Differential and integral methods for multidimensional inverse scattering problems

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A layer stripping procedure for solving three-dimensional Schrödinger equation inverse scattering problems is developed. This procedure operates by recursively reconstructing the Radon transform of the potential from the jump in the Radon transform of the scattered field at the wave front. This reconstructed potential is then used to propagate the wave front and scattered field differentially further into the support of the potential. The connections between this differential procedure and integral equation procedures are then illustrated by the derivations of two well known exact integral equation procedures using the Radon transform and a generalized Radon transform. These procedures, as well as the layer stripping procedure, are then reduced to the familiar Born approximation result for this problem by neglecting multiple scattering events. This illustrates the central role of the Radon transform in both exact and approximate inversion procedures.

I. INTRODUCTION

The inverse scattering problem for the Schrödinger equation in three dimensions with a time-independent, local, nonspherically symmetric potential has a wide variety of applications. In particular, the inverse seismic problem of reconstructing the density and wave speed of an inhomogeneous isotropic acoustic medium from surface measurements of the medium response to an excitation can be formulated as a Schrödinger equation inverse scattering problem, as was done by Coen *et al.*[1] Other applications include quantum mechanical particle scattering problems, in which particles are treated as wave functions,[2] and the propagation of electromagnetic waves in the ionosphere.[3]

There are several methods available for solving the inverse scattering problem. The most important of the *exact* methods are generalized Gel'fand–Levitan and Marchenko integral equation procedures of Newton,[4] and the coupled integral equation procedure of Moses.[5] Newton's Marchenko integral equation procedure has been applied to an inverse seismic problem in (Ref. 1). Moses[5] gave the first exact (in principle) solution to the inverse scattering problem, but Moses's procedure cannot be implemented in closed form. Other exact methods have been given in Refs. 6–12; this paper focuses on the exact procedures given in Refs. 4 and 5.

An alternative approach is to use the *first Born approximation*, in which the wave field inside the support of the potential is approximated by the incident field being used to probe the potential. This approach has been applied to the variable-velocity wave equation by Cohen and Bleistein,[13] Devaney,[14] and others.

All of these methods have shortcomings. Newton's integral equation procedure requires that the scattering amplitude (the far-field response) be measured for all incident and outgoing directions and all frequencies. This makes it unsuitable for inverse seismic problems, for which data are only available in the near field and in backscattered directions. Furthermore, the complete specification of the scattering amplitude results in an overdetermined problem, so that a slight corruption of the data may result in an inadmissible scattering amplitude. Moses's coupled integral equations cannot be solved in closed form; power series expansions are required for various quantities, and as a result a considerable amount of computation is required to determine each higher-order correction term. The Born approximation methods, although requiring less computation, employ a single scattering approximation, and thus are only valid for problems with weakly scattering potentials.

A completely different approach to solving the Schrödinger equation inverse scattering problem is *layer stripping*. Layer stripping is a *differential* procedure, in contrast to the above *integral* equation procedures. A layer stripping algorithm works by recursively reconstructing the potential as the probing wave penetrates it. By employing causality and the inherent structure of an inverse scattering problem, a layer stripping algorithm requires much less computation than the integral equation procedures of Newton and Moses. It also requires only near-field, backscattered data, making it ideal for applications to inverse seismic problems, and avoiding the overdetermined problem to which Newton's procedure is applicable. A layer stripping algorithm has been proposed in Yagle and Levy[15]; however, this algorithm is numerically untested.

The objectives of this paper are twofold: (1) to present a new layer stripping algorithm for solving the Schrödinger equation inverse scattering problem; and (2) to present an approach, based on the Radon transform, for interpreting all of the various methods mentioned above for solving the inverse scattering problem. We thus show, for the first time, how the integral-equation methods of Newton and Moses, the Born approximation approach, and the layer stripping method presented in this paper are all related to each other. In this way, the common basis of all of these seemingly unrelated approaches is exposed, resulting in new insight into their operation.

The paper is organized as follows. The Radon transform is quickly reviewed in Sec. II, including a generalized Radon transform noted in Rose *et al.*[9] The basic Schrödinger equation inverse scattering problem that is the subject of this paper is set up in Sec. III, and the basic integral equation procedures of Newton and Moses for solving this problem are specified. In Sec. IV a new layer stripping procedure for solving this problem is presented and discussed. In Sec. V the same basic equations used in deriving the layer stripping algorithm are used to derive the integral equation procedures of Newton and Moses. This illustrates that all three procedures have a common basis. The basic Born approximation result is also derived from all three procedures by neglecting multiple scattering events. Finally, Sec. VI concludes by summarizing the results of the paper and noting directions in which further research is needed.

II. THE RADON TRANSFORM

The Radon transform of a function in three-dimensional space is the integral of the function over a plane. It is thus a slice or sample of the function. Specifically, the Radon transform $\mathscr{R}\{f(\mathbf{x})\}$ of a function $f(\mathbf{x})$ is given by

$$\mathscr{R}\{f(\mathbf{x})\} = \hat{f}(\tau,\mathbf{e}) = \int f(\mathbf{x})\delta(\tau - \mathbf{e}\cdot\mathbf{x})d\mathbf{x}. \quad (2.1)$$

Given the projections $\hat{f}(\tau,\mathbf{e})$ for all τ and all angles $\mathbf{e}$, the function $f(\mathbf{x})$ may be recovered by the inverse Radon transform

$$f(\mathbf{x}) = \mathscr{R}^{-1}\{\hat{f}(\tau,\mathbf{e})\} = -(8\pi^2)^{-1}\int_{S^2}\frac{\partial^2}{\partial\tau^2}\hat{f}(\tau = \mathbf{e}\cdot\mathbf{x},\mathbf{e})d^2\mathbf{e}, \quad (2.2)$$

where S^2 is the unit sphere in R^3. This result is originally due to Radon[16]; a good treatment is Deans.[17]

Following Rose *et al.*,[9] a *generalized Radon transform* can be defined from the fact that the solutions of the Schrödinger equation in the absence of bound states form a complete set. If $u(\mathbf{x},k,\mathbf{e})$ is a solution of the Schrödinger equation, where $\mathbf{e}$ is the direction of initial probing, and $f(\mathbf{x})$ is square integrable, then we may write

$$f(\mathbf{x}) = (2\pi)^{-3}\int_0^\infty\int_{S^2} u(\mathbf{x},k,\mathbf{e}) \times \int u^*(\mathbf{y},k,\mathbf{e})f(\mathbf{y})d\mathbf{y}\,d^2\mathbf{e}k^2\,dk, \quad (2.3)$$

and if $u(\mathbf{x},k,\mathbf{e})$ is extended to negative k by $u(\mathbf{x},-k,\mathbf{e}) = u^*(\mathbf{x},k,\mathbf{e})$ then an inverse Fourier transform from k to t

$$u(\mathbf{x},t,\mathbf{e}) = \mathscr{F}^{-1}\{u(\mathbf{x},k,\mathbf{e})\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\mathbf{x},k,\mathbf{e})e^{ikt}\,dk \quad (2.4)$$

results in

$$f(\mathbf{x}) = -(8\pi^2)^{-1}\int_{-\infty}^{\infty}\int_{S^2}\int u(\mathbf{x},t,\mathbf{e}) \times \frac{\partial^2}{\partial t^2}u(t,\mathbf{e},\mathbf{y})f(\mathbf{y})d\mathbf{y}\,d^2\mathbf{e}\,dt, \quad (2.5)$$

which can be written as the pair of equations

$$\hat{f}(t,\mathbf{e}) = \mathscr{G}\{f(\mathbf{x})\} = \int u(\mathbf{y},t,\mathbf{e})f(\mathbf{y})d\mathbf{y}, \quad (2.6a)$$

$$f(\mathbf{x}) = \mathscr{G}^{-1}\{\hat{f}(t,\mathbf{e})\} = -(8\pi^2)^{-1}\int_{-\infty}^{\infty}\int_{S^2} u(\mathbf{x},t,\mathbf{e})\frac{\partial^2}{\partial t^2}\hat{f}(t,\mathbf{e})d^2\mathbf{e}\,dt, \quad (2.6b)$$

which is the generalized Radon transform pair.

In the particular case where $u(\mathbf{x},t,\mathbf{e})$ is chosen to be the impulse $\delta(t - \mathbf{e}\cdot\mathbf{x})$, then it is clear that the generalized Radon transform pair (2.6) reduces to the standard Radon transform pair (2.1) and (2.2). This explains why (2.6) is termed a generalized Radon transform.

III. THE INVERSE SCATTERING PROBLEM

The inverse scattering problem considered in this paper is as follows. The wave field $u(\mathbf{x},k)$ satisfies the Schrödinger equation

$$(\Delta + k^2 - V(\mathbf{x}))u(\mathbf{x},k) = 0, \quad (3.1)$$

where the potential $V(\mathbf{x})$ is real-valued, smooth, and has compact support. It is also assumed that $V(\mathbf{x})$ does not induce bound states; a sufficient condition for this is for $V(\mathbf{x})$ to be non-negative.

Scattering solutions of (3.1) are given by the Lippman–Schwinger equation

$$u(\mathbf{x},k,\mathbf{e}_i) = e^{-ik\mathbf{e}_i\mathbf{x}} - \int(4\pi|\mathbf{x}-\mathbf{y}|)^{-1} \times e^{-ik|\mathbf{x}-\mathbf{y}|}V(\mathbf{y})u(\mathbf{y},k,\mathbf{e}_i)d\mathbf{y}, \quad (3.2)$$

where the incident field is an impulsive plane wave propagating in the direction of the unit vector $\mathbf{e}_i$. Letting $\mathbf{x} = |\mathbf{x}|\mathbf{e}_s$ and taking $|\mathbf{x}|\to\infty$ we have, in the far field,

$$u(\mathbf{x},k,\mathbf{e}_i) = e^{-ik\mathbf{e}_i\mathbf{x}} + (e^{-ik|\mathbf{x}|}/4\pi|\mathbf{x}|) \times A(k,\mathbf{e}_s,\mathbf{e}_i) + O(|\mathbf{x}|^{-2}), \quad (3.3)$$

where

$$A(k,\mathbf{e}_s,\mathbf{e}_i) = -\int e^{ik\mathbf{e}_s\mathbf{y}}V(\mathbf{y})u(\mathbf{y},k,\mathbf{e}_i)d\mathbf{y} \quad (3.4)$$

is the *scattering amplitude* for incident direction $\mathbf{e}_i$ and scattered direction $\mathbf{e}_s$.

An inverse Fourier transform of (3.1) yields the *plasma wave equation*

$$\left(\Delta - \frac{\partial^2}{\partial t^2} - V(\mathbf{x})\right)u(\mathbf{x},t) = 0. \quad (3.5)$$

This equation models the propagation of electromagnetic waves in the ionosphere.[3] An inverse Fourier transform of (3.3) results in

$$u(\mathbf{x},t,\mathbf{e}_i) = \delta(t - \mathbf{e}_i\cdot\mathbf{x}) + (4\pi|\mathbf{x}|)^{-1} \times R(t - \mathbf{e}_s\cdot\mathbf{x},\mathbf{e}_s,\mathbf{e}_i) + O(|\mathbf{x}|^{-2}), \quad (3.6)$$

where $R(t,\mathbf{e}_s,\mathbf{e}_i)$ is the inverse Fourier transform of $A(k,\mathbf{e}_s,\mathbf{e}_i)$. Since $R(\cdot)$ represents the time response in the far field to the probing impulse $\delta(t - \mathbf{e}_i\cdot\mathbf{x})$, it is termed the impulse response.

Exact solutions to this inverse scattering problem have been given by Newton[4] and Moses[5] (and others as well).

Both of these methods involve the solution of integral equations. Newton's method is to solve the *Marchenko integral equation*

$$u_s(\mathbf{x},t,\mathbf{e}_i) = \int_{S^2}\int_{-\mathbf{e}_s\mathbf{x}}^{\infty} M(t+\tau,\mathbf{e}_s,\mathbf{e}_i)u_s(\mathbf{x},\tau,-e_s) \times d\tau\, d^2\mathbf{e}_s + \int_{S^2} M(t-\mathbf{e}_s\cdot\mathbf{x},\mathbf{e}_s,\mathbf{e}_i)d^2\mathbf{e}_s \tag{3.7}$$

for the scattered field $u_s(\mathbf{x},t,\mathbf{e}_i)$, which is defined by

$$u(\mathbf{x},t,\mathbf{e}_i) = \delta(t-\mathbf{e}_i\cdot\mathbf{x}) + u_s(\mathbf{x},t,\mathbf{e}_i). \tag{3.8}$$

In (3.7) the quantity $M(t,\mathbf{e}_s,\mathbf{e}_i)$ is obtained from the scattering data using

$$M(t,\mathbf{e}_s,\mathbf{e}_i) = -\frac{1}{8\pi^2}\frac{\partial}{\partial t}R(t,\mathbf{e}_s,\mathbf{e}_i). \tag{3.9}$$

Finally, the potential $V(\mathbf{x})$ is recovered from the scattered field using the *miracle* equation[4]

$$V(\mathbf{x}) = -2\mathbf{e}_i\cdot\nabla u_s(\mathbf{x},t=\mathbf{e}_i\cdot\mathbf{x},\mathbf{e}_i). \tag{3.10}$$

Note the redundancy in this equation. Newton[4] notes that the right side of this equation characterizes admissible scattering amplitudes: only a subset of all possible $A(k,\mathbf{e}_s,\mathbf{e}_i)$ (five independent variables) can result from all possible $V(\mathbf{x})$ (three independent variables). Thus the inverse scattering problem solved by this method is overdetermined; clearly there is a great amount of unnecessary computation to reconstruct $V(\mathbf{x})$. In addition, the use of far-field data and transmission data makes this procedure unsuitable for solving inverse seismic problems, as noted in (Ref. 1).

Moses's method is to solve the coupled set of integral equations:

$$T(\mathbf{k},\mathbf{k}') = V(\mathbf{k},\mathbf{k}') + \int V(\mathbf{k},\mathbf{k}'') \times\gamma(k'^2-k''^2)T(\mathbf{k}'',\mathbf{k}')d\mathbf{k}'', \tag{3.11}$$

$$W(\mathbf{k}) = b(\mathbf{k}) + \int T(-\mathbf{k},\mathbf{k}')[1(k)\gamma^*(k'^2-k^2) + 1(-k)\gamma(k'^2-k^2)]T^*(\mathbf{k},\mathbf{k}')d\mathbf{k}', \tag{3.12}$$

$$V(\mathbf{k},\mathbf{k}') = W((\mathbf{k}'-\mathbf{k})/2), \tag{3.13}$$

where $b(\mathbf{k})$ is the backscattering amplitude

$$b(\mathbf{k}) = b(k,\mathbf{e}) = A(k,-\mathbf{e},\mathbf{e}),\quad k\geqslant 0, \tag{3.14}$$

$1(k)$ is the Heaviside or unit step function, and $\gamma(k)$ is defined by

$$\gamma(k) = -i\pi\delta(k) + P/k = \lim_{\epsilon\to+0}(1/k+i\epsilon), \tag{3.15}$$

which is the Fourier transform of $1(t)$ (P denotes the Cauchy principal value). The potential $V(\mathbf{x})$ is recovered from $W(\mathbf{k})$ using the inverse Fourier transform

$$V(\mathbf{x}) = (2\pi)^{-3}\int W(\mathbf{k})e^{-2i\mathbf{k}\mathbf{x}}d\mathbf{k}. \tag{3.16}$$

Note that Moses's method is not overdetermined, since only the backscattering amplitude $b(\mathbf{k})$, not the entire scattering amplitude $A(k,\mathbf{e}_s,\mathbf{e}_i)$, is used to reconstruct the potential $V(\mathbf{x})$. However, the coupled integral equations cannot be solved in closed form. Moses[5] employs power series expansions for $T(\mathbf{k},\mathbf{k}')$, $V(\mathbf{k},\mathbf{k}')$, and $W(\mathbf{k})$; clearly an easier procedure for solving this problem is desirable. However, Moses's approach has been suggested for solving an inverse seismic problem in (Ref. 18).

An alternative to these integral equation methods is a differential or layer stripping approach. Such an approach was used to derive a layer stripping algorithm in (Ref. 15). A different layer stripping algorithm, employing the Radon transform, is derived below.

IV. A LAYER STRIPPING SOLUTION TO THE INVERSE SCATTERING PROBLEM

A *layer stripping* procedure recursively reconstructs the potential as the probing wave penetrates it. It is thus a *differential* procedure, in contrast to the *integral* equation procedures described above. By taking full advantage of the inherent structure of the inverse scattering problem, and of time causality, a layer stripping algorithm requires significantly less computation to reconstruct a scattering potential than the above methods. This is important in a three-dimensional inverse problem, since the number of points to be reconstructed in a discretized potential increases as the cube of the number of discrete points in a single dimension.

The essence of a layer stripping procedure is to differentially reconstruct the Radon transform of $V(\mathbf{x})$ from the jump in the scattered field at the wave front, and then use this reconstructed slice of $V(\mathbf{x})$ to propagate the wave front and scattered field differentially further. The jump in the scattered field at the new location of the wave front yields another slice of $V(\mathbf{x})$, which can be used to propagate the wave front and scattered field still further. This differential, layer-by-layer reconstruction contrasts with the batch operation of the integral equation approach.

There are several advantages to using a layer stripping technique. Only one direction of probing is required, and only backscattered data in the near field is used. This makes the procedure more applicable to inverse seismic problems than the integral equation procedures, which require far-field data and, in Newton's[4] procedure, transmission data. The procedure is in principle exact, since all multiple reflection, refraction, and diffraction effects are accounted for. Approximation is inherent only in the discretization necessary to implement the algorithm numerically, and data at all frequencies are used. However, the applicability of this approach to problems with bound states is not clear at present.

The layer stripping concept has been used to obtain fast algorithm solutions for the one-dimensional Schrödinger equation inverse scattering problem by Corones *et al.*,[19] Symes,[20] Bruckstein *et al.*,[21] and Yagle and Levy.[22] This approach has also been applied to various one-dimensional inverse seismic problems by Bube and Burridge,[23] and Yagle and Levy.[24–26] Similar approaches have been used by other authors. Results of computer runs of these one-dimensional problem algorithms have been encouraging (see Bube and Burridge[23] and Yagle[27]). The numerical performance of the multidimensional problem algorithms proposed in Yagle and Levy[15] and this paper are unknown at present, but are subjects of current research.

The layer stripping procedure given in this section differs from that of Ref. 15 in the use of the Radon transform. To use this transform, we operate in the time domain. Recall from Sec. III that an inverse Fourier transform of the Schrödinger equation results in the plasma wave equation

$$\left(\Delta - \frac{\partial^2}{\partial t^2} - V(\mathbf{x})\right) u(\mathbf{x},t) = 0 \tag{4.1}$$

and the scattered field $u_s(\mathbf{x},t,\mathbf{e}_i)$ is defined by

$$u(\mathbf{x},t,\mathbf{e}_i) = \delta(t - \mathbf{e}_i \cdot \mathbf{x}) + u_s(\mathbf{x},t,\mathbf{e}_i). \tag{4.2}$$

Taking the Radon transform of (4.1) results in

$$\left\{\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial t^2}\right\} U(\tau,t,\mathbf{e}) = \mathscr{R}\{V(\mathbf{x})u(\mathbf{x},t)\}, \tag{4.3}$$

where $U(\tau,t,\mathbf{e})$ is the Radon transform of $u(\mathbf{x},t,\mathbf{e}_i)$, and the parametric dependence on the direction of probing $\mathbf{e}_i$ is no longer listed. Equation (4.3) may be written as the coupled first-order system

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial t}\right) U(\tau,t,\mathbf{e}) = Q(\tau,t,\mathbf{e}), \tag{4.4a}$$

$$\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial t}\right) Q(\tau,t,\mathbf{e}) = \mathscr{R}\{V(\mathbf{x})u(\mathbf{x},t)\}. \tag{4.4b}$$

The crucial step is to recognize that the scattered field $u_s(\mathbf{x},t,\mathbf{e}_i)$ is *causal*: until the probing impulsive plane wave $\delta(t - \mathbf{e}_i \cdot \mathbf{x})$ reaches $\mathbf{x}$, the scattered field at $\mathbf{x}$ is zero. This can be written explicitly as

$$u(\mathbf{x},t,\mathbf{e}_i) = \delta(t - \mathbf{e}_i \cdot \mathbf{x}) + u_s(\mathbf{x},t,\mathbf{e}_i) 1(t - \mathbf{e}_i \cdot \mathbf{x}). \tag{4.5}$$

Taking the Radon transform of (4.5) and considering only $\mathbf{e} = \mathbf{e}_i$ gives

$$U(\tau,t,\mathbf{e} = \mathbf{e}_i) = \delta(t - \tau) + U_s(\tau,t,\mathbf{e} = \mathbf{e}_i) 1(t - \tau). \tag{4.6}$$

A mental picture of the Radon transform will make the meaning of (4.6) clear: Since the Radon transform is being taken over planes parallel to the probing impulsive plane wave ($\mathbf{e} = \mathbf{e}_i$), it must be zero if t is less than τ, since in this case the plane lies entirely with the region that the probing impulsive plane wave has not yet penetrated. From the form of (4.4a) it may be seen that $Q(\tau,t,\mathbf{e} = \mathbf{e}_i)$ is also causal. Specifically,

$$Q(\tau,t,\mathbf{e} = \mathbf{e}_i) = Q_s(\tau,t,\mathbf{e} = \mathbf{e}_i) 1(t - \tau). \tag{4.7}$$

Inserting (4.5)–(4.7) into the coupled system (4.4) results in

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial t}\right) U_s(\tau,t,\mathbf{e}) = Q_s(\tau,t,\mathbf{e}), \tag{4.8a}$$

$$\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial t}\right) Q_s(\tau,t,\mathbf{e}) = \mathscr{R}\{V(\mathbf{x})u_s(\mathbf{x},t)\}, \tag{4.8b}$$

$$\mathscr{R}\{V(\mathbf{x})\} = -2Q_s(\tau,t = \tau,\mathbf{e}), \tag{4.8c}$$

where equating the coefficients of $\delta(t - \tau)$ in (4.4b) has been used to obtain (4.8c), and $\mathbf{e} = \mathbf{e}_i$ throughout.

Equations (4.8) suggest a *recursive procedure for reconstructing* $V(\mathbf{x})$: Starting with known $U_s(\tau = 0,t, \mathbf{e} = \mathbf{e}_i)$ and $Q_s(\tau = 0,t, \mathbf{e} = \mathbf{e}_i)$, propagate Eqs. (4.8) recursively in increasing τ, yielding $\mathscr{R}\{V(\mathbf{x})\}$ recursively in τ. Once $\mathscr{R}\{V(\mathbf{x})\}$ has been computed for all τ, and for a hemisphere of angles of probing $\mathbf{e}_i$, then the inverse Radon transform (2.2) can be used to compute $V(\mathbf{x})$ [only a hemisphere of incident directions is needed, since $V(\tau, -\mathbf{e}) = V(-\tau,\mathbf{e})$].

However, there is a complication: The right side of (4.8b) seems to require that $u_s(\mathbf{x},t)$ and $V(\mathbf{x})$ be computed recursively as well. Since it is assumed in (4.8) that $\mathbf{e} = \mathbf{e}_i$, there is insufficient information to compute these quantities, since the inverse Radon transform requires projection at all angles, not just the angle of probing $\mathbf{e}_i$, and it also requires knowledge for all positive τ.

The solution to this problem is to recognize that (4.8b) requires not $u_s(\mathbf{x},t)$, but only $\mathscr{R}\{V(\mathbf{x})u_s(\mathbf{x},t)\}$. Writing this out gives

$$\mathscr{R}\{V(\mathbf{x})u_s(\mathbf{x},t)\} = -\frac{1}{8\pi^2}\int_{S^2}\int \frac{\partial^2}{\partial \tau_1^2} U_s(\tau_1 = \mathbf{e}_1 \cdot \mathbf{x},\mathbf{e}_1) \times \frac{\partial^2}{\partial \tau_2^2} V(\tau_2 = \mathbf{e}_2 \cdot \mathbf{x},\mathbf{e}_2) \times \delta(\tau - \mathbf{e}_i \cdot \mathbf{x}) d\mathbf{x}\, d^2\mathbf{e}_1 d^2\mathbf{e}_2. \tag{4.9}$$

Note that this quantity is only required for $\mathbf{e} = \mathbf{e}_i$ and for a specific value of τ. The integrand is nonzero only for $\mathbf{x}$ such that $\tau = \mathbf{e}_i \cdot \mathbf{x}$, and $U_s(\tau,\mathbf{e})$ and $V(\tau,\mathbf{e})$ are only required for $\tau_1 = \mathbf{e}_1 \cdot \mathbf{x}$ and $\tau_2 = \mathbf{e}_2 \cdot \mathbf{x}$. These three planes intersect in a point unless at least two of them coincide; virtually all of the contribution to the integral occurs when all three planes coincide (this point is made and discussed in Ref. 28). Fortunately, those values of $U_s(\tau,\mathbf{e})$ that make this contribution are precisely those available at each recursion: $U_s(\tau = \mathbf{e}_i \cdot \mathbf{x}, \mathbf{e} = \mathbf{e}_i)$. Similar comments hold for $V(\mathbf{x})$; however, since $V(\mathbf{x})$ is independent of the direction of probing $\mathbf{e}_i$, it can be completely reconstructed using the inverse Radon transform (2.2) once the algorithm is complete. The second partial derivatives required in the inverse Radon transforms can be implemented numerically.

The procedure is initialized as follows. Assume without loss of generality that the support of $V(\mathbf{x})$ is contained inside a sphere of radius R, and that the backscattered field $u_s(\mathbf{x},t,\mathbf{e}_i)$ is measured on the plane $R = -\mathbf{e}_i \cdot \mathbf{x}$. Then it is possible to compute $U_s(\tau = -R,t,\mathbf{e}_i)$, and from this compute $Q_s(\tau = R,t,\mathbf{e}_i)$, and then propagate the algorithm in increasing τ from $-R$ to R. Since the support of $V(\mathbf{x})$ lies inside a sphere of radius R, $\mathscr{R}\{V(\mathbf{x})\}$ is zero for $\tau \geqslant R$.

The layer stripping procedure can be summarized as follows. (1) *Initialize* the procedure by computing

$$U_s(\tau = -R,t,\mathbf{e}_i) = \mathscr{R}\{u_s(\tau = -R,t, \mathbf{e} = \mathbf{e}_i)\} \tag{4.10}$$

from measurements of the backscattered field on the plane $-R = \mathbf{e}_i \cdot \mathbf{x}$, which by hypothesis lies outside the support of $V(\mathbf{x})$. Compute $Q_s(\tau = -R,t,\mathbf{e}_i)$ from $R_s(\tau = -R,t,\mathbf{e}_i)$ using (4.8a) above.

(2) *Recursively* compute $U_s(\cdot)$ and $Q_s(\cdot)$ in τ using (4.8) above, for each $\mathbf{e}_i$, yielding $\mathscr{R}\{V(\mathbf{x})\}$ on the plane $t = \tau = \mathbf{e}_i \cdot \mathbf{x}$ from (4.8c) at each recursion. This is used along with $U_s(\cdot)$ in (4.9) to compute the right-hand side of (4.8b). The recursion in τ runs from $-R$ to R.

(3) After the recursion is complete, an inverse Radon transform may be used to reconstruct $V(\mathbf{x})$, since its support lies inside a sphere of radius R.

Some comments are in order here. First, note that the recursive, layer-by-layer (in τ) reconstruction of $V(\mathbf{x})$ sharply contrasts with the batch reconstructions of the integral equation procedures. Newton's[4] procedure first reconstructs the scattered field for *all* angles of probing *all at once* [note the coupling in $\mathbf{e}_i$ in (3.7)]. Computationally, this is a tremendous undertaking. The layer stripping procedure decouples the computations for different angles of probing, so that they can be run concurrently on an array processor, and results from different angles are combined only in (2.2).

Second, note that the simpler form of the layer stripping algorithm is a result of the exploitation of *causality* and the structure of the inverse scattering problem. By examining the jump in the scattered field at the wave front, which is measured by the first nonzero value of the causal quantity $Q_s(\cdot)$, we avoid problems with multiple scattering events, which aids in recovering values of $V(\mathbf{x})$. This structure is manifested by the Hankel structure of the kernel of the Marchenko integral equation (3.7); but this structure can be exploited more directly by appealing to the physical nature of the problem. The concept of exploiting the jump in the scattered field at the wave front in order to determine the potential has been noted in Morawetz,[29] and is the basis of the *miracle* equation of Newton[4] [Eq. (3.10) above] and the *fundamental identity* of Rose *et al.*[9] This is discussed in more detail below.

Finally, note that the layer stripping algorithm uses only near-field, backscattered data, unlike the integral equation procedures. This makes it more suitable for solving inverse seismic problems, since for these problems data are measured in the near field and transmission data are not available. In Yagle and Levy,[15] a layer stripping algorithm is applied to an inverse seismic problem formulated in Coen *et al.*[1] The issue of overdetermination arising in Newton's procedure also does not arise in the present procedure since only backscattered data are used.

V. INTEGRAL EQUATION METHODS AND THE RADON TRANSFORM

In this section it is shown that the same basic equations that led to the layer stripping procedure in Sec. IV also lead to the integral equation procedures of Moses and Newton described in Sec. III. This shows that the layer stripping and integral equation approaches are related to each other. Similar connections between layer stripping and integral equation approaches were demonstrated for the one-dimensional inverse problem in Bruckstein *et al.*[21] It is also demonstrated in this section that basic Born approximation results may be derived easily from all three methods by employing a single scattering approximation. The results of this section are not intended to be rigorous derivations; they are heuristic derivations that illustrate why the equations have the forms they have. They are intended to aid in understanding and interpreting the various inverse problem solution procedures.

A. The integral equation procedure of Moses

In Moses,[5] $T(\mathbf{k}',\mathbf{k})$ is defined as [Eq. (5.12) in Ref. 5]

$$T(\mathbf{k}',\mathbf{k}) = \int e^{-i\mathbf{k}'\mathbf{x}} V(\mathbf{x}) u(\mathbf{x},\mathbf{k}) d\mathbf{x}, \tag{5.1}$$

where $\mathbf{k} = k\mathbf{e}_i$, so that $u(\mathbf{x},\mathbf{k}) = u(\mathbf{x},k,\mathbf{e}_i)$. Therefore, $T(\mathbf{k}',\mathbf{k})$ can be interpreted as a *generalized scattering amplitude* [Eq. (5.1) reduces to the definition (3.4) of scattering amplitude if $|\mathbf{k}| = |\mathbf{k}'|$]. In addition, if we write $\mathbf{k} = |\mathbf{k}|\mathbf{e}_i = k\mathbf{e}_i$ and $\mathbf{k}' = |\mathbf{k}'|\mathbf{e}_s = k'\mathbf{e}_s$, and extend k and k' to negative values by writing $\mathbf{k} = (-k)(-\mathbf{e}_i)$ and $\mathbf{k}' = (-k')(-\mathbf{e}_s)$, then we may regard $T(\mathbf{k}',\mathbf{k})$ as the double Fourier transform of $\mathscr{R}\{V(\mathbf{x})u(\mathbf{x},t,\mathbf{e}_i)\}$:

$$\begin{aligned} T(\mathbf{k}',\mathbf{k}) &= \mathscr{F}\mathscr{F}\{\mathscr{R}\{V(\mathbf{x})u(\mathbf{x},t,\mathbf{e}_i)\}\} \\ &= \int\int\int V(\mathbf{x})u(\mathbf{x},t,\mathbf{e}_i)\delta(\tau - \mathbf{e}_s\cdot x) \\ &\quad \times e^{-ikt}e^{-ik'\tau}\, d\mathbf{x}\, dt\, d\tau. \end{aligned} \tag{5.2}$$

Using this observation, taking the double Fourier transform of the Radon transform (4.3) of the plasma wave equation (4.1) gives

$$(k^2 - k'^2)u(\mathbf{k}',\mathbf{k}) = T(\mathbf{k}',\mathbf{k}), \tag{5.3}$$

which leads to

$$u_s(\mathbf{k}',\mathbf{k}) = \gamma(k^2 - k'^2)T(\mathbf{k}',\mathbf{k}), \tag{5.4}$$

where $\gamma(\cdot)$ is defined by (3.15). In Moses[5] (5.4) was derived directly from the definition (3.4) of the scattering amplitude, but this lends no insight into *why* (5.3) has the form it does, whereas the present derivation shows that (5.3) is a direct consequence of the application of the Radon transform to the plasma wave equation. Multiplying the trivial definition (4.2) of the scattered field $u_s(\cdot)$ by $V(\mathbf{x})$ and taking the double Fourier transform from t to k and $\mathbf{x}$ to $\mathbf{k}' = k'\mathbf{e}_s$ (recall $\mathbf{e}_s = \mathbf{x}/|\mathbf{x}|$) results in

$$\begin{aligned} T(\mathbf{k}',\mathbf{k}) &= V(\mathbf{k}',\mathbf{k}) + \int V(\mathbf{k}',\mathbf{k}'')u_s(\mathbf{k}'',\mathbf{k})d\mathbf{k}'' \\ &= V(\mathbf{k}',\mathbf{k}) + \int V(\mathbf{k}',\mathbf{k}'') \\ &\quad \times \gamma(k^2 - k''^2)T(\mathbf{k}'',\mathbf{k})d\mathbf{k}'', \end{aligned} \tag{5.5}$$

where $V(\mathbf{k}',\mathbf{k})$ is defined by

$$V(\mathbf{k}',\mathbf{k}) = \int V(\mathbf{x})e^{i(\mathbf{k}-\mathbf{k}')\mathbf{x}}\, d\mathbf{x}, \tag{5.6}$$

and (5.4) has been used. Note that (5.5) is the same as (3.11), the first of the coupled integral equations of Moses's procedure.

The other equations of Moses's procedure may be derived using the *generalized Radon transform* (2.6) and a Fourier transform $\mathscr{F}$ that takes time t into k'', where $\mathbf{k}'' = k''\mathbf{e}$ and k'' is extended to negative values as before. We may write

$$\begin{aligned} &V(\mathbf{x})e^{-i\mathbf{k}'\mathbf{x}} \\ &\quad = (\mathscr{F}\mathscr{G})^{-1}(\mathscr{F}\mathscr{G})\{V(\mathbf{x})e^{-i\mathbf{k}'\mathbf{x}}\} \\ &\quad = (\mathscr{F}\mathscr{G})^{-1}\{T(\mathbf{k}',\mathbf{k}'')\} \\ &\quad = \int_{S^2}\int_0^\infty T(\mathbf{k}',\mathbf{k}'')u^*(\mathbf{x},\mathbf{k}'')k''^2\, d^2\mathbf{e}\, dk'' \\ &\quad = \int T(\mathbf{k}',\mathbf{k}'')u^*(\mathbf{x},\mathbf{k}'')d\mathbf{k}'' \end{aligned} \tag{5.7}$$

and a Fourier transform taking $\mathbf{x}$ into $\mathbf{k}$ results in

$$V(\mathbf{k}',\mathbf{k}) = \int T(\mathbf{k}',\mathbf{k}'')u^*(\mathbf{k},\mathbf{k}'')d\mathbf{k}''. \tag{5.8}$$

Taking a double Fourier transform of (4.2), inserting into (5.8), and setting $\mathbf{k}' = -\mathbf{k}$ results in

$$W(\mathbf{k}) = V(-\mathbf{k},\mathbf{k}) = T(-\mathbf{k},\mathbf{k}) + \int T(-\mathbf{k},\mathbf{k}'')\gamma^*(k''^2 - k^2)T^*(\mathbf{k},\mathbf{k}'')d\mathbf{k}''. \tag{5.9}$$

Using (3.14), $T(-\mathbf{k},\mathbf{k})$ can be obtained from the back-scattering amplitude $A(k,-\mathbf{e},\mathbf{e})$ for $k \geqslant 0$. For $k \leqslant 0$, replace k with $-k$ in (5.9). This equation, combined with (5.9), gives (3.12), the second of Moses's equations. The last of the coupled integral equations (3.13) follows immediately from the definition of $W(\mathbf{k})$ in (5.9).

Thus it may be seen that the coupled integral equations (3.11)–(3.13) of Moses[5] may be interpreted as merely various Fourier and Radon transforms of elementary equations like (4.1) and (4.2). Note that at no point in the above derivations was time causality used. Indeed, this solution procedure does not exploit the structure of the inverse scattering problem at all. This is why the layer stripping algorithm, which does exploit this structure, is simpler.

B. The integral equation procedure of Newton

The generalized Radon transform may also be used to derive Newton's Marchenko integral equation (3.7). Let $u^+(\mathbf{x},t,\mathbf{e}_i)$ be a solution of the Schrödinger equation (3.1) with an outgoing radiation condition, and let $u^-(\mathbf{x},t,\mathbf{e}_i)$ be a solution with an incoming radiation condition. By reversing time we have that

$$u^-(\mathbf{x},t,\mathbf{e}_i) = u^+(\mathbf{x},-t,-\mathbf{e}_i). \tag{5.10}$$

Consider

$$\mathscr{G}\{u^+(\mathbf{x},t,\mathbf{e}_i) - u^-(\mathbf{x},t,\mathbf{e}_i)\} = \int u^-(\mathbf{y},\tau,\mathbf{e}_s)(u^+(\mathbf{y},t,\mathbf{e}_i) - u^-(\mathbf{y},t,\mathbf{e}_i))d\mathbf{y}. \tag{5.11}$$

We show first that this quantity can only depend on the time difference $t-\tau$. To do this, we apply a double Fourier transform taking t into k and τ into k' to the right side of (5.11). The result is shown to be the product of some function and $\delta(k+k')$. Since

$$\mathscr{F}^{-1}\{\mathscr{F}^{-1}\{\hat{f}(k)\delta(k+k')\}\} = f(t-\tau), \tag{5.12}$$

this will demonstrate that the right side of (5.11) depends only on the difference $t-\tau$.

Proceeding as discussed above, the double Fourier transform of (5.11) is

$$\mathscr{F}\{\mathscr{F}\{\mathscr{G}\{u^+ - u^-\}\}\} = \int u^-(\mathbf{y},k',\mathbf{e}_s)(u^+(\mathbf{y},k,\mathbf{e}_i) - u^-(\mathbf{y},k,\mathbf{e}_i))d\mathbf{y}. \tag{5.13}$$

We know that $u^+(\mathbf{y},k,\mathbf{e}_i)$ and $u^-(\mathbf{y},k,\mathbf{e}_i)$ are related by some *scattering operator* S by[4]

$$u^+(\mathbf{y},k,\mathbf{e}_i) = u^-(\mathbf{y},k,\mathbf{e}_i)S, \tag{5.14}$$

where the application of the operator S has the form[4]

$$u^+(\mathbf{y},k,\mathbf{e}_i) = \int u^-(\mathbf{y},k,\mathbf{e})S(k,\mathbf{e},\mathbf{e}_i)d^2\mathbf{e}. \tag{5.15}$$

Note that only the *existence* of the operator S is being used here; nothing need be known about it except that its inverse operator S^{-1} also exists. Using (5.14) in (5.13) along with the double Fourier transform of (5.10) results in

$$\mathscr{F}\{\mathscr{F}\{\mathscr{G}\{u^+ - u^-\}\}\} = \int u^{+*}(\mathbf{y},k',-\mathbf{e}_s)u^+(\mathbf{y},k,\mathbf{e}_i)(I - S^{-1})d\mathbf{y} = \delta(k\mathbf{e}_i + k'\mathbf{e}_s)(I - S^{-1}), \tag{5.16}$$

where I is the identity operator and the last equality follows from the completeness property of the solutions $u^+(\mathbf{y},k,\mathbf{e}_i)$.[8,9] Equation (5.16) has the form of the left side of (5.12); hence (5.11) depends only on $t-\tau$. In point of fact we have[4]

$$S - I = (k/2\pi i)A, \tag{5.17}$$

where A is the scattering amplitude operator defined similarly to (5.15), and S is Hermitian. Although these facts are not used here, they explain the use of $u^+ - u^-$ and the presence of $R(t,\mathbf{e}_s,\mathbf{e}_i)$ below.

Next, we evaluate (5.11) using this time invariance property. Without loss of generality, we may let t and τ approach infinity. Then the field in the vicinity of the scatterer will have decayed to zero, and virtually all of the contribution to the integral (5.11) will be in the far field. The incoming wave $u^-(\mathbf{y},\tau,\mathbf{e}_s)$ is simply the probing plane wave $\delta(\tau - \mathbf{e}_s \cdot \mathbf{y})$, and the outgoing wave is given by (3.6). Inserting these into (5.11), defining $\mathbf{e}_y = y/|y|$, and noting that $d\mathbf{y} = |\mathbf{y}|^2 d|\mathbf{y}|\, d^2\mathbf{e}_y$ gives

$$\begin{aligned}
\mathscr{G}\{u^+(\mathbf{x},t,\mathbf{e}_i) - u^-(\mathbf{x},t,\mathbf{e}_i)\}
&= \int \delta(\tau - \mathbf{e}_s \cdot \mathbf{y}) \frac{R(t-|\mathbf{y}|,\mathbf{e}_y,\mathbf{e}_i)}{4\pi|\mathbf{y}|} |\mathbf{y}|^2 d|\mathbf{y}|\, d^2\mathbf{e}_y \\
&= \frac{1}{4\pi}\int_{S^2}\int_0^\infty \delta\left(\frac{\tau}{|y|} - \mathbf{e}_s \cdot \mathbf{e}_y\right) \times R(t-|\mathbf{y}|,\mathbf{e}_y,\mathbf{e}_i)d|\mathbf{y}|\, d^2\mathbf{e}_y \\
&= \int_0^{t-\tau} R(t-\tau-|\mathbf{y}|,\mathbf{e}_s,\mathbf{e}_i)d|\mathbf{y}|.
\end{aligned} \tag{5.18}$$

The final equality in (5.18) is a result of letting $\tau,|\mathbf{y}| \to \infty$: the scattered field is significant only in the vicinity of the wave front $\tau = |\mathbf{y}|$ (the speed of propagation is unity) so that the only contribution to the integral occurs for $\tau/|\mathbf{y}| = 1$. Note that the upper limit at the end of (5.18) results from the causality of $R(\cdot)$.

Taking an inverse generalized Radon transform of both sides results in

$$\begin{aligned}
u^+(\mathbf{x},t,\mathbf{e}_i) - u^-(\mathbf{x},t,\mathbf{e}_i)
&= -\left(\frac{1}{8\pi^2}\right)\int_{S^2}\int_{-\infty}^{\infty} u^-(\mathbf{x},\tau,\mathbf{e}_s)\frac{\partial^2}{\partial\tau^2} \times \int_0^{t-\tau} R(t-\tau-|\mathbf{y}|,\mathbf{e}_s,\mathbf{e}_i)d|\mathbf{y}|\, d^2\mathbf{e}_s \\
&= -\left(\frac{1}{8\pi^2}\right)\int_{S^2}\int_{-\infty}^{\infty} u^-(\mathbf{x},\tau,\mathbf{e}_s)\frac{\partial}{\partial t} \times R(t-\tau,\mathbf{e}_s,\mathbf{e}_i)d\tau\, d^2\mathbf{e}_s,
\end{aligned} \tag{5.19}$$

which is (4.16) in Ref. 9. Using (3.8) and (5.10) in (5.19), and noting that $u^+(\mathbf{x},t,\mathbf{e}_i)$ is zero for $t \leqslant \mathbf{e}_i \cdot \mathbf{x}$ yields the Marchenko integral equation (3.7). Thus this integral equation is essentially an inverse generalized Radon transform. This interpretation of the Marchenko integral equation differs greatly from the functional analysis derivation in Newton[4] and Rose *et al.*,[9] and the representation theorem derivation in Rose *et al.*[30]

To recover the potential from the scattered field, we simply insert (4.5), which expresses the causality of the scattered field, into the plasma wave equation (4.1). Equating the coefficient of $\delta(t-\mathbf{e}_i\cdot\mathbf{x})$ to zero gives the miracle equation (3.10), since $(\partial/\partial t)u_s(\mathbf{x},t,\mathbf{e}_i)$ does not jump at the wave front. In Rose *et al.*[9] (3.10) is called the *fundamental identity*, and it is pointed out that this equation and the miracle equation are identical. An inverse Radon transform of (4.8c), from the layer stripping algorithm, also gives this equation. In all three cases, this equation is used to recover the potential from the jump in the scattered field at the wave front. However, the methods used to recover the scattered field itself differ widely.

Solving the Marchenko integral equation (3.7) is very difficult, due to the coupling between the $u_s(\mathbf{x},t,\mathbf{e}_i)$ in $\mathbf{e}_i$—it is necessary to solve for all of the scattered fields, due to probings in all directions, in one huge batch operation. The reason for this can be found by noting from (5.2) that knowledge of the complete Radon transform of $V(\mathbf{x})u(\mathbf{x},t,\mathbf{e}_i)$ is equivalent to knowledge of the generalized scattering amplitude $T(\mathbf{k},\mathbf{k}')$. However, this quantity is known only for $|\mathbf{k}|=|\mathbf{k}'|$, so that the scattering amplitude for one direction of probing $\mathbf{e}_i$ is not sufficient to reconstruct the scattered field for that $\mathbf{e}_i$. It is necessary to utilize the scattering amplitude for *all* $\mathbf{e}_i$ to reconstruct the scattered field for *any* $\mathbf{e}_i$.

C. The Born approximation

The (first) Born approximation is a *single scattering* approximation that greatly simplifies the solution to the inverse scattering problem. It consists of approximating the total wave field $u(\mathbf{x},t)$ inside the support of $V(\mathbf{x})$ by the probing impulse $\delta(t-\mathbf{e}_i\cdot\mathbf{x})$ alone—the scattered field $u_s(\mathbf{x},t)$ is neglected. This amounts to neglecting all multiple scattering events, an assumption that is reasonable for weak potentials or large values of k. Applying this approximation to the definition (3.4) of scattering amplitude and taking an inverse Fourier transform from k to t yields

$$\int V(\mathbf{x})\delta(t-(\mathbf{e}_s-\mathbf{e}_i)\cdot\mathbf{x})d\mathbf{x} = \mathscr{R}\{V(\mathbf{x})\}|_{\mathbf{e}=(\mathbf{e}_s-\mathbf{e}_i)} = -R^B(t,\mathbf{e}_s,\mathbf{e}_i), \qquad (5.20)$$

where $R^B(\cdot)$ is the impulse response in the Born approximation. Thus the potential $V(\mathbf{x})$ can be recovered by an inverse Radon transform of the impulse response.

It is elucidating to note how the three exact methods discussed in this paper all reduce to this result when a single scattering approximation is imposed on each of them. This illustrates where multiple scattering events are being accounted for in each method, and thus further illuminates their operation.

D. Moses's integral equation procedure

In Moses's procedure the second term on the right side of (3.12) accounts for multiple scattering events. To see this, note that if this term is neglected (3.12) reduces to

$$W(\mathbf{k})=W(k,\mathbf{e}_i)=\int V(\mathbf{x})e^{-2ik\mathbf{e}_i\mathbf{x}}\,d\mathbf{x} = -b(k,\mathbf{e}_i)=-A^B(k,-\mathbf{e}_i,\mathbf{e}_i) \qquad (5.21)$$

and an inverse Fourier transform from k to t (with k extended to negative values in the usual way) results in (5.20) with $\mathbf{e}_s=-\mathbf{e}_i$. Note that this is sufficient information to invert the Radon transform; backscattered data alone suffices.

E. Newton's integral equation procedure

The Born approximation applied to Newton's procedure amounts to neglecting the first term in the Marchenko integral equation (3.7). This leaves

$$u_s(\mathbf{x},t,\mathbf{e}_i)=-\frac{1}{8\pi^2}\int\frac{d}{dt}R^B(t-\mathbf{e}_s\cdot\mathbf{x},\mathbf{e}_s,\mathbf{e}_i)d^2\mathbf{e}_s. \qquad (5.22)$$

Applying the miracle equation (3.10), which we write here as

$$V(\mathbf{x})=-2\mathbf{e}_i\cdot\nabla u_s(\mathbf{x},t=\mathbf{e}_i\cdot\mathbf{x},\mathbf{e}_i) +2\frac{\partial}{\partial t}u_s(\mathbf{x},t=\mathbf{e}_i\cdot\mathbf{x},\mathbf{e}_i) \qquad (5.23)$$

(recall that the second term is zero) results in

$$V(\mathbf{x})=-\frac{1}{8\pi^2}\int\frac{\partial^2}{\partial t^2}\times R^B(t=(\mathbf{e}_i-\mathbf{e}_s)\mathbf{x},\mathbf{e}_s,\mathbf{e}_i)|\mathbf{e}_i-\mathbf{e}_s|^2d^2\mathbf{e}_s =\mathscr{R}^{-1}\{-R^B(t,\mathbf{e}_s,\mathbf{e}_i-\mathbf{e}_s)\} \qquad (5.24)$$

and a Radon transform of both sides results in (5.20).

F. Layer stripping procedure

In the layer stripping procedure the coupling in the system of equations (4.8) accounts for multiple scattering events. To see this, neglect this coupling, so that the algorithm becomes simply (4.8c), backpropagated to the far field as

$$\mathscr{R}\{V(\mathbf{x})\}=-2Q_s(\tau,t=\tau,\mathbf{e}) =-2Q_s(-R,t=2\tau+R,\mathbf{e}). \qquad (5.25)$$

Taking the Radon transform of (3.6) and utilizing the definition (4.8a) of $Q_s(\cdot)$ in terms of $U_s(\cdot)$ yields (5.20).

VI. CONCLUSION

A layer stripping algorithm for solving Schrödinger equation inverse scattering problems has been proposed. This algorithm is differential in nature, in contrast to the other integral equation procedures discussed. By exploiting the inherent structure of the inverse scattering problem (time causality), this algorithm appears to require much less computation time than the integral equation procedures, which reconstruct the potential in one huge batch operation without taking advantage of the structure of the problem. In addition, this algorithm requires near-field, backscattered data, making it more suitable for inverse seismic problems and other situations in which transmission data are not available.

The proposed algorithm differs from that of Yagle and Levy[15] in its use of the Radon transform. While this makes the reconstruction of $V(\mathbf{x})$ more complicated, it also simplifies the propagation of the scattered field, since the transverse Laplacian required at each step of the algorithm in Ref. 15 is no longer required. Both algorithms are in principle exact, with approximation inherent only in the discretization needed to implement them.

A significant consequence of the use of the Radon transform in this algorithm is that it made direct mathematical comparison between the layer stripping and integral equation procedures possible, which was not the case in Ref. 15. Indeed, the integral equation procedures of Newton and Moses were *derived* heuristically using the Radon transform and the generalized Radon transform. In addition, it was shown how all three inversion procedures reduce to the Born approximation when single scattering approximations are made. This showed the important role these transforms play in both exact and approximate procedures.

Considerable work remains to be done in making the layer stripping procedure a practical method for solving inverse scattering problems. Their numerical performance on synthetic data is a subject of current research. Other topics on which research is needed include numerical performance on noisy data, improved ways of implementing Eq. (4.9), and investigation of the applicability of this procedure to problems with bound states.

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