Propagators from integral representations of Green's functions for the N-dimensional free-particle, harmonic oscillator and Coulomb problems

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The radial Green's functions for the N-dimensional free-particle, isotropic harmonic oscillator and Coulomb problems all contain a product of two Bessel or Whittaker functions. After integral representations for these respective products are introduced, each Green's function exhibits the structure of a Fourier transform. One obtains thereby the Feynman propagators $K(r_1,r_2,t)$ for the free particle and harmonic oscillator. In the Coulomb case, the Fourier transform involves the quantum number variable and leads instead to the recently defined Sturmian propagator. The well-known connection between Coulomb and oscillator eigenstates of various dimensionality is manifested in a new way by the structure of the propagators derived here.

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1. GREEN'S FUNCTIONS AND PROPAGATORS

Consider a particle moving in an N-dimensional Euclidian space in a "central" potential $V(r)$, where r represents the N-dimensional radius $[\Sigma_i x_i^2]^{1/2}$. The L th "partial-wave" Green's function satisfies the radial equation[1]

$$\left[E+\frac{1}{2r^{N-1}}\frac{\partial}{\partial r}r^{N-1}\frac{\partial}{\partial r}-\frac{L(L+N-2)}{2r^2}-V(r)\right] \times G_L^{(N)}(r,r',E)=\delta(r-r')/(rr')^{N/2-1/2} \quad (L=0,1,2,\cdots). \tag{1.1}$$

Atomic units will be used throughout, with $\hbar=m=e=1$. We will consider, in turn, the free particle, isotropic harmonic oscillator and Coulomb system, corresponding to $V(r)=0$, $\frac{1}{2}\omega^2r^2$, and $-Z/r$, respectively. Equation (1.1), rearranged to standard Sturm–Liouville form, becomes

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{2}r^{N-1}\right)\frac{\partial}{\partial r}+r^{N-1}\left(E+\frac{L(L+N-2)}{2r^2}-V(r)\right)\right] \times G_L^{(N)}(r,r',E)=\delta(r-r'). \tag{1.2}$$

The usual procedure for constructing Green's functions then gives[2]

$$G(r,r',E)=u(r_<)v(r_>)/\tfrac{1}{2}r^{N-1}W[u,v]. \tag{1.3}$$

Here, $u(r)$ and $v(r)$ are solutions of the homogeneous equation (1.2), when $r\neq r'$, appropriate to the boundary conditions at $r=0$ and $r=\infty$, respectively, while $W[u,v]$ is the Wronskian

$$W[u,v]\equiv u(r)v'(r)-v(r)u'(r). \tag{1.4}$$

The particular solution $G^+(r,r',E)$ behaves like an outgoing spherical wave as $r_>\to\infty$. It is associated with the contour along the E axis such that $\operatorname{Im} E>0$. This Green's function is related by a Fourier transform to the Feynman propagator $K(r,r',t)$, as follows[3]:

$$G^+(r,r',E)=-i\int_0^\infty K(r,r',t)e^{iEt}\,dt. \tag{1.5}$$

This can be shown, most readily, from the respective spectral representations

$$G^+(r,r',E)=\sum_n\frac{R_n(r)R_n^*(r')}{E+i\epsilon-E_n} \tag{1.6}$$

and

$$K(r,r',E)=\sum_n R_n(r)R_n^*(r')e^{-iE_nt}. \tag{1.7}$$

2. N-DIMENSIONAL FREE PARTICLE

Applying the procedure outlined above to the free particle, we write Eq. (1.2) with $V(r)=0$ and $E=k^2/2$. The solution of the homogeneous equation analytic at $r=0$ is readily shown to be

$$u(r)=r^{1-N/2}J_{L+N/2-1}(kr). \tag{2.1}$$

For the outgoing wave Green's function G^+, the appropriate form of the outer solution is

$$v(r)=r^{1-N/2}H^{(1)}_{L+N/2-1}(kr). \tag{2.2}$$

The Wronskian of (2.1) and (2.2) is given by[4]

$$W[u,v]=r^{2-N}W[J_{L+N/2-1}(kr),H^{(1)}_{L+N/2-1}(kr)] =(2i/\pi)r^{1-N}. \tag{2.3}$$

Thus, by (1.3)

$$G_L^{(N)+}(r_1,r_2,k) =-i\pi(r_1r_2)^{1-N/2}J_{L+N/2-1}(kr_<)H^{(1)}_{L+N/2-1}(kr_>). \tag{2.4}$$

For odd dimension ($N=3,5,\ldots$), one can introduce the corresponding spherical Bessel functions, to give

$$G_L^{(N)+}(r_1,r_2,k) =-2ik(r_1r_2)^{3/2-N/2}j_{L+N/2-3/2}(kr_<) \times h^{(1)}_{L+N/2-3/2}(kr_>). \tag{2.5}$$

We now make use of an integral representation for a product of two Bessel functions[5]

$$J_\nu(z)H_\nu^{(1)}(Z) =\frac{1}{i\pi}\int_0^{c+i\infty}\exp\left[\frac{1}{2}t-\frac{Z^2+z^2}{2t}\right]I_\nu\left(\frac{zZ}{t}\right)\frac{dt}{t}. \tag{2.6}$$

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With the substitutions $t \to ik^2 t$, $c = 0$, $\nu = L + N/2 - 1$, $z = kr_1$, $Z = kr_2$, we obtain

$$J_{L+N/2-1}(kr_<)H^{(1)}_{L+N/2-1}(kr_>) = \frac{(-i)^{L+N/2}}{\pi}\int_0^\infty e^{ik^2t/2}e^{i(r_1^2+r_2^2)/2t} \times J_{L+N/2-1}\left(\frac{r_1r_2}{t}\right)\frac{dt}{t}. \tag{2.7}$$

Therefore

$$G_L^{(N)+}(r_1,r_2,k) = (-i)^{L+N/2+1}(r_1r_2)^{1-N/2} \times \int_0^\infty e^{ik^2t/2}e^{i(r_1^2+r_2^2)/2t}J_{L+N/2-1}\left(\frac{r_1r_2}{t}\right)\frac{dt}{t}. \tag{2.8}$$

We note that the energy parameter $E = k^2/2$ is now isolated in the first exponential factor of the integrand. Thus (2.8) can be identified with the Fourier transform (1.5), which immediately gives the free-partial propagator

$$K_L^{(N)}(r_1,r_2,t) = (-i)^{L+N/2}(r_1r_2)^{1-N/2}t^{-1} \times e^{i(r_1^2+r_2^2)/2t}J_{L+N/2-1}(r_1r_2/t). \tag{2.9}$$

From the derivative formula[6] $J_{\nu+1}(z) = -z^\nu(\partial/\partial z)[z^{-\nu}J_\nu(z)]$, the following recursive relation for N with $L = 0$ can be demonstrated

$$K_0^{(N+2)} = \frac{it}{\eta}\left(\frac{\partial K_0^{(N)}}{\partial \eta}\right)_\xi, \quad \eta \equiv r_1r_2, \quad \xi \equiv (r_1^2 + r_2^2)/2. \tag{2.10}$$

It is readily verified that (2.9) is a solution of the partial differential equation

$$i\frac{\partial K}{\partial t} + \frac{1}{2r^{N-1}}\frac{\partial}{\partial r}r^{N-1}\frac{\partial K}{\partial r} - \frac{L(L+N-2)}{2r^2}K = 0, \tag{2.11}$$

with the initial condition

$$K_L^{(N)}(r_1,r_2,0) = \delta(r_1 - r_2)/(r_1r_2)^{N/2-1/2}. \tag{2.12}$$

If the Bessel function in (2.9) is expressed in terms of a confluent hypergeometric function,[7] viz.,

$$J_\nu(z) = \frac{(z/2)^\nu e^{-iz}}{\Gamma(\nu+1)}M\left(\nu + \frac{1}{2};2\nu+1;2iz\right), \tag{2.13}$$

then the propagator (2.9) exhibits the structure[8]

$$K = F e^{iS}, \tag{2.14}$$

with S representing the one-dimesional free-particle action

$$S(r_1,r_2,t) = (r_1 - r_2)^2/2t. \tag{2.15}$$

We cite, in particular, the two- and three-dimensional cases. For $N = 2$, with the customary notation $L = m$ and $r = \rho$:

$$K_m^{(2)}(\rho_1,\rho_2,t) = (-i)^{m+1}t^{-1}e^{i(\rho_1^2+\rho_2^2)/2t}J_m(\rho_1\rho_2/t) \quad (m = 0, \pm 1, \pm 2,\cdots). \tag{2.16}$$

Summation over m gives

$$K^{(2)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,t) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}K_m^{(2)}(\rho_1,\rho_2,t)e^{im(\phi_1-\phi_2)}. \tag{2.17}$$

Making use of the generating function

$$e^{z(u-1/u)/2} = \sum_{m=-\infty}^{\infty}u^mJ_m(z), \tag{2.18}$$

with $u = -ie^{i(\phi_1-\phi_2)}$, $z = \rho_1\rho_2/t$, we obtain

$$K^{(2)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,t) = (2\pi it)^{-1}e^{i(\rho_1^2+\rho_2^2)/2t}e^{-i(\rho_1\rho_2/t)\cos(\phi_1-\phi_2)} = (2\pi it)^{-1}e^{i(\boldsymbol{\rho}_1-\boldsymbol{\rho}_2)^2/2t}. \tag{2.19}$$

For $N = 3$ with $L = l$:

$$K_l^{(3)}(r_1,r_2,t) = (-1)^{l+3/2}(2/\pi t^3)^{1/2} \times e^{i(r_1^2+r_2^2)/2t}j_l(r_1r_2/t). \tag{2.20}$$

The sum over partial waves gives

$$K^{(3)}(\mathbf{r}_1,\mathbf{r}_2,t) = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}P_l(\cos\theta)K_l^{(3)}(r_1,r_2,t). \tag{2.21}$$

With use of the addition theorem[9]

$$e^{iz\cos\theta} = \sum_{l=0}^{\infty}(2l+1)i^lj_l(z)P_l(\cos\theta), \tag{2.22}$$

we obtain the familiar three-dimensional free-particle propagator[10]

$$K^{(3)}(\mathbf{r}_1,\mathbf{r}_2,t) = (2\pi it)^{-3/2}e^{i(r_1^2+r_2^2)/2t}e^{-ir_1r_2\cos\theta/t} = (2\pi it)^{-3/2}e^{i(\mathbf{r}_1-\mathbf{r}_2)^2/2t}. \tag{2.23}$$

3. N-DIMENSIONAL HARMONIC OSCILLATOR

For the N-dimensional isotropic harmonic oscillator, Eq. (1.1) takes the form

$$\left[E + \frac{1}{2}\left(\frac{1}{r^{N-1}}\frac{\partial}{\partial r}r^{N-1}\frac{\partial}{\partial r} - \frac{L(L+N-2)}{r^2} - \omega^2r^2\right)\right] \times G_L^{(N)}(r,r',E) = \delta(r - r')/(rr')^{N/2-1/2}. \tag{3.1}$$

All results in this section reduce to the corresponding free-particle formulas in the limit $\omega \to 0$. The following solutions of the homogeneous equation can be demonstrated[11]

$$u(r) = r^{-N/2}M_{E/2\omega}^{(L+N/2-1)/2}(\omega r^2) \tag{3.2}$$

and

$$v(r) = r^{-N/2}W_{E/2\omega}^{(L+N/2-1)/2}(\omega r^2), \tag{3.3}$$

where M and W are Whittaker functions as defined by Buchholz. However, for compactness of notation, we write $M_\kappa^{\mu/2}(z)$ in place of $\mathscr{M}_{\kappa,\mu/2}(z)$ and $W_\kappa^{\mu/2}(z)$ in place of $W_{\kappa,\mu/2}(z)$.[12] Using the Wronskian[13]

$$W[M_\kappa^{\mu/2}(z),W_\kappa^{\mu/2}(z)] = \frac{-1}{\Gamma[(\mu+1)/2-\kappa]}, \tag{3.4}$$

we obtain

$$W[u,v] = \frac{-2\omega r^{1-N}}{\Gamma(L/2+N/4-E/2\omega)} \tag{3.5}$$

and thereby, by (1.3),

$$G_L^{(N)}(r_1,r_2,E) = -\omega^{-1}(r_1r_2)^{-N/2}\Gamma(L/2+N/4-E/2\omega) \times M_{E/2\omega}^{(L+N/2-1)/2}(\omega r_<^2)W_{E/2\omega}^{(L+N/2-1)/2}(\omega r_>^2). \tag{3.6}$$

Note that this Green's function is nonpropagating, as is indeed expected for a purely discrete spectrum. The eigenvalues for this system follow simply from the poles of the gamma function, viz.,

$$E^{(N)}_{n,L} = (2n + L + N/2)\omega \quad (n = 0,1,2,\cdots). \tag{3.7}$$

We next make use of an integral representation for a product of two Whittaker functions given by Buchholz[14]:

$$\Gamma\left(\frac{\mu+1}{2} - \kappa\right) W^{\mu/2}_{\kappa}(a_1 t) M^{\mu/2}_{\kappa}(a_2 t) = t\sqrt{a_1 a_2}\int_0^\infty \exp -\tfrac{1}{2}(a_1 + a_2)t \cosh v \times I_\mu(t\sqrt{a_1 a_2}\sinh v)\coth^{2\kappa}(v/2)dv \tag{3.8}$$

restricted, however, by the condition that

$$\mathrm{Re}\left(\frac{\mu+1}{2} - \kappa\right) > 0. \tag{3.9}$$

In order to make this representation applicable, we temporarily turn ω into a pure imaginary

$$\omega = -i\sigma. \tag{3.10}$$

With the substitutions in (3.8): $\mu = L + N/2 - 1$, $t = -i\sigma$, $\kappa = iE/2\sigma$, $a_1 = r^2_>$, $a_2 = r^2_<$ and the variable transformation $\sinh v = \operatorname{csch}\sigma t$, $\cosh v = \coth \sigma t$, $\coth(v/2) = e^{\sigma t}$, $dv = -\sigma \operatorname{csch}\sigma t\, dt$, we obtain

$$G^{(N)}_L(r_1,r_2,E)|_{\omega=-i\sigma} = -(-i)^{L+N/2-1}\sigma(r_1r_2)^{1-N/2} \times \int_0^\infty dt\, e^{iEt}\operatorname{csch}\sigma t \times e^{(1/2)i\sigma(r_1^2+r_2^2)\coth\sigma t} \times J_{L+N/2-1}(\sigma r_1 r_2 \operatorname{csch}\sigma t). \tag{3.11}$$

Again, this can be identified with the Fourier transform (1.5). After reverting back to real $\omega(\sigma = i\omega)$, we obtain the harmonic-oscillator propagators

$$K^{(N)}_L(r_1,r_2,t) = (-i)^{L+N/2}\omega(r_1r_2)^{1-N/2}\csc\omega t \times e^{(1/2)i\omega(r_1^2+r_2^2)\cot\omega t} J_{L+N/2-1}(\omega r_1 r_2 \csc\omega t). \tag{3.12}$$

Again, using (2.13), we find that (3.12) shows the structure (2.14) with the one-dimensional harmonic-oscillator action[15]

$$S(r_1,r_2,t) = \tfrac{1}{2}\omega(r_1^2 + r_2^2)\cot\omega t - \omega r_1 r_2 \csc\omega t. \tag{3.13}$$

The spectral representation of the propagator (3.12) follows from the Hille–Hardy formula[16,17]:

$$\frac{e^{(x+y)h/(1+h)}}{1+h}\frac{J_\mu[2\sqrt{xy}h^{1/2}/(1+h)]}{(xyh)^{\mu/2}} = \sum_{\lambda=0}^\infty \frac{\lambda!}{\Gamma(\lambda+\mu+1)}(-h)^\lambda L^{(\mu)}_\lambda(x) L^{(\mu)}_\lambda(y). \tag{3.14}$$

Expressing the Laguerre functions $L^{(\mu)}_\lambda$ in terms of Whittaker functions[18] and rearranging, we obtain

$$h^{-\mu/2}(1+h)^{-1}\sqrt{xy}e^{(x+y)(1+h)/(1-h)/2} \times J_\mu[2\sqrt{xy}h^{1/2}/(1+h)] = \sum_{\lambda=0}^\infty (-h)^\lambda \frac{\Gamma(\lambda+\mu+1)}{\lambda!} \times M^{\mu/2}_{\lambda+(\mu+1)/2}(x) M^{\mu/2}_{\lambda+(\mu+1)/2}(y). \tag{3.15}$$

With the substitutions: $\mu = L + N/2 - 1$, $x = \omega r_1^2$, $y = \omega r_2^2$, $h = -e^{-2i\omega t}$, $\lambda = n$, we obtain

$$K^{(N)}_L(r_1,r_2,t) = \sum_{n=0}^\infty R^{(N)}_{n,L}(r_1) R^{(N)}_{n,L}(r_2) e^{-iE^{(N)}_{n,L}t} \tag{3.16}$$

with the eigenvalues $E^{(N)}_{n,L}$ given by (3.7) and the radial eigenfunctions by

$$R^{(N)}_{n,L}(r) = \left[\frac{2\Gamma(n+L+N/2)}{n!}\right]^{1/2} r^{-N/2} M^{(L+N/2-1)/2}_{n+L/2+N/4}(\omega r^2) \quad (n = 0,1,2,\cdots). \tag{3.17}$$

Again we note the special cases $N = 2$ and 3. For $N = 2$,[19]

$$K^{(2)}_m(\rho_1,\rho_2,t) = (-i)^{m+1}\omega\csc\omega t \times e^{(1/2)i\omega(\rho_1^2+\rho_2^2)\cot\omega t} J_m(\omega\rho_1\rho_2\csc\omega t) \quad (M = 0, \pm 1, \pm 2,\cdots). \tag{3.18}$$

The summation analogous to (2.17) results in

$$K^{(2)}(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, t) = (-i/2\pi)\omega\csc\omega t \times \exp[\tfrac{1}{2}i\omega(\rho_1^2+\rho_2^2)\cot\omega t - i\omega\boldsymbol{\rho}_1\cdot\boldsymbol{\rho}_2\csc\omega t]. \tag{3.19}$$

For $N = 3$:

$$K^{(3)}_l(r_1,r_2,t) = (-i)^{l+3/2}(2/\pi)^{1/2}(\omega\csc\omega t)^{3/2} \times e^{(1/2)i\omega(r_1^2+r_2^2)\cot\omega t} j_l(\omega r_1 r_2\csc\omega t). \tag{3.20}$$

The sum over partial waves as in (2.21) gives[20]

$$K^{(3)}(\mathbf{r}_1,\mathbf{r}_2,t) = (\omega\csc\omega t/2\pi i)^{3/2} \times \exp[\tfrac{1}{2}i\omega(r_1^2+r_2^2)\cot\omega t - i\omega\mathbf{r}_1\cdot\mathbf{r}_2\csc\omega t]. \tag{3.21}$$

4. N-DIMENSIONAL COULOMB PROBLEM

We consider finally the N-dimensional hydrogenic radial equation

$$\frac{1}{2}\left[k^2 + \frac{1}{r^{N-1}}\frac{\partial}{\partial r}r^{N-1}\frac{\partial}{\partial r} - \frac{L(L+N-2)}{r^2} + \frac{2Z}{r}\right] \times G^{(N)}_L(r,r',k) = \delta(r-r')/(rr')^{N/2-1/2}. \tag{4.1}$$

The appropriate solutions to the homgeneous equation are, in this case[21]

$$u(r) = r^{1/2-N/2}M^{L+N/2-1}_{i\nu}(-2ikr), \quad v(r) = r^{1/2-N/2}W^{L+N/2-1}_{i\nu}(-2ikr), \tag{4.2}$$

where

$$\nu \equiv Z/k. \tag{4.3}$$

Using (3.4) and (1.3) once again, we obtain the Coulomb Green's functions

$$G_L^{(N)}(r_1,r_2,k) = (ik)^{-1}\Gamma(L+N/2-1/2-iv) \times (r_1r_2)^{1/2-N/2} M_{iv}^{L+N/2-1}(-2ikr_<) \times W_{iv}^{L+N/2-1}(-2ikr_>), \quad (4.4)$$

a result previously given by Hostler.[22] The poles of the gamma function at $v = -i(L+N/2-1/2+n')$, $n' = 0,1,2,...,$ determine the N-dimensional hydrogenic spectrum:

$$E_{n,L}^{(N)} = Z^2/2v^2 = \frac{-Z^2}{2(n+N/2-3/2)^2}, \quad n = L+1, L+2,\cdots. \quad (4.5)$$

The integral representation (3.8) is again applicable, now with $\mu/2 = L + N/2 - 1$, $\kappa = iv$, $t = -2ik$, $a_1 = r_>$, $a_2 = r_<$, and the variable transformation $\sinh v = \operatorname{csch} q$. We obtain

$$G_L^{(N)}(r_1,r_2,k) = -2(-i)^{2L+N-2}(r_1r_2)^{1-N/2} \times \int_0^\infty dq\, e^{2ivq} \operatorname{csch} q\, e^{ik(r_1+r_2)\coth q} \times J_{2L+N-2}(2k\sqrt{r_1r_2}\operatorname{csch} q). \quad (4.6)$$

Unfortunately, this is not a Fourier transform wrt time and energy variables. In fact, no closed form for the Coulomb propagator is known, as yet. Equation (4.6) does, however, represent a Fourier transform wrt the quantum number variable v. In a similar instance, we have introduced the *Sturmian propagator*,[23] defined by the transform

$$G(r_1,r_2,v) = \frac{-2i}{k}\int_0^\infty dq\, e^{2ivq} S(r_1,r_2,q). \quad (4.7)$$

For the Green's function (4.6), we identify the corresponding Sturmian propagator

$$S_L^{(N)}(r_1,r_2,q) = (-i)^{2L+N-1} k(r_1r_2)^{1-N/2} \operatorname{csch} q \times e^{ik(r_1+r_2)\operatorname{csch} q} J_{2L+N-2}(2k\sqrt{r_1r_2}\operatorname{csch} q). \quad (4.8)$$

By substituting $Z = kv$ [cf. Eq. (4.3)] in (4.1) and using the Fourier transform (4.7), we obtain a partial differential equation for S:

$$\frac{1}{2}\left[k^2 + \frac{1}{r^{N-1}}\frac{\partial}{\partial r} r^{N-1}\frac{\partial}{\partial r} - \frac{L(L+N-2)}{r^2} + \frac{ik}{r}\frac{\partial}{\partial q}\right]S = 0, \quad (4.9)$$

$$r = r_1, r_2$$

subject to the boundary condition

$$S_L^{(N)}(r_1,r_2,0) = \delta(r_1-r_2)/(r_1r_2)^{N/2-1}. \quad (4.10)$$

The propagator (4.8), with k and v real, pertains to the Coulomb continuum. Of more significance is the discrete spectrum Sturmian propagator, obtained by the substitutions: $k \to ik$, $q \to iq$, viz.,

$$S_L^{(N)}(r_1,r_2,q) = (-i)^{2L+N-1} k(r_1r_2)^{1-N/2} \csc q \times e^{ik(r_1+r_2)\cot q} J_{2L+N-2}(2k\sqrt{r_1r_2}\csc q). \quad (4.11)$$

The spectral representation of (4.11) follows again from (3.15), with $\mu = 2L+N-2$, $x = 2kr_1$, $y = 2kr_2$, $h = -e^{-2iq}$, $\lambda = n-L-1$. The result is

$$S_L^{(N)}(r_1,r_2,q) = \sum_{n=L+1}^{\infty} R_{n,L}^{(N)}(r_1) R_{n,L}^{(N)}(r_2) \times e^{-2iq(n+N/2-3/2)}, \quad (4.12)$$

with

$$R_{n,L}^{(N)}(r) = \left[\frac{(n+L+N-3)!}{(n-L-1)!}\right]^{1/2} \times r^{1/2-N/2} M_{n+N/2-3/2}^{L+N/2-1}(2kr), \quad (4.13)$$

$$n = L+1, L+2,\cdots.$$

If $k = Z/n$ then (4.13) gives the N-dimensional Coulomb radial eigenfunctions. For k arbitrary, as is the case here, the $R_{n,L}^{(N)}(r)$ represent Sturmian functions,[24] hence our designation for the propagator $S_L^{(N)}$.

Relationships between Coulomb eigenstates and those of harmonic oscillators of various dimension have been known for a long time.[25] This connection manifests itself in the similarity of the propagator $S_L^{(N)}(r_1,r_2,q)$ to the harmonic-oscillator propagator $K_\lambda^{(v)}(\rho_1,\rho_2,t)$. Specifically, under the substitutions

$$r = \rho^2/2, \quad k = \omega, \quad q = \omega t \quad (4.14)$$

and

$$2L + N - 2 = \lambda + v/2 - 1 \quad (4.15)$$

the two propagators are related by

$$(r_1r_2)^{N/2-1} S_L^{(N)}(r_1,r_2,q) = (\rho_1\rho_2)^{v/2-1} K_\lambda^{(v)}(\rho_1,\rho_2,t). \quad (4.16)$$

A formula equivalent to (4.15) was found by Giovannini and Tonietti.[26] Two realizations of (4.15) for the three-dimensional Coulomb problem ($N = 3$, $L = l$) have been given. Schwinger[27] set $v = 2$, so that $2l+1 = \lambda = |m|$, thus connecting hydrogenic states to those of a two-dimensional oscillator. Bergmann and Frishman[28] set $\lambda = 0$, so that $4l + 4 = v$, thus establishing a connection with states of a v-dimensional oscillator. Further, comparison of (4.13) with (3.17) shows that the hydrogenic principal quantum number n corresponds to the oscillator quantum number $n' = n - L - 1$.

For the three-dimensional case, the sum over partial waves according to

$$S(\mathbf{r}_1,\mathbf{r}_2,q) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} P_l(\cos\theta) S_l^{(3)}(r_1,r_2,q) \quad (4.17)$$

can be evaluated using Neumann's formula[29]

$$\tfrac{1}{2}zJ_0(z\cos\theta/2) = \sum_{l=0}^{\infty}(-)^l(2l+1)P_l(\cos\theta)J_{2l+1}(z). \quad (4.18)$$

The result is the Coulomb Sturmian propagator[23]

$$S(\mathbf{r}_1,\mathbf{r}_2,q) = -(4\pi)^{-1}k^2\csc^2 q \times e^{ik\xi\cot q} J_0(k\eta\csc q), \quad (4.19)$$

$$\xi \equiv r_1 + r_2, \quad \eta \equiv 2\sqrt{r_1r_2}\cos\theta/2.$$

Substituting (4.19) back into (4.7), we recover an integral representation for the Coulomb Green's function first derived by Hostler.[30]

[1]The form of the N-dimensional Laplacian is given by J. D. Louck, J. Mol. Spectrosc. **4**, 298 (1960).
[2]See, for example, S. I. Vetchinkin and V. L. Bachrach, Int. J. Quantum Chem. **6**, 143 (1972).
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