A Transformation for Free-Surface Flow in Porous Media

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Two methods for solving two-dimensional free-surface flows in porous media are presented. Both are based on the fact that the free surface can be transformed into a straight line. The first method utilizes the principle of images, whereas in the second method the Schwarz–Christoffel transformation is used. The solution for free-surface flow into a sink is given to illustrate the first method. The solution of the problem of water wedging is given to illustrate the second method.

1. INTRODUCTION

SEEPAGE flow in porous media is governed by Darcy's law

$$(u, v, w) = -\frac{k}{\mu}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi, \tag{1}$$

in which x, y, and z are Cartesian coordinates, u, v, and w are velocity components in the directions of increasing x, y, and z, k is the permeability of the porous medium, and μ is the viscosity of the fluid. The potential ϕ is $p + \rho g y$, in which p is the pressure, ρ is the density of the fluid, g is the gravitational acceleration, and y is measured in a direction opposite to that of g. If the fluid is incompressible, the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2}$$

If k and μ are constant throughout the field of flow, (1) and (2) yield

$$\nabla^2\phi = 0, \tag{3}$$

in which ∇^2 is the Laplacian operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

On a free surface p = constant, so that

$$\phi = \rho g y. \tag{4}$$

The differential equation (3), the free-surface condition (4), and the other boundary conditions govern free-surface flows of a homogeneous fluid in a homogeneous medium. For steady flows, not only must (4) be satisfied on the free surface, but the velocity must be tangent to the surface everywhere. In two-dimensional steady flows, the free surface is a streamline.

Since the free surface is not given, but is to be determined, the solution of the problem is much more difficult than one in which the boundaries are specified. Even for two-dimensional flows, the usual method for dealing with potential flows with free streamlines in classical hydrodynamics cannot be used because the Schwarz–Christoffel transformation cannot be applied to the logarithmic hodograph plane. However, (4) is, after all, linear, and should not present an insurmountable difficulty for two-dimensional flows. Already it is known that in the hodograph plane the free surface is a circular arc. (See Muskat, 1937,[1] and later development in this paper.) But the hodograph of the boundary of the flow is still too inconvenient for a solution. The only solution known to the writer is the one presented by Muskat and it is by no means a simple one.

In this paper two methods for solving two-dimensional free-surface seepage flows will be presented. In both, a transformation turns the free surface into a straight line, facilitating the use of the method of images (the first method) or of the Schwarz–Christoffel transformation (the second method). Flow into a two-dimensional sink is given to illustrate the first method, and the problem of water wedging is treated by the second.

2. FREE-SURFACE FLOW INTO A SINK

Consider the case of steady two-dimensional flow into a sink. The fluid is supposed to extend from the free surface downward to infinity. First, the free-surface condition (4) will be expressed in terms of the velocity components u and v. Since the flow is steady, the free-surface is a streamline, so that the speed along the free surface is

$$q = -(k/\mu)\partial\phi/\partial s.$$

Differentiating (4) by s and multiplying the result by $-(k/\mu)q$, one has

$$q^2 = -(k\rho g/\mu)q\ \partial y/\partial s. \tag{5}$$

[1] M. Muskat, *The Flow of Homogeneous Fluids through Porous Media* (McGraw-Hill Book Company, Inc., New York, 1937).

Since

$$q\, \partial y/\partial s = v \quad \text{and} \quad q^2 = u^2 + v^2,$$

(5) becomes

$$u^2 + v^2 + \alpha v = 0, \tag{6}$$

in which

$$\alpha = k\rho g/\mu.$$

Thus, the free surface is a circular arc in the hodograph plane, as is well known.

Now define the complex variable ζ by

$$\zeta = 1/[i(-u + iv)] = \xi + i\eta. \tag{7}$$

Since the derivative of the complex potential w with respect to the complex variable $z(=x + iy)$ is

$$dw/dz = -u + iv, \tag{8}$$

and is an analytic function of z, so is ζ. But

$$\zeta = (-v + iu)/(u^2 + v^2),$$

and (6) shows that on the free surface the real part of ζ is constant $(=1/\alpha)$. Consequently, in the ζ plane the free surface is the straight line

$$\xi = \xi_0(=1/\alpha). \tag{9}$$

Equation (9) has been obtained from (4) and

$$\psi = \text{constant} \tag{10}$$

on the free surface, in which ψ is the stream function conjugate to ϕ, so that

$$w = \phi + i\psi$$

is the complex potential. Although (10) has been used once to obtain (9), (4) and (10) are not represented solely by (9). Another relationship must be obtained so that (4) and (10) can be adequately replaced. Taking the differentials of (4), one has

$$dw = \rho g\, dy, \tag{11}$$

since

$$dw = d\phi$$

on the free surface. Equation (11) can be replaced by

$$-\frac{i}{\zeta}\, dz = \rho g\, dy,$$

or

$$\frac{1}{\zeta}\frac{dz}{d\zeta} = i\rho g \frac{dy}{d\zeta}, \tag{12}$$

on the free surface. Now, on the free surface, $d\zeta = i\, d\eta$; hence,

$$\operatorname{Im}\left(\frac{1}{\zeta}\frac{d\zeta}{dz}\right) = 0. \tag{13}$$

This is the equation we are looking for. Equations (9) and (13) replace (4) and (10).

For a sink situated at the origin,

$$\zeta = -\frac{i}{(dw/dz)} = -\frac{iz}{m}\,(\text{near } z = 0), \tag{14}$$

in which $m = m'\mu/k$, and $2\pi m'$ is the total discharge (strength of the sink). Consequently,

$$\frac{1}{\zeta}\frac{dz}{d\zeta} \sim mi\zeta^{-1} \quad \text{near} \quad z = 0. \tag{15}$$

But (13) demands that $(1/\zeta)dz/d\zeta$ be real on $\xi = \xi_0$. Thus

$$\frac{1}{\zeta}\frac{dz}{d\zeta} = mi\left(\frac{1}{\zeta} - \frac{1}{2\xi_0 - \zeta}\right), \tag{16}$$

or

$$\frac{dz}{d\zeta} = mi\left(1 - \frac{\zeta}{2\xi_0 - \zeta}\right) = 2mi\left(1 + \frac{\xi_0}{\zeta - 2\xi_0}\right).$$

Hence

$$z = 2mi\left(\zeta + \xi_0 \ln \frac{2\xi_0 - \zeta}{2\xi_0}\right), \tag{17}$$

the constant of integration being so determined as to make z approach zero as ζ approaches zero. Equation (17) gives the solution to the problem. It can be made dimensionless by dividing by $m\xi_0$. The result is

$$z_1 = 2i[\zeta_1 + \ln \tfrac{1}{2}(2 - \zeta_1)], \tag{18}$$

in which

$$z_1 = z/m\xi_0 = x_1 + iy_1, \qquad \zeta_1 = \zeta/\xi_0 = \xi_1 + i\eta_1.$$

On the free surface $\xi = \xi_0$, hence $\xi_1 = 1$. Thus (18) gives

$$\tfrac{1}{2}x_1 = -\eta_1 + \tan^{-1}\eta_1 \tag{19}$$

and

$$\tfrac{1}{2}y_1 = 1 + \ln\left[\tfrac{1}{2}(1 + \eta_1^2)^{\frac{1}{2}}\right] \tag{20}$$

as parametric equations of the free surface. The cusp point at which the free surface streamlines $\psi = 0$ and $\psi = m$ meet is given by

$$\frac{dx}{dy} = 0, \qquad \text{or} \qquad \frac{dx/d\eta_1}{dy/d\eta_1} = 0.$$

L'Hospitals' rule gives $\eta_1 = 0$, which gives $x_1 = 0$ and $y_1 = 2(1 - \ln 2)$ for the coordinates of the cusp point.

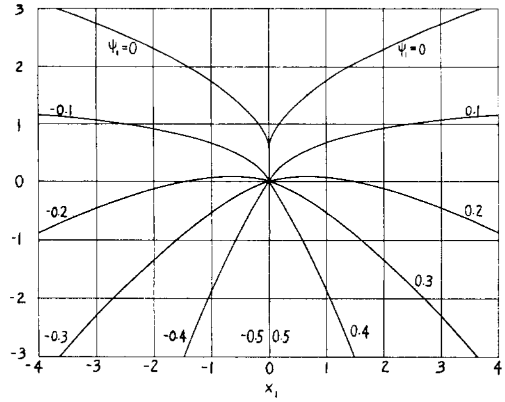

FIG. 1. Pattern of free-surface flow into a sink.

The other streamlines are determined in the following way. Equation (18) can be written as

$$\tfrac{1}{2}x_1 = -\eta_1 + \tan^{-1}[\eta_1/(2-\xi_1)], \tag{21}$$

$$\tfrac{1}{2}y_1 = 1 + \ln\{\tfrac{1}{2}[(2-\xi_1)^2 + \eta_1^2]^{\frac{1}{2}}\}. \tag{22}$$

Assume a fixed x_1, and for this fixed x_1 assume various values of η_1. Compute ξ_1 from (21), then y_1 from (22). Thus, one has $\xi_1 + i\eta_1$ for many values of y_1 for a fixed x_1. Since

$$\xi_1 + i\eta_1 = -i/w_1', \tag{23}$$

with $w_1' = \xi_0(dw/dz)$, at these values of y_1 one also has the values of

$$w_1' = -u_1 + iv_1,$$

with $u_1 = u\xi_0$, $v_1 = v\xi_0$. The dimensionless stream function ψ_1 can then be determined from

$$-\partial\psi_1/\partial y_1 = u_1, \tag{24}$$

by integration with respect to y_1. With ψ_1 so determined for various values of x_1, the streamlines can be drawn through points with the same value of ψ_1. The flow pattern is shown in Fig. 1. Note that (24) implies

$$w_1 = (\xi_0/m\xi_0)w = w/m = (\phi/m) + i(\psi/m).$$

Hence

$$\phi_1 = \phi/m, \qquad \psi_1 = \psi/m.$$

3. PROBLEMS OF WATER WEDGING

When oil is pumped from the ground the problem of water coning is often encountered. If the rate of pumping is too great, water as well as oil will be pumped out. At low rates of pumping presumably a stagnant "cone" of water can exist beneath the flowing oil. The main problem of water coning is therefore to determine the maximum rate of pumping under given conditions, above which water will be pumped out. The axisymmetric problem can at present only be solved by the relaxation method. But an analytical solution for the two-dimensional problem, or the problem of water wedging, is possible. This solution will be presented in this section.

With reference to Fig. 2(a), the density of the flowing fluid (oil) is denoted by ρ_1, the density of the stagnant fluid (water) is denoted by ρ_2, and the difference $(\rho_2 - \rho_1)$ is denoted by $\Delta\rho$. By a development similar to that employed in Sec. 2, it can be shown that on the free surface BC the velocity components satisfy the condition

$$u^2 + v^2 - \alpha v = 0, \tag{25}$$

in which $\alpha = k\Delta\rho g/\mu$. This equation differs from (6) only in the sign preceding α, because the flowing fluid is now *above* the stagnant fluid. At the point B, $v = 0$. Hence $u = 0$ according to (25), and B is a stagnation point. At the point C the velocity, if not zero, is in the direction of the vertical. We shall assume first that the velocity at C is not zero, but will later deal with the case in which C is a stagnation point. It appears that the first case is the critical case, in the sense that any increase in discharge (m) will result in the appearance of water in the fluid pumped out. A more detailed discussion will be given after the solutions are obtained. Note that a rigid horizontal boundary at the level of AB is tacitly assumed.

Case 1. Critical case

With ζ defined as before, the boundary of the flow region (or one-half of it) is shown in Fig. 2(b)

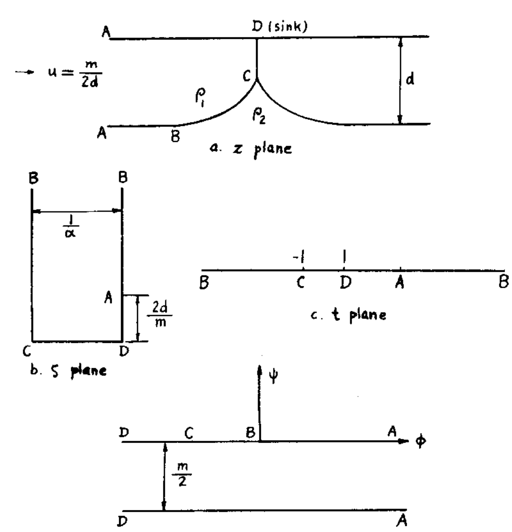

FIG. 2. Planes showing conformal mapping for the problem of water wedging. Case 1: critical case.

in the ζ plane, and in Fig. 2(d) in the w plane. With the coordinates B, C, and D in the t plane as shown in Fig. 2(c), the transformation of Schwarz and Christoffel gives

$$\zeta = (i/\alpha\pi) \cosh^{-1} t \qquad (26)$$

and

$$w = -(m/2\pi)[\ln (t - t_A) - \ln (t - 1)], \qquad (27)$$

in which the constants of integration and the constant factors on the right-hand sides have been determined from the coordinates given in the ζ plane and w plane. The letter m represents the discharge into the sink.

Since

$$\zeta = -i(dz/dw),$$

it follows that

$$dz = i\zeta\, dw = \frac{m}{2\alpha\pi^2} \cosh^{-1} t\left(\frac{1}{t - t_A} - \frac{1}{t - 1}\right) dt. \qquad (28)$$

Since t is negative between B and C, and since the imaginary part of $\cosh^{-1} t$ is then πi for negative t, integration of the imaginary part of (28) yields

$$y = \frac{m}{2\alpha\pi} \ln \frac{t_A - t}{1 - t}, \qquad (29)$$

and

$$y_c = \frac{m}{2\alpha\pi} \ln \frac{t_A + 1}{2}. \qquad (30)$$

In (29), the constant of integration is zero because y_B is zero by choice. The constant t_A is determined from the fact that $u = m/2d$ at A, so that $\zeta_A = (2d/m)i$. Equation (26) then determines t_A to be $\cosh (2d\alpha\pi/m)$. Thus,

$$y_c = \frac{m}{2\alpha\pi}\left[\ln\left(\cosh \frac{2d\alpha\pi}{m} + 1\right) - \ln 2\right]$$
$$= \frac{m}{\alpha\pi} \ln \cosh \frac{d\alpha\pi}{m}. \qquad (31)$$

The shape of the free streamline is given parametrically by (28). The expression for y is given by (29). The expression for x can be determined numerically at least. It is interesting to see whether it is horizontal at B, or intersects AB at an angle. Since at B

$$\frac{dy}{dx} = \lim_{t\to-\infty} \frac{\pi}{\cosh^{-1} |t|} = 0,$$

the free streamline is tangent to AB at B. This is rather strange at first sight, since B is a stagnation point. But the transformations are consistent with stagnancy *and* tangency at B. Evidently the velocity at a smooth corner can be reduced to zero.

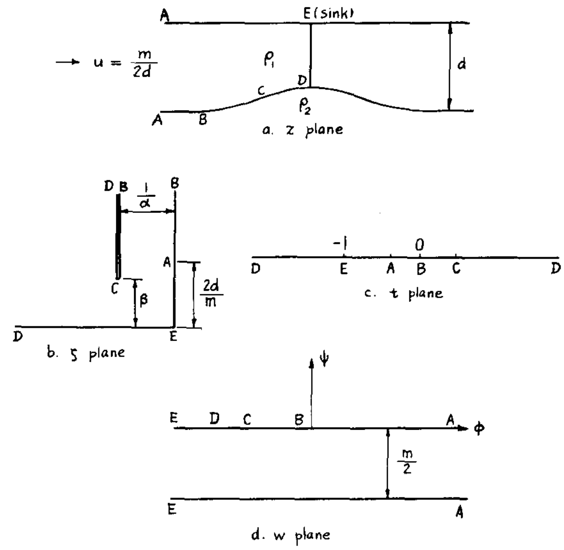

FIG. 3. Planes showing conformal mapping for the problem of water wedging. Case 2: subcritical case.

Case 2. Subcritical case

The graphs of the boundary in the four planes are shown in Fig. 3. The point D is now assumed to be a stagnation point. Between B and D there is a point at which the speed is highest. This point is designated by C. The transformation between ζ and t is

$$\zeta = M \int \frac{t - t_c}{t(t + 1)^{\frac{1}{2}}}\, dt + N$$
$$= 2M[(t + 1)^{\frac{1}{2}} + t_c \tanh^{-1} (t + 1)^{\frac{1}{2}}] + N. \qquad (32)$$

The constant N is zero because $\zeta_E = 0$ and $t_E = -1$. The determination of M depends on the fact that the imaginary part of $\tanh^{-1} (t + 1)^{\frac{1}{2}}$ for any positive t is $\frac{1}{2}\pi i$. As t crosses B it changes from negative to positive. Hence the change in ζ is

$$Mt_c\pi i.$$

On the other hand, this change is exactly $-1/\alpha$. Hence,

$$M = i/\alpha t_c\pi. \qquad (33)$$

The quantity t_c is determined from the equation

$$-\frac{1}{\alpha} + \beta i = \frac{2i}{\alpha\pi} \frac{(t_c + 1)^{\frac{1}{2}}}{t_c} + \tanh^{-1} [(t_c + 1)^{\frac{1}{2}}]. \qquad (34)$$

The real part of this equation is automatically satisfied because M has been determined to give the real part of ζ_c (or of ζ_B for $t = +0$) the value $-1/\alpha$.

The imaginary part of (34) determines t_c in terms of the parameter β:

$$\beta = \frac{2}{\alpha\pi}\left\{\frac{(t_c+1)^{\frac{1}{2}}}{t_c} + \coth^{-1}[(t_c+1)^{\frac{1}{2}}]\right\}, \qquad (35)$$

in which, as is well known,

$$\coth^{-1}[(t_c+1)^{\frac{1}{2}}] = \tanh^{-1}[(t_c+1)^{\frac{1}{2}}] - \tfrac{1}{2}\pi i$$

Thus, we have

$$\zeta\left(= -i\frac{dz}{dw}\right)$$
$$= \frac{2i}{\alpha\pi}\left\{\frac{(t+1)^{\frac{1}{2}}}{t_c} + \tanh^{-1}[(t+1)^{\frac{1}{2}}]\right\}, \qquad (36)$$

whereas w is still given by (27). Thus the free streamline is given parametrically by

$$dz = i\zeta\, dw = \frac{m}{\alpha\pi^2}\left\{\frac{(t+1)^{\frac{1}{2}}}{t_c} + \coth^{-1}[(t+1)^{\frac{1}{2}}] + \tfrac{1}{2}\pi i\right\}\left\{\frac{1}{t-t_A} - \frac{1}{t+1}\right\} dt. \qquad (37)$$

The imaginary part of this equation can be integrated to yield

$$y = \frac{m}{2\alpha\pi}\left[\ln\frac{t-t_A}{t+1} - \ln(-t_A)\right], \qquad (38)$$

the constant of integration being determined by the condition $y_B = 0$. In particular

$$y_D = \frac{m}{2\alpha\pi}\ln\left(\frac{1}{-t_A}\right), \qquad (39)$$

in which t_A is determined from

$$\frac{i2d}{m} = \frac{2i}{\alpha\pi}\frac{(t_A+1)^{\frac{1}{2}}}{t_c} + \tanh^{-1}[(t_A+1)^{\frac{1}{2}}]. \qquad (40)$$

Again at B, dy/dx of the free streamline is zero. Furthermore, at D the same is true. This is consistent with the fact that D is a stagnation point.

Now if the discharge is m_1 the solution for Case 1 gives a volume of water underneath the oil. If m is increased above m_1 to m_2, say, some water will flow out until its volume beneath the oil is that consistent with the solution for $m = m_2$. [In this connection it may be remarked that y_c for Case 1 is zero for $m = \infty$ and is 1 for $m = 0$. It can be assumed that y_c increases monotonically as m decreases. This can perhaps be proved from (31). But the proof turns out to be much more difficult than expected.] If the discharge increases further, the volume decreases further. However, if the discharge is now decreased to m_1 again, say, the flow cannot return to that for Case 1, for $m = m_1$, because the water, once lost, cannot be regained, unless it is added artificially. The flow will then be given by a solution for Case 2, with the volume of water equal to that which is now available, but with $m = m_1$. If the discharge is further reduced, the solution for Case 2 continues to govern the flow, with the free surface presumably becoming flatter and flatter and y_D smaller and smaller. It should be noted that given an m and a water volume, there is one value of β, and hence, from (35) one value of t_c. Then (40) gives the value of t_A and (39) the value of y_D. The situation is too complicated to enable one to see that, with the water volume fixed, y_D decreases with m. The writer believes this to be the case. The verification can only be furnished by numerical computations. The foregoing discussion justifies the use of the terms "critical case" and "subcritical case," because for a prescribed water volume, if the discharge is higher than that on which the solution for Case 1 is based, water will be pumped out.

The case in which the sink is located below the upper boundary can be treated similarly.

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