations are given by

$$\delta e_\mu^m = \frac{1}{4}\bar\epsilon^i\Gamma^m\psi_{\mu i},\tag{2.2}$$

$$\delta\phi = \frac{\mathrm{i}}{\sqrt{2}}\bar\epsilon^i\chi_i,\tag{2.3}$$

$$\delta A_\mu^{ij} = -\frac{1}{\sqrt{3}}\,\mathrm{e}^{-\frac{1}{\sqrt{6}}\phi}\left(\bar\epsilon^i\Gamma_\mu\chi^j + \frac{1}{4}\Omega^{ij}\bar\epsilon^k\Gamma_\mu\chi_k\right) - \mathrm{i}\,\mathrm{e}^{-\frac{1}{\sqrt{6}}\phi}\left(\bar\epsilon^i\psi_\mu^j + \frac{1}{4}\Omega^{ij}\bar\epsilon^k\psi_{\mu k}\right),\tag{2.4}$$

$$\delta B_\mu = \frac{1}{2\sqrt{3}}\,\mathrm{e}^{\frac{2}{\sqrt{6}}\phi}\bar\epsilon^i\Gamma_\mu\chi_i - \frac{\mathrm{i}}{4}\,\mathrm{e}^{\frac{2}{\sqrt{6}}\phi}\bar\epsilon^i\psi_{\mu i},\tag{2.5}$$

$$\delta\psi_{\mu i} = \nabla_\mu\epsilon_i + \frac{\mathrm{i}}{12}F_{\rho\sigma ij}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\left(\Gamma_\mu{}^{\rho\sigma} - 4\delta_\mu^\rho\Gamma^\sigma\right)\epsilon^j + \frac{\mathrm{i}}{24}G_{\rho\sigma}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}\left(\Gamma_\mu{}^{\rho\sigma} - 4\delta_\mu^\rho\Gamma^\sigma\right)\epsilon_i,\tag{2.6}$$

$$\delta\chi_i = -\frac{\mathrm{i}}{2\sqrt{2}}\partial_\mu\phi\Gamma^\mu\epsilon_i + \frac{1}{4\sqrt{3}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}F_{\rho\sigma ij}\Gamma^{\rho\sigma}\epsilon^j - \frac{1}{4\sqrt{3}}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}G_{\rho\sigma}\Gamma^{\rho\sigma}\epsilon_i, \tag{2.7}$$

up to three-fermi terms in the gravitino and dilatino variations. Here Ω_{ij} is the real antisymmetric $USp(4)$ invariant tensor satisfying $\Omega^{ij}\Omega_{jk} = -\delta^i_k$. In particular, Ω_{ij} is used to raise and lower the $USp(4)$ indices according to the northwest–southeast rule

$$V^i = \Omega^{ij}V_j, \qquad V_i = V^j\Omega_{ji}. \tag{2.8}$$

All spinors are symplectic Majorana, obeying

$$\bar\lambda^i = (\lambda^i)^T C, \tag{2.9}$$

where $\bar\lambda^i \equiv (\lambda_i)^*\Gamma_0$, and C is the real antisymmetric charge conjugation matrix. It is also useful at this stage to note the Majorana flip condition

$$\overline{\chi}^i\Gamma^{\mu_1\cdots\mu_n}\lambda^j = -(-)^{n(n-1)/2}\overline{\lambda}^j\Gamma^{\mu_1\cdots\mu_n}\chi^i. \tag{2.10}$$

In what follows, we find it convenient to use the isomorphism between the $USp(4)$ and $SO(5)$ groups to convert the $USp(4)$ valued indices $i = 1, \ldots, 4$ to $SO(5)$ ones $a = 1, \ldots, 5$. This may be accomplished by introducing a set of matrices $T^{ai}{}_j$ satisfying the $SO(5)$ Clifford algebra $\{T^a, T^b\}^i{}_j = 2\delta^{ab}\delta^i_j$. This allows us to convert expressions with $USp(4)$ index pairs into ones involving purely vectorial $SO(5)$ quantities.

2.1. Construction of the invariant tensors

Following [3–6], for a purely bosonic background, we first assume the existence of a commuting Killing spinor ϵ^i satisfying the Killing spinor equations

$$\delta\psi_{\mu i} = 0, \qquad \delta\chi_i = 0, \tag{2.11}$$

where the gravitino and dilatino expressions are given by (2.6) and (2.7), respectively. Given such a spinor, we may construct a complete set of invariant tensors formed out of bilinears of ϵ^i. In terms of irreducible $USp(4)$ representations, we define the following bispinors:

$$\begin{aligned} f^{[ij]|} + f\Omega^{ij} &= i\bar\epsilon^i\epsilon^j, \\ V_\mu^{[ij]|} + K_\mu\Omega^{ij} &= \bar\epsilon^i\Gamma_\mu\epsilon^j, \\ \Phi^{(ij)}_{\mu\nu} &= i\bar\epsilon^i\Gamma_{\mu\nu}\epsilon^j. \end{aligned} \tag{2.12}$$

The factors of i have been inserted so that the bispinors are real-valued tensors, and the (anti-)symmetry properties follow from the Majorana flip relation (2.10).

We note that the total number of bispinor components is given by counting the number of elements in the matrix $\epsilon^i_\alpha\epsilon^j_\beta$, which is symmetric under the interchange of the combined indices (i, α) with (j, β). This comes out to 136, which equals the number of components in (2.12). More explicitly, the scalar f and 1-form K are $USp(4)$ singlets, while the scalars f^{ij} and 1-forms V^{ij} transform in the **5** of $USp(4)$, and the 2-form Φ^{ij} belongs to the **10** representation. As indicated above, we prefer to use a $SO(5)$ notation for the bispinors

$$\begin{aligned} f^a &= \frac{\mathrm{i}}{4}T^a_{ij}(\bar\epsilon^i\epsilon^j), & f &= -\frac{\mathrm{i}}{4}(\bar\epsilon^i\epsilon_i), \\ V^a_\mu &= \frac{1}{4}T^a_{ij}(\bar\epsilon^i\Gamma_\mu\epsilon^j), & K_\mu &= -\frac{1}{4}(\bar\epsilon^i\Gamma_\mu\epsilon_i), \\ \Phi^{ab}_{\mu\nu} &= \frac{\mathrm{i}}{4}T^{ab}_{ij}(\bar\epsilon^i\Gamma_{\mu\nu}\epsilon^j), \end{aligned} \tag{2.13}$$

in which case the **5** and **10** of $SO(5)$ is manifest. Since the underlying spinor ϵ^i contains only 16 real components, not all 136 components of the above tensors are independent. As a result, there are numerous algebraic identities (derived through Fierzing) relating the above tensors to each other. An important set of such algebraic identities is presented in appendix A.

2.2. The G-structure classification

For a pure geometry solution, any time the background admits a Killing spinor ϵ satisfying $\nabla_\mu\epsilon = 0$, there is an associated Killing vector of the form $(\bar\epsilon\Gamma^\mu\epsilon)$. This guarantees the existence of at least one isometry associated with (partially) unbroken supersymmetry. In the present case, it is easy to show that, even in the presence of additional fields, the vector K^μ defined in (2.13) remains a Killing vector, as it satisfies the Killing equation (B.28). Furthermore, as demonstrated at the end of appendix B, this isometry of the metric extends to the entire solution.

The preferred Killing vector K^μ plays a fundamental role in the identification of the structure group from the invariant tensors. To proceed, we note that the norm of K^μ is easily obtained through the Fierz identities. In particular, expression (A.3), namely

$$K^\mu K_\mu = -(f^a f^a), \tag{2.14}$$

demonstrates that the Killing vector is either timelike or null (as expected for supersymmetric backgrounds). The classification thus splits into two cases.

2.2.1. The timelike case. For the timelike case, we take $|K|^2 = -(f^a)^2 < 0$. In this case, the $SO(5)$ vector defined by f^a is non-vanishing, and may be used to parametrize the breaking of $SO(5)$ into $SO(4)$. We make this explicit by defining the projection

$$\Pi_4^{ab} = \delta^{ab} - u^a u^b, \tag{2.15}$$

where u^a is the normalized $SO(5)$ vector $u^a = f^a/|f^a|$. This projection allows us to decompose the 1-forms V^a under $\mathbf{5} \to \mathbf{4} + \mathbf{1}$ as

$$V_\mu^a = u^a V_\mu^{(1)} + V_\mu^{(4)a}, \tag{2.16}$$

where $V_\mu^{(1)} = u^a V_\mu^a$ and $V_\mu^{(4)a} = \Pi_4^{ab} V_\mu^b$. Use of the Fierz identity (A.6), namely $f K_\mu = f^a V_\mu^a$, then shows that $V_\mu^{(1)}$ is aligned with K_μ. This gives

$$V_\mu^a = \frac{f f^a}{(f^b)^2} K_\mu + V_\mu^{(4)a}. \tag{2.17}$$

Furthermore, projecting (A.8) onto $SO(4)$ demonstrates that $\mathrm{i}_K V^{(4)a} = 0$. This indicates that the above decomposition of V_μ^a is onto the timelike direction specified by K^μ and its orthogonal spacelike hyperplanes.

The $SO(4)$ valued 1-forms $V_\mu^{(4)a}$ furthermore satisfy the conditions

$$\begin{aligned} V_\mu^{(4)a} V^{(4)\mu b} &= ((f^c)^2 - f^2)\Pi_4^{ab}, \\ V_\mu^{(4)a} V_\nu^{(4)a} &= ((f^c)^2 - f^2)(g_{\mu\nu} - K_\mu K_\nu/|K|^2), \end{aligned} \tag{2.18}$$

which arise from the Fierz identities (A.12) and (A.14). Note that, by taking one additional contraction of either equation, we see that $\left|V_\mu^{(4)a}\right|^2 = 4((f^c)^2 - f^2)$. Since $V_\mu^{(4)a}$ are everywhere spacelike (or vanishing), we are led to deduce that

$$(f^a)^2 \geqslant f^2. \tag{2.19}$$

Furthermore, the $V_\mu^{(4)a}$ must vanish identically when this inequality is saturated.

If desired, we may make an explicit choice of coordinates so that $K^\mu\partial_\mu = \partial_t$. This allows us to express the five-dimensional metric as a foliation of four-dimensional hypersurfaces

$$\mathrm{d}s^2 = -(f^a)^2(\mathrm{d}t + \omega)^2 + \frac{1}{\sqrt{(f^a)^2}} h_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n. \tag{2.20}$$

In this case, the 1-form associated with K_μ is simply $K_\mu\, dx^\mu = -(f^a)^2(dt+\omega)$. However, we note that the following discussion is completely general, and does not depend on any particular choice of coordinates.

Turning next to the 2-form Φ^{ab}, we again use u^a to project it onto invariant $SO(4)$ components

$$\Phi^{ab} = u^a\Phi^{(4)b} - u^b\Phi^{(4)a} + \Phi^{(6)ab}, \tag{2.21}$$

under $\mathbf{10} \to \mathbf{4}+\mathbf{6}$. As above, the $\mathbf{4}$ and $\mathbf{6}$ can be disentangled by projecting with combinations of Π_4^{ab}. Combining this $SO(5)$ decomposition with the Fierz identity (A.21), we find that

$$\Phi^{ab} = \frac{f^a}{(f^c)^2}K\wedge V^{(4)b} - \frac{f^b}{(f^c)^2}K\wedge V^{(4)a} + \Phi^{(6)ab}. \tag{2.22}$$

The components valued in the $\mathbf{6}$ of $SO(4)$ satisfies $\mathrm{i}_K\Phi^{(6)ab} = 0$, and hence live on surfaces orthogonal to K^μ. In fact, contraction of (A.21) with K^μ yields a condition

$$(f^c)^2\Phi^{(6)ab} - \tfrac{1}{2}\epsilon^{abcde} f f^c \Phi^{(6)de} = *(K\wedge V^{(4)a}\wedge V^{(4)b}), \tag{2.23}$$

while the identity (A.19) gives directly

$$f\Phi^{(6)ab} + *(K\wedge\Phi^{(6)ab}) = 0. \tag{2.24}$$

Additional use of (A.21) then leads to the expressions

$$\begin{aligned}((f^c)^2 - f^2)\Phi^{(6)ab} &= -fV^{(4)a}\wedge V^{(4)b} + *(K\wedge V^{(4)a}\wedge V^{(4)b}),\\ ((f^c)^2 - f^2)\tfrac{1}{2}\epsilon^{abcde}f^c\Phi^{(6)de} &= -(f^c)^2V^{(4)a}\wedge V^{(4)b} + f*(K\wedge V^{(4)a}\wedge V^{(4)b}).\end{aligned} \tag{2.25}$$

Combining the two above equations demonstrates that the 2-form $V^{(4)a}\wedge V^{(4)b}$ must satisfy the joint spacetime and internal $SO(4)$ anti-self-duality condition

$$((f^c)^2 - f^2)\left(\delta_\mu^{[\rho}\delta_\nu^{\sigma]}\delta^{ac}\delta^{bd} + \left(\tfrac{1}{2}\epsilon^{abcde}u^e\right)\left(\tfrac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma\lambda}K_\lambda\right)\right)V_\rho^{(4)c}V_\sigma^{(4)d} = 0. \tag{2.26}$$

$SU(2)$ structure. Until now, we have not placed any further restrictions on f and f^a other than the inequality (2.19). It ought to be clear from the above, however, that we ought to distinguish between two subcases of the general timelike case, depending on whether the inequality is saturated or not. Consider first the case

$$(f^a)^2 = f^2. \tag{2.27}$$

In this case, $V^{(4)a}$ vanishes, and we are left with

$$V^a = u^aK \qquad \text{and} \qquad \Phi^{ab} = \Phi^{(6)ab}, \tag{2.28}$$

where

$$\mathrm{i}_K\Phi^{(6)ab} = 0, \qquad \Phi^{(6)ab} = \tfrac{1}{2}\epsilon^{abcde}u^c\Phi^{(6)de}. \tag{2.29}$$

This indicates that the 1-forms V^a are aligned with K, and that the 2-forms Φ^{ab} are both transverse to K^μ and take values in the self-dual $SU(2)_+$ inside $SO(4)$. The hyper-Kähler structure may be obtained from the Fierz identity (A.26), specialized to the present case:

$$\begin{aligned}\Phi^{ab}_{mn}\Phi^{ncd}_{p} = &-h_{mp}\left[\Pi_4^{ac}\Pi_4^{bd} - \Pi_4^{ad}\Pi_4^{bc} + \epsilon^{abcde}u^e\right]\\ &+\left[\delta_e^a\delta_f^c\Pi_4^{bd} + \delta_e^b\delta_f^d\Pi_4^{ac} - \delta_e^a\delta_f^d\Pi_4^{bc} - \delta_e^b\delta_f^c\Pi_4^{ad}\right]\Phi^{ef}_{mn},\end{aligned} \tag{2.30}$$

where space indices are raised with the four-dimensional metric h_{mn} of (2.20). Moreover, from (2.24), we see that the 2-forms are anti-self-dual on the four-dimensional base. This presents a curious connection between the spacetime and internal indices of $\Phi^{(6)ab}_{\mu\nu}$, as it resides in both the tangent space group $SU(2)_- \subset SO(4) \subset SO(4,1)$ and the internal symmetry group $SU(2)_+ \subset SO(4) \subset SO(5)$.

This case is nearly identical to the corresponding one for timelike configurations of minimal $\mathcal{N}=2$ supergravity [4]. The combination of a timelike Killing vector K^μ along with an $SU(2)_+$ triplet of 2-forms $\Phi^{(6)ab}$ with $i_K\Phi^{(6)ab}=0$ guarantees the existence of a preferred $SU(2)$ structure. However, the distinction between the $\mathcal{N}=2$ theory with $USp(2)\simeq SU(2)$ and the $\mathcal{N}=4$ theory with $USp(4)\simeq SO(5)$ is clear: in order to identify an $SU(2)$ structure in the $\mathcal{N}=4$ case, we had to impose the additional constraint (2.27) on the bilinears. Furthermore, as we will see in section 4, unlike for the $\mathcal{N}=2$ case, where the base had actually $SU(2)$ holonomy, here we find only the weaker condition of $SU(2)$ structure.

Identity structure. Finally, we note that if $(f^a)^2 > f^2$, then the nature of the solution is strikingly different. In particular, from (2.18), we see that the $SO(4)$ valued 1-forms $V^{(4)a}$ serve as vielbeins on the four-dimensional base transverse to K^μ. More precisely, by defining

$$e^a = \frac{1}{\sqrt{((f^b)^2 - f^2)}} V^{(4)a} \tag{2.31}$$

we are led to a natural five-dimensional metric (2.20) of the form $\mathrm{d}s^2 = -(e^0)^2 + (e^a)^2$ with $e^0 = \sqrt{(f^a)^2}(\mathrm{d}t+\omega)$ and e^a given above. So long as $(f^a)^2 > f^2$, the 2-forms $\Phi^{(6)ab}$ are completely determined by

$$\Phi^{(6)ab} = -f e^a \wedge e^b - |f^a| * (e^0 \wedge e^a \wedge e^b), \tag{2.32}$$

where we have taken $K = -(f^a)^2(\mathrm{d}t+\omega) = -|f^a| e^0$. Note that these 2-forms do not have any particular (anti-)self-duality properties on the base, as $|f^a| \neq |f|$.

Since the local frame fünfbein is completely determined in terms of the Killing spinor bilinears, then the structure group G has been reduced to identity. Furthermore, this $G = Id$ structure case has no counterpart in the $\mathcal{N}=2$ analysis. As a result, supersymmetric configurations falling in this class presumably would not admit a purely $\mathcal{N}=2$ interpretation. Of course, a more detailed examination would be in order to see if this is really the case.

2.2.2. The null case. The null case is given by $|K|^2 = -(f^a)^2 = 0$. From this, we infer that the five scalars f^a are all vanishing. Additionally, from the identity (A.6), namely $fK_\mu = f^a V^a_\mu$, we find that f vanishes as well (since we assume K_μ to be everywhere non-vanishing). Next, we may use the identity (A.17), given in form notation as $K\wedge V^a = -\Phi^{ab} f^b$ (=0 when $f^b=0$), to demonstrate that V^a is aligned with K. This allows us to write

$$V^a_\mu = u^a K_\mu. \tag{2.33}$$

The norm of the $SO(5)$ vector u^a is determined from (A.14)

$$V^a_\mu V^a_\nu = K_\mu K_\nu + g_{\mu\nu}((f^a)^2 - f^2) \tag{2.34}$$

to be equal to 1:

$$u^a u^a = 1. \tag{2.35}$$

Therefore (just as in the timelike case) the u^a's parametrize an $SO(5)/SO(4)$ coset.

Tuning now to the 2-forms Φ^{ab}, we recall that they generically transform as the **10** of $SO(5)$. However (A.20)

$$V^a_\mu \Phi^{ab}_{\nu\lambda} = -2g_{\mu[\nu}\big(fV^b_{\lambda} - f^b K_{\lambda]}\big) + \epsilon_{\mu\nu\lambda}{}^{\rho\sigma} V^b_\rho K_\sigma \tag{2.36}$$

reveals that the number of independent 2-forms Φ^{ab} is in fact constrained by

$$u^a \Phi^{ab} = 0, \tag{2.37}$$

and so we are left with only six 2-forms, corresponding to keeping the **6** in the decomposition **10** $\to$ **6** + **4** under $SO(5) \to SO(4)$. In fact, these six components are not all independent, as can be seen by consideration of the additional Fierz identity (A.21)

$$K_\mu \Phi^{ab}_{\nu\lambda} = -2g_{\mu[\nu}\left(f^a V^b_{\lambda} - f^b V^a_{\lambda]}\right) - \epsilon_{\mu\nu\lambda}{}^{\rho\sigma} V^a_\rho V^b_\sigma + \tfrac{1}{2}\epsilon^{abcde} V^c_\mu \Phi^{de}_{\nu\lambda}. \qquad (2.38)$$

This further constrains Φ^{ab} to satisfy a self-duality condition in group space

$$\Phi^{ab} = \tfrac{1}{2}\epsilon^{abcde} u^c \Phi^{de}. \qquad (2.39)$$

And so we are left with the $\mathbf{3}_+$ under the complete decomposition $\mathbf{10} \to \mathbf{6} + \mathbf{4} \to (\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2})$ of $SO(5) \to SO(4) \to SU(2)_+ \times SU(2)_-$. (Note that this $SU(2)_+$ is an internal symmetry group, and is at least superficially unrelated to the structure of the space.)

Finally, using (A.16) and (A.18)

$$\begin{aligned} K^\mu \Phi^{ab}_{\mu\nu} &= -2f^{[a} V^{b]}_\nu, \\ \epsilon_{\mu\nu\rho\sigma\lambda} K^\rho \Phi^{\sigma\lambda ab} &= -\epsilon^{abcde} f^c \Phi^{de}_{\mu\nu}, \end{aligned} \qquad (2.40)$$

we conclude that

$$\mathrm{i}_K \Phi^{ab} = \mathrm{i}_K * \Phi^{ab} = 0. \qquad (2.41)$$

This combination of a null Killing vector K^μ along with three independent 2-forms Φ^{ab} satisfying (2.41) demonstrates that there is an $\mathbb{R}^3$ structure associated with this null Killing vector case.

Although the above is sufficient to demonstrate $\mathbb{R}^3$ structure, we find it useful to make more explicit choices for the purpose of constructing solutions, which is taken up in section 4. In particular, using the appropriate differential identities below, we may demonstrate that K is hypersurface orthogonal, and that it can be chosen to be $K^\mu \partial_\mu = \partial_v$. This allows us to write the five-dimensional metric as

$$\mathrm{d}s^2 = H^{-1}(\mathcal{F}\,\mathrm{d}u^2 + 2\mathrm{d}u\,\mathrm{d}v) - H^2 h_{mn}(\mathrm{d}y^m + a^m\,\mathrm{d}u)(\mathrm{d}y^n + a^n\,\mathrm{d}u). \qquad (2.42)$$

We can now use the fact that the 2-forms Φ^{ab} are aligned with K to introduce a set of 1-forms X^{ab} according to

$$\Phi^{ab} = K \wedge X^{ab}. \qquad (2.43)$$

These three independent 1-forms X^{ab} reside on the three-dimensional base (with metric h_{mn}), and satisfy the multiplication rule

$$X^{ac}_m X^{bc}_n = -\epsilon_{mnp} X^{pab} + \Pi^{ab}_4 h_{mn}, \qquad (2.44)$$

which may be obtained from (A.25). Note that Π^{ab}_4 is given by (2.15), and is a projection onto $SO(4) \subset SO(5)$. Thus we are led to the conclusion that, in the null Killing vector case, the five-dimensional metric can be written in terms of a three-dimensional base (hypersurfaces orthogonal to the Killing vector), with three independent 1-forms X^{ab} satisfying $SO(4)$ self-duality with respect to u^a residing on it.

Note that it is important to keep in mind that $\mathbb{R}^3$ structure does not guarantee the closure of these 1-forms X^{ab}. Of course, in the minimal $\mathcal{N} = 2$ theory, the corresponding 1-forms constructed from the original 2-forms through similar steps as above were found to be closed [4]. In the $\mathcal{N} = 4$ case, however, this will happen only in special circumstances; this issue will be addressed at length in section 4, when we take up the differential identities in the null case.

2.3. Differential identities

Until now, we have mainly focused on the algebraic identities and the resulting structure equations. As is well known, the existence of an appropriate set of invariant tensors is sufficient to demonstrate the appropriate G-structure of the system, and this is what we have accomplished above using the algebraic identities. Integrability of these structures, however, falls into the realm of the differential identities, which we turn to next.

The differential identities encode the content of the Killing spinor equations, and hence depend explicitly on the model under investigation. For us, this is the minimal $\mathcal{N} = 4$ supergravity with Killing spinor equations corresponding to the vanishing of (2.6) and (2.7). However, we anticipate that this analysis could easily be extended to include the coupling to $\mathcal{N} = 4$ Maxwell multiplets as well. These Killing spinor equations may be converted into differential identities on the bispinors either by multiplication on the left with $\bar{\epsilon}\Gamma^{\mu_1\cdots\mu_n}$ or by taking the Hermitian conjugate and then multiplying on the right with $\Gamma^{\mu_1\cdots\mu_n}\epsilon$. As a result, these equations are (at most) first order and linear in the bispinors.

Note that, unlike in the case of minimal supergravity, where there was only the gravitino variation, here we also have the dilatino equation (2.7) to consider. The identities resulting from this dilatino condition are not truly differential, as they are only algebraic in the bilinears. We nevertheless denote all such expressions as 'differential identities' to distinguish them from the algebraic structure equations related to the Fierz identities. This notation of differential identities also fits a Kaluza–Klein interpretation, where the dilatino may be viewed as internal components of the higher dimensional gravitino. The loss of the derivative acting on the dilatino is then attributed to the zero-mode nature of the higher dimensional gravitino living on the compactification manifold.

The complete set of differential identities are tabulated in appendix B. This will provide a basis of the analysis in the following section for the timelike Killing vector case and section 4 for the null Killing vector case.

3. The timelike case

As indicated above, the timelike case falls into two categories, depending on the structure being either $SU(2)$ or Id. We focus mainly on the $SU(2)$ structure case, but will say a few words about the identity structure solutions at the end of this section.

3.1. Timelike with SU(2) structure

The $SU(2)$ structure case arises when $f^2 = (f^a)^2$, and is the most direct generalization of the analogous $\mathcal{N} = 2$ situation. To arrive at the complete solution, we start with the five-dimensional metric of the form (2.20)

$$\mathrm{d}s^2 = -f^2(\mathrm{d}t + \omega)^2 + f^{-1}h_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n, \qquad (3.1)$$

where $f, \omega = \omega_m\,\mathrm{d}x^m$, and h_{mn} are independent of time t. This metric admits a natural fünfbein basis

$$e^0 = f(\mathrm{d}t + \omega), \qquad e^{\overline{m}} = f^{-1/2}\hat{e}_m^{\overline{m}}\,\mathrm{d}x^m, \qquad (3.2)$$

where $h_{mn} = \hat{e}_m^{\overline{m}}\hat{e}_{n\overline{m}}$. We also note that, with our metric signature, we have $K^\mu\partial_\mu = \partial_t$ and $K_\mu\,\mathrm{d}x^\mu = -fe^0$.

We now proceed to derive the gauge field strengths F^a and G. To do so, we start with the decomposition

$$G = \alpha \wedge K + \overline{G}, \qquad F^a = \alpha^a \wedge K + \overline{F}^a, \qquad (3.3)$$

where α and α^a are 1-forms, and $\overline{G}$ and $\overline{F}^a$ are 2-forms on the four-dimensional base satisfying $\mathrm{i}_K\overline{G} = \mathrm{i}_K\overline{F}^a = 0$. Contractions with the Killing vector then gives

$$\begin{aligned} \mathrm{i}_K G &= f^2\alpha, & \mathrm{i}_K * G &= f *_4 \overline{G}, \\ \mathrm{i}_K F^a &= f^2\alpha^a, & \mathrm{i}_K * F^a &= f *_4 \overline{F}^a, \end{aligned} \tag{3.4}$$

where $*_4$ is defined with respect to the metric h_{mn} on the base.

The 1-forms α and α^a may be obtained in terms of the scalar quantities

$$\mathcal{H}_1^{-1} \equiv \mathrm{e}^{\frac{2}{\sqrt{6}}\phi} f, \qquad \mathcal{H}_2^{-1} \equiv \mathrm{e}^{-\frac{1}{\sqrt{6}}\phi} f, \tag{3.5}$$

through the use of (B.20) and (B.22), respectively. The result is

$$\alpha = f^{-2} d\big(\mathcal{H}_1^{-1}\big), \qquad \alpha^a = -f^{-2} d\big(u^a\mathcal{H}_2^{-1}\big). \tag{3.6}$$

The 'magnetic' components $\overline{G}$ and $\overline{F}^a$ are somewhat harder to disentangle. Nevertheless, use of the 2-form differential identities (B.24) and (B.26) allows us to solve for the (four-dimensional) self-dual and anti-self-dual components

$$\begin{aligned} &\mathcal{H}_1\overline{G}^- = -\mathcal{F}^-, \\ &\sqcap^a H_2\overline{F}^{a-} = \mathcal{F}^- \\ &\mathcal{H}_1\overline{G}^+ - 2u^a\mathcal{H}_2\overline{F}^{a+} = -\mathcal{F}^+, \\ &\Pi_4^{ab}\overline{F}^{b+} = 0. \end{aligned} \tag{3.7}$$

Here $\mathcal{F} = \mathrm{d}\omega$, and $\Pi_4^{ab} = \delta^{ab} - u^a u^b$ is the projection onto $SO(4)$. One further restriction on $\overline{F}^{a-}$ may be obtained from the identity (B.18), which gives the additional condition $\Phi^{ab}_{\mu\nu}F^{\mu\nu b} = 0$. Noting that the $SU(2)$ structure along with the Fierz identities ensure that the 2-form $\Phi^{ab}_{\mu\nu}$ is anti-self-dual on the base, we may deduce that the anti-self-dual component of $\overline{F}^a$ must vanish when projected with $\Phi^{ab}_{\mu\nu}$. This gives simply $\Pi_4^{ab}\overline{F}^{b-} = 0$, and when combined with (3.7), we see that $\overline{F}^a$ points only along the u^a direction.

The above relations, (3.7) along with the condition $\overline{F}^a \equiv u^a\overline{F}$, allow us to write the gauge field strengths in terms of $\mathcal{F} = \mathrm{d}\omega$ along with an undetermined self-dual 2-form $\overline{F}^+$. In particular, we may see that

$$\begin{aligned} G &= -\mathrm{d}\big[\mathcal{H}_1^{-1}(\mathrm{d}t + \omega)\big] + 2\frac{\mathcal{H}_2}{\mathcal{H}_1}\overline{F}^+, \\ F^a &= \mathrm{d}\big[u^a\mathcal{H}_2^{-1}(\mathrm{d}t + \omega)\big] + u^a\big(\overline{F}^+ - \mathcal{H}_2^{-1}\mathcal{F}^+\big). \end{aligned} \tag{3.8}$$

The Bianchi identities $dG = dF^a = 0$ immediately give

$$\mathrm{d}\left(\frac{\mathcal{H}_2}{\mathcal{H}_1}\overline{F}^+\right) = 0, \qquad \mathrm{d}\big(u^a\big(\overline{F}^+ - \mathcal{H}_2^{-1}\mathcal{F}^+\big)\big) = 0. \tag{3.9}$$

At the same time, the two form equations of motion

$$\mathrm{d}\big(\mathrm{e}^{\frac{2}{\sqrt{6}}\phi} * F^a\big) = F^a \wedge G, \qquad \mathrm{d}\big(\mathrm{e}^{-\frac{4}{\sqrt{6}}\phi} * G\big) = F^a \wedge F^a, \tag{3.10}$$

yield the conditions

$$\begin{aligned} \Box_4\mathcal{H}_1 &= \frac{1}{2}\big(\overline{F}^+ - \mathcal{H}_2^{-1}\mathcal{F}^+\big)^2, \\ \Box_4\mathcal{H}_2 &= -\frac{\mathcal{H}_2}{\mathcal{H}_1}\overline{F}^+\big(\overline{F}^+ - \mathcal{H}_2^{-1}\mathcal{F}^+\big) + \mathcal{H}_2 u^a\Box_4 u^a, \end{aligned} \tag{3.11}$$

along with the $SO(5)$ sigma-model equation of motion

$$\Box_4 u^a = u^a u^b \Box_4 u^b, \tag{3.12}$$

on the unit-norm $SO(5)$ vector u^a. Note that $\Box_4$ is the scalar Laplacian with respect to the metric h_{mn} on the base.

Until now, we have not paid much attention to the conditions on u^a. In addition to the sigma-model equation of motion given above, u^a must also satisfy a first-order condition

$$\partial_m u^a = \Phi^{ab}_{mn} h^{np} \partial_p u^b. \tag{3.13}$$

This condition, along with the expressions for F^a and G given in (3.8), guarantees that all 1-form and 2-form differential identities (B.20) through (B.27) are satisfied.

Finally, the remaining differential identities, and in particular (B.16) for $\nabla_\mu \Phi^{ab}_{\nu\lambda}$, demand that

$$\hat{\nabla}_m \Phi^{ab}_{np} = \tfrac{4}{3}\left[\Phi^{bc}_{m[p} u^a \partial_{n]} u^c + \Phi^{bc}_{np} u^a \partial_m u^c + \tfrac{1}{2}\epsilon_{mnp}{}^q u^a \partial_q u^b\right], \tag{3.14}$$

where all quantities are defined in terms of the metric h_{mn}. Making use of (3.13), along with anti-self-duality of Φ^{ab}_{mn} and the projection $u^a \Phi^{ab}_{mn} = 0$, the above expression can be written in the form

$$D_m \Phi^{ab}_{pq} \equiv \hat{\nabla}_m \Phi^{ab}_{pq} + \mathcal{A}^{ac}_m \Phi^{cb}_{pq} + \mathcal{A}^{bc}_m \Phi^{ac}_{pq} = 0, \tag{3.15}$$

where $\mathcal{A}^{ab}_m$ is the composite $SO(5)$ connection

$$\mathcal{A}^{ab} = 2u^{[a}\, \mathrm{d}u^{b]}. \tag{3.16}$$

This clearly shows that in the rigid case (where u^a is constant, so that $\mathcal{A}^{ab}_m$ vanishes), we have $\hat{\nabla}_m \Phi^{ab}_{pq} = 0$, which implies that the four-dimensional base has $SU(2)$ holonomy. However, in the general non-rigid case, the base only has $SU(2)$ structure. Note, also, that the fully antisymmetrized components of (3.14) may be written as

$$d\Phi^{ab} = *_4 \mathcal{A}^{ab}, \tag{3.17}$$

which demonstrates that the composite connection may also be given in terms of the 2-form Φ^{ab}.

Integrability of the covariant derivative D_m gives rise to

$$\hat{R}_{mnp}{}^r \Phi^{ab}_{rq} + \hat{R}_{mnq}{}^r \Phi^{ab}_{pr} + \mathcal{F}^{ac}_{mn} \Phi^{cb}_{pq} + \mathcal{F}^{bc}_{mn} \Phi^{ac}_{pq} = 0, \tag{3.18}$$

where the composite field strength is given by

$$\mathcal{F}^{ab}_{mn} \equiv 2\partial_{[m} \mathcal{A}^{ab}_{n]} + 2\mathcal{A}^{ac}_{[m} \mathcal{A}^{cb}_{n]} = \partial_m u^a \partial_n u^b - \partial_m u^b \partial_n u^a. \tag{3.19}$$

Contracting (3.18) with Φ^{ab}_{mq} and using the structure equations

$$\begin{aligned} \Phi^{ab}_{mn} \Phi^{ab}_{pq} &= 4\epsilon_{mnpq} + 4(h_{mp} h_{nq} - h_{mq} h_{np}), \\ \Phi^{ab}_{mn} \Phi^{nbc}_{p} &= 3\Pi^{ac}_4 h_{mp} - 2\Phi^{ac}_{mp}, \end{aligned} \tag{3.20}$$

results in the integrability condition $\hat{R}_{mn} = -\frac{1}{2}\mathcal{F}^{ab}_{mp} \Phi_n{}^{pab}$. Using (3.13), this gives the Einstein equation

$$\hat{R}_{mn} = -\partial_m u^a \partial_n u^a, \tag{3.21}$$

on the base. The combination of the equation of motion (3.12) along with the Einstein equation is suggestive of an $SO(5)$ sigma model coupled to gravity. However, in the present situation, the gravitational coupling in (3.21) is of the 'wrong sign', corresponding to a negative stress–energy tensor.

The above Einstein equation indicates that in the non-rigid case the four-dimensional base can no longer be Ricci-flat. This is an explicit demonstration that such solutions only have $SU(2)$ structure, and not holonomy. Before proceeding, we note that taking the trace of the

Einstein equation (3.21) and making use of the fact that u^a has unit norm (so that $u^a \partial_m u^a = 0$) gives us a simple expression for the four-dimensional curvature scalar

$$\hat{R} = u^a \Box_4 u^a. \tag{3.22}$$

If desired, this allows us to rewrite the scalar equations (3.11) and (3.12) as

$$\Box_4 \mathcal{H}_1 = \tfrac{1}{2}(G_2^+)^2, \qquad (\Box_4 - \hat{R})\mathcal{H}_2 = \tfrac{1}{2} G_1^+ G_2^+, \qquad (\Box_4 - \hat{R}) u^a = 0, \tag{3.23}$$

where we have defined the self-dual 2-forms G_1^+ and G_2^+ by

$$G_1^+ = -2(\mathcal{H}_2/\mathcal{H}_1)\overline{F}^+, \qquad G_2^+ = \overline{F}^+ - \mathcal{H}_2^{-1}\mathcal{F}^+. \tag{3.24}$$

This demonstrates that $\mathcal{H}_1$ behaves as a minimally coupled scalar, while $\mathcal{H}_2$ behaves as a non-minimally coupled scalar on the four-dimensional base.

Of course, in addition to the second-order equations (3.12) and (3.21), supersymmetry demands the stronger first-order condition (3.13) as well. As this condition is somewhat awkward to work with directly (since the 2-form Φ^{ab} is incompletely specified), it is instructive to directly examine the Killing spinor equations (2.6) and (2.7) for this $SU(2)$ structure timelike solution. Substituting in the expressions for the gauge fields (3.8), as well as the definitions for the scalars (3.5), we obtain the slightly cumbersome expressions

$$\begin{aligned}
\sqrt{3} f^{-1/2}\delta\chi_i = {} & \left[i\delta_i^j \hat{\gamma}^m \partial_m \log\mathcal{H}_1 - \frac{1}{4} f^{3/2}\mathcal{H}_2\big(\big(2\overline{F}^+_{mn} - \mathcal{H}_2^{-1}\mathcal{F}^+_{mn}\big)\delta_i^j + \overline{F}^+_{mn} u^a T_i^{a\,j}\big)\hat{\gamma}^{mn}\right] P_1\epsilon_j \\
& - \big[i\hat{\gamma}^m \partial_m \log\mathcal{H}_2\big] P_2\epsilon_i + \left[\frac{1}{2} f^{3/2}\mathcal{F}^-_{mn}\hat{\gamma}^{mn}\right] P_3\epsilon_i + \frac{1}{2}\Gamma^0\big(\hat{\gamma}^m \partial_m u^a T_i^{a\,j}\big)\epsilon_j, \\
f^{-1/2}\delta\psi_{ti} = {} & \left[-\frac{1}{3}\delta_i^j \Gamma_0 \hat{\gamma}^m \partial_m \log\mathcal{H}_1 + \frac{1}{12} f^{3/2}\mathcal{H}_2\big(\big(\overline{F}^+_{mn} - 2\mathcal{H}_2^{-1}\mathcal{F}^+_{mn}\big)\delta_i^j + \overline{F}^+_{mn} u^a T_i^{a\,j}\big)\hat{\gamma}^{mn}\right] P_1\epsilon_j \\
& - \left[\frac{2}{3}\Gamma_0 \hat{\gamma}^m \partial_m \log\mathcal{H}_2\right] P_2\epsilon_i - \left[\frac{1}{6} f^{3/2}\mathcal{F}^-_{mn}\hat{\gamma}^{mn}\right] P_3\epsilon_i - \frac{\mathrm{i}}{3}\big(\hat{\gamma}^m \partial_m u^a T_i^{a\,j}\big)\epsilon_j, \\
\delta\psi_{mi} - \omega_m \delta\psi_{ti} = {} & \big(\mathcal{H}_1\mathcal{H}_2^2\big)^{-1/6}\left[\hat{\nabla}_m \delta_i^j - \frac{1}{2}\partial_m u^a T_i^{a\,j}\right]\big(\mathcal{H}_1\mathcal{H}_2^2\big)^{1/6}\epsilon_j \\
& + \left[\frac{1}{6}\delta_i^j\big(\hat{\gamma}_m{}^n - 2\delta_m^n\big)\partial_n \log\mathcal{H}_1 + \partial_m u^a T_i^{a\,j} - \frac{1}{3} f^{3/2}\big(\mathcal{F}^+_{mn} + 3\mathcal{F}^-_{mn}\big)\delta_i^j \hat{\gamma}^n \Gamma^0\right] P_1\epsilon_j \\
& + \left[\frac{1}{3}\delta_i^j\big(\hat{\gamma}_m{}^n - 2\delta_m^n\big)\partial_n \log\mathcal{H}_2\right] P_2\epsilon_i + \left[\frac{1}{3} f^{3/2}\mathcal{H}_2\big(-\overline{F}^+_{mn}(1 + 2i\Gamma^0)\right. \\
& \left. + \mathcal{H}_2^{-1}\mathcal{F}^-_{mn}(1 - 2i\Gamma^0)\big)\hat{\gamma}^n \Gamma^0\right] P_3\epsilon_i - \frac{\mathrm{i}}{6}\Gamma^0 \hat{\gamma}_m\big(\hat{\gamma}^n \partial_n u^a T_i^{a\,j}\big)\epsilon_j.
\end{aligned} \tag{3.25}$$

Here we have defined the projections

$$P_1 = \tfrac{1}{2}(1 + i\Gamma^0), \qquad P_{2i}{}^j = \tfrac{1}{2}\big(\delta_i^j + i\Gamma^0 u^a T_i^{a\,j}\big), \qquad P_{3i}{}^j = \tfrac{1}{2}\big(\delta_i^j - u^a T_i^{a\,j}\big). \tag{3.26}$$

Note, also, that the Dirac matrices $\hat{\gamma}^m$ are defined with respect to the base metric h_{mn}.

The three projections in (3.26) are mutually commuting, and are furthermore degenerate, with $P_{2i}{}^j = P_{3i}{}^j + u^a T_i^{a\,j} P_1$. As a result, the generic solution preserves at most $1/4$ of the supersymmetries, with $1/2$ also possible in special cases (when some of the fields are not active). Note, however, that preservation of supersymmetry demands the additional requirement

$$\big(\hat{\gamma}^m \partial_m u^a T_i^{a\,j}\big)\epsilon_j = 0, \tag{3.27}$$

which is trivially satisfied only in the rigid case. In fact, the rigid case is particularly simple; so long as ϵ_i is projected out by (3.26), the surviving requirement on ϵ_i for it to be a Killing spinor is simply the parallel spinor equation

$$\hat{\nabla}_m \epsilon_i = 0. \tag{3.28}$$

In this case, the base has $SU(2)$ holonomy, and the solution is either $1/2$ or $1/4$ supersymmetric, depending on the set of active fields.

In the non-rigid case, however, the situation is rather more involved. For ϵ_i to be a Killing spinor, it must not only be projected out by (3.26), but must also satisfy the sigma-model requirement (3.27). Provided this is the case, the content of the supersymmetry variations (3.25) reduces to

$$\left[\hat{\nabla}_m \delta_i{}^j - \tfrac{1}{2}\partial_m u^a T_i^{a\,j}\right]\varepsilon_j = 0, \tag{3.29}$$

where $\varepsilon_i = \left(\mathcal{H}_1\mathcal{H}_2^2\right)^{1/6}\epsilon_i$. It is easily shown that integrability of this Killing spinor equation gives rise to an Einstein equation identical to (3.21). In order to count the number of preserved supersymmetries, we have to identify $USp(4)$ symplectic-Majorana spinors ϵ_i which simultaneously satisfy the conditions given above. Generically, (3.27) may be considered as a sum of four terms, one for each direction on the base ($m = 1, 2, 3, 4$). Schematically, the Killing spinor condition is then of the form $\pm a \pm b \pm c \pm d = 0$, with all possible combinations of signs. With 16 possibilities, and the observation that if one choice of signs satisfies this condition, then the completely opposite choice would too, we see that this generically yields a $1/8$ supersymmetric projection. Combining (3.27) with any single projection from (3.26) leaves the solution $1/8$ supersymmetric, while combining this with two projections gives a solution that is $1/16$ supersymmetric (i.e. with a single supersymmetry out of the original 16).

Although we have not done so, it would be noteworthy to tabulate all possible fractions of preserved supersymmetries. This would entail a somewhat more sophisticated investigation of (3.27) to identify special cases away from the generic $1/8$ fraction of supersymmetry and to ensure their compatibility with the projections of (3.26). (Kinematically, the projection (3.27) alone gives the fractions 0, 1, 2, 3, 4, 8 out of 8. However, it remains to be seen whether all such possibilities can be realized.) In this respect, the tools of generalized holonomy [19, 20] may also be useful in enumerating the possibilities.

To summarize, the supersymmetric timelike solutions with $SU(2)$ structure are given by the bosonic fields

$$\begin{aligned}
&\mathrm{d}s^2 = -\left(\mathcal{H}_1\mathcal{H}_2^2\right)^{-2/3}(\mathrm{d}t+\omega)^2 + \left(\mathcal{H}_1\mathcal{H}_2^2\right)^{1/3} h_{mn}\,\mathrm{d}x^m\,\mathrm{d}x^n,\\
&G = -\mathrm{d}\left[\mathcal{H}_1^{-1}(\mathrm{d}t+\omega)\right] - G_1^+, \qquad F^a = d\left[u^a\mathcal{H}_2^{-1}(\mathrm{d}t+\omega)\right] + u^a G_2^+,\\
&\mathrm{e}^{\frac{3}{\sqrt{6}}\phi} = \mathcal{H}_2/\mathcal{H}_1,
\end{aligned} \tag{3.30}$$

where self-duality (the + superscript) is with respect to the four-dimensional base metric h_{mn}. The solution is specified by the set of functions (fields)

$$u^a, \qquad \mathcal{H}_1, \qquad \mathcal{H}_2, \qquad G_1^+, \qquad G_2^+, \qquad h_{mn}, \tag{3.31}$$

which satisfy the Bianchi identities (3.9)

$$\mathrm{d}G_1^+ = 0. \qquad \mathrm{d}\left(u^a G_2^+\right) = 0, \tag{3.32}$$

scalar equations of motion (3.23)

$$\Box_4\mathcal{H}_1 = \tfrac{1}{2}\left(G_2^+\right)^2, \qquad (\Box_4 - \hat{R})\mathcal{H}_2 = \tfrac{1}{2}G_1^+G_2^+, \qquad (\Box_4 - \hat{R})u^a = 0. \tag{3.33}$$

Einstein equation on the base (3.21)

$$\hat{R}_{mn} = -\partial_m u^a \partial_n u^a, \tag{3.34}$$

the relation

$$(\mathrm{d}\omega)^+ = -\tfrac{1}{2}\mathcal{H}_1 G_1^+ - \mathcal{H}_2 G_2^+, \tag{3.35}$$

and also the sigma-model supersymmetry conditions (3.27) and (3.29). (Actually, the Killing spinor condition (3.29) implies the Einstein equation (3.34) on the base.)

In the rigid case (u^a = constant), the base metric h_{mn} has $SU(2)$ holonomy, and this $\mathcal{N} = 4$ solution becomes a straightforward generalization of the timelike $\mathcal{N} = 2$ case analysed in [4]. Viewed from an $\mathcal{N} = 2$ perspective, the rigid case is essentially that of $\mathcal{N} = 2$ supergravity coupled with a single vector multiplet. This results in a 'two-charge' extension of the 'one-charge' (graviphoton only) solution given in [4], and is the origin of the second harmonic function $\mathcal{H}_2$ along with a second self-dual 2-form G_2^+. From an $\mathcal{N} = 2$ point of view, these solutions preserve either 0, 1/2 or all of the supersymmetries, while under $\mathcal{N} = 4$ they may preserve either 0, 1/4, 1/2 or all of the supersymmetries.

Of course, in the non-rigid case, additional fractions (such as 1/16 and 1/8) are also allowed. To better understand this non-rigid case, we note that the $\mathcal{N} = 4$ supergravity multiplet

$$\left(g_{\mu\nu}, A_\mu^{[ij]|}, B_\mu, \phi, \chi^i, \psi_\mu^i\right) \tag{3.36}$$

admits the decomposition into an $\mathcal{N} = 2$ supergravity multiplet coupled to one vector and one gravitino multiplet

$$\left(g_{\mu\nu}, A_\mu, \psi_\mu^i\right) + (A_\mu, \phi, \chi^i) + \left(A_\mu^{iI}, \chi^I, \psi_\mu^I\right) \tag{3.37}$$

(where $i = 1, 2$ and $I = 1, 2$). The graviphoton along with the vector in the vector multiplet is a linear combination of B_μ and $u^a A_\mu^a$ (i.e. the component of A_μ^a along u^a). These two $U(1)$ fields carry electric components characterized by $\mathcal{H}_1$ and $\mathcal{H}_2$ as well as magnetic components given by G_1^+ and G_2^+,

$$-G = d\left[\mathcal{H}_1^{-1}(\mathrm{d}t + \omega)\right] + G_1^+, \qquad u^a F^a = d\left[\mathcal{H}_2^{-1}(\mathrm{d}t + \omega)\right] + G_2^+. \tag{3.38}$$

The remaining four field strengths in the gravitino multiplet are given by projection with Π_4^{ab},

$$\Pi_4^{ab} F^b = \mathrm{d}u^a \wedge \left[\mathcal{H}_2^{-1}(\mathrm{d}t + \omega)\right], \tag{3.39}$$

and are only active in the non-rigid case. Thus, from an $\mathcal{N} = 2$ point of view, the non-rigid case corresponds to excitations of the gravitino multiplet. Because of this, such non-rigid solutions are true $\mathcal{N} = 4$ configurations without corresponding realization within an $\mathcal{N} = 2$ truncation.

3.2. Timelike with Id structure

We now turn to the identity structure case, which occurs when $(f^a)^2 > f^2$. As demonstrated in section 2.2.1, the Killing spinor bilinear 1-forms $V^{(4)a}$ define a natural vielbein basis for the metric of the form

$$\begin{aligned}\mathrm{d}s^2 &= -(e^0)^2 + (e^a)^2 \\ &= -(f^a)^2(\mathrm{d}t + \omega)^2 + ((f^b)^2 - f^2)^{-1} V_\mu^{(4)a} V_\nu^{(4)a}.\end{aligned} \tag{3.40}$$

Recall that, although the $SO(5)$ index a runs from 1 through 5, the constraint $u^a V_\mu^{(4)a} = 0$ ensures that it only takes values in the **4** of $SO(4)$. From (3.40), it is clear that a fünfbein local frame is completely determined by the Killing spinor.

In order to obtain the full solution with Id structure, we make use of the fact that all spinor bilinears except f and f^a are fully specified in terms of the metric and vielbein elements through (2.31) and (2.25). In this case, we may solve directly for the gauge fields F^a and G by noting that an arbitrary 2-form $\mathcal{F}$ obeys the relation

$$\mathrm{i}_K * \mathrm{i}_K * \mathcal{F} = -K^2\mathcal{F} + K \wedge \mathrm{i}_K\mathcal{F}. \tag{3.41}$$

Taking $K^2 = -(f^a)^2$ then allows us to write

$$(f^a)^2\mathcal{F} = -K \wedge (\mathrm{i}_K\mathcal{F}) + \mathrm{i}_K * (\mathrm{i}_K * \mathcal{F}), \tag{3.42}$$

which essentially splits $\mathcal{F}$ into components along K and orthogonal to K.

In fact, the differential identities (B.20), (B.22), (B.24) and (B.26) provide sufficient information for disentangling all components of F^a and G through use of the above relation. In this manner, we obtain

$$\begin{aligned}
&\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}(f^a)^2G - 2\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}f(f^aF^a)\\
&\quad= \mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}K \wedge \mathrm{d}\big(\mathrm{e}^{\frac{2}{\sqrt{6}}\phi}f\big) - \mathrm{e}^{\frac{2}{\sqrt{6}}\phi}\mathrm{i}_K * \mathrm{d}\big(\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}K\big) + 2\,\mathrm{e}^{-\frac{1}{\sqrt{6}}\phi}f^a\mathrm{d}\big(\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}V^a\big),\\
&\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}ff^aG + \mathrm{e}^{\frac{1}{\sqrt{6}}\phi}[((f^a)^2 - f^2)\delta^{ab} - 2f^af^b]F^b\\
&\quad= \mathrm{e}^{\frac{1}{\sqrt{6}}\phi}K \wedge \mathrm{d}\big(\mathrm{e}^{-\frac{1}{\sqrt{6}}\phi}f^a\big) - \mathrm{e}^{-\frac{1}{\sqrt{6}}\phi}\mathrm{i}_K * \mathrm{d}\big(\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}V^a\big) + \mathrm{e}^{\frac{2}{\sqrt{6}}\phi}f^a\mathrm{d}\big(\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}K\big)\\
&\qquad+ \mathrm{e}^{-\frac{1}{\sqrt{6}}\phi}f\mathrm{d}\big(\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}V^a\big).
\end{aligned}\tag{3.43}$$

Solving this for F^a and G gives

$$\begin{aligned}
((f^a)^2 - f^2)(f^b)^2G &= 2f\big[\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}f^a\mathrm{i}_K * \mathrm{d}\big(\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}V^a\big) - \mathrm{e}^{\frac{4}{\sqrt{6}}\phi}(f^a)^2\mathrm{d}\big(\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}K\big)\\
&\quad- \mathrm{e}^{\frac{3}{\sqrt{6}}\phi}f^aK \wedge \mathrm{d}\big(\mathrm{e}^{-\frac{1}{\sqrt{6}}\phi}f^a\big)\big] + 2\,\mathrm{e}^{\phi/\sqrt{6}}(f^a)^2f^b\mathrm{d}\big(\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}V^b\big)\\
&\quad+ ((f^a)^2 + f^2)\big[K \wedge \mathrm{d}\big(\mathrm{e}^{\frac{2}{\sqrt{6}}\phi}f\big) - \mathrm{e}^{\frac{4}{\sqrt{6}}\phi}\mathrm{i}_K * \mathrm{d}\big(\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}K\big)\big],\\
((f^b)^2 - f^2)(f^c)^2F^a &= f\big[-\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}f^a\mathrm{i}_K * \mathrm{d}\big(\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}K\big) + \mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}(f^b)^2\mathrm{d}\big(\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}V^a\big)\\
&\quad+ \mathrm{e}^{-\frac{3}{\sqrt{6}}\phi}f^aK \wedge \mathrm{d}\big(\mathrm{e}^{\frac{2}{\sqrt{6}}\phi}f\big)\big] - \mathrm{e}^{\frac{1}{\sqrt{6}}\phi}f^a(f^b)^2\mathrm{d}\big(\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}K\big)\\
&\quad+ (2f^af^b - \delta^{ab}(f^c)^2)\big[-K \wedge \mathrm{d}\big(\mathrm{e}^{-\frac{1}{\sqrt{6}}\phi}f^b\big) + \mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}\mathrm{i}_K * \mathrm{d}\big(\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}V^b\big)\big].
\end{aligned}\tag{3.44}$$

By decomposing the vector V^a according to (2.17), we finally arrive at the expressions

$$\begin{aligned}
((f^a)^2 - f^2)(f^b)^2G &= -((f^a)^2 - f^2)\big[K \wedge \mathrm{d}\big(\mathrm{e}^{\frac{2}{\sqrt{6}}\phi}f\big) + \mathrm{e}^{\frac{2}{\sqrt{6}}\phi}\mathrm{i}_K * \mathrm{d}K\big]\\
&\quad+ 2\,\mathrm{e}^{\frac{2}{\sqrt{6}}\phi}[(f^b)^2(V^{(4)a} \wedge df^a) + f\mathrm{i}_K * (V^{(4)a} \wedge df^a)],\\
((f^b)^2 - f^2)(f^c)^2F^a &= ((f^c)^2 - f^2)\big[(f^c)^2K \wedge \mathrm{d}\big(\mathrm{e}^{-\frac{1}{\sqrt{6}}\phi}f^a(f^c)^{-2}\big) + \mathrm{e}^{-\frac{1}{\sqrt{6}}\phi}f^a\,\mathrm{d}K\big]\\
&\quad+ 2\,\mathrm{e}^{-\frac{1}{\sqrt{6}}\phi}f^a\mathrm{i}_K * (V^{(4)b} \wedge df^b)\\
&\quad+ \mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}(f^b)^2\big[f\mathrm{d}\big(\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}V^{(4)a}\big) - \mathrm{i}_K * \mathrm{d}\big(\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}V^{(4)a}\big)\big].
\end{aligned}\tag{3.45}$$

Note that, just as in section 2.2.1, these expressions become trivial when $(f^a)^2 = f^2$.

To obtain a complete solution, we must demand that the Bianchi identities and equations of motion hold for the gauge fields given by (3.45). We have left this as an exercise to the ambitious reader. Nevertheless, we expect the procedure to be similar to that of the $SU(2)$ structure case, and hence we expect to obtain second-order equations of a form similar to (3.11). Note, however, that here a decomposition of the magnetic components of F^a and G into self-dual and anti-self-dual components on the base does not appear natural; instead the Hodge duality in (3.45) implies something more along the lines of taking $\widetilde{\mathcal{F}} = (f + |f^a|*_4)\mathcal{F}$, which is not a projection.

In addition, we must still ensure that the remaining differential identities are satisfied. Presumably this will lead to a sigma-model equation identical to (3.12) for the unit-norm vector $u^a = f^a/|f^a|$, as well as first-order conditions of the form (3.27)

$$(\hat{\gamma}^m \partial_m u^a T_i^{aj})\epsilon_j = 0. \tag{3.46}$$

From this point of view, the supersymmetry analysis of the Id structure case is rather similar to that of the $SU(2)$ structure case given in the previous section. A potentially important distinction, however, is that in the present case the Killing spinor ϵ_i does *not* satisfy the simple time-direction projection $P_1\epsilon_i = 0$ with the $SU(2)$ structure projection P_1 given by (3.26). (A simple way to see this is to realize that $P_1\epsilon_i = 0$ implies that $K^{\overline{\mu}}$ points only in the 0 direction, and that this in turn gives $K^2 = -f^2$. When combined with the Fierz identity $K^2 = -(f^a)^2$, one obtains the $SU(2)$ structure case $f^2 = (f^a)^2$.) As a result, the counting of supersymmetries will presumably differ from that of the $SU(2)$ structure case.

4. The null case

In this section we study the implications of having a null Killing vector, and in particular use the differential identities to construct the general class of supersymmetric backgrounds with $\mathbb{R}^3$ structure. We first observe that, since in this case all scalar bispinors vanish ($f = f^a = 0$), the differential identity (B.24) reduces to

$$\mathrm{d}\big(\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi} K\big) = \mathrm{i}_K\big(\mathrm{e}^{-\frac{4}{\sqrt{6}}\phi} * G\big). \tag{4.1}$$

Contracting with K^μ in turn implies that

$$K \cdot \mathrm{d}K = 0. \tag{4.2}$$

Moreover, with $f = 0$, we have $\mathrm{i}_K G = 0$ from (B.20). Thus

$$K \wedge \mathrm{d}K = 0. \tag{4.3}$$

We now infer from (4.2) and (4.3) that the Killing vector K^μ is such that it is hypersurface-orthogonal and may be written as

$$K_\mu\, \mathrm{d}x^\mu = H^{-1}\,\mathrm{d}u, \qquad K^\mu \partial_\mu = \partial_v, \tag{4.4}$$

where we have parametrized the five-dimensional spacetime in terms of the coordinates (u, v, y^m) with $m = 1, 2, 3$. The coordinate v is the affine parameter along the geodesics of constant u. In particular, the five-dimensional metric can be written as

$$\mathrm{d}s^2 = H^{-1}(\mathcal{F}\,\mathrm{d}u^2 + 2\mathrm{d}u\,\mathrm{d}v) + H^2 h_{mn}(\mathrm{d}y^m + a^m\,\mathrm{d}u)(\mathrm{d}y^n + a^n\,\mathrm{d}u). \tag{4.5}$$

Given that ∂_v is an isometry generator, all the functions that appear in the metric are v-independent. For later convenience, we note that this metric admits a natural vielbein basis

$$e^+ = H^{-1}\,\mathrm{d}u, \qquad e^- = \mathrm{d}v + \tfrac{1}{2}\mathcal{F}\,\mathrm{d}u, \qquad \mathrm{e}^{\bar{m}} = H\hat{e}_m^{\bar{m}}(\mathrm{d}y^m + a^m\,\mathrm{d}u), \tag{4.6}$$

where the dreibeins $\hat{e}^{\bar{m}}$ are related to the three-dimensional base according to

$$\hat{e}_m^{\bar{m}}\hat{e}_n^{\bar{m}} = h_{mn}. \tag{4.7}$$

Furthermore, although a u dependent coordinate transformation may be used to eliminate the shift vectors a^m, just as in [4] we find it useful to keep this metric general, at least for the moment.

We now recall some of the results derived in section 2.2.2 for $\mathbb{R}^3$ structure, namely that in the null case the 1-forms V^a as well as the 2-forms Φ^{ab} are all aligned with K,

$$V^a = u^a K, \qquad \Phi^{ab} = K \wedge X^{ab}. \tag{4.8}$$

In order to construct the supersymmetric solutions of $\mathcal{N} = 4$ supergravity characterized by a null Killing vector, we must go beyond the structure equations and use the differential identities tabulated in appendix B to express the solutions in terms of the spinor bilinears, and then to solve for as many of the bispinors as possible.

From (B.20) and (B.22), we find the gauge field strengths of the six abelian gauge fields are such that $\mathrm{i}_K F^a = \mathrm{i}_K G = 0$. This allows us to introduce the decomposition

$$F^a = F^a_{+\bar{m}}\, \mathrm{e}^+ \wedge \mathrm{e}^{\bar{m}} + \tfrac{1}{2} F^a_{\bar{m}\bar{n}}\, \mathrm{e}^{\bar{m}} \wedge \mathrm{e}^{\bar{n}}, \qquad G = G_{+\bar{m}}\, \mathrm{e}^+ \wedge \mathrm{e}^{\bar{m}} + \tfrac{1}{2} G_{\bar{m}\bar{n}}\, \mathrm{e}^{\bar{m}} \wedge \mathrm{e}^{\bar{n}}. \tag{4.9}$$

Furthermore, the components $F^a_{\bar{m}\bar{n}}$ and $G_{\bar{m}\bar{n}}$ lying on the three-dimensional base can be found from the $(m+)$ components of (B.24) and (B.25). Concretely, we obtain

$$\hat{F}^a_{mn} = H^{-2}\epsilon_{mn}{}^p(u^a\partial_p\mathcal{H}_2 - \mathcal{H}_2\partial_p u^a), \qquad \hat{G}_{mn} = -H^{-2}\epsilon_{mn}{}^p\partial_p\mathcal{H}_1, \tag{4.10}$$

where the hatted quantities are defined with respect to the three-dimensional base

$$\hat{F}^a_{mn} \equiv \hat{e}^{\bar{m}}_m \hat{e}^{\bar{n}}_n F^a_{\bar{m}\bar{n}}, \qquad \hat{G}_{mn} \equiv \hat{e}^{\bar{m}}_m \hat{e}^{\bar{n}}_n G_{\bar{m}\bar{n}}. \tag{4.11}$$

The new functions $\mathcal{H}_1, \mathcal{H}_2$ showing up in (4.10) are defined as

$$\mathcal{H}_1 = \mathrm{e}^{\frac{2}{\sqrt{6}}\phi} H, \qquad \mathcal{H}_2 = \mathrm{e}^{-\frac{1}{\sqrt{6}}\phi} H. \tag{4.12}$$

Enforcing the Bianchi identities leads to the second-order equations

$$\Box_3\mathcal{H}_1 = 0, \qquad u^a\Box_3\mathcal{H}_2 - \mathcal{H}_2\Box_3 u^a = 0, \tag{4.13}$$

as well as the constraints

$$\frac{1}{\sqrt{h}}\partial_u(\sqrt{h}h^{mn}\partial_n\mathcal{H}_1) = -\epsilon^{mnp}\partial_n(\hat{G}_{+p} + \epsilon_{pq}{}^r a^q\partial_r\mathcal{H}_1),$$
$$\frac{1}{\sqrt{h}}\partial_u(\sqrt{h}h^{mn}(u^a\partial_n\mathcal{H}_2 - \mathcal{H}_2\partial_n u^a)) = \epsilon^{mnp}\partial_n(\hat{F}^a_{+p} - \epsilon_{pq}{}^r a^q(u^a\partial_r\mathcal{H}_2 - \mathcal{H}_2\partial_r u^a)), \tag{4.14}$$

where $\hat{F}^a_{+m} \equiv \hat{e}^{\bar{m}}_m F^a_{+\bar{m}}$ and $\hat{G}_{+m} \equiv \hat{e}^{\bar{m}}_m G_{+\bar{m}}$. The second-order equations (4.13) demonstrate that $\mathcal{H}_1$ is harmonic as a function of y^m. At the same time, the equation for $\mathcal{H}_2$ decomposes into the system

$$\Box_3\mathcal{H}_2 = \mathcal{H}_2 u^a\Box_3 u^a, \qquad \Box_3 u^a = u^a u^b\Box_3 u^b. \tag{4.15}$$

Note the similarity with the corresponding equations (3.11) and (3.12) in the timelike case. In particular, this reveals that the u^a's define an O(5) vector model (this time on the three-dimensional base as opposed to a four-dimensional base in the timelike case).

The equations of motion for the field strengths provide additional constraints on the null components $\hat{F}^a_{+m}$ and $\hat{G}_{+m}$. However we defer these to later, and instead focus first on the 1-forms X^{ab} on the three-dimensional base. As in the timelike case, these turn out to be closely related to the behaviour of the O(5) vector u^a. Starting from (B.23), we see that $\Phi^{ab} \wedge F^b = 0$, which yields the condition

$$X^{ab}_m h^{mn}\partial_n u^b = 0. \tag{4.16}$$

Furthermore, from (B.27), we find the relation

$$\mathrm{d}u^a = - *_3 X^{ab} \wedge \mathrm{d}u^b, \tag{4.17}$$

where $*_3$ is defined with respect to the metric h_{mn}. In addition, the $(+[mn])$ component of the differential identity obeyed by the 2-form Φ^{ab} (B.16) yields

$$dX^{ab} = 2 *_3 u^{[a}\,\mathrm{d}u^{b]}. \tag{4.18}$$

The previous two equations can be combined into

$$dx^{ab} + \mathcal{A}^{ac} \wedge X^{cb} + \mathcal{A}^{bc} \wedge X^{ac} = 0, \qquad \mathcal{A}^{ab} = 2u^{[a}\, du^{b]}, \tag{4.19}$$

where we have introduced the composite O(5) connection $\mathcal{A}^{ab}$ (3.16). Note that, in contrast to the minimal $\mathcal{N}=2$ supergravity, the 1-forms X^{ab} are not generically closed (i.e. for non-trivial u^a configurations). A bit more work is required to extract

$$\hat{\nabla}_m X^{ab}_n = 2X^{c[a}_{(m} u^{b]} \partial_{n)} u^c + \epsilon_{mn}{}^p u^{[a} \partial_p u^{b]}, \tag{4.20}$$

from the same (B.16). Using (4.17), we obtain the direct analogue of (3.15)

$$D_m X^{ab}_n \equiv \hat{\nabla}_m X^{ab}_n + \mathcal{A}^{ac}_m X^{cb}_n + \mathcal{A}^{bc}_m X^{ac}_n = 0. \tag{4.21}$$

We now return to the null components of the field strengths. The conditions of interest follow most directly from $\nabla_+ V^a_+$:

$$\nabla_+ u^a = \mathcal{H}_2^{-1} X^{ab}_{\bar{m}} F^b_{+\bar{m}} \tag{4.22}$$

and from $\nabla_+ \Phi^{ab}_{+\bar{m}}$:

$$\begin{aligned} \nabla_+ \Phi^{ab}_{+\bar{m}} &= \nabla_+ X^{ab}_{\bar{m}} - \hat{e}^m_{\bar{m}} \partial_{[m} a_{n]} X^{nab} \\ &= \tfrac{1}{2}\epsilon_{\bar{m}\bar{n}\bar{p}} \left[\mathcal{H}_1^{-1} X^{ab}_{\bar{n}} G_{+\bar{p}} - 2\mathcal{H}_2^{-1}\left(2u^{[a} X^{b]c}_{\bar{n}} + u^c X^{ab}_{\bar{n}}\right) F^c_{+\bar{p}}\right] - 2\mathcal{H}_2^{-1} u^{[a} F^{b]}_{+\bar{m}}. \end{aligned} \tag{4.23}$$

Note that $\nabla_+ = \partial_u - a^m \hat{\nabla}_m$. We decompose $\hat{F}^a_{+m}$ into $SO(4)$ components according to

$$\hat{F}^a_{+m} = u^a \hat{F}_{+m} + \hat{F}^{(4)a}_{+m}, \tag{4.24}$$

where $\hat{F}_{+m} = u^a \hat{F}^a_{+m}$ and $\hat{F}^{(4)a}_{+m} = \Pi^{ab}_4 \hat{F}^b_{+m}$. In this case, (4.22) and (4.23) give rise to

$$\begin{aligned} &X^{mab}\left(\mathcal{H}_2^{-1} \hat{F}^{(4)b}_{+m}\right) = \nabla_+ u^a, \\ &\mathcal{H}_1^{-1} \hat{G}_{+m} - 2\mathcal{H}_2^{-1} \hat{F}_{+m} = -\epsilon_m{}^{np}\left[\partial_n a_p + \tfrac{1}{4} X^{ab}_n \partial_u X^{ab}_p\right]. \end{aligned} \tag{4.25}$$

The $SO(4)$ singlet term was given by multiplying both sides of (4.23) with $X^{ab}_{\bar{q}}$ and using the relation

$$X^{ab}_m X^{ab}_n = 4h_{mn}, \tag{4.26}$$

which follows from (2.44). In addition, we have used the fact that $X^{ab}_p \hat{\nabla}_m X^{ab}_n = 0$ (which follows from contracting (4.20) with X^{ab}_p) to write $X^{ab}_n \nabla_+ X^{ab}_p = X^{ab}_n \partial_u X^{ab}_p$ in (4.25).

In contrast to the null solution of $\mathcal{N}=2$ supergravity [4], here the null components of the field strengths are only incompletely determined. Additional requirements on these components may be obtained from the 2-form equations of motion. With some manipulation, the equations of motion (3.10) give rise to

$$\begin{aligned} \hat{\nabla}^m\left(\mathcal{H}_1^{-1} \hat{G}_{+m}\right) &= -2\left(\mathcal{H}_{-1} \hat{G}_{+m} + \mathcal{H}_2^{-1} \hat{F}_{+m}\right) h^{mn} \partial_n \log \mathcal{H}_2 + 2\mathcal{H}_2^{-1} \hat{F}^a_{+m} h^{mn} \partial_n u^a, \\ \hat{\nabla}^m\left(\mathcal{H}_2^{-1} \hat{F}^a_{+m}\right) &= -\left(u^a \mathcal{H}_1^{-1} \hat{G}_{+m} + \mathcal{H}_2^{-1} \hat{F}^a_{+m}\right) h^{mn} \partial_n \log \mathcal{H}_2 + \mathcal{H}_1^{-1} \hat{G}_{+m} h^{mn} \partial_n u^a. \end{aligned} \tag{4.27}$$

Note that subtracting twice the $SO(5)$ singlet component of the $\hat{F}^a_{+m}$ equation from the $\hat{G}_{+m}$ equation gives rise to the divergence free condition

$$\hat{\nabla}^m\left(\mathcal{H}_1^{-1} \hat{G}_{+m} - 2\mathcal{H}_2^{-1} \hat{F}_{+m}\right) = 0. \tag{4.28}$$

Combining this with (4.25) yields the consistency requirement

$$\epsilon^{mnp} \hat{\nabla}_m\left(X^{ab}_n \partial_u X^{ab}_p\right) = 0. \tag{4.29}$$

It turns out, however, that this is automatically satisfied based on the properties of u^a and X^{ab}_m. This allows us to conclude that the right-hand side of the second expression in (4.25) may be written as a pure curl, as it is automatically divergence free. Because the shift vectors a^m

were introduced purely as a convenience, we may thus absorb the somewhat awkward term $\frac{1}{4}X_m^{ab}\partial_u X_p^{ab}$ into a redefinition of a^m. This then gives us

$$\mathcal{H}_1^{-1}\hat{G}_{+m} - 2\mathcal{H}_2^{-1}\hat{F}_{+m} = -\epsilon_m{}^{np}\partial_n a_p, \tag{4.30}$$

which is analogous to the corresponding expression in the null $\mathcal{N} = 2$ case [4]. Finally, for completeness, we note that the projection of (4.23) onto the anti-self-dual $SU(2)_-$ in $SO(4)$ gives the condition

$$\left[\Pi_4^{ac}\Pi_4^{bd} - \tfrac{1}{4}X_n^{ab}X^{ncd}\right]\nabla_+ X_m^{cd} = 0. \tag{4.31}$$

To summarize what we have obtained for the field strengths, the $\hat{G}_{+m}$ and $\hat{F}_{+m}$ components cannot be solved for independently. Instead, a linear combination of the two is determined via (4.30). This is similar to what happens for the magnetic components in the timelike case, where (3.7) demonstrates that $\overline{G}^+$ and $\overline{F}^+$ only enter through the combination $\mathcal{H}_1\overline{G}^+ - 2\mathcal{H}_2\overline{F}^+$. The components of $\hat{F}_{+m}^{(4)a}$ taking values in the **4** of $SO(4)$ are determined only so far as their projection onto X_m^{ab}, as given in (4.25). Of course, in all cases, the Bianchi identities (4.14) and equations of motion (4.27) still need to be satisfied.

Turning now to the Killing spinor equations, from the dilatino supersymmetry variation we find the projectors

$$\gamma^+\epsilon = 0, \qquad (1 - u^a T^a)\epsilon = 0, \tag{4.32}$$

as well as the constraint

$$\hat{\gamma}^m\partial_m u^a T^a\epsilon = 0, \tag{4.33}$$

which coincides with (3.27) in the timelike case. The supersymmetry variation of the gravitino yields one more constraint

$$\hat{\nabla}_m\epsilon = \tfrac{1}{2}\partial_m u^a T^a\epsilon, \tag{4.34}$$

which also has a direct analogue in the timelike case, namely (3.29). The integrability conditions which follow from this equation are the same as those derived from the covariant derivative D_m defined in (4.21). Namely, we find that

$$R_{mn} = -\partial_m u^a\partial_n u^a. \tag{4.35}$$

Equation (4.35), together with (4.15), can be interpreted as the Einstein equation of a three-dimensional O(5) vector model coupled to gravity. (However, just as in the timelike case, this model has an unconventional sign for the stress tensor.)

If the u^a's are taken to be rigid O(5) vectors, then the three-dimensional base is not only Ricci-flat, as indicated by (4.35), but is actually flat. This can be derived from (4.19); with the 1-forms X^{ab} closed, we can choose coordinates on the base such that dy^m are identified with the three independent 1-forms X^{ab}. That these independent 1-forms define a dreibein basis follows from the multiplication rule (2.44) obeyed by X^{ab}. The situation is rather more involved for the non-rigid case. For one thing, most quantities can then be functions of the null coordinate u. In this case, a slight simplification may arise by setting the vectors $a^i = 0$ through an appropriate choice of coordinates. Nevertheless, a complete analysis of the non-rigid case appears somewhat formidable, and still remains to be completed.

Finally, for solutions in the null category, it should be noted that the R_{++} Einstein equation remains to be solved independently of the supersymmetry conditions. For the $\mathcal{N} = 4$ model, this component of the Einstein equation turns out to be

$$R_{++} + \tfrac{1}{4}\,e^{\frac{2}{\sqrt{6}}\phi}F_{+\rho}^{ij}F_+{}^{\rho ij} + \tfrac{1}{2}\,e^{-\frac{4}{\sqrt{6}}\phi}G_{+\rho}G_+{}^{\rho} + \tfrac{1}{2}\partial_+\phi\partial_+\phi = 0. \tag{4.36}$$

Given the null metric (4.5) with vielbeins (4.6), we find the expression for R_{++},

$$R_{++} = -\frac{1}{2H}\Box_3\mathcal{F} - H\nabla_+ W_{\bar{m}\bar{m}} - W_{(\bar{m}\bar{n})}W_{(\bar{m}\bar{n})}, \tag{4.37}$$

where

$$W_{\bar{m}\bar{n}} = \nabla_+ H\delta_{\bar{m}\bar{n}} + H\delta_{\bar{m}\bar{p}}\left(\nabla_+\hat{e}^{\bar{p}}_m\right)\hat{e}^m_{\bar{n}} - H\delta_{\bar{m}\bar{p}}\hat{e}^{\bar{p}}_m\hat{e}^n_{\bar{n}}\hat{\nabla}_n a^m. \tag{4.38}$$

The actual Einstein equation, (4.36), is rather cumbersome as the null components $\hat{F}^a_{+m}$ and $\hat{G}_{+m}$ are only partially determined in the present analysis.

5. Solutions

As discussed above in section 3.1, the field content of $\mathcal{N} = 4$ five-dimensional supergravity can be decomposed in $\mathcal{N} = 2$ representations as follows: the minimal supergravity multiplet (the metric, one gauge field, and two gravitini transforming in the **2** of $USp(2)$), one vector multiplet (one gauge field and one scalar, the dilaton) and a gravitino multiplet (the remaining two gravitini and four gauge fields). Thus, by setting the matter multiplets to zero, we shall reproduce the supersymmetric solutions of minimal five-dimensional supergravity found in [4]. To do so requires rigid $SO(5)/SO(4)$ vectors u^a. Furthermore, truncating the set of gauge fields must be done such that (i) for the $SU(2)$ structure case we demand $G^+_1 = G^+_2$; or (ii) for the null case $F^{(4)a}$ must vanish. Lastly, setting the dilaton to zero, which amounts to $\mathcal{H}_1 = \mathcal{H}_2$, leads to the set of equations and constraints which determine the supersymmetric backgrounds of minimal five-dimensional supergravity with either $SU(2)$ holonomy or $\mathbb{R}^3$ structure, respectively [4].

If, on the other hand, we impose the conditions that u^a is rigid but allow $\mathcal{H}_1$ and $\mathcal{H}_2$ (as well as G^+_1 and G^+_2 in the timelike case) to be independent, then we fall back onto the two-charge solutions of minimal supergravity coupled to one vector multiplet described in [10]. In this class of rigid solutions, we are also able to reproduce a subset of the black ring solutions of [21], which are characterized by two electric and two (magnetic) dipole charges. To see this, select the case of a timelike Killing vector and begin again by choosing rigid u^a. Then simply identify the three harmonic functions of [21] as $Z_1 = \mathcal{H}_1$, $Z_2 = Z_3 = \mathcal{H}_2$; these harmonic functions determine the electric charge distributions. The magnetic fields of [21] are to be identified with $G_1 = G^+_1$, $G_2 = G_3 = G^+_2$.

Note that in all the previous examples we began by selecting a rigid five-dimensional unit norm vector u^a. As discussed in section 3.1, having a non-trivial u^a amounts to turning on the gravitino multiplet. In this case, the starting point in constructing the five-dimensional supersymmetric backgrounds must be solving a gravitating $SO(5)$ vector sigma model, in three or four dimensions. The worldvolume of the sigma model is a Riemannian manifold (positive definite metric). We proceed next to construct a few solutions of the gravitating vector model

$$\Box u^a = Ru^a, \qquad R_{mn} = -\partial_m u^a \partial_n u^a. \tag{5.1}$$

At the same time, to ensure that these solutions lead to five-dimensional backgrounds, we must enforce the supersymmetry constraint

$$\gamma^m \partial_m u^a T^a \epsilon = 0. \tag{5.2}$$

The simplest case has the u^a's defining maps from a one-dimensional manifold into a circle. However, this is at odds with the supersymmetry constraint, as $\gamma^1 T^1$ has no zero eigenvalues. The first non-trivial case corresponds to maps from a two-dimensional manifold

$$\mathrm{d}s^2 = \mathrm{e}^{2\Psi(z,\bar{z})}\,\mathrm{d}z\,\mathrm{d}\bar{z} \tag{5.3}$$

into a two-sphere S^2. These maps define the stereographic projection

$$u^a = \left(\frac{\psi+\bar\psi}{1+\psi\bar\psi}, i\frac{\psi-\bar\psi}{1+\psi\bar\psi}, \frac{1-\psi\bar\psi}{1+\psi\bar\psi}, 0, 0\right), \tag{5.4}$$

where we have assumed that $\psi = \psi(z)$ are holomorphic functions. The supersymmetry constraint is satisfied since

$$\partial u^a T^a = \frac{\partial\psi}{(1+\psi\bar\psi)^2}((1-\bar\psi^2)T^1 + i(1+\bar\psi^2)T^2 + 2\bar\psi T^3), \tag{5.5}$$

and the $SO(5)$ matrix which appears between the brackets has zero eigenvalues. The $\gamma^{\bar z}\bar u^a T^a\epsilon$ term vanishes because $\gamma^{\bar z}$ has zero eigenvalues and we require that $\gamma^{\bar z}\epsilon = 0$. According to the supersymmetry analysis done in section 3.1, in this case the Killing spinor condition is of the form $a \pm a \pm b \pm b$ with $a \neq b$ (corresponding to the two different projectors). This means that the Killing spinor preserves 2 out of 8 supersymmetries, and the solution is 1/4 BPS. The solution to (5.1) yields

$$\mathrm{e}^{2\Psi} = (1+\psi\bar\psi)^2|\xi(z)|^2, \tag{5.6}$$

where $\xi(z)$ is an arbitrary holomorphic function of z.

To construct the corresponding five-dimensional solution, we first extend the two-dimensional base to a three or four-dimensional manifold. Then we need to solve for the functions $\mathcal{H}_1, \mathcal{H}_2$ such that $\Box\mathcal{H}_1 = 0$ and $\Box\mathcal{H}_2 = R\mathcal{H}_2$ in the null case, as well as in the $SU(2)$ structure case in the absence of fluxes. Recall that in the timelike case the warp factor of the five-dimensional metric is $f = \left(\mathcal{H}_1\mathcal{H}_2{}^2\right)^{-1/3}$, with $\mathrm{d}s_5^2 = -f^2\,\mathrm{d}t^2 + f^{-1}\,\mathrm{d}s_4^2$, while in the null case the warp factor is $H = \left(\mathcal{H}_1\mathcal{H}_2{}^2\right)^{1/3}$, with $\mathrm{d}s_5^2 = H^{-1}(\mathcal{F}\,\mathrm{d}u^2 + 2\mathrm{d}u\,\mathrm{d}v) + H^2\mathrm{d}s_3^2$. Since $\mathcal{H}_1$ is harmonic, we can take $\mathcal{H}_1 = 1$. On the other hand, $\mathcal{H}_2$ satisfies the same equation as u^a. By identifying $\mathcal{H}_2$ with u^3 we generate a warp factor which has only a radial dependence on the two-dimensional base. Even though the base is regular, the five-dimensional solution may be singular. The reason why this could happen is that zeros of $\mathcal{H}_2$ translate into singularities. In this case, the singularities of the five-dimensional background are at the locus of $\psi\bar\psi = 1$. Noticing that the volume of the base manifold vanishes when $\psi\bar\psi = 1$, we conclude that the singularity is pointlike. Other choices of $\mathcal{H}_2$ (such as turning on some Fourier modes, which can be done by identifying $\mathcal{H}_2$ with u^1 or u^2) could give a different picture in terms of the location of the singularity, but they cannot remove it. For instance $\mathcal{H}_2 = u^1$ vanishes when $\mathrm{Re}(\psi) = 0$.

We find a similar story unfolding when considering higher dimensional maps from conformally flat spaces to spheres. For u^a spanning an S^3,

$$u^a = (\sin(\psi(r))\sin\theta\cos\phi, \sin(\psi(r))\sin\theta\sin\phi, \sin(\psi(r))\cos\theta, \cos(\psi(r)), 0), \tag{5.7}$$

and with the three-dimensional base given by

$$\mathrm{d}s_3^2 = \mathrm{e}^{2\Psi(r)}\left(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2\right), \tag{5.8}$$

we find that the supersymmetry constraint yields

$$\left(\gamma^r\frac{\mathrm{d}\psi}{\mathrm{d}r}T^1(r,\theta,\phi) + \gamma^\theta\sin\psi\, T^2(r,\theta,\phi) + \gamma^\phi\sin\psi\sin\theta\, T^3(r,\theta,\phi)\right)\epsilon = 0, \tag{5.9}$$

where $T^{1,2,3}(r,\theta,\phi)$ are $SO(4)$-rotated $SO(5)$ matrices. Given that $[\gamma^r T^1, \gamma^\theta T^2] = 0$, etc., these matrices can be diagonalized simultaneously. The existence of zero eigenvalues requires either

$$\text{(i)}\quad \psi = \text{const}, \qquad \text{or} \qquad \text{(ii)}\quad \frac{\mathrm{d}\psi}{\mathrm{d}r} = \pm 2\frac{\sin\psi}{r}. \tag{5.10}$$

In the case (i), the Killing spinor equation is of the form $1 \pm 1 \pm 0 \pm 0$, and the (would-be) solution is $1/2$ BPS. In the second case, (ii), we find $2 \pm 1 \pm 1 \pm 0$, and the amount of supersymmetry being preserved is 2 out of 8 (1/4 BPS).

Next we proceed to solve the gravitating vector sigma-model equations. We find that the second case is the only possibility, leading to

$$\cos\psi = \frac{1 - r^4}{1 + r^4}. \tag{5.11}$$

The first option, ψ = constant, is excluded since spheres, while compatible with supersymmetry, have positive curvature. On the other hand, we are looking for solutions to a gravitating $SO(5)$ sigma model with a negative contribution to the stress–energy tensor. Therefore we are looking for manifolds of negative curvature.

For maps from conformally flat four-dimensional manifolds into S^4, supersymmetry requires that either

$$\text{(i)}\quad \frac{\mathrm{d}\psi}{\mathrm{d}r} = \pm\frac{\sin\psi}{r}, \qquad \text{or} \qquad \text{(ii)}\quad \frac{\mathrm{d}\psi}{\mathrm{d}r} = \pm 3\frac{\sin\psi}{r}, \tag{5.12}$$

with the latter being realized as a solution of the gravitating sigma model

$$\cos\psi = \frac{1 - r^6}{1 + r^6}. \tag{5.13}$$

The case (i), which solves the supersymmetry constraint, without leading to a solution of the gravitating sigma model, preserves 3 out of 8 supersymmetries $(1 \pm 1 \pm 1 \pm 1)$. In the second case, (ii), which leads to a solution of the sigma model, the Killing spinor equation is of the form $3 \pm 1 \pm 1 \pm 1$. This is a $1/8$ BPS solution.

As discussed before, to construct the corresponding five-dimensional solutions requires solving for the harmonic function $\mathcal{H}_1$ as well as for $\mathcal{H}_2$. Note that since we may identify $\mathcal{H}_2$ with any of u^a, and since u^a's are unit vectors spanning a sphere, they will vanish: u^4 has zeros at $r = 1$, and the rest vanish when $r = 0$. In addition u^1, u^2 and u^3 have zeros coming from the angular dependence. At the location of the zeros of $\mathcal{H}_2$, the five-dimensional solution will be singular.

It is worth asking whether by turning on fluxes we can improve the current predicament. In the null case, this will have no repercussions, since the equation for $\mathcal{H}_2$ is insensitive to any flux. In the timelike case with $SU(2)$ structure, it appears at first, that by adding fluxes G_1^+, G_2^+ one could make a difference. However, the flux G_2^+ is constrained by $d(u^a G_2^+) = 0$. With the u^a's spanning at least a two-sphere, all components of the self-dual 2-form G_2^+ are set to zero, and no additional source term is generated for $\mathcal{H}_2$.

We have explored a few other solutions. Another simple way to generate negative curvature spaces is to consider cones over spheres. For instance

$$\mathrm{d}s_3^2 = r^B(\mathrm{d}r^2 + Ar^2\mathrm{d}\Omega_2), \tag{5.14}$$

with u^a spanning S^2 is compatible both with supersymmetry and with the gravitating vector sigma-model equations.

$$R_{rr} = 0, \qquad R_{\theta\theta} = -1, \tag{5.15}$$

provided that

$$A(B + 2)^2 = 8, \qquad A > 0. \tag{5.16}$$

In this case, solving for $\mathcal{H}_2 = \mathcal{H}_2(r)$ yields

$$\mathcal{H}_2 = \mathrm{Re}\big(r^{\frac{(B+2)}{4}(1\pm i\sqrt{3})}\big), \tag{5.17}$$

which has an infinite number of nodes. As has been explained before, the five-dimensional solution built on the three-dimensional manifold (5.14) will be singular at the location of these nodes.

Lastly, we have investigated a warped three-dimensional manifold

$$ds_3^2 = dy^2 + y^2 e^{2\Psi(r)}(dr^2 + r^2 d\phi^2) \tag{5.18}$$

and with $u^a = (\sin(\psi(r))\sin\phi, \sin(\psi(r))\cos\phi, \cos(\psi(r)), 0, 0)$. Given that the supersymmetry constraint is satisfied, we move onto the gravitating vector sigma-model equations. The warp factor y^2 is the only choice up to y-translations which solves

$$R_{yy} = 0 \tag{5.19}$$

other than a trivial warping $y^0 = 1$. This time $\mathcal{H}_2$ can be a function of both r and y. We found solutions using separation of variables, $\mathcal{H}_2 = h(r)\tilde{h}(y)$. It turns out, however, that if $\tilde{h}(y)$ has no zeros, then $h(r)$ will, and *vice versa*.

As a final comment, we would like to mention that we have inquired about the existence of Id structure solutions. Under the simple assumptions of a rigid f^a and of a flat four-dimensional base with all fields depending on a single variable, the only solutions to the Bianchi identities and equations of motion compatible with the supersymmetry constraints turned out to be trivial, with $f^a =$ constant and $f =$ constant. It remains to be seen whether there are any large classes of solutions with identity structure yet to be found.

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Appendix A. Fierz identities

The determination of the structure groups, as well as the explicit construction of the solutions, requires consideration of the algebraic identities satisfied by the spinor bilinears. These identities are essentially Fierz identities, and are obtained by using the five-dimensional Fierz relation

$$4(\bar{\epsilon}_1\epsilon_2)(\bar{\epsilon}_3\epsilon_4) = (\bar{\epsilon}_1\epsilon_4)(\bar{\epsilon}_3\epsilon_2) + (\bar{\epsilon}_1\Gamma_\rho\epsilon_4)(\bar{\epsilon}_3\Gamma^\rho\epsilon_2) - \tfrac{1}{2}(\bar{\epsilon}_1\Gamma_{\rho\sigma}\epsilon_4)(\bar{\epsilon}_3\Gamma^{\rho\sigma}\epsilon_2), \tag{A.1}$$

where the $USp(4)$ indices have been hidden.

Although a great number of identities may be obtained, we only highlight some of the more useful ones here. Furthermore, as was done in the body of the paper, we use a $SO(5)$ notation for the bispinors

$$f, \qquad f^a, \qquad K_\mu, \qquad V_\mu^a, \qquad \Phi_{\mu\nu}^{ab}, \tag{A.2}$$

which were defined in (2.13). Note that (when considered as tangent space indices) the spacetime indices $\mu, \nu, \ldots$ take values in SO(1,4), while indices $a, b, \ldots$ are valued in $SO(5)$. Because of this similarity in groups, the Fierz identities exhibit a formal symmetry under the interchange of spacetime and internal space indices along with the exchange $f^a \leftrightarrow K_\mu$.

We organize the algebraic identities according to the number of open spacetime and internal space indices. For the scalar-singlet combination, we have

$$(K_\mu)^2 = -(f^a)^2, \tag{A.3}$$

$$\left(V_\mu^a\right)^2 = -5f^2 + 4(f^a)^2, \tag{A.4}$$

$$\tfrac{1}{8}\left(\Phi^{ab}_{\mu\nu}\right)^2 = 5f^2 + (f^a)^2. \tag{A.5}$$

The first identity above demonstrates that the Killing vector K^μ is nowhere spacelike. For the vector-singlet case, we have

$$fK_\mu = f^a V^a_\mu, \tag{A.6}$$

$$fK_\mu = \tfrac{1}{96}\epsilon_\mu{}^{\nu\rho\lambda\sigma}\Phi^{ab}_{\nu\rho}\Phi^{ab}_{\lambda\sigma} \tag{A.7}$$

while the scalar-**5** case yields

$$ff^a = -K^\mu V^a_\mu, \tag{A.8}$$

$$ff^a = \tfrac{1}{96}\epsilon^{abcde}\Phi^{bc}_{\mu\nu}\Phi^{\mu\nu de}. \tag{A.9}$$

Turning to cases with additional open indices, we start with the vector-**5** relations

$$\tfrac{1}{96}\epsilon_\mu{}^{\nu\rho\lambda\sigma}\epsilon^{abcde}\Phi^{bc}_{\nu\rho}\Phi^{de}_{\lambda\sigma} = fV^a_\mu + f^a K_\mu, \tag{A.10}$$

$$\tfrac{1}{4}\Phi^{ab}_{\mu\nu}V^{\nu b} = fV^a_\mu - f^a K_\mu. \tag{A.11}$$

Next, we find that the scalar-symmetric tensor (**1** + **14**) combination gives

$$V^a_\mu V^{\mu b} = -f^a f^b + \delta^{ab}((f^c)^2 - f^2) \tag{A.12}$$

$$\tfrac{1}{4}\Phi^{ac}_{\mu\nu}\Phi^{\mu\nu bc} = -3f^a f^b + \delta^{ab}((f^c)^2 + 2f^2). \tag{A.13}$$

Note that contraction with the $SO(5)$ invariant tensor δ^{ab} gives the singlet identities (A.4) and (A.5) above. The flipped version of (A.12) is the tensor-singlet combination

$$V^a_\mu V^a_\nu = K_\mu K_\nu + g_{\mu\nu}((f^a)^2 - f^2). \tag{A.14}$$

Turning next to the vector-antisymmetric tensor (**10**) combination, we find

$$K^\mu\Phi^{ab}_{\mu\nu} = -\tfrac{1}{6}\epsilon^{abcde}V^{\mu c}\Phi^{de}_{\mu\nu}, \tag{A.15}$$

$$K^\mu\Phi^{ab}_{\mu\nu} = -2f^{[a}V^{b]}_\nu. \tag{A.16}$$

The latter equation has a flipped tensor-**5** version

$$f^a\Phi^{ab}_{\mu\nu} = 2K_{[\mu}V^b_{\nu]}. \tag{A.17}$$

Finally, a couple of useful tensor-**10** relations are

$$\epsilon_{\mu\nu}{}^{\rho\lambda\sigma}K_\rho\Phi^{ab}_{\lambda\sigma} = -\epsilon^{abcde}f^c\Phi^{de}_{\mu\nu}, \tag{A.18}$$

$$\epsilon_{\mu\nu}{}^{\rho\lambda\sigma}K_\rho\Phi^{ab}_{\lambda\sigma} = 4V^{[a}_\mu V^{b]}_\nu - 2f\Phi^{ab}_{\mu\nu}. \tag{A.19}$$

For more complicated combinations, we do not perform a complete decomposition into irreducible representations, but merely list the number of spacetime and internal space indices according to (# spacetime, # internal). In the (3, 1) and (3, 2) categories, we have

$$V^a_\mu\Phi^{ab}_{\nu\lambda} = -2g_{\mu[\nu}\left(fV^b_\lambda - f^b K_{\lambda]}\right) + \epsilon_{\mu\nu\lambda}{}^{\rho\sigma}V^b_\rho K_\sigma, \tag{A.20}$$

$$K_\mu\Phi^{ab}_{\nu\lambda} = -2g_{\mu[\nu}\left(f^a V^b_\lambda - f^b V^a_{\lambda]}\right) - \epsilon_{\mu\nu\lambda}{}^{\rho\sigma}V^a_\rho V^b_\sigma + \tfrac{1}{2}\epsilon^{abcde}V^c_\mu\Phi^{de}_{\nu\lambda}. \tag{A.21}$$

These identities are useful for deducing the basic properties of the 2-form Φ^{ab}. Additional information on Φ^{ab} and its relation to $SU(2)$ or $\mathbb{R}^3$ structure can be obtained from the $(1, 4)$ identity

$$\frac{1}{4}\epsilon_\mu{}^{\nu\rho\lambda\sigma}\Phi^{ab}_{\nu\rho}\Phi^{cd}_{\lambda\sigma} = \epsilon^{abcde}\left(f^e K_\mu + f V^e_\mu\right) + 2fK_\mu(\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc}) - 2\left[f^{(a}V^{c)}_\mu\delta^{bd} + f^{(b}V^{d)}_\mu\delta^{ac} - f^{(a}V^{d)}_\mu\delta^{bc} - f^{(b}V^{c)}_\mu\delta^{ad}\right], \tag{A.22}$$

as well as the (2, 2) identity

$$\Phi^{ab}_{\mu\lambda}\Phi^{\lambda bc}_\nu = \delta^{ac}[3K_\mu K_\nu + g_{\mu\nu}(f^2 + 2(f^d)^2)] - 3g_{\mu\nu}f^a f^c - 3V^{(a}_\mu V^{c)}_\nu + V^{[a}_\mu V^{c]}_\nu - 2f\Phi^{ac}_{\mu\nu}. \tag{A.23}$$

A contraction on the internal indices results in the (2, 0) counterpart of (A.13)

$$\frac{1}{4}\Phi^{ab}_{\mu\lambda}\Phi^{\lambda ab}_\nu = -3K_\mu K_\nu - g_{\mu\nu}((f^c)^2 + 2f^2). \tag{A.24}$$

The identities with more open indices are rather tedious, but useful for completing the determination of the structure. In the (4, 2) category, we have

$$\begin{aligned}\Phi^{ac}_{\mu\nu}\Phi^{bc}_{\rho\sigma} = {}& \epsilon_{\mu\nu\rho\sigma}{}^\lambda(f^{(a}V^{b)}_\lambda - \delta^{ab}fK_\lambda) + (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})(-f^a f^b + \delta^{ab}f^2)\\ &+ \delta^{ab}[K_\mu K_\rho g_{\nu\sigma} + K_\nu K_\sigma g_{\mu\rho} - K_\nu K_\rho g_{\mu\sigma} - K_\mu K_\sigma g_{\nu\rho}]\\ &+ \tfrac{1}{2}\left[K_{[\mu}\epsilon_{\nu]\rho\sigma}{}^{\alpha\beta} - K_{[\rho}\epsilon_{\sigma]\mu\nu}{}^{\alpha\beta}\right]\Phi^{ab}_{\alpha\beta}\\ &- f\left[\Phi^{ab}_{\mu\rho}g_{\nu\sigma} + \Phi^{ab}_{\nu\sigma}g_{\mu\rho} - \Phi^{ab}_{\nu\rho}g_{\mu\sigma} - \Phi^{ab}_{\mu\sigma}g_{\nu\rho}\right]\\ &- \left[V^b_\mu V^a_\rho g_{\nu\sigma} + V^b_\nu V^a_\sigma g_{\mu\rho} - V^b_\nu V^a_\rho g_{\mu\sigma} - V^b_\mu V^a_\sigma g_{\nu\rho}\right],\end{aligned} \tag{A.25}$$

while the opposite (2, 4) case gives a similar expression

$$\begin{aligned}\Phi^{ab}_{\mu\lambda}\Phi^{\lambda cd}_\nu = {}& -\epsilon^{abcde}\left(K_{(\mu}V^e_{\nu)} + g_{\mu\nu}ff^e\right) + (\delta^{ac}\delta^{bd} - \delta^{ad}\delta^{bc})(-K_\mu K_\nu + g_{\mu\nu}K^2)\\ &+ g_{\mu\nu}[f^a f^c\delta^{bd} + f^b f^d\delta^{ac} - f^c f^d\delta bc - f^b f^c\delta^{ad}]\\ &+ \tfrac{1}{2}(f^{[a}\epsilon^{b]cdef} - f^{[c}\epsilon^{d]abef})\Phi^{ef}_{\mu\nu} + f\left[\Phi^{ac}_{\mu\nu}\delta^{bd} + \Phi^{bd}_{\mu\nu}\delta^{ac} - \Phi^{ad}_{\mu\nu}\delta^{bc} - \Phi^{bc}_{\mu\nu}\delta^{ad}\right]\\ &+ \left[V^c_\mu V^a_\nu\delta^{bd} + V^d_\mu V^b_\nu\delta^{ac} - V^c_\mu V^b_\nu\delta^{ad} - V^d_\mu V^a_\nu\delta^{bc}\right].\end{aligned} \tag{A.26}$$

Appendix B. Differential identities

Here we provide the complete set of differential identities obtained from the action of the dilation and gravitino variations on the $USp(4)$ bispinors. We recall that the $USp(4)$ valued scalar, vector and tensor bispinors were defined in (2.12) as

$$\begin{aligned} f^{[ij]} &= i\bar\epsilon^i\epsilon^j,\\ V^{[ij]}_\mu &= \bar\epsilon^i\gamma_\mu\epsilon^j,\\ \Phi^{(ij)} &= i\bar\epsilon^i\gamma_{\mu\nu}\epsilon^j.\end{aligned} \tag{B.1}$$

These may be split into irreducible $SO(5)$ representations according to

$$\begin{aligned} f^{ij} &= f\Omega^{ij} + f^a T^{aij},\\ V^{ij}_\mu &= K_\mu\Omega^{ij} + V^a_\mu T^{aij}\\ \Phi^{ij}_{\mu\nu} &= \tfrac{1}{2}\Phi^{ab}_{\mu\nu}T^{abij}.\end{aligned} \tag{B.2}$$

Although the dilatino variation (2.7) does not lead to derivatives on ϵ^i, we nevertheless consider the resulting expressions as 'differential' identities to distinguish them from kinematical or Fierzing relations. By taking $\bar\epsilon^i\{1, \Gamma_\mu, \Gamma_{\mu\nu}\}\delta\chi^j = 0$, we obtain

$$0 = K^{\mu}\partial_{\mu}\phi, \tag{B.3}$$

$$0 = V^{\mu a}\partial_{\mu}\phi + \frac{1}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\Phi^{ab}_{\mu\nu}F^{\mu\nu b}, \tag{B.4}$$

$$0 = \mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}\Phi^{ab}_{\mu\nu}G^{\mu\nu} + \frac{1}{2}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\epsilon^{abcde}\Phi^{cd}_{\mu\nu}F^{\mu\nu e}, \tag{B.5}$$

$$0 = f\partial_{\mu}\phi + \frac{2}{\sqrt{6}}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}G_{\mu\nu}K^{\nu} + \frac{2}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}F^{a}_{\mu\nu}V^{\nu a}, \tag{B.6}$$

$$0 = f^{a}\partial_{\mu}\phi + \frac{2}{\sqrt{6}}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}G_{\mu\nu}V^{\nu a} + \frac{2}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}F^{a}_{\mu\nu}K^{\nu} - \frac{1}{2\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\epsilon^{\mu\nu\lambda\rho\sigma}F^{b}_{\nu\lambda}\Phi^{ab}_{\rho\sigma}, \tag{B.7}$$

$$0 = \Phi^{ab}_{\mu\nu}\partial^{\nu}\phi + \frac{1}{2\sqrt{6}}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}\epsilon_{\mu\nu\lambda\rho\sigma}G^{\nu\lambda}\Phi^{ab}_{\rho\sigma} + \frac{1}{4\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\epsilon_{\mu\nu\lambda\rho\sigma}\epsilon^{abcde}\Phi^{cd}_{\rho\sigma}F^{e}_{\nu\lambda} + \frac{4}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}F^{[a}_{\mu\nu}V^{\nu b]}, \tag{B.8}$$

$$\begin{aligned}0 = {}& \frac{2}{\sqrt{6}}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}fG_{\mu\nu} + 2K_{[\mu}\partial_{\nu]}\phi - \frac{1}{\sqrt{6}}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}\epsilon_{\mu\nu\lambda\rho\sigma}G^{\lambda\rho}K^{\sigma}\\ &+ \frac{2}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}f^{a}F^{a}_{\mu\nu} - \frac{1}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\epsilon_{\mu\nu\lambda\rho\sigma}F^{a}_{\lambda\rho}V^{a}_{\sigma},\end{aligned} \tag{B.9}$$

$$\begin{aligned}0 = {}& \frac{2}{\sqrt{6}}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}f^{a}G_{\mu\nu} + 2V^{a}_{[\mu}\partial_{\nu]}\phi - \frac{1}{\sqrt{6}}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}\epsilon_{\mu\nu\lambda\rho\sigma}G^{\lambda\rho}V^{a}_{\sigma} + \frac{2}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}fF^{a}_{\mu\nu}\\ &- \frac{1}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\epsilon_{\mu\nu\lambda\rho\sigma}F^{\lambda\rho a}K_{\sigma} - \frac{4}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\Phi^{ab}_{[\mu|\lambda|}F^{\lambda b}_{\nu]},\end{aligned} \tag{B.10}$$

$$\begin{aligned}0 = {}& \frac{1}{4}\epsilon_{\mu\nu}{}^{\lambda\rho\sigma}\Phi^{ab}_{\rho\sigma}\partial_{\lambda}\phi + \frac{2}{\sqrt{6}}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}\Phi_{[\mu}{}^{\lambda ab}G^{\lambda}{}_{\nu]} - \frac{2}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}f^{[a}F^{b]}_{\mu\nu}\\ &- \frac{1}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\epsilon_{\mu\nu}{}^{\lambda\rho\sigma}F^{[a}_{\lambda\rho}V^{b]}_{\sigma} + \frac{1}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\epsilon^{abcde}\Phi^{cd}_{[\mu|\lambda|}F^{\lambda e}{}_{\nu]}.\end{aligned} \tag{B.11}$$

The true differential identities are obtained by taking a covariant derivative of the bilinears (B.2), and then using the gravitino variation (2.6) to re-express $\nabla_{\mu}\epsilon^{i}$ in terms of algebraic expressions. The result is

$$\nabla_{\mu}f = -\frac{1}{3}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}G_{\mu\nu}K^{\nu} + \frac{2}{3}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}F^{a}_{\mu\nu}V^{\nu a}, \tag{B.12}$$

$$\nabla_{\mu}f^{a} = -\frac{1}{3}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}G_{\mu\nu}V^{\nu a} + \frac{1}{12}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\epsilon_{\mu}{}^{\nu\lambda\rho\sigma}F^{b}_{\nu\lambda}\Phi^{ab}_{\rho\sigma} + \frac{2}{3}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}F^{a}_{\mu\nu}K^{\nu}, \tag{B.13}$$

$$\nabla_{\mu}K_{\nu} = \frac{1}{12}\epsilon_{\mu\nu}{}^{\rho\lambda\sigma}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}G_{\rho\lambda}K_{\sigma} + \frac{1}{3}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}fG_{\mu\nu} - \frac{1}{6}\epsilon_{\mu\nu}{}^{\rho\lambda\sigma}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}F^{a}_{\rho\lambda}V^{a}_{\sigma} - \frac{2}{3}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}F^{a}_{\mu\nu}f^{a}, \tag{B.14}$$

$$\begin{aligned}\nabla_{\mu}V^{a}_{\nu} = {}& \frac{1}{12}\epsilon_{\mu\nu}{}^{\rho\lambda\sigma}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}G_{\rho\lambda}V^{a}_{\sigma} + \frac{1}{3}\,\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}f^{a}G_{\mu\nu} - \frac{1}{6}\epsilon_{\mu\nu}{}^{\rho\lambda\sigma}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}F^{a}_{\rho\lambda}K_{\sigma}\\ &- \frac{2}{3}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}fF^{a}_{\mu\nu} - \frac{1}{6}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\left(g_{\mu\nu}\Phi^{ab}_{\rho\sigma} - 2g_{\nu\rho}\Phi_{\mu\sigma}{}^{ab} - 4g_{\mu\rho}\Phi^{ab}_{\nu\sigma}\right)F^{\rho\sigma b},\end{aligned} \tag{B.15}$$

$$\begin{aligned}\nabla_{\mu}\Phi^{ab}_{\nu\lambda} = {}& \frac{1}{12}\left(-g_{\mu[\nu}\epsilon_{\lambda]}{}^{\rho\sigma\alpha\beta} + 2\delta^{\rho}_{[\nu}\epsilon_{\lambda]\mu}{}^{\sigma\alpha\beta} - 2\delta^{\rho}_{\mu}\epsilon_{\nu\lambda}{}^{\sigma\alpha\beta}\right)\\ &\times\left(\mathrm{e}^{-\frac{2}{\sqrt{6}}\phi}G_{\rho\sigma}\Phi^{ab}_{\alpha\beta} - \mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\epsilon^{abcde}\Phi^{cd}_{\alpha\beta}F^{e}_{\rho\sigma}\right) - \frac{1}{3}\epsilon_{\nu\lambda\mu}{}^{\rho\sigma}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}f^{[a}F^{b]}_{\rho\sigma}\\ &+ \frac{2}{3}\left(-2g_{\mu[\nu}\delta^{\rho}_{\lambda]}g^{\sigma\alpha} - \delta^{\rho}_{[\nu}\delta^{\sigma}_{\lambda]}\delta^{\alpha}_{\mu} + 4\delta^{\rho}_{\mu}\delta^{\sigma}_{[\nu}\delta^{\alpha}_{\lambda]}\right)\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}V^{[a}_{\alpha}F^{b]}_{\rho\sigma}.\end{aligned} \tag{B.16}$$

The above dilatino and gravitino equations simplify when combined. Using a form notation, we first have the 'zero form' expressions

$$\mathrm{i}_{K}\,\mathrm{d}\phi = 0, \tag{B.17}$$

$$\mathrm{i}_{V^{a}}\,\mathrm{d}\phi = -\frac{1}{\sqrt{6}}\,\mathrm{e}^{\frac{1}{\sqrt{6}}\phi}\Phi^{ab}_{\mu\nu}F^{\mu\nu b}, \tag{B.18}$$

$$0 = \Phi^{ab}_{\mu\nu} G^{\mu\nu} + \tfrac{1}{2}\, e^{\frac{3}{\sqrt{6}}\phi} \epsilon^{abcde} \Phi^{cd}_{\mu\nu} F^{\mu\nu e}. \tag{B.19}$$

The 1-form differential identities are

$$\mathrm{d}\big(e^{\frac{2}{\sqrt{6}}\phi} f\big) = \mathrm{i}_K G, \tag{B.20}$$

$$\mathrm{d}\big(e^{-\frac{1}{\sqrt{6}}\phi} f\big) = -\mathrm{i}_{V^a} F^a, \tag{B.21}$$

$$\mathrm{d}\big(e^{-\frac{1}{\sqrt{6}}\phi} f^a\big) = -\mathrm{i}_K F^a, \tag{B.22}$$

$$\mathrm{d}\big(e^{\frac{2}{\sqrt{6}}\phi} f^a\big) = \mathrm{i}_{V^a} G + e^{\frac{3}{\sqrt{6}}\phi} * (\Phi^{ab} \wedge F^b), \tag{B.23}$$

while the 2-form differential identities are

$$\mathrm{d}\big(e^{-\frac{2}{\sqrt{6}}\phi} K\big) = \mathrm{i}_K \big(e^{-\frac{4}{\sqrt{6}}\phi} * G\big) - 2\big(e^{-\frac{1}{\sqrt{6}}\phi} f^a\big) F^a, \tag{B.24}$$

$$\mathrm{d}\big(e^{\frac{1}{\sqrt{6}}\phi} K\big) = -\mathrm{i}_{V^a} \big(e^{\frac{2}{\sqrt{6}}\phi} * F^a\big) + e^{-\frac{1}{\sqrt{6}}\phi} f G - e^{\frac{2}{\sqrt{6}}\phi} f^a F^a, \tag{B.25}$$

$$\mathrm{d}\big(e^{\frac{1}{\sqrt{6}}\phi} V^a\big) = -\mathrm{i}_K \big(e^{\frac{2}{\sqrt{6}}\phi} * F^a\big) + \big(e^{-\frac{1}{\sqrt{6}}\phi} f^a\big) G - e^{\frac{2}{\sqrt{6}}\phi} f F^a, \tag{B.26}$$

$$\mathrm{d}\big(e^{-\frac{2}{\sqrt{6}}\phi} V^a\big) = \mathrm{i}_{V^a} \big(e^{-\frac{4}{\sqrt{6}}\phi} * G\big) - 2\, e^{-\frac{1}{\sqrt{6}}\phi} f F^a + e^{-\frac{1}{\sqrt{6}}\phi} \Phi^{ab}_{\mu\lambda} F^{\lambda b}_{\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{B.27}$$

In addition, the symmetrized rank two equations are

$$\nabla_{(\mu} K_{\nu)} = 0, \tag{B.28}$$

$$\nabla_{(\mu} V^a_{\nu)} = -\tfrac{1}{6}\big(g_{\mu\nu} \Phi^{ab}_{\rho\sigma} - 3 g_{\mu\rho} \Phi^{ab}_{\nu\sigma} - 3 g_{\nu\rho} \Phi^{ab}_{\mu\sigma}\big)\, e^{\frac{1}{\sqrt{6}}\phi} F^{\rho\sigma b}. \tag{B.29}$$

In particular, we see that K^μ identically satisfies the Killing equation. Finally, we may obtain expressions involving the 2-form Φ^{ab}

$$\begin{aligned} &\mathrm{d}\big(e^{\frac{3}{\sqrt{6}}\phi} \Phi^{ab}\big) = *\big[\big(e^{\frac{1}{\sqrt{6}}\phi} \Phi^{ab}_{\mu\lambda} G^\lambda{}_\nu + \tfrac{1}{2}\, e^{\frac{4}{\sqrt{6}}\phi} \epsilon^{abcde} \Phi^{cd}_{\mu\lambda} F^{\lambda e}_\nu\big)\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu - 4\, e^{\frac{4}{\sqrt{6}}\phi} f^{[a} F^{b]}\big], \\ &\mathrm{d}\big(e^{-\frac{1}{\sqrt{6}}\phi} * \Phi^{ab}\big) = -\tfrac{1}{2} \epsilon^{abcde} \Phi^{cd} \wedge F^e. \end{aligned} \tag{B.30}$$

Equations (B.17) through (B.30), along with the covariant derivative on $\Phi^{ab}_{\mu\nu}$ given in (B.16) form a complete set of differential identities.

Note that, by taking an exterior derivative of (B.20), (B.22), (B.24) and (B.26), and by using the relation $\mathcal{L} = \mathrm{d}\mathrm{i}_K + \mathrm{i}_K d$ for the Lie derivative, we may obtain

$$\mathcal{L}_K G = 0, \tag{B.31}$$

$$\mathcal{L}_K F^a = 0, \tag{B.32}$$

$$\mathcal{L}_K \big(e^{-\frac{4}{\sqrt{6}}\phi} * G\big) = \mathrm{i}_K \big[\mathrm{d}\big(e^{-\frac{4}{\sqrt{6}}\phi} * G\big) - F^a \wedge F^a\big], \tag{B.33}$$

$$\mathcal{L}_K \big(e^{\frac{2}{\sqrt{6}}\phi} * F^a\big) = \mathrm{i}_K \big[\mathrm{d}\big(e^{\frac{2}{\sqrt{6}}\phi} * F^a\big) - F^a \wedge G\big]. \tag{B.34}$$

The last two lines vanish by the gauge field equations of motion. These expressions, along with (B.17) and the Killing equation (B.28), ensure that the isometry generated by K extends to the entire solution. Finally, using (B.26) along with the Fierz identity (A.8), we may also deduce that

$$\mathcal{L}_K \big(e^{\frac{1}{\sqrt{6}}\phi} V^a\big) = 0. \tag{B.35}$$

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