Supersymmetry and Lie algebras

R. Y. Levine and Y. Tomozawa
Department of Physics, University of Michigan, Ann Arbor, Michigan 48109

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Starting from the standard supersymmetry algebra, an infinite Lie algebra is constructed by introducing commutators of fermionic generators as members of the algebra. From this algebra a finite Lie algebra results for fixed momentum analogous to the Wigner analysis of the Poincaré algebra. It is shown that anticommutation of the fermionic charges plays the role of a constraint on the representation. Also, it is suggested that anticommuting parameters can be avoided by using this infinite Lie algebra with fermionic generators modified by a Klein transformation.

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I. INTRODUCTION

Supersymmetry is unique as a symmetry of nature in that bosons and fermions are grouped together in the same multiplet.[1,2] This feature is essential for the construction of a sensible supergravity[3] theory, but also means that any low energy theory has to incorporate supersymmetry breaking. A deeper understanding of how supersymmetry may arise could certainly shed light on its breaking. The existence of a fermionic charge in supersymmetry requires that the algebra be defined by anticommutation as well as commutation relations. While this allows the evasion of the Coleman–Mandula no go theorem,[4] the resulting algebra is not a Lie algebra and the parameters for infinitesimal transformations are anticommuting numbers (Grassmann variables). This leads naturally to an extension of Minkowski space, known as superspace, in which spinors are attached to each space-time point.[2] With anticommuting parameters, one has the conceptual problem of nilpotent translation parameters. Further, all continuous symmetries in nature have been represented by a Lie algebra. It is therefore natural to ask whether supersymmetry can be represented by a Lie algebra.

In this paper fermionic anticommutation relations are used to construct the commutation relations of an infinite Lie algebra. In this algebra successive multiplication by the momentum operator defines new generators. The algebra thus obtained has both the Wigner representation of the Poincaré algebra and the standard supersymmetry representations. The latter arises when anticommutation of the fermionic charges is used as a constraint. The standard superalgebra requires the parameters of infinitesimal supersymmetry transformations to be anticommuting c-numbers in order to have a finite closed algebra. With our formalism commuting c-numbers close the algebra due to the added generators. However, in order to preserve the spin statistics relationship it is necessary to modify the fermionic generators with a Klein transformation.

In Sec. II, an explicit construction of the infinite Lie algebra is presented. Section III contains the resulting finite Lie algebra for fixed momentum, which is analogous to the Wigner analysis of the Poincaré algebra. This section also contains constructions of massive and massless representations of the finite algebra with the anticommutation constraint. A discussion of the modification of fermionic generators by the Klein transformation is in Sec. IV followed by general discussion in Sec. V.

II. THE INFINITE LIE ALGEBRA

The superalgebra is defined by the following commutation and anticommutation relations[1,2]:

$$\begin{aligned}
&[J_{\mu\nu},J_{\lambda\rho}] = i(\delta_{\mu\lambda}J_{\nu\rho} - \delta_{\mu\rho}J_{\nu\lambda} + \delta_{\nu\rho}J_{\mu\lambda} - \delta_{\nu\lambda}J_{\mu\rho}),\\
&[J_{\mu\lambda},P_\nu] = i(\delta_{\mu\nu}P_\lambda - \delta_{\lambda\nu}P_\mu),\\
&[P_\mu,P_\nu] = 0,\\
&[S^\alpha,P_\mu] = 0,\\
&[S^\alpha,J_{\lambda\nu}] = \tfrac{1}{2}(\sigma_{\lambda\nu})_{\alpha\beta}S^\beta,\\
&\{S^\alpha,S^\beta\} = i(\gamma_\mu C)_{\alpha\beta}P^\mu.
\end{aligned} \tag{2.1}$$

Here $J_{\mu\nu}$ and P_λ are the generators of the Poincaré algebra, $S^\alpha (\alpha = 1,2,3,4)$ are the fermionic Majorana generators of supersymmetry, and C is the charge conjugation matrix ($C^+C = 1$, $C^T = -C$, $C^{-1}\gamma_\mu C = -\gamma_\mu^T$). In this paper only this simplest algebra is considered although the extension to the case of a fermionic Dirac generator or multiple Majorana generators is trivial. In order to form the Lie algebra we consider the commutator

$$[S^\alpha,S^\beta] \equiv T^{\alpha\beta}. \tag{2.2}$$

If $T^{\alpha\beta}$ was expressible as a linear combination of the generator of the superalgebra we would have

$$T^{\alpha\beta} = a(\gamma_5\gamma_\mu C)^{\alpha\beta}P^\mu, \tag{2.3}$$

where a is a dimensionless number. Note that C, $(\gamma_5 C)$, and $(\gamma_5\gamma_\mu C)$ are antisymmetric and $(\gamma_\mu C)$ and $(\sigma_{\mu\nu}C)$ are symmetric. Using the Jacobi identity (2.4), where

$$[[A,B],C] = \{\{B,C\},A\} - \{\{C,A\},B\} \tag{2.4}$$

$$= -[[B,C],A] - [[C,A],B], \tag{2.5}$$

with $A = S^\alpha$, $B = S^\beta$, and $C = S^\delta$, it is easy to show that (2.3) is inconsistent. Therefore, we conclude that $T^{\alpha\beta}$ is a new generator. Furthermore, there is no consistent second-order operator in the Poincaré algebra that $T^{\alpha\beta}$ could equal. For example,

$$T^{\alpha\beta} = a(\gamma_5\gamma_\mu C)^{\alpha\beta}W_\mu, \tag{2.6}$$

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where $W^\mu = -i/2\, \epsilon^{\mu\lambda\nu\rho} J_{\lambda\nu} P_\rho$ is the Pauli–Lubanski vector,[5] is contradicted by the Jacobi identity (2.4) with $A = S^\alpha$, $B = S^\beta$, and $C = S^\delta$. (This is proven with an appropriate Fierz transformation of the γ-matrices.)

Because $T^{\alpha\beta}$ is an independent generator we consider the commutators with $J_{\mu\nu}$, P_μ, S^α, and itself:

$$\begin{aligned}
&[T^{\alpha\beta},S^\delta] = 2iS^\alpha(\gamma_\mu C)_{\beta\delta}P^\mu - 2i(\gamma_\mu C)_{\alpha\delta}P^\mu S^\beta,\\
&[T^{\alpha\beta},T^{\eta\delta}] = 2iT^{\eta\alpha}(\gamma_\mu C)_{\delta\beta}P^\mu - 2iT^{\delta\alpha}(\gamma_\mu C)_{\eta\beta}P^\mu\\
&\qquad - 2iT^{\eta\beta}(\gamma_\mu C)_{\delta\alpha}P^\mu + 2iT^{\delta\beta}(\gamma_\mu C)_{\eta\alpha}P^\mu,\\
&[T^{\alpha\beta},P_\mu] = 0,\\
&[T^{\alpha\beta},J_{\mu\nu}] = \tfrac{1}{2}(\sigma_{\mu\nu})_{\alpha\delta}T_{\delta\beta} - \tfrac{1}{2}(\sigma_{\mu\nu})_{\beta\delta}T_{\delta\alpha}.
\end{aligned} \tag{2.7}$$

The first of these is constructed using the anticommutation relation in (2.1) and the identity of (2.4). The rest follow from the commutation relations in (2.1) and the identity (2.5). From (2.7) it can be seen that there are new generators $S^\alpha P_\mu$ and $T^{\alpha\beta}P_\mu$. Again the commutators of these new members with all the previous generators and with themselves must be considered. We exhibit these in Appendix A. From these relations one must include as new members of the algebra the operators on the right-hand side of (A1):

$$S^\alpha P_\mu P_\nu, \quad T^{\alpha\beta}P_\mu P_\nu,$$
$$S^\alpha P_\mu P_\nu P_\lambda, \quad T^{\alpha\beta}P_\mu P_\nu P_\lambda.$$

Obviously, there are a finite number of generators of the form $S^\alpha P_\mu \cdots P_\omega$ and $T^{\alpha\beta}P_\mu \cdots P_\omega$ added for each order of commutation. Thus, we obtain an infinite Lie algebra with generators $J_{\mu\nu}$, P_μ, S_α, $T^{\alpha\beta}$, $S^\alpha P_\mu$, $T^{\alpha\beta}P_\mu$,..., $S^\alpha P_\mu \cdots P_\omega$, $T^{\alpha\beta}P_\mu \cdots P_\omega$,... . The added generators are of a geometrical series type and the resulting algebra is called an affine Lie algebra.[6] Note that the Casimir operators of this algebra are identical to the Casimir operators of the superalgebra. Also, it will be shown that the representations of the superalgebra are those representations of the infinite algebra satisfying the anticommutation relation in (2.1) as a constraint. Instead of studying the infinite Lie algebra directly, we will consider in the next section the finite Lie algebra which results from a fixed momentum condition. For this to be consistent we must use the operator W_μ instead of $J_{\mu\lambda}$, since W_μ commutes with P_λ.

Incidently, note with W_μ alone we have the commutators

$$\begin{aligned}
&[W^\mu,W^\nu] = \epsilon^{\mu\nu\lambda\rho}W_\lambda P_\rho,\\
&[W^\mu P_\lambda,W^\nu] = \epsilon^{\mu\nu\kappa\rho}W_\kappa P_\rho P_\lambda,
\end{aligned} \tag{2.8}$$

and so on. This forms an infinite Lie algebra with generators W_μ, $W_\mu P_\lambda$, $W_\mu P_\lambda P_\nu$,... similar to the structure above. A finite SU(2) algebra follows for fixed timelike momentum.It is important to realize that the analysis of the finite algebra for fixed momentum is equivalent to an analysis of the Poincaré algebra.

III. THE FINITE ALGEBRA FOR FIXED MOMENTUM

The algebra generated by W_μ, $T^{\alpha\beta}$, and S^α for fixed momentum is defined by the commutation relations

$$\begin{aligned}
&[W^\mu,W^\nu] = \epsilon^{\mu\nu\lambda\rho}W_\lambda P_\rho,\\
&[W^\mu,S^\alpha] = (i/4)\epsilon^{\mu\lambda\nu\rho}(\sigma_{\lambda\nu})_{\alpha\beta}S^\beta P_\rho,\\
&[T^{\alpha\beta},W^\mu] = (i/4)\epsilon^{\mu\lambda\nu\rho}(\sigma_{\lambda\nu})_{\beta\delta}T^{\delta\alpha}P_\rho\\
&\qquad - (i/4)\epsilon^{\mu\lambda\nu\rho}(\sigma_{\lambda\nu})_{\alpha\delta}T^{\delta\beta}P_\rho,\\
&[S^\alpha,S^\beta] = T^{\alpha\beta},\\
&[T^{\alpha\beta},T^{\eta\delta}] = 2iT^{\eta\alpha}(\gamma_\mu C)_{\delta\beta}P^\mu - 2iT^{\delta\alpha}(\gamma_\mu C)_{\eta\beta}P^\mu\\
&\qquad - 2iT^{\eta\beta}(\gamma_\mu C)_{\delta\alpha}P^\mu + 2iT^{\delta\beta}(\gamma_\mu C)_{\eta\alpha}P^\mu,\\
&[T^{\alpha\beta},S^\delta] = 2iS^\alpha(\gamma_\mu C)_{\beta\delta}P^\mu - 2i(\gamma_\mu C)_{\alpha\delta}P^\mu S^\beta.
\end{aligned} \tag{3.1}$$

For fixed momentum we have the finite algebra generated by $\{W_\mu, T^{\alpha\beta}, S^\delta\}$ (denoted A_{WTS}) and a subalgebra generated by $\{W_\mu, T^{\alpha\beta}\}$ (denoted A_{WT}). A Casimir operator of the superalgebra (and therefore the infinite algebra) is also a Casimir operator of A_{WTS}. Denoting A_{JPS} to be the superalgebra and $C(A)$ to be the set of Casimir operators of an algebra A we have the inclusion relations

$$C(A_{JPS}) \subset C(A_{WTS}) \subset C(A_{WT}). \tag{3.2}$$

Also, the irreducible representations of A_{JPS} correspond to irreducible representations of A_{WTS} in the same way that the irreducible representations of A_W for fixed momentum correspond to irreducible representations of the Poincaré algebra. For both statements the converse is not true. In fact the representations of A_{JPS} are obtained by enforcing the anticommutation relation

$$\{S^\alpha,S^\beta\} = i(\gamma_\mu C)_{\alpha\beta}P^\mu \tag{3.3}$$

as a constraint on the representations of A_{WTS} (as well as the infinite algebra). Similar to Wigner,[5] we give an explicit construction of A_{WTS} in the rest frame and for mass zero.

A. Timelike momentum (massive particle representation)

Choosing the rest frame $P_\mu = (0,0,0,iP_0)$, we first rearrange the generators so that the group structure of A_{WT} is transparent.[7] Define

$$\begin{aligned}
&L_3 = \frac{(T^{23} - T^{14})}{4P_0}, \quad M_3 = \frac{(T^{23} + T^{14})}{4P_0},\\
&N_3 = \frac{W_3}{P_0} - \frac{(T^{23} + T^{14})}{4P_0},\\
&L_+ = \frac{T^{12}}{2P_0}, \quad M_+ = \frac{T^{24}}{2P_0},\\
&N_+ = \left(\frac{W_+}{P_0} - \frac{T^{24}}{2P_0}\right),\\
&L_- = \frac{-T^{34}}{2P_0}, \quad M_- = -\frac{T^{13}}{2P_0},\\
&N_- = \left(\frac{W_-}{P_0} + \frac{T^{13}}{2P_0}\right)
\end{aligned} \tag{3.4}$$

(note that that $W_j = \frac{1}{2}\epsilon_{jkl}J^{kl}P_0$, $W_0 = 0$). From (3.1) with $P_\mu = (\mathbf{0},iP_0)$ one has that the operators **L, M, N** all commute with each other and each generate an SU(2) algebra;

$$[N_3,N_\pm] = \pm N_\pm, \quad [N_+,N_-] = 2N_3, \text{ etc.} \tag{3.5}$$

In other words, $A_{WT} = \mathrm{SU(2)} \times \mathrm{SU(2)} \times \mathrm{SU(2)}$. Trivially then, the Casimirs of A_{WT} are given by $\mathbf{N}^2$, $\mathbf{L}^2$, $\mathbf{M}^2$, where $N_\pm = N_1 \pm iN_2$, etc. We note that N_i $(i = 1,2,3)$ commutes with S^α $(\alpha = 1,2,3,4)$.

To construct A_{WTS} we normalize $\tilde{S}^\alpha = S^\alpha/(2P_0)^{1/2}$ and have the commutation relations

$$[L_3,\tilde{S}^\alpha] = \pm\tfrac{1}{2}\tilde{S}^\alpha\begin{cases} +, & \alpha = 1,2\\ -, & \alpha = 3,4,\end{cases}$$
$$[M_3,\tilde{S}^\alpha] = \pm\tfrac{1}{2}\tilde{S}^\alpha\begin{cases} +, & \alpha = 2,4\\ -, & \alpha = 1,3,\end{cases}$$
$$[L_+,\tilde{S}^3] = \tilde{S}^1,\quad [L_-,\tilde{S}^2] = \tilde{S}^4,$$
$$[L_+,\tilde{S}^4] = \tilde{S}^2,\quad [L_-,\tilde{S}^1] = \tilde{S}^3,$$
$$[M_+,\tilde{S}^1] = -\tilde{S}^2,\quad [M_-,\tilde{S}^2] = -\tilde{S}^1, \tag{3.6}$$
$$[M_+,\tilde{S}^3] = -\tilde{S}^4,\quad [M_-,\tilde{S}^4] = -\tilde{S}^3,$$
$$[\tilde{S}^1,\tilde{S}^2] = L_+,\quad [\tilde{S}^2,\tilde{S}^4] = M_+,$$
$$[\tilde{S}^3,\tilde{S}^4] = -L_-,\quad [\tilde{S}^1,\tilde{S}^3] = -M_-,$$
$$[\tilde{S}^1,\tilde{S}^4] = (M_3 - L_3),\quad [\tilde{S}^2,\tilde{S}^3] = (L_3 + M_3),$$

with the rest zero. From the root vector diagram exhibited in Fig. 1 for these relations, one has that the algebra generated by **L**, **M** and S^α is $Sp(4)\equiv C_2$ or, equivalently, $SO(5)\equiv B_2$. Therefore, the algebra A_{WTS} is $Sp(4)\times SU(2)$, where SU(2) is generated by **N**.

The irreducible representations of A_{WTS} are determined by (λ_1,λ_2,N), where (λ_1,λ_2) are the highest weight values in the Sp(4) representation.[8] Members of the representation are designated by $(\lambda_1,\lambda_2,L,L_3,M,M_3;N,N_3)$. However, in the following it is shown that the anticommutation relation (3.3) restricts the values of λ_1 and λ_2 while leaving N unconstrained due to its commuting with S^α.

In the rest frame, (3.3) is given by

$$\{S^1,S^4\} = -P_0,\quad \{S^2,S^3\} = P_0,\quad (S^\alpha)^2 = 0, \tag{3.7}$$

and all other anticommutators zero. From the Majorana condition,

$$S = C\bar{S}^T \tag{3.8}$$

or

$$(S^4)^\dagger = -S^1 \quad\text{and}\quad (S^3)^\dagger = S^2. \tag{3.9}$$

We are led to identify the following operators[2]:

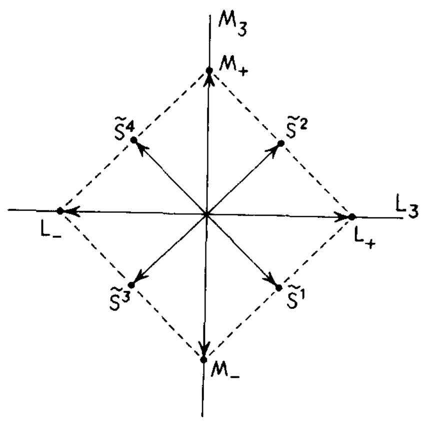

FIG. 1. Root vector diagrams for Sp(4) algebra. The generators are identified at the head of the corresponding root vector. The axes refer to the Cartan subalgebra with $H_1 = L_3$ and $H_2 = M_3$.

$$a_1 = \frac{S^1}{\sqrt{P_0}},\quad a_1^* = -\frac{S^4}{\sqrt{P_0}},$$
$$a_2 = \frac{S^2}{\sqrt{P_0}},\quad a_2^* = \frac{S^3}{\sqrt{P_0}}. \tag{3.10}$$

From the commutation relations between J_{12} and S^α in Eq. (2.1), we have

$$[J_{12},a_i] = \mp\tfrac{1}{2}a_i \quad\text{for}\quad i = \begin{cases}1\\2\end{cases}. \tag{3.11}$$

Therefore, we identify $a_1(a_2)$ as the annihilation operator for $J_z = \frac{1}{2}(-\frac{1}{2})$ and $a_1^*(a_2^*)$ as the creation operator for $J_z = \frac{1}{2}(-\frac{1}{2})$. Defining number operators $n_1 = a_1^*a_1$ and $n_2 = a_2^*a_2$ we have

$$N_3 = J_{12} + \frac{n_2 - n_1}{2},\quad N_+ = J_+ - a_1^*a_2,$$
$$N_- = J_- + a_1a_2^*, \tag{3.12}$$
$$L_3 = \frac{1 - n_1 - n_2}{2},\quad M_3 = \frac{n_1 - n_2}{2},$$

and[9]

$$\mathbf{L}^2 + \mathbf{M}^2 = \tfrac{3}{4}. \tag{3.13}$$

Equation (3.13) implies that the irreducible representation is restricted to $(L,M) = (\frac{1}{2},0)$ or $(0,\frac{1}{2})$ or, equivalently, $(\lambda_1,\lambda_2) = (1,0)$ with arbitrary N. This gives the identification of n_1 and n_2 as follows:

$$n_1 = 0,\quad n_2 = 0,\quad L_3 = \tfrac{1}{2},\quad M_3 = 0,$$
$$n_1 = 1,\quad n_2 = 1,\quad L_3 = -\tfrac{1}{2},\quad M_3 = 0,$$
$$n_1 = 1,\quad n_2 = 0,\quad L_3 = 0,\quad M_3 = \tfrac{1}{2}, \tag{3.14}$$
$$n_1 = 0,\quad n_2 = 1,\quad L_3 = 0,\quad M_3 = -\tfrac{1}{2}.$$

We can, therefore, replace L and M by n_1 and n_2 and arrive at the set $\{N,N_3,n_1,n_2\}$ or $\{N,J_{12},n_1,n_2\}$ as the commuting operators. These bases are used by Salam and Strathdee to construct the explicit representations of the superalgebra.[2] For completeness we reconstruct this representation. Using Eq. (3.12) one constructs the $N = 0$ representations containing $J = \{\frac{1}{2},0,0\}$ and the $N = \frac{1}{2}$ representation with $J = \{1,\frac{1}{2},\frac{1}{2},0\}$. For arbitrary $N > 0$ the representation contains $J = \{N + \frac{1}{2},N,N,N - \frac{1}{2}\}$ with a total of $4(2N + 1)$ states. The two states with $J = N$ correspond to $(n_1,n_2) = (0,0)$ and $(1,1)$. The parity operation

$$S \rightarrow S' = e^{i\eta}\gamma_4 S, \tag{3.15}$$

with the Majorana condition $S' = C\bar{S}'^T$, requires that $\eta = \pi/2$ (or $-\pi/2$) and

$$P\,|N,N_3,n_1,n_2\rangle = (-1)^{n_1+n_2}|N,N_3,n_1,n_2\rangle, \tag{3.16}$$

where P is the parity operator. Thus the $J = N$ states are of opposite parity.

As described above, among the representations of $A_{WTS} = Sp(4)\times SU(2)$ only the Sp(4) spinor representation is allowed by the constraint (3.3). Therefore, the only nonconstant Casimir is $\mathbf{N}^2$. Note from Eq. (3.4) this can be written

$$N_j = \frac{1}{P_0}\,[\,W_j - (i/4)SC^{-1}\gamma_j\gamma_5 S\,] \tag{3.17}$$

and can be generalized relativistically by defining

$$K_\mu = W_\mu - (i/4)SC^{-1}\gamma_\mu\gamma_5 S, \tag{3.18}$$

with

$$\mathbf{N}^2 = \left(K_\mu^2 - \frac{(K\cdot P)^2}{P^2}\right)\bigg|_{P=(0,iP_0)}. \tag{3.19}$$

From the commutators

$$[K_\mu S^\alpha] = -\tfrac{1}{2}(\gamma_5)_{\alpha\beta}S^\beta P_\mu \tag{3.20}$$

and

$$[K_\mu, P_\lambda] = 0, \tag{3.21}$$

we have $(K_\mu P_\nu - K_\nu P_\mu)^2$ commuting with all operators S^α, $J_{\mu\nu}$, and P_μ. This is the relativistic expression of the Casimir operator $\mathbf{N}^2$. This operator, with P^2, forms the set of Casimir operators for A_{JPS}.

B. Lightlike momentum (massless particles)

Taking $P_\mu = (0,0,p,ip)$ in Appendix B, all commutation relations for $A_{W\hat{T}\hat{S}}$ are given, where

$$\hat{S}^1 = \frac{(S^1+S^3)}{2}, \quad \hat{S}^2 = \frac{(S^2+S^4)}{2},$$
$$\hat{S}^3 = \frac{(S^3-S^1)}{2}, \quad \hat{S}^4 = \frac{(S^4-S^2)}{2}, \tag{3.22}$$

and

$$\hat{T}^{\alpha\beta} = [\hat{S}^\alpha, \hat{S}^\beta].$$

Note from Eq. (B7) that the group structure of $A_{W\hat{T}\hat{S}}$ is $G\times U(1)$, where U(1) is generated by $\hat{T}^{23}$, which commutes with all generators of $A_{W\hat{T}\hat{S}}$. From Eq. (B6) the Casimir operator is

$$K_{\mu\nu}^2 = -2(\hat{T}^{23})^2 p^2. \tag{3.23}$$

Instead of analyzing G and its representations, we consider the constraint condition (3.3),

$$\{\hat{S}^1, \hat{S}^4\} = -p, \tag{3.24}$$

and all other $\{\hat{S}^i, \hat{S}^j\} = 0$. Define creation and annihilation operators

$$\frac{\hat{S}^1}{\sqrt{p}} = a, \quad -\frac{\hat{S}^4}{\sqrt{p}} = a^*,$$
$$\frac{\hat{S}^2}{\sqrt{p}} = b, \quad -\frac{\hat{S}^3}{\sqrt{p}} = b^*, \tag{3.25}$$

with

$$\{a, a^*\} = 1, \tag{3.26}$$

$$\{b, b^*\} = 0, \tag{3.27}$$

and all others vanish. Equation (3.26) leads us to identify $a(a^*)$ as the annihilation (creation) operator of a spin up state. (Note $[J_{12}, \hat{S}^1] = -\tfrac{1}{2}\hat{S}^1$ and $[J_{12}, \hat{S}^4] = \tfrac{1}{2}\hat{S}^4$). Equation (3.27) implies the operation of b on any state $|\psi\rangle$ is zero:

$$b\,|\psi\rangle = b^*|\psi\rangle = 0. \tag{3.28}$$

It immediately follows that

$$\hat{T}^{23} = p[b, b^*] = 0. \tag{3.29}$$

In fact, all $\hat{T}^{ij}$ except $\hat{T}^{14}$ vanish. With this result it follows that K_3 and $K\pm$, defined in Eq. (B5), form the Euclidean algebra E_2. As seen from Eq. (B9),

$$[K_+, K_-] = 0,$$
$$[K_3, K_\pm] = \pm K_\pm\, p. \tag{3.30}$$

As is shown in Appendix C, this implies that $K_+ = K_- = 0$ for a finite dimensional representation. This is analogous to the $W^2 = 0$ condition in the Wigner analysis of lightlike momentum. In this case the representation is characterized by generalized helicity Λ,

$$\Lambda = \frac{\mathbf{K}\cdot\mathbf{P}}{p^2} = \frac{K_3}{p} = \frac{(W_3 - \tfrac{1}{2}\hat{T}^{14})}{p}$$
$$= (J_{12} - n + \tfrac{1}{2}), \tag{3.31}$$

where $n = a^*a$. As is shown by Salam and Strathdee,[2] the representation is characterized by two states, $|j_3\rangle$ and $|j_3 + \tfrac{1}{2}, n = 1\rangle$, where $a|j_3\rangle = 0$ and $|j_3 + \tfrac{1}{2}, n = 1\rangle = a^*|j_3\rangle$. Both have $\Lambda = (j_3 + \tfrac{1}{4})$. It is obvious that the parity operator acting on these states gives a basis set of the opposite helicity. It should be emphasized that, as for the massive case, among all possible representations of $A_{W\hat{T}\hat{S}}$ the supersymmetry representation obtained above is selected by the constraint (3.3).

IV. SPIN STATISTICS IN THE INFINITE LIE ALGEBRA

In the previous sections it has been shown that the representations of superalgebra are equivalent to the representations of the infinite Lie algebra with the anticommutation relations among the fermionic charges as a constraint. From states in the irreducible representations of A_{WTS}, field operators can be constructed by a standard method.[10] The infinitesimal tranformation of the field operator $\Phi(x)$ by fermionic generators is given by

$$\delta\Phi = i\,\bar{\epsilon}^\alpha[S^\alpha, \Phi], \tag{4.1}$$

where ϵ^α is a constant spinor. The choice of ϵ^α to be a commuting parameter contradicts the spin statistics relation in the case of fermionic fields $\Phi \equiv \psi$. The standard procedure is to use anticommuting c-numbers as parameters for supersymmetry transformations. These parameters also anticommute with fermionic fields.

For the infinite Lie algebra we require commuting parameters because the commutator of two supersymmetry transformations should be a generator of the algebra,

$$[\bar{\epsilon}_1 S, \bar{\epsilon}_2 S] = \bar{\epsilon}_1^\alpha \bar{\epsilon}_2^\beta T^{\alpha\beta}. \tag{4.2}$$

In order to resolve the spin statistics problem in Eq. (4.1), we use the Klein transformation[11] to define a new fermionic operator,

$$S'^\alpha = i(-1)^{N_F} S^\alpha, \tag{4.3}$$

where N_F is the fermionic number operator. Note that the fermionic content of S^α is not defined; that is, being a Majorana spinor, S^α is a mixture of the ± 1 eigenvectors of N_F. However, the Klein operator $(-1)^{N_F}$ has definite anticommutation relations with all fermionic operators regardless of Dirac or Majorana properties:

$$\{(-1)^{N_F},\psi\} = 0 \tag{4.4}$$

and

$$\{(-1)^{N_F},S^\alpha\} = 0, \tag{4.5}$$

where ψ and S^α are Dirac or Majorana. This is seen by the following argument. Letting ψ be a Dirac field satisfying

$$[N_F,\psi] = \pm\psi, \tag{4.6}$$

we have

$$e^{i\pi N_F}\psi e^{-i\pi N_F} = e^{\pm i\pi}\psi = -\psi. \tag{4.7}$$

This equation implies (4.4) for Dirac fields with definite fermion number. A Majorana field ψ_1 or ψ_2 can be expressed in terms of a Dirac field ψ by

$$\psi_1 = \frac{(\psi+\psi^C)}{\sqrt{2}} \tag{4.8}$$

or

$$\psi_2 = \frac{(\psi-\psi^C)}{\sqrt{2}\,i}. \tag{4.9}$$

It is easy to see that Eq. (4.7) and therefore (4.4) is valid for Majorana fields without definite fermion number.

With these new operators, Eq. (4.3), we have

$$\delta\psi = [\bar\epsilon^\alpha S'^\alpha,\psi] = -\bar\epsilon^\alpha\{S^\alpha,\psi\}(-1)^{N_F} \tag{4.10}$$

for fermionic fields ψ. Therefore, we have

$$[\bar\epsilon^\alpha S'^\alpha,\bar\epsilon^\beta S'^\beta] = \bar\epsilon^\alpha\bar\epsilon^\beta T^{\alpha\beta} \tag{4.11}$$

because $(-1)^{2N_F} = 1$. In terms of operators we have the equation

$$[S'^\alpha,S'^\beta] = T^{\alpha\beta} \equiv T'^{\alpha\beta}. \tag{4.12}$$

The constraint is given by

$$\{S'^\alpha,S'^\beta\} = i(\gamma_\mu C)_{\alpha\beta}P^\mu. \tag{4.13}$$

The infinite Lie algebra should be modified using S'. Equations (2.1), (2.7), (3.3), and those in Appendix A are altered by the replacement

$$S^\alpha \to S'^\alpha,$$
$$T^{\alpha\beta} \to T'^{\alpha\beta}, \tag{4.14}$$

This modification alters neither the structure nor the physical content of A_{WTS}. Note that the Majorana condition for S' implies

$$S' = i(-1)^{N_F}C\bar S^T = C\bar S'^T. \tag{4.15}$$

Therefore, the sign change in the constraint Eq. (4.13) does not alter the definition of creation and annihilation operators in Eq. (3.10).

V. CONCLUSION

The standard supersymmetry transformation is exponentiated[12] using infinitesimal anticommuting parameters. We have avoided this by constructing an infinite Lie algebra and using the Klein transformation. The anticommutation relations of fermionic generators becomes a constraint on the representations of the infinite Lie algebra which yields the standard supersymmetry representations.

While anticommuting parameters are natural in superspace and the superspace formalism is convenient for constructing field theories, it is extremely difficult to comprehend a physical reality in such a space. Our formalism replaces this difficult concept with an infinite Lie algebra which uses commuting parameters. This new viewpoint of supersymmetry may help in understanding the nature of the symmetry, its breaking, and supergravity.

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APPENDIX A

Consider the commutation relations for $\{S^\alpha P_\mu, T^{\alpha\beta}P_\mu\}$ which appear on the right-hand side of Eq. (2.7). These operators are generators for the infinite Lie algebra.

$$
\begin{aligned}
&[S^\alpha P_\mu,S^\beta P_\lambda] = T^{\alpha\beta}P_\mu P_\lambda,\\
&[S^\alpha P_\mu,T^{\beta\eta}P_\lambda] = -2iS^\beta(\gamma_\rho C)_{\eta\alpha}P^\rho P_\mu P_\lambda\\
&\qquad + 2i(\gamma_\rho C)_{\beta\alpha}S^\eta P^\rho P_\mu P_\lambda,\\
&[T^{\alpha\beta}P_\lambda,T^{\eta\delta}P_\nu] = [2iT^{\eta\alpha}(\gamma_\mu C)_{\delta\beta}P^\mu\\
&\qquad - 2iT^{\delta\alpha}(\gamma_\mu C)_{\eta\beta}P^\mu\\
&\qquad - 2iT^{\eta\beta}(\gamma_\mu C)_{\delta\alpha}P^\mu\\
&\qquad + 2iT^{\delta\beta}(\gamma_\mu C)_{\eta\alpha}P^\mu]P_\lambda P_\nu,\\
&[S^\alpha P_\mu,S^\beta] = T^{\alpha\beta}P_\mu,\\
&[T^{\alpha\beta}P_\mu,S^\eta] = 2iS^\alpha(\gamma_\rho C)_{\beta\eta}P^\rho P_\mu\\
&\qquad - 2i(\gamma_\rho C)_{\alpha\eta}S^\beta P^\rho P_\mu,\\
&[S^\alpha P_\mu,P_\lambda] = 0,\\
&[T^{\alpha\beta}P_\mu,P_\lambda] = 0,\\
&[S^\alpha P_\mu,T^{\beta\eta}] = -2iS^\beta(\gamma_\rho C)_{\eta\alpha}P^\rho P_\mu\\
&\qquad + 2i(\gamma_\rho C)_{\beta\alpha}S^\eta P^\rho P_\mu,\\
&[T^{\alpha\beta}P_\lambda,T^{\eta\delta}] = [2iT^{\eta\alpha}(\gamma_\mu C)_{\delta\beta}P^\mu - 2iT^{\delta\alpha}(\gamma_\mu C)_{\eta\beta}P^\mu,\\
&\qquad - 2iT^{\eta\beta}(\gamma_\mu C)_{\delta\alpha}P^\mu\\
&\qquad + 2iT^{\delta\beta}(\gamma_\mu C)_{\eta\alpha}P^\mu]P_\lambda,\\
&[S^\alpha P_\mu,J_{\lambda\nu}] = iS^\alpha(-\delta_{\lambda\mu}P_\nu + \delta_{\mu\nu}P_\lambda)\\
&\qquad + \tfrac{1}{2}(\sigma_{\lambda\nu})_{\alpha\beta}S^\beta P_\mu,
\end{aligned} \tag{A1}
$$

$$[T^{\alpha\beta}P_\mu,J_{\lambda\nu}] = iT^{\alpha\beta}(-\delta_{\lambda\mu}P_\nu + \delta_{\mu\nu}P_\lambda) + \tfrac{1}{2}(\sigma_{\lambda\nu})_{\alpha\delta}T_{\delta\beta}P_\mu - \tfrac{1}{2}(\sigma_{\lambda\nu})_{\beta\delta}T_{\delta\alpha}P_\mu. \tag{A2}$$

Note that Eqs. (A1) are obtained from Eq. (2.7) by multiplying by the appropriate momentum operators. Equation (A2) has no new generators on the right-hand side.

APPENDIX B: MASSLESS CASE $P_\mu = (0,0,p,ip)$

Introduce the variables

$$\hat S^1 = \left(\frac{S^1+S^3}{2}\right),\quad \hat S^2 = \left(\frac{S^2+S^4}{2}\right),$$
$$\hat S^3 = \left(\frac{S^3-S^1}{2}\right),\quad \hat S^4 = \left(\frac{S^4-S^2}{2}\right). \tag{B1}$$

With the constraint equation (3.3) we have

$$\{\hat S^1,\hat S^4\} = -p,$$
$$\{\hat S^1,\hat S^2\} = \{\hat S^1,\hat S^3\} = \{\hat S^2,\hat S^4\} = \{\hat S^2,\hat S^3\} = \{\hat S^3,\hat S^4\} = 0. \tag{B2}$$

In terms of the $\hat S$ variables the commutators $\hat T^{\alpha\beta}$ are given by

$$\hat T^{12} = [\hat S^1,\hat S^2] = \tfrac{1}{4}(T^{12} + T^{14} - T^{23} + T^{34}),$$
$$\hat T^{13} = \tfrac{1}{2}T^{13},$$
$$\hat T^{14} = \tfrac{1}{4}(-T^{12} + T^{14} + T^{23} + T^{34}),$$
$$\hat T^{23} = \tfrac{1}{4}(T^{12} + T^{14} + T^{23} - T^{34}),$$
$$\hat T^{24} = \tfrac{1}{2}T^{24},$$
$$\hat T^{34} = \tfrac{1}{4}(T^{12} - T^{14} + T^{23} + T^{34}). \tag{B3}$$

Also,

$$W_1 = (J_{23} + iJ_{24})p,\quad W_3 = (J_{12})p,$$
$$W_2 = -(J_{13} + iT_{14})p,\quad W_4 = i(J_{12})p. \tag{B4}$$

Expressing the components of K_μ [Eq. (3.18)] in terms of $\hat T$ amd W,

$$K_+ = (W_+ - \tfrac{1}{2}T^{24}) = (W_+ - \hat T^{24}),$$
$$K_- = (W_- + \tfrac{1}{2}T^{13}) = (W_- + \hat T^{13}),$$
$$K_3 = W_3 - \tfrac{1}{4}(T^{14} + T^{23}) = W_3 - \tfrac{1}{2}(\hat T^{14} + \hat T^{23}),$$
$$K_4 = i(W_3 - \tfrac{1}{4}(-T^{12} + T^{34})) = i(W_3 - \tfrac{1}{2}(\hat T^{14} - \hat T^{23})), \tag{B5}$$

and the Casimir operator is given by

$$K^2_{\mu\nu} = -2(K_\mu P^\mu)^2 = -2(\hat T^{23})^2p^2. \tag{B6}$$

The commutation relations for $A_{W\hat T\hat S}$ are given by the following:

$\hat T^{23}$ commutes with all $\hat S^\alpha$ and W^i as seen from Eq. (B6), (B7)

K_+,K_- commute with all $\hat S^\alpha$, $\hat T^{\alpha\beta}$, (B8)

$$[K_+,K_-] = -2p\hat T^{23}, \tag{B9}$$
$$[K_3,K_\pm] = \pm K_\pm p, \tag{B10}$$
$$[K_3/p,\hat T^{12}] = \hat T^{12},$$
$$[K_3/p,\hat T^{34}] = -\hat T^{34}, \tag{B11}$$
$$[K_3/p,\hat S^1] = \tfrac{1}{2}\hat S^1,$$
$$[K_3/p,\hat S^2] = \tfrac{1}{2}\hat S^2,$$
$$[K_3/p,\hat S^3] = -\tfrac{1}{2}\hat S^3,$$
$$[K_3/p,\hat S^4] = -\tfrac{1}{2}\hat S^4, \tag{B12}$$
$$[\hat T^{j4},\hat S^1] = -2p\hat S^j,\quad j=1,2,3$$
$$[\hat T^{1j},\hat S^4] = 2p\hat S^j,\quad j=2,3,4 \tag{B13}$$
$$[\hat T^{14},\hat T^{1k}] = -2p\hat T^{1k},\quad k=2,3$$
$$[\hat T^{14},\hat T^{j4}] = 2p\hat T^{j4},\quad j=2,3$$
$$[\hat T^{12},\hat T^{34}] = -2p\hat T^{23},$$
$$[\hat T^{13},\hat T^{24}] = 2p\hat T^{23}, \tag{B14}$$

the rest is zero.

APPENDIX C: PROOF THAT $K_\pm = 0$ FOR FINITE REPRESENTATION OF E_2

Consider the Euclidean algebra E_2 commutation relations for $\{K_+,K_-,K_3\}$:

$$[K_+,K_-] = 0, \tag{C1}$$
$$[K_3,K_\pm] = \pm K_\pm, \tag{C2}$$

where $K_+^\dagger, = K_-$. The Casimir operator is K_-K_+ and Eq. (C2) implies $K_+(K_-)$ is the raising (lowering) operator for the K_3 eigenvalues. Consider the minimum K_3 eigenstates, $|k_{\min}\rangle$, defined by $K_-|k_{\min}\rangle = 0$ and $K_3|k_{\min}\rangle = k_{\min}|k_{\min}\rangle$. From (C1),

$$K_-K_+|k_{\min}\rangle = 0, \tag{C3}$$

and therefore $K_-K_+ = 0$ for the entire representation because it is a Casimir operator. This implies $K_+ = K_- = 0$ for any finite representation.

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