

York-Peng Yao

Department of Physics  
 University of Michigan  
 Ann Arbor, Michigan 48109

## I. Introduction

I want to talk about two problems in grand unification and indicate their solutions. These are the decoupling problem and the gauge hierarchy problem. They are interrelated. This work has been performed with Kazama and Unger<sup>1</sup>.

As is well-known, in grand unification<sup>2</sup>, we assume the existence of a symmetry  $G$ , such that some scalar acquires a big vacuum expectation value  $V$  and then subsequently another scalar develops a vacuum expectation value  $v$  to bring the symmetry to  $SU(3)_{\text{color}} \times (SU(2) \times U(1))_{\text{E.M. weak}}$ .

$$G \xrightarrow{V} G' \xrightarrow{v} SU(3)_{\text{color}} \times (SU(2) \times U(1))_{\text{E.M. weak}}$$

Now, the problems are:

(1) Gauge hierarchy issue:<sup>3</sup> There are two aspects:

(a) why is  $V \gg v$ . We have no deep answer for this. It is in our opinion the same kind of question "why is  $m_\mu \gg m_e$ ". They are just input parameters.

(b) To us, a more serious problem is "can we define a light sector"? In other words, can we maintain light particles to have small mass to all orders in perturbation without fine tuning. The answer is yes, to all orders in coupling expansion.

(2) Decoupling theorem:<sup>4</sup> Having said that we know how to separate particles naturally into light and heavy sectors, we may ask "does there exist an effective local Lagrangian theory, such that we can use it to reproduce all the light particle physical matrix elements at energy and  $|P_i| \ll M$ ? The answer is again yes, to all orders of perturbation in couplings, i.e.

$$\mathcal{L}(g, M, m, \mu) \rightarrow \Gamma^n \text{ (1 light particle irreducible)}$$

$$\mathcal{L}^*(g^*, m^*, \mu) \rightarrow \Gamma^{*n} \text{ (1 particle irreducible)}$$

we can show that

$$(1) \Gamma^n = z^{n/2} \Gamma^{*n} + O\left(\frac{1}{M^2}\right)$$

$$(2) g^* = g^*(g, \ell n M/\mu)$$

$$m^* = m f(g, \ell n M/\mu)$$

Besides, we know how to calculate, to all orders in  $g$ ,  $g^*$  and  $m^*$  via improved perturbation, i.e. renormalization group equation. The operator structure of  $\mathcal{L}^*$  is obtained from  $\mathcal{L}$  by deleting all terms involving heavy fields.

Note that we do our physics in the low energy region throughout. We consider  $\ell n M/\mu$  effects to be radiative correction due to heavy particles. This is in contradistinction to some other people's attitudes<sup>5</sup>, in which they boost up the energy of the external particles and devise methods so that their effective theories, which may not be local, will join smoothly with the full theory.

## II. Model

We need a model to make my statements more concrete. Let me now be more specific. Let us consider an  $O(3)$  gauge model with two scalar triplets

$$\mathcal{L} = -\frac{1}{4} (\partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu - e \vec{A}_\mu \times \vec{A}_\nu)^2$$

$$- \frac{1}{2} (\partial_\mu \vec{\phi}_1 - e \vec{A}_\mu \times \vec{\phi}_1)^2 - \frac{1}{2} (\partial_\mu \vec{\phi}_2 - e \vec{A}_\mu \times \vec{\phi}_2)^2$$

$$- \frac{1}{2} m^2 \vec{\phi}_1^2 - \frac{1}{2} m^2 \vec{\phi}_2^2 - \frac{1}{4} \lambda_1 (\vec{\phi}_1^2)^2$$

$$- \frac{1}{4} \lambda_2 (\vec{\phi}_1^2)^2 - \frac{1}{2} \lambda_3 \vec{\phi}_1^2 \vec{\phi}_2^2 - \frac{1}{2} \lambda_4 (\vec{\phi}_1 \cdot \vec{\phi}_2)^2$$

We assume that  $\lambda_1, \lambda_2 > 0$  so that there is a lower bound for the potential. Now, we shall assume that the vacuum is unstable, such that

$$\vec{\phi}_1 = \begin{pmatrix} v_1 + \sigma \\ \pi_2 \\ \pi_3 \end{pmatrix} \quad \vec{\phi}_2 = \begin{pmatrix} \psi_1 \\ v_2 + \phi \\ \psi_3 \end{pmatrix}$$

with  $v_1 \gg v_2$  (renormalized  $v_1$  and  $v_2$  are input parameters). The other parameters are adjusted to make the potential reach its absolute minimum.

At the tree level, we have  $\lambda_4 > 0$  and

$$\max \left( -\sqrt{\lambda_1 \lambda_2}, \frac{-\lambda \frac{v_1^2}{11}}{\frac{v_2^2}{2}}, \frac{-\lambda \frac{v_2^2}{22}}{\frac{v_1^2}{1}} \right) < \lambda_3 < \sqrt{\lambda_1 \lambda_2}$$

The minimum vacuum conditons are

$$-m_1^2 = \lambda \frac{v_1^2}{11} + \lambda \frac{v_2^2}{32}$$

$$-m_2^2 = \lambda \frac{v_2^2}{22} + \lambda \frac{v_1^2}{31}$$

In this way,  $m_1^2$  and  $m_2^2$  are determined parameters. In fact, we can show that a pertrubation series can be organized such that  $m_1^2$  and  $m_2^2$  never appear in any calculation, to all orders.

Let us now talk about the spectrum which will bring out another problem which must be treated in discussing decoupling. We find that  $\sigma$  and  $\phi$  mix.

The mass eigenstates are

$$H = \sigma \cos \theta - \phi \sin \theta$$

$$h = \sigma \sin \theta + \phi \cos \theta$$

with masses

$$m_H^2 \cong 2 \left( \lambda \frac{v_1^2}{11} + \left( \frac{\lambda}{\lambda} \right) \frac{v_2^2}{31} \right) \quad \text{Heavy}$$

$$m_h^2 \cong 2 \left( \lambda \frac{v_2^2}{22} - \left( \frac{\lambda}{\lambda} \right) \frac{v_1^2}{31} \right) \quad \text{Light}$$

$$\sin \theta \cong - \frac{\lambda \frac{v_2^2}{32}}{\lambda \frac{v_1^2}{11}}$$

Now, we quantize the theory in the 't Hooft-Feynman gauge

$$\begin{aligned}
 \mathcal{L}_{\text{gauge}} = & -\frac{1}{2\alpha} (\partial_{\mu} A_1^{\mu} - \alpha e v_2 \psi_3)^2 \\
 & -\frac{1}{2\alpha} (\partial_{\mu} A_2^{\mu} - \alpha e v_1 \pi_3)^2 \\
 & -\frac{1}{2\alpha} (\partial_{\mu} A_3^{\mu} - \alpha e (v_1 \pi_2 - v_2 \psi_1))^2 \\
 & + \mathcal{L}_{\text{ghost}}, \quad \alpha_{\text{tree}}=1
 \end{aligned}$$

Then we have the following spectrum

Physical Particles	(mass) <sup>2</sup>	would be Goldstone partner
A <sub>1</sub>	$e^2 v_2^2$	$\psi_3$
A <sub>2</sub>	$e^2 v_1^2$	$\pi_3$
A <sub>3</sub>	$e^2 (v_1 + v_2)^2$	$\xi = (v_1 \pi_2 - v_2 \psi_1) / (v_1 + v_2)^{1/2}$
H	$\cong 2(\lambda_1 v_1^2 + (\lambda_3 / \lambda_1) v_2^2)$	
h	$\cong 2(\lambda_2 - \lambda_3 / \lambda_1) v_2^2$	
$\eta = (v_2 \pi_2 + v_1 \psi_1) / (v_1 + v_2)^{1/2}$	$\lambda_4 (v_1 + v_2)^2$	

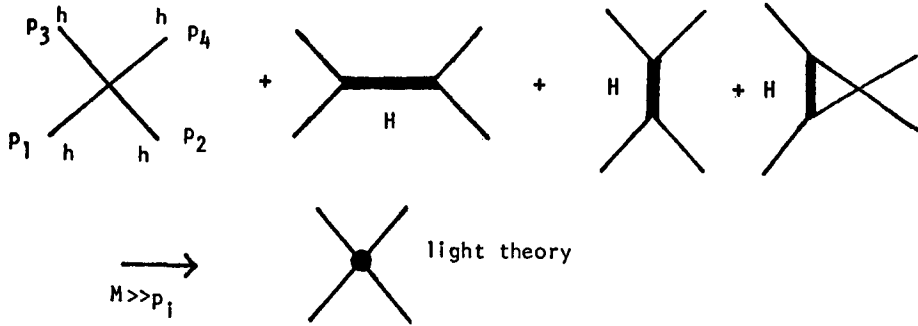
The light particles are A<sub>1</sub>, h, and  $\psi_3$  and the ghost of A<sub>1</sub>. Let us generically call these tree (masses)<sup>2</sup> m<sup>2</sup>. Our solution to the gauge hierarchy problem is

$$m_{\text{true}}^2 = m^2 \left( 1 + \sum_{i=\text{loop}} [a_i (\ell_n v_1^2 / \mu^2)^{n_i} + b_i (\ell_n v_2^2 / \mu^2)^{n_i-1} + \dots] \right)$$

to all orders.

Now h and H will mix further when we carry out the loop expansion. We shall devise a set of Green's functions, such that mixing is automatically taken into account. To illustrate the natural choice, let us consider h-h

scattering. In the full theory we have



So, the Green's functions in the full theory that we want are the one light particle irreducible Green's functions. The decoupling theorem that we can prove to all orders in coupling expansion is the following

$$\Gamma_{1LPI}^n(p_i) = Z^{n/2} \Gamma_{1PI}^{*n}(p_i) + O(1/v_1^2)$$

$$p_i \ll M$$

in which  $\Gamma^n$  is calculated via the full Lagrangian  $\mathcal{L}(g, \lambda, v_1, v_2, \mu, \alpha)$ , while  $\Gamma^*$  is calculated from a light Lagrangian, the operator structure of which is that obtained when all heavy fields in  $\mathcal{L}$  are all deleted. The coupling, mass and gauge parameters are all functions of  $g, \lambda, v_1, v_2, \mu$  and  $\alpha$ .

$$\mathcal{L}_{light}^* = \mathcal{L}^*(g^*, \lambda^*, v^*, \mu, \alpha^*)$$

$$g^* = g^*(g, \lambda, \alpha, \ln v_1^2 / \mu^2)$$

$$v^* = v_2 f(g, \lambda, \alpha, \ln v_1^2 / \mu^2)$$

$$\lambda^* = \lambda^*(g, \lambda, \alpha, \ln v_1^2 / \mu^2)$$

$$\alpha^* = \alpha^*(g, \lambda, \alpha, \ln v_1^2 / \mu^2)$$

We do minimal subtraction in both theories.

### III. Methodology

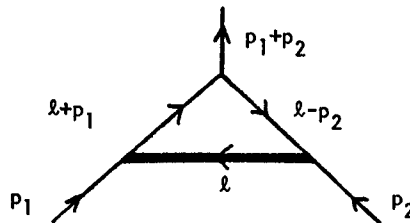
How do we prove such results?

(1) We first show that given any diagram with heavy internal lines, the corresponding integral can be rearranged so that the  $O(1)$  terms all have the heavy lines shrunk into vertices with no more than 4 light lines entering and/or leaving. The rest are negligible. This establishes the local renormalizable nature of the effective vertices. In fact, we can identify what these effective vertices are in relation to the light Lagrangian. However, a more economical way to show what the resulting local theory ensues is via BRS identities.

(2) We can show that the relevant BRS identities in the full theory ( $v_1 \gg |p_1|, v_2 \gg 0(2)$ ) spontaneously broken BRS identities.

Thus, the limit of this  $O(3)$  theory is just the Abelian Higgs model in its asymmetrical phase. Besides, only  $v_2$  appears explicitly in the resulting identities, which is a confirmation of the stability of gauge hierarchy.

To illustrate the first part, it is best to give an example. Consider a three point function



$$I = M^2 \int d^4 l \frac{1}{(l+p_1)^2 + m^2} \frac{1}{(l-p_2)^2 + m^2} \frac{1}{l^2 + M^2}$$

we define

shrinking a (sub) graph to a point

≡ setting all external momenta which go into/out of this graph to zero

≡ localization

There are two (sub)graphs which contains a heavy line

1.



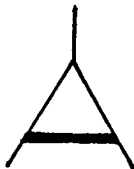
$$(\text{shrink operator } \tau_1) \frac{1}{\ell^2 + M^2} = \frac{1}{M^2}$$

2.



$$\begin{aligned} (\text{shrink operator } \tau_2) \frac{1}{\ell^2 + M^2} \frac{1}{(\ell + p_1)^2 + m^2} \frac{1}{(\ell - p_2)^2 + m^2} \\ = \frac{1}{\ell^2 + M^2} \frac{1}{\ell^2 + m^2} \frac{1}{\ell^2 + m^2} \end{aligned}$$

Then, we have an identity



$$\begin{aligned} &= (1 - \tau_2)(1 - \tau_1) \left[ 1 \sim \begin{array}{c} \text{triangle with thick bottom edge} \\ \text{and vertical line} \end{array} \right] \\ &+ (1 - \tau_2) \left[ \tau_1 \mid \sim \begin{array}{c} \text{circle with thick bottom edge} \\ \text{and vertical line} \end{array} \right] \\ &+ \left[ \tau_2 \mid \sim \begin{array}{c} \text{triangle with thick bottom edge} \\ \text{and vertex} \end{array} \right] \end{aligned}$$

Note that the resulting integrals are automatically renormalized. Only the second and third terms, which have been localized, contribute to order 1. The first term is negligible.

The proof of reduction of BRS identities depends heavily on power counting.

#### IV. Renormalization Group Equations

I want to devote the remainder of the discussion to the renormalization group equations which will be used to perform leading  $\ell n$  sum of the dependence on  $\ell n v_1^2 / \mu^2$  of the effective parameters.

As I said earlier,

we stay in low energy region to do our physics. Now when we work with the full theory, we have the renormalization group equation

$$\left( \mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g} + \gamma_{v_1} v_1 \frac{\partial}{\partial v_1} + \gamma_{v_2} v_2 \frac{\partial}{\partial v_2} + \gamma_\alpha \frac{\partial}{\partial \ln \alpha} - n\gamma \right) \Gamma^n = 0$$

with

$$\beta_g = \mu \frac{d}{d\mu} g, \quad \gamma_{v_1} = \mu \frac{d}{d\mu} \ln v_1, \text{ etc.},$$

where  $\Gamma^n$  has only light external lines.

When we work with the effective theory, we have

$$\left( \mu \frac{\partial}{\partial \mu} + \beta^* \frac{\partial}{\partial g^*} + v^* \gamma^* \frac{\partial}{\partial v^*} + \gamma^* \frac{\partial}{\partial \ln \alpha^*} - n\gamma^* \right) \Gamma^{*n} = 0$$

Now, we put the decoupling equation, which is

$$\Gamma^n = z^{n/2} \Gamma^{*n} + O\left(\frac{1}{M^2}\right),$$

into the renormalization group equation of the full theory and demand that the resulting equation agree with that from the effective theory. We obtain many relations, which are basically chain rules for differentiation. For example

$$\beta^*(g^*) = \left( \mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g} + \gamma_{v_1} \frac{\partial}{\partial \ln v_1} + \gamma_\alpha \frac{\partial}{\partial \ln \alpha} \right) g^*$$

This rather humble equation in fact allows us to sum up the leading contributions. For example, returning to 0(3) model, we have for the gauge coupling,

$$e^* = e f(e^2 \ln v_1^2 / \mu^2) + e^3 f'(e^2 \ln v_1^2 / \mu^2) + \dots$$

$e$  is considered to be small and  $e^2 \ln v_1^2 / \mu^2$  is  $O(1)$ . In leading order

$$e^* \cong e f(e^2 \ln v_1^2 / \mu^2)$$



$$\beta^*(e^*) \cong 1/3 \frac{e^{*3}}{16\pi^2}$$

$$\beta(e) \cong -20/3 \frac{e^3}{16\pi^2}$$

we have an equation

$$\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial e} \right) e^* = \beta^*(e^*)$$

or

$$\left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial e} \right) (-24\pi^2/e^{*2}) = 1$$

To solve this equation, we define a new running coupling constant with respect to the full theory where  $\kappa = \ell_n v_1/\mu$ .

$$\frac{d}{d\kappa} \bar{e} = \beta(\bar{e})$$

The solution is

$$\frac{24\pi^2}{e^{*2}} = \frac{24\pi^2}{\bar{e}^2} + \ell_n v_1/\mu$$

or

$$e^{*2} = \frac{e^2}{1 + \frac{7}{16\pi^2} e^2 \ell_n v_1/\mu}$$

This result, when generalized to an appropriate group, agrees with solutions of other people.

In conclusion, we have shown that

(a) Gauge hierarchy is a non-issue, by which we mean that there is no need for fine tuning to separate out the light sector. Once we divide heavy and light

sectors at the tree level, then at every loop order only  $g^2 \ln v_1^2 / \mu^2$  corrections contribute to the light effective parameters.

(b) We have a decoupling theorem. i.e. there exists a light Lagrangian,  $\mathcal{L}^*(g^*, \lambda^*, v^*, \alpha^*, \mu)$ , which can be used to deal with low energy physics.

Besides, the dependence of  $g^*$ ,  $v^*$ ,  $\lambda^*$ ,  $\alpha^*$  on the parameters in the full theory  $(g, \lambda, v_1, v_2, \alpha)$  can be determined by staying in low energy regions.

There exist natural renormalization group equations to sum up the

$\ln v_1^2 / \mu^2$  powers.

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## REFERENCES

1. Yoichi Kasama, David G. Unger and York-Peng Yao, UM HE 80-36; Yoichi Kazama and York-Peng Yao, in preparation.
2. J.C. Pati and A. Salam, Phys. Rev. D8, 1240 (1973); H. Georgi and S.L. Glashow, Phys. Rev. Letts. 32, 438 (1974); H. Georgi, H.R. Quinn and S. Weinberg, Phys. Rev. Letts. 33, 451 (1974).
3. The original gauge hierarchy problem was pointed out by E. Gildener, Phys. Rev. D14, 1667 (1976). For further discussion and references see I. Bars, invited talk at Orbis Scientiae 1979, Coral Gables, Florida; E. Gildener, Phys. Letts. 92B, 111 (1980); K. T. Mahanthappa and D. G. Unger, preprint COLO-HEP-14 and UM HE 79-31, (revised 1980).
4. K. Symanzik, Comm. Math. Phys. 34, 7 (1973); T. Appelquist and J. Carrazone, Phys. Rev. D11, 2856 (1975). There has been a flurry of activity in demonstrating decoupling at the one loop or two loop level. A partial list is B. Ovrut and H. Schnitzer, Phys. Rev. D21, 3369 (1980), D22, 2518 (1980) and Brandeis preprints; T. Hagiwara and N. Nakazawa, preprint HUTP-80/A012; F.T. Hadjiioannou, A.B. Lahanas and C.E. Vayonakis, Phys. Lett. 84B, 427 (1979); G. Senjanovic and A. Sokorac, Nucl. Phys. B164, 305 (1980); M. Yoshimura, Prog. Theor. Phys. 64, 353 and 594 (1980). For different approaches, see S. Weinberg, Phys. Rev. 22, 1414 (1980); P. Binetruy and T. Schucker, CERN preprints TH-2802-CERN, TH-2857-CERN; L. Hall, preprint HUTP-80/A024.
5. A partial list is: D.A. Ross, Nucl. Phys. B140, 1 (1975); A.J. Buras, J. Ellis, M.K. Gaillard, and D.V. Nanopoulos, Nucl. Phys. B135, 66 (1978); T. Goldman and D.A. Ross, Phys. Letts. 84B, 208 (1979); C.H. Llewellyn Smith, G.G. Ross and J.F. Wheeler, Oxford Preprint 1980; I. Antoniadis, C. Bouchiat, J. Iliopoulos, LPTENS 80/21 (1980). For a review and further references, see J. Ellis, M.K. Gaillard, D.V. Nanopoulos and S. Rudaz, Nucl. Phys. B176, 61 (1980).