

Propositions and Paradoxes

by

Dustin Tucker

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Doctoral Committee:

Professor Richmond H. Thomason, Chair
Professor Andreas R. Blass
Associate Professor James P. Tappenden
Assistant Professor Eric P. Swanson

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For Brad, Dad, Mom, Rex, and Wendy

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ABSTRACT

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Dustin Tucker

Chair: Richmond H. Thomason

Propositions are more than the bearers of truth and the meanings of sentences: they are also the objects of an array of attitudes including belief, desire, hope, and fear. This variety of roles leads to a variety of paradoxes, most of which have been sorely neglected. Arguing that existing work on these paradoxes is either too heavy-handed or too specific in its focus to be fully satisfactory, I develop a basic intensional logic and pursue and compare three strategies for addressing the paradoxes, one employing truth-value gaps, one restricting propositional quantification, and one restricting our ability to have attitudes like belief and desire. This results in four distinct resolutions of the paradoxes, all but one of which are novel and all of which receive novel and general implementations. While resolving the paradoxes is of course the ultimate goal, I do not here argue that any one of the resolutions is superior. These paradoxes have been so little studied that my primary goal is only to identify the most fundamental costs and benefits of the various approaches one can take to addressing them. Each resolution I develop has significant drawbacks, which I argue highlight tensions between the different roles propositions play. Past researchers have skirted these tensions, and the issues raised by these paradoxes more generally, by focusing on non-propositional paradoxes, such as the most familiar forms of the Liar paradox. At the least, then, I hope this dissertation establishes that the propositional paradoxes deserve attention not only because of their consequences for intensional logic, but also because of their consequences for our understanding of content, truth, quantification, and a host of mental attitudes.

CHAPTER 1

Propositions and paradoxes

1.1 Paradoxes

Most philosophers are aware that (1) leads to problems, but working through the details of the paradox will be worthwhile.

Epimenides, a Cretan, said that everything any Cretan says is false. (1)

We can suppose that (1) is true. If Epimenides has said that everything any Cretan says is false—if (1) is true—then he has said something false: surely some Cretan sometime has said something true. We encounter problems only when we suppose that (2) is also true.

Everything else any Cretan says is false. (2)

We can again ask whether Epimenides has said something true. Now, however, there is no obvious answer. Epimenides has said (3).¹

Everything any Cretan says is false. (3)

Is (3) true? Suppose that it is—suppose that everything any Cretan says is false. We know from (1) that (3) has been said by a Cretan, so (3) is false. Thus, if (3) is true, then it is false. Anything whose truth implies its falsity must be false, so we have proved that (3) is false. Thus, at least one thing a Cretan says must be true. From our supposition of (2), that one thing must be what Epimenides says according to

¹This is not quite correct: (3) is a sentence of English, and Epimenides did not speak English. I address this below.

(1). That’s (3), so (3) must be true. But now we have proved that (3) is both true and false; something has gone wrong.

1.2 Propositions

The Epimenides paradox is often taken to be merely an unnecessarily complicated form of the Liar paradox, by which I mean the paradox involving (4), the Liar sentence.

(4) is false. (4)

The Liar is similar in structure to the Epimenides paradox. Suppose that (4) is true. Then it is true that (4) is false. That is, (4) is false. So if (4) is true, then it is false. Anything whose truth implies its falsity must be false, so (4) is false. Thus, in virtue of what (4) says, it is false that (4) is false—(4) is true. So we have proved that (4) is both false and true. Again, something has gone wrong somewhere.

In both of these cases, we have a genuine paradox. Beginning with intuitively consistent assumptions—e.g., that (1) and (2) are both true—and taking intuitively valid steps, we have proved contradictions. Thus, there can be no completely satisfactory resolution: some of our intuitions must be reined in somewhere. The task in constructing a resolution of these paradoxes is to minimize such concessions.

One might hope that this task could be at least mostly accomplished for the Epimenides by adapting existing work on the Liar. But the Liar deals only in sentences, while the Epimenides says nothing about sentences at all, and this difference is crucial. The Liar paradox shows that no language can contain its own sentential truth predicate—a predicate that holds of all and only those sentences that are, in fact, true, or satisfied by a model. Because of this, and in order to distinguish this form of the Liar from others, I call this form the *satisfaction Liar*.²

The Epimenides, on the other hand, is not concerned with the satisfaction of sentences at all. (1) involves indirect, rather than direct, discourse: the complement of ‘said’ is a *used*, not mentioned, phrase of English, and thus (1) means something very different from the sentence

Epimenides, a Cretan, said, “Everything a Cretan says is false.”

²Though this is the most commonly addressed analysis of (4), it is not the only one: some authors claim that ‘true’ and ‘false’ are not properly predicated of sentences, but of propositions, which sentences express. (4) then means something along the lines of “(4) expresses a false proposition.” I discuss this form of the Liar in Section 2.3.

Properly speaking, I should have said not that Epimenides said (3), but rather that Epimenides said what (3) says. Or, more awkwardly but more precisely: Epimenides said the meaning (or denotation) of (3). (1) does not claim that Epimenides uttered a sentence of English; it claims that he has said (in the indirect sense) whatever English that-clauses denote.³ I call those denotations *propositions*, and I use slanted type to emphasize that a certain sentence or clause refers to a proposition. Thus, when I say that Epimenides said the denotation of (3), I mean to say that Epimenides said the proposition *that everything a Cretan says is false*.

I remain as neutral as possible on the question of what propositions are. Perhaps they are sets of possible worlds. Perhaps they are primitive abstract entities, or structured objects made up of more primitive entities, such as properties and individuals. Perhaps they are something else entirely. All I suppose is that they are truth-bearers and can be the objects of propositional (or intensional) attitudes, such as speech, belief, thought, desire, and fear. I call the Epimenides and its ilk (as well as some other paradoxes that are crucially concerned with propositions) *intensional* paradoxes. I sometimes call the non-intensional paradoxes *extensional* paradoxes.

Nevertheless, I assume throughout that talk about propositions is never a mistake. The easiest way for this to be true is for propositions to exist, but there may be other ways. All I mean to say is that I am, for the purposes of this dissertation, setting aside the possibility of some sort of nominalist or anti-realist theory of propositions. Perhaps talk about propositions is merely one useful but imperfect way of describing certain phenomena, such as certain mental states. In that case, the paradoxes do nothing more than highlight those circumstances in which such talk breaks down, and the present project is merely an investigation into which portions of such talk we can salvage when as a whole it can be taken no further. But again, for the purposes of this dissertation, I ignore this possibility.

1.3 My goals

These intensional paradoxes have been largely (though by no means entirely) neglected. This dissertation is concerned mainly with five intensional paradoxes, which I present in Chapter 2. In Chapter 3, I discuss three different approaches to the paradoxes. The first brings truth-value gaps to bear on the problem, as Anthony Anderson and Tyler Burge have suggested; the second considers various ways of restricting

³This is a very simplistic treatment of sentences of indirect discourse, but I think that it is harmlessly so. As I explain in the next paragraph, I do not assume very much about the denotations of that-clauses.

the domain of propositional quantification, loosely following many authors, including Russell, William Kneale, Charles Parsons, George Bealer, and Michael Glanzberg; and the third follows Arthur Prior in insisting that sometimes certain propositions simply cannot be the objects of certain propositional attitudes. In Chapter 4, I detail the logic that I use throughout the dissertation and formalize the first four paradoxes. In Chapter 5, I provide novel implementations of the approaches using that logic. In Chapter 6, I turn to the fifth and final paradox. Finally, in Chapter 7, I discuss the benefits and detriments of the different resolutions.

I do not mean to argue for one resolution or another. The intensional paradoxes are, I think, symptomatic of a fundamental inconsistency in our intuitions and assumptions about propositions. By showing how the approaches can be rigorously implemented at a very basic level, we better isolate the problematic intuitions and better understand how they can be reined in to avoid contradictions. I argue that ultimately, tensions between propositions' roles as the objects of attitudes, the bearers of truth, and the meanings of sentences force us to make choices that do not arise when studying only the extensional paradoxes. If, for instance, it is constitutive of a proposition that it has a truth value, then the truth-value gap approach is nonsense. But if one takes the object-of-attitude role to be fundamental, truth-value gaps might seem to be the best option, while the other two approaches begin to lose their appeal.

Neither do I pretend to construct even one satisfactory account of the intensional paradoxes, let alone intensionality more generally. In general, I raise many more questions about these resolutions than I attempt to answer. But very little has been done with these paradoxes, and my aim is to provide the beginning of more systematic research into their resolution, not the end.

CHAPTER 2

Intensional paradoxes

2.1 The complex Epimenides paradox

One particular two-person version of the Epimenides will be useful. Suppose that (5)–(8) are true.

I fear *that everything you hope is false*. (5)

You hope *that everything I fear is true*. (6)

Everything else I fear is true. (7)

Everything else you hope is false. (8)

(5) says that I am, in virtue of my fears, related somehow to the proposition denoted by (9), and (6) says that you are, in virtue of your hopes, related somehow to the proposition denoted by (10).

Everything you hope is false. (9)

Everything I fear is true. (10)

Adapting notation from Section 4.2, I abbreviate “the proposition denoted by (9)” with “[[(9)]]” and so on, and in the interest of simplicity, I use ‘fear’ and ‘hope’ as slightly awkward transitive verbs. Thus, I say things like “I fear [[(9)]] and you hope [[(10)]]” with the understanding that we can, if we wish, rewrite them so as to not presuppose a potentially simplistic and archaic understanding of attitudes.

A situation in which (5)–(8) are true seems to be easy to imagine.⁴ Perhaps (5) I am afraid that all your hopes are bound to be disappointed, (6) you don’t like me

⁴Of course, the particular attitudes and agents are unimportant.

much and hope that I'm living in a nightmare, and both (8) my fear and (7) your hope are at least *almost* correct. Unfortunately for our imaginations, however, we can prove from these four assumptions that (the propositions denoted by) (9) and (10) are each both true and false.

Suppose that $\llbracket(9)\rrbracket$ (the proposition denoted by (9)) is true—that everything you hope is false. By (6), you hope $\llbracket(10)\rrbracket$, so it must then be false. But $\llbracket(10)\rrbracket$ is the proposition *that everything I fear is true*, so since it's false, something I fear must be false. By (7), the only possible witness to this is $\llbracket(9)\rrbracket$, so $\llbracket(9)\rrbracket$ must be false. Thus, on the supposition that $\llbracket(9)\rrbracket$ is true, we have proved that it is false. Anything whose truth implies its falsity must be false,⁵ and so we know that $\llbracket(9)\rrbracket$ is false. $\llbracket(9)\rrbracket$ is the proposition *that everything you hope is false*, so if it's false, then something you hope is true. By (8), the only possible witness to this is $\llbracket(10)\rrbracket$, so $\llbracket(10)\rrbracket$ is true. But, once again, $\llbracket(10)\rrbracket$ says that everything I fear is true. By (5), I fear $\llbracket(9)\rrbracket$, and so it must be true. And now we have proved that $\llbracket(9)\rrbracket$ is both true and false.

We can here see one main difference between intensional and extensional paradoxes. It will not do to simply prohibit sentences like (9) and (10) or insist that they do not denote propositions. As long as the propositions themselves are there to be the objects of attitudes and the bearers of truth, we can derive the contradiction.

2.2 The complex Prior paradox

Arthur Prior focuses exclusively on one family of intensional paradoxes in [Pri61]. One of the paradoxes he briefly discusses involves a situation in which he says *that either what Prior is now saying or whatever Tarski says immediately after, but not both, is false* and Tarski says *that snow is white* and nothing else immediately after Prior speaks [Pri61, p. 29]. We then ask whether Prior has said something true or false: we have a paradox whenever Tarski says something true, but when he says something false, what Prior says becomes a sort of truth-teller, and can be either true or false without trouble.

I will be concerned with a slightly more problematic paradox. To the best of my

⁵Barring truth-value gaps, which we will do until Chapter 3.

knowledge, (11) was true a few years ago.⁶

I am now thinking and thinking only *that everything I am thinking is false if and only if someone else bears a propositional attitude towards something at some time in the future.* (11)

Now we have a paradox as long as anybody else has thought, feared, known, said, asserted, etc. any proposition whatever in the last few years. More specifically, when (11) and (12) are true—and they seem to me to be consistent, at least, perhaps with some rewriting to fix the tenses—we can prove that $\llbracket(13)\rrbracket$ is both true and false.

You said *that snow is black.* (12)

Everything I was then thinking is false if and only if someone else bore a propositional attitude towards something. (13)

Snow is black. (14)

To prove this, we prove that the (proposition denoted by the) left-hand side of the biconditional in (13) is both true and false given that the (proposition denoted by the) right-hand side is true. This derivation parallels the one for the complex Epimenides paradox, so I omit it.

2.3 The expression Liar paradox

One intensional form of the Liar that has received some attention in the Liar literature⁷ is what I call the (strengthened) *expression Liar*.

(15) does not express a true proposition. (15)

Here, we seem to be able to prove that it is both true and false that (15) expresses a true proposition. As with the other intensional paradoxes, though, this does not involve any satisfaction predicates—we are talking about the truth of a proposition, not of a sentence. Again, the derivation is almost identical to what we have seen, so I omit it.

I will not say much about this paradox directly until Section 7.1 because the truth-value gap and quantifier domain restriction resolutions that I develop transfer

⁶This thought was inspired by a form of the paradox presented in [Tho88], which was itself inspired by Prior's paradox.

⁷It is, for instance, the main focus of [Kne72, Par74, Gla04].

immediately to this paradox, once we include an expression relation in our logic. But this paradox is still important, not least because it does not obviously involve any propositional attitudes, and thus poses a problem for any resolution like Prior’s, which relies on restricting propositional attitudes to explain the paradoxes.

This paradox is also important because it is a strengthened paradox. Perhaps we can comfortably say that the sentence, “This sentence expresses a false proposition,” fails to express a proposition at all, or expresses a proposition that lacks a truth value. But no such easy solutions are available for (15). I ignore this issue until Section 7.2.

2.4 The proposition Liar paradox

This paradox is the main concern of [BE87] and [Gro94]. I call it the *proposition* Liar, rather than the much more natural *propositional* Liar, because there are other ways to introduce propositions into the Liar paradox (cf. the expression Liar). It involves a proposition that is identical to its own negation, which of course leads directly to a paradox so long as propositions and their negations must have different truth values. As I explained above, I do not want to make any assumptions about the nature of propositions if I can help it, so while such a proposition might be unusual, I do not want to assume without argument that it is impossible.

2.5 The Appendix B paradox

This paradox was first presented by Russell in the end of Appendix B of [Rus03].⁸ For each set S of propositions, there is the proposition *that every proposition in S is true*; call this proposition s . It is plausible that each s is unique to its S —that if $s = s'$ then $S = S'$. Of course, if this is true—if we can, in fact, construct a unique proposition for each set of propositions—then we have violated Cantor’s theorem, and a contradiction must be waiting in the wings. As Russell observes, we can consider the set of all the ss that are not contained in their corresponding S ; call this set T . We can prove that the proposition *that every proposition in T is true* both is and is not a member of T .

Variations of this paradox replace sets with properties, pluralities, and propositional functions. The last will be my initial focus in Chapter 6, but I discuss other

⁸John Myhill later presented a related paradox in [Myh58], but I restrict myself to Russell’s original paradox and variations thereon. This paradox has always received some attention, including recently [Deu08, Kle01], but it has been largely ignored by authors attempting to construct resolutions of paradoxes.

versions as well.

2.6 Other paradoxes

There are several other intensional paradoxes that I will not directly address because they are, for my purposes, equivalent to the first four paradoxes.

In [And87], Anthony Anderson considers a situation in which Church's favorite proposition is the proposition *that Church's favorite proposition is false*. But he formalizes this with a definite description—the proposition which is Church's favorite is the proposition *that the proposition ...*—and once we have that, any resolution of the complex Epimenides and Prior paradoxes, which involve quantification, should carry over so long as we treat quantifiers and other determiners in relevantly similar ways.

We can also construct an intensional version of Grelling's classic paradox, according to which a predicate is heterological if and only if it denotes a function from predicates to propositions and the result of applying it to itself denotes a false proposition. But this is not importantly different from the expression Liar.

Finally, Yablo's paradox from [Yab93] involves an infinite sequence of sentences, each of which says that all the later sentences are false. As with the proposition Liar, I want to focus on a purely intensional form of Yablo's paradox—a paradox that arises from assumptions about propositional identity. Roughly, the idea is to have an infinite series of propositions, each of which is the proposition that every proposition later in the series is false. One might think that this involves propositional quantification, and is thus not importantly different than the complex Epimenides and Prior paradoxes. Formally, though, we can quantify over the arguments to a propositional function and thereby talk generically about a class of propositions without quantifying over propositions directly. I revisit this issue in Section 5.3.3.

CHAPTER 3

Three strategies for resolving the paradoxes

3.1 Truth-value gaps

In light of similar approaches to the satisfaction Liar, one might say that the relevant propositions in the intensional paradoxes are neither true nor false. I am not the first to consider truth-value gaps as a resolution of the intensional paradoxes. Of course, truth-value gaps have been heavily discussed in the literature on the satisfaction Liar—for some of the earliest discussions, see, e.g., [vF68, Kri75, MW76, Her76]—but their application to the intensional paradoxes has also been proposed, though not developed in any detail, by Tyler Burge [Bur79, Bur84],⁹ and developed in great detail by Anthony Anderson in [And87].

Nevertheless, I do not want to work with Anderson’s version, because he uses a version of Alonzo Church’s Logic of Sense and Denotation. This is not a problem in itself, of course, but I think that it disguises the generality of the resolution. One need not assume anything like a Fregean treatment of propositions in order to resolve the paradoxes with truth-value gaps; my insistence on assuming as little as possible about the nature of propositions is intended to show this.

Once we give that up, though, we lose all of Anderson’s developments. We need a principled, systematic way of ensuring that all and only the problematic propositions lack truth values, even in the more complex multi-agent cases. Anderson himself does not discuss these paradoxes, but I do not mean to say that Anderson has not provided, or at any rate could not construct such a resolution of the complex Epimenides and Prior paradoxes. But once we have left the Logic of Sense and Denotation behind, we must start from scratch. These details will be all the more important because I use the truth-value gap treatment of these more involved paradoxes as the basis of

⁹Burge also proposes some contextualist machinery, which I discuss in Chapter 7.

the other resolutions.

At an informal level, once we allow propositions to lack truth values, it is easy to see how even the multi-agent paradoxes should be resolved. Recall, for instance, the complex Epimenides paradox.

I fear *that everything you hope is false*. (5)

You hope *that everything I fear is true*. (6)

Everything else I fear is true. (7)

Everything else you hope is false. (8)

Everything you hope is false. (9)

Everything I fear is true. (10)

In Section 2.1, we began our derivation of a contradiction from (5)–(8) by supposing that the object of my fear, $\llbracket(9)\rrbracket$, was true, which entailed that $\llbracket(10)\rrbracket$ was false, which entailed that $\llbracket(9)\rrbracket$ was false. At this point, we concluded that $\llbracket(9)\rrbracket$ was not true, on pain of contradiction, and so had to be false. It is the last step that this approach blocks—if propositions can lack truth values, then we cannot conclude that $\llbracket(9)\rrbracket$ is false from our proof that it is not true. Indeed, we can prove that it is not false through reasons very similar to the second half of the proof in Section 2.1, and so, according to this approach, $\llbracket(9)\rrbracket$ is gappy (and so is $\llbracket(10)\rrbracket$).

There are a number of ways one can treat truth-value gaps. I use the strong Kleene scheme extended to quantification following [Kri75], so that a conjunction is true iff both of its conjuncts are true, false iff one of its conjuncts is false, and neither otherwise; an existential quantification is true iff one of its instances is true, false iff all of its instances are false, and neither otherwise; and so on. I spell this out precisely in Section 4.2, but for now it is worth observing its consequences for the four paradoxes under immediate consideration. The proposition Liar is immediately resolved, because no proposition that is identical to its own negation can have a truth value. And the expression Liar is almost as simple: $\llbracket(15)\rrbracket$ will always lack a truth value as well.

(15) does not express a true proposition. (15)

Since a universal quantification cannot be true unless every instance is true, and can be false only when there is a witness to its falsity, it is also nearly immediate, given the reasoning above, that both $\llbracket(9)\rrbracket$ and $\llbracket(10)\rrbracket$ will always lack truth values when (5)–(8) are true. And finally, we will never be able to assign a truth value to

[[13]] when (11) and (12) are true, though of course there will be no trouble assigning a truth value to [[14]].

I am now thinking and thinking only *that everything I am thinking is false if and only if someone else bears a propositional attitude towards something at some time in the future.* (11)

You said *that snow is black.* (12)

Everything I was then thinking is false if and only if someone else bore a propositional attitude towards something. (13)

Snow is black. (14)

One important foundational task facing any truth-value gap resolution, which I mentioned above when discussing Anderson’s resolution, and which is easy to overlook if one begins with the work on the satisfaction Liar, is to systematically determine which propositions lack truth values and which ones do not. In the interest of finding the least intrusive resolution, the last part is important: we do not want more truth-value gaps than are necessary. Kripke addressed this task, among others, for the satisfaction Liar in [Kri75]. But we cannot straightforwardly adopt his fixed-point construction, because it relies on the structure of sentences. For instance, he relies on being able to distinguish the sentences “Snow is white” and “‘Snow is white’ is true” from each other, but I do not want to assume that the proposition *that snow is white* is different from the proposition *that the proposition that snow is white is true*, because as I said in Section 1.2, I want to be as silent as possible on the nature of propositions. Happily, we will see in Section 5.2 that we can avoid such an assumption, given the setup in Chapter 4.

Of course, the truth-value gap approach raises other questions. When, in response to the satisfaction Liar, one proposes that certain sentences lack truth values, one might imagine those sentences simply failing to express propositions in the first place. In contrast, when one proposes that propositions themselves lack truth values, one risks resolving the problems for propositions *qua* objects of attitudes only by introducing problems for them *qua* bearers of truth. But I want to set this issue aside until Section 7.3, as I explained in Section 1.3. Certainly any fully satisfactory account of intensionality that relies on truth-value gaps to address the intensional paradoxes must have an explanation of what it means for truth-bearers to lack truth values. But the more foundational task from the preceding paragraph is no less important, and answering it also turns out to be crucial for my implementations of the

other resolutions.

3.2 Quantifier domain restrictions

An alternative to allowing truth-value gaps is insisting that the domain of propositional quantification is restricted (perhaps contextually, perhaps in some other fashion; I briefly discuss this issue in Section 7.2). One of the earliest earliest instances of this approach is the ramified theory of types, which I discuss at length below. This theory was first developed by Bertrand Russell, with the most influential statement of it appearing in [WR13], and a handful of authors [Chu93, Kap95, Tho88] have since proposed ramification as a resolution of (some of the) intensional paradoxes.

Other authors have considered restricting propositional quantification to avoid paradoxes, but their suggestions are consistently either too narrow in their focus or not detailed enough to guarantee the right results in the trickier multi-agent paradoxes. George Bealer [Bea82, pp. 99–100] and Sten Lindström in [Lin03b] fall into the latter category: while they suggest that the correct resolution of the paradoxes is to say that the propositional quantifiers are restricted, they do not explain how this restriction goes. Bealer, for instance, says that context fixes the domain, but does not explain how. Lindström provides more detail in [Lin03a], but the resulting theory is similar enough to ramification that the concerns I raise below about the latter apply to his proposal as well. Bealer later says that he actually prefers an approach “based on fixed-point ideas developed by [Fitch, Gilmore, Feferman, and Kripke]” [Bea94, p. 162],¹⁰ which sounds something like the resolution I ultimately develop (as well as something like Michael Glanzberg’s resolution of the expression *Liar*, which I explain below). Again, though, he does not provide any details of a resolution based on these ideas.

Michael Glanzberg’s resolution of the expression *Liar* [Gla04] makes use of Kripkean fixed points to implement a restriction on the domain of quantification, and so might be thought of as a spiritual successor to Bealer’s suggestion. But Glanzberg’s approach is actually more of a spiritual successor to ideas presented by William Kneale and Charles Parsons [Kne72, Par74], and is thus much more narrow in its focus than Bealer’s. Kneale and Parsons both think that truth is ultimately properly predicated of propositions, not sentences, and thus that the *Liar* sentence (4) is really to be taken

¹⁰See [Bea94] for the references.

as the expression Liar sentence (15).

(4) is false. (4)

(15) does not express a true proposition. (15)

Glanzberg takes the same approach. But all three authors focus heavily on propositions as what sentences express. This fact plays a crucial role in Glanzberg's resolution of the expression Liar: he proposes that we identify the propositions that must be kept out of the domain of quantification by way of a predicate that holds of *sentences* and Kripkean fixed points.

Glanzberg does not think that this is a problem, writing,

If we apply truth to propositions, then for most applications of the truth predicate we will make, we will have to work with a proposition presented via a sentence that expresses it. Though we might sometimes manage to name a proposition in some more or less direct way, we usually wind up saying something like 'the proposition expressed by sentence *s* is true', or more informally, 'what she said when he [sic] said *s* is true'. [Gla04, p. 31]

But here I think that Glanzberg is underestimating our ability to talk directly about propositions. We can easily do so with indirect discourse, as illustrated by the abundance of that-clauses in this dissertation, often written in slanted text to avoid confusion. Unfortunately, because of this focus on sentences and the expression relation, Glanzberg's resolution is silent on all the non-expression paradoxes—one cannot even formulate them in his logic. Although Kneale and Parsons do not provide as much detail as Glanzberg, their proposals are similarly limited.

I do not mean to say that Glanzberg's proposal is hopeless in the face of the other intensional paradoxes. He might very well extend his system to address them. But I think that to do so he would have to make assumptions about the structure of mental states, or a language of thought, or something along those lines, and these would not be in line with my attempt to remain as neutral as possible about propositions and the surrounding issues. I do not think that we need to make such assumptions simply to resolve the paradoxes, and so I do not want to pursue Glanzberg's approach.

That said, Glanzberg's emphasis of sentences is not without its benefits. In particular, it allows him to give a nice contextualist explanation of several important issues surrounding the expression Liar. I discuss this in more detail in Section 7.2.

3.2.1 Ramification

Ramification¹¹ is fundamentally a theory of quantification. It says that no proposition can quantify over itself (or over propositions that can quantify over it, etc.).¹² Slightly more carefully, so as to not assume that propositions themselves contain quantifiers, it says that there is an infinite hierarchy of orders of propositions, and that if a sentence (or, even more carefully, a formula \mathbf{P}) denotes a proposition of order n , quantifiers in the sentence (\mathbf{P}) can range over only orders $m < n$. I often speak loosely of propositions themselves quantifying with the understanding that such talk can be avoided if necessary. I also assume, contrary to Russell’s version of ramification but in line with Church’s, that orders are cumulative, so that propositions of order n also appear in all orders $m > n$.

Ramification avoids the first three intensional paradoxes: the complex Epimenides, the complex Prior, and the expression Liar. Consider the complex Epimenides once again.

I fear *that everything you hope is false*. (5)

You hope *that everything I fear is true*. (6)

Everything else I fear is true. (7)

Everything else you hope is false. (8)

Everything you hope is false. (9)

Everything I fear is true. (10)

In the argument in Section 2.1, it was crucial that $\llbracket(9)\rrbracket$ was in the range of the quantifier in (10) and $\llbracket(10)\rrbracket$ was in the range of the quantifier in (9). But this is not possible in a ramified theory of propositions. Ramification requires that all quantifiers be restricted to an order, so we must replace (9) and (10) with (9′) and (10′). (We must replace (5)–(8) as well, but for simplicity’s sake I omit the replacements here and suppose that the quantifiers in (7) and (8) range over orders higher than n and m .)

Everything of order n you hope is false. (9′)

Everything of order m I fear is true. (10′)

¹¹Throughout, I work with Church’s formulation of ramified type theory [Chu76]. I do not mean to claim that this is a perfectly accurate reconstruction of Russell’s own theory, but it is close enough for my purposes.

¹²It also concerns quantification over propositional functions, which I address in Chapter 6.

$\llbracket(9')\rrbracket$ and $\llbracket(10')\rrbracket$ are of at least orders $n + 1$ and $m + 1$ respectively, and so one will definitely be outside the range of the quantifier in the other. Suppose, for instance, that $n > m$ (the other two cases are equally straightforward). Since I am letting orders be cumulative, anything of order $m + 1$ will be of order n , and so the quantifier in $(9')$ ranges over $\llbracket(10')\rrbracket$. When we suppose that $\llbracket(9')\rrbracket$ is true, then, we can still conclude that $\llbracket(10')\rrbracket$ is false—that something I fear of order m is false. But the proof can go no farther: $\llbracket(9')\rrbracket$ cannot be in order m (because it must be of at least order $n + 1$ and $n > m$).

I am not quite satisfied with ramification in its traditional form, as I explain in the next subsection. But before moving on, I want to set aside (until Section 7.3) two worries about ramification of any form. One major issue involves quantification. In a ramified theory of propositions, even the more flexible one I describe in the next subsection, it could very well be impossible to quantify over every proposition—there might be no universal order, which contains every proposition. This should be cause for concern, because we seem to be able to make claims about every proposition. According to ramification, for instance, every proposition has an order. But if there is no universal order, then the preceding sentence cannot have expressed what I intended, and the object of that intention is not what you might have thought, and the content of that thought is not what you tried to make it, and so on.

This is not a new objection to ramification, but it is still an important one. It is, in a way, the problem of providing a universal metalanguage for our theory combined with the challenges facing unrestricted quantification.¹³ But the problems are compounded when we are dealing with propositions, supposedly the fundamental bearers of truth.

One might also worry about the orders. Where do the n and m in $(9')$ and $(10')$ come from? Are they contextually supplied? If so, what is contextually sensitive? Something in our mental state? And what can propositions look like if they can sometimes differ in only the orders over which they quantify (more carefully: in only the orders over which quantifiers in sentences that express them quantify)?

Both of these issues are important, and a proponent of ramification must ultimately address them. But as I have emphasized, my purpose here is to see how far the approach can be taken (and in particular to show that we can give ramification a lighter, more discerning touch), and to identify the costs such an approach must pay at the most fundamental level. I do not mean to argue that ramification is definitely

¹³For an early discussion of universal metalanguages, see [Fit64]. For the problems with quantification, see, e.g., [RU06].

the correct approach to the paradoxes. As with the concerns about truth-value gaps, I revisit these issues in Section 7.3.

3.2.2 Compressed ramification

As we saw, ramification resolves the complex Epimenides, and it equally well handles the complex Prior and the expression Liar. (More on the proposition Liar in Section 3.2.4.) But it is heavy-handed. It prohibits all circular quantification all the time, and most circular quantification is unproblematic in most circumstances. This observation is analogous to one of Kripke's [Kri75, §II], and the present proposal, compressed ramification, stands to traditional ramification in much the same way that Kripke's single truth predicate stands to Tarski's hierarchy. One of the troubles with Tarski's hierarchy of languages is that it prohibits all talk about truth from within an object language. It is, in a way, the ultimate truth-value gap theory: within a language, the truth predicate for that language is *entirely* gappy. This makes it impossible to capture the contingent nature of certain paradoxical circumstances that Kripke discusses, and his solution is to compress the hierarchy of truth predicates into one, leaving gaps only as necessary.

Similarly, ramification is the ultimate restriction on propositional quantification: within an order, *nothing* can quantify over that order. This makes it impossible to capture the contingent nature of the present paradox: the fear and hope reported in (5) and (6) are *risky* but not inherently paradoxical. Any restrictions on propositional quantification that are motivated by paradoxes like the complex Epimenides ought to be similarly responsive to the facts; we should not block (9) and (10) from quantifying over each others' denotations except when we must to avoid contradictions—when (5)–(8) are true.

The present idea is to compress the orders as much as possible, forcing propositions up the hierarchy only when leaving them lower in it leads to contradictions.¹⁴ We still insist that quantifiers be restricted to particular orders, but allow the contents of the orders to be contingent and variable, with each proposition appearing (with a few exceptions explained below) in the lowest order it can. Most propositions will thus be of order 0, the lowest possible order. This includes propositions that involve propositional quantification, such as the proposition that every proposition of order

¹⁴This is not quite analogous to Kripke's approach, as he has only one truth predicate, while I still have an infinite hierarchy of orders. The true analogue is the special case of my proposal in which we restrict ourselves to quantification over order 0 and ignore (or even eliminate) all other orders.

17 is self-identical. It also includes $\llbracket(9')\rrbracket$ and $\llbracket(10')\rrbracket$ whenever at least one of (5)–(8) is false.

Speaking very loosely about propositions and quantification, here is one way to think about these orders: in a standard ramified hierarchy, if a proposition x is in a higher order, it means both (i) that x can quantify over more propositions and (ii) that fewer propositions can quantify over x . In compressed ramification, only the second is retained. The domains x can quantify over are no longer tied to x 's own order; x being in order 0 means only that every proposition can unproblematically quantify over x .

Even when (5)–(8) are true, compressed ramification will usually disagree with traditional ramification. According to the latter, $\llbracket(9')\rrbracket$ is of order $n + 1$ and $\llbracket(10')\rrbracket$ is of order $m + 1$. But this is not actually necessary to resolve the paradoxes. Recall how ramification avoided the paradox when $n > m$: $\llbracket(9')\rrbracket$ could not be in the domain of the quantifier in (10'), and so we could not carry the argument out all the way. But this only requires that $\llbracket(9')\rrbracket$ be of order $m + 1$, not $n + 1$. More generally, irrespective of the relationship between n and m , to resolve this paradox we need to require only that $\llbracket(9')\rrbracket$ and $\llbracket(10')\rrbracket$ both be of order $\min(n, m) + 1$; this is what compressed ramification ensures.

Actually, there are other ways we can avoid this paradox with compressed orders; this is why I had to qualify my above gloss of the compressed orders as making each proposition appear “in the lowest order it can.” The derivation of a contradiction will be blocked as long as one of $\llbracket(9')\rrbracket$ and $\llbracket(10')\rrbracket$ is kept out of the range of the other sentence's quantifier, so any of the three following options will work (along with infinitely many uninteresting others).

- (i) $\llbracket(9')\rrbracket$ and $\llbracket(10')\rrbracket$ are both of order $\min(n, m) + 1$.
- (ii) $\llbracket(9')\rrbracket$ is of order 0 and $\llbracket(10')\rrbracket$ is of order $n + 1$.
- (iii) $\llbracket(9')\rrbracket$ is of order $m + 1$ and $\llbracket(10')\rrbracket$ is of order 0.

Of these three, I think (i) is clearly the best option. After all, (9') and (10') are perfectly symmetrical; it would be strange if our resolution allowed us to treat $\llbracket(9')\rrbracket$ and $\llbracket(10')\rrbracket$ differently.

Still, we must be cautious because of paradoxes like the complex Prior.

I am now thinking and thinking only *that everything I am
thinking is false if and only if someone else bears a propositional
attitude towards something at some time in the future.* (11)

You said *that snow is black*. (12)

*Everything I was then thinking is false if and only if
someone else bore a propositional attitude towards something.* (13)

Snow is black. (14)

In a ramified theory of types, we must replace (13) with (13'). (Again, I suppress the replacement for (11), assuming that the quantifier in the first “only” ranges over an order higher than n and m .)

*Everything of order n I was then thinking is false if and only if
someone else bore a propositional attitude towards something of order m .* (13')

We have a contradiction as long as $\llbracket(13')\rrbracket$ is of order n or less and $\llbracket(14)\rrbracket$ is of order m or less. Thus, as before, we can avoid the paradox in any of three ways:

- (iv) $\llbracket(13')\rrbracket$ and $\llbracket(14)\rrbracket$ are both of order $\min(n, m) + 1$,
- (v) $\llbracket(13')\rrbracket$ is of order 0 and $\llbracket(14)\rrbracket$ is of order $n + 1$, and
- (vi) $\llbracket(13')\rrbracket$ is of order $m + 1$ and $\llbracket(14)\rrbracket$ is of order 0.

Here, though, I think (vi) is the clear favorite: making the object of my thought be order $n + 1$ is preferable to making the object of your speech—and indeed the objects of every attitude every other person has had for the past three years—be order $m + 1$.

Getting these results—respecting the symmetry in the complex Epimenides case and the asymmetry in the complex Prior case—is, I think, crucial for any satisfactory refinement of ramification. This is why we cannot do something simple, like beginning with the ramified orders and then pushing every proposition down as far as it can go (that process would, it seems, select (iii) when $n > m$, and could perform similarly badly if (14) contained a quantifier). And it is why Sections 5.3 and 5.4 take a detour through truth-value gaps.

3.2.3 Two domains

A different version of the quantifier domain restriction approach, which I briefly develop in Chapter 5, is to return to the unramified theory, and to say that there are only two domains of quantification. The main domain, which contains every proposition, is the domain that most quantifiers range over. Only quantifiers in problematic propositions, such as $\llbracket(9)\rrbracket$ when (5)–(8) are true, are restricted, and then they are restricted to quantifying over only the unproblematic propositions. In

a way, this approach is more conservative than compressed ramification, because it leaves the vast majority of propositions completely untouched. But it does not have even the ad hoc machinery of orders to explain why, for instance, the quantifier in (9) suddenly ceases to range over $\llbracket(9)\rrbracket$ and $\llbracket(10)\rrbracket$ when (5)–(8) are true. These issues only compound the questions raised at the end of Section 3.2.1.

3.2.4 The proposition Liar paradox

The quantifier domain restriction approach has nothing to say about the proposition Liar paradox, as there are no quantifiers involved in imagining a proposition that is identical to its own negation. (Again, intensional forms of Yablo’s paradox can be similar—while they must involve quantifiers, they need not involve *propositional* quantifiers, as I show in Section 5.3.3.) Unfortunately, this is one place where I think that a proponent of a quantifier domain restriction approach must appeal to a theory of propositions, and insist that such a proposition is not possible. If we allow such a proposition, then we must have some other resolution of the paradox, such as a truth-value gap resolution, and it is likely that any such resolution will resolve all the intensional paradoxes while leaving the domains of quantification untouched, obviating the need for the domain restriction approach in the first place.

Perhaps this is not such a large price to pay. Certainly Russell would not have been pleased with an inherently circular proposition like the one in the proposition Liar, and most contemporary theories of propositions, structured or otherwise, also prohibit it. The Yablo propositions are less straightforward, but the sequence itself is still circularly defined, and plausibly viciously so.¹⁵ One might hope to find grounds for prohibiting viciously circular definitions without prohibiting circular quantification that is merely potentially vicious, so as to retain the quantifier domain restriction approach to the contingent paradoxes while ruling out the necessary ones from the start.

Still, I am not convinced that this is the right approach to take to the proposition Liar and Yablo paradoxes. My purpose in this subsection is merely to argue out that we cannot be silent on the nature of propositions forever, if we want to embrace a domain-restriction resolution.

¹⁵Not all circularity is vicious, however—see, e.g., [Ant00]—and I do not pretend to have a good test.

3.3 Restrictions on attitudes

Arthur Prior's answer to the paradoxes is to say that sometimes people cannot bear certain propositional attitudes to certain propositions, despite our intuitions otherwise. That is, Prior says that what people can say, think, assert, fear, etc. depends in part on matters of fact. In one important sense, this is the least radical approach of them all, because it says absolutely nothing about propositions themselves, and it embraces the derivations in Chapter 2: for Prior, (5)–(8) really are inconsistent. Thus, where the other approaches are forced to look for flaws in the reasoning in the proofs that led to the contradictions, Prior needs only to explain why our intuitions about the consistency of the paradoxical assumptions are incorrect.

Unfortunately, Prior's approach is in other respects quite radical indeed. He writes,

How, we all want to cry out, can what a man is thinking and even what a man *can* be thinking on a given occasion, depend on what number is written on the other side of a door? [Pri61, p. 30].¹⁶

But he is quite explicit about his position on these paradoxes:

At this point I must confess that all I can say to allay the misgivings expressed in the past four sections is that so far as I have been able to find out, my terms are the best at present offering. I have been driven to my conclusion very unwillingly, and have as it were wrested from Logic the very most that I can for myself and others who feel as I do. So far as I can see, we must just accept the fact that thinking, fearing, etc., because they are attitudes in which we put ourselves in relation to the real world, must from time to time be oddly blocked by factors in that world, and we must just let Logic teach us where these blockages will be encountered. [Pri61, p. 32]

The position, then, is that the connection between mental states and propositions can sometimes fail, and that such failures can depend on just about anything. How these failures occur is entirely obscure, but this obscurity is probably necessary in the absence of a satisfactory theory from the philosophy of mind of how the connection succeeds in non-paradoxical cases, and at any rate seems similar to obscurity about the bases of truth-value gaps and restricted domains of propositional quantification.

One strange (but non-fatal, I think) consequence of this approach is that even *attempts* to think, fear, etc. must sometimes be blocked: if, for instance, everything

¹⁶Here, Prior is referring to a paradox in which Mr. X thinks in Room 7 at 6 o'clock *that everything thought in Room 7 at 6 o'clock is false*, thinks nothing else, and is the only person in the room at that time.

else a Cretan tries to say is false, then Prior must claim that Epimenides cannot try to say *that everything a Cretan tries to say is false*. Similarly, we cannot describe all paradoxical circumstances in terms of what would usually happen. It may be true in the complex Epimenides paradox, for instance, that although I do not fear *that everything you hope is false*, I am nevertheless in a fear-like mental state that would in most circumstances relate me to that proposition. But talk about such mental states will not always work: I cannot be in that mental state if you hope *that every proposition to which I would be related in most circumstances in virtue of my fear-like mental states is true* (so long as every other proposition to which I am related in that way is true).

Indeed, sometimes I cannot even act as though I have a certain fear, if you hope something about the propositions that I act as though I fear; sometimes Epimenides cannot even say words that would in normal circumstances express a certain proposition. Of course Prior is not committed to saying that I cannot perform actions or that Epimenides cannot utter words, but he is, I think, committed to saying that sometimes, certain actions are not even what we would normally do if we had certain attitudes, and certain words are not even such that they would normally express certain propositions. This last consideration about words not even normally expressing propositions might be especially worrying, because it seems to involve semantics rather than mental attitudes, but I think it is just another version of the expression Liar paradox, which I discuss at the end of this section.

Perhaps we can swallow these consequences. Prior, at any rate, seemed to be able to, and the approach does still allow us to retain whatever theory of propositions we want. But now we need to know which of (5)–(8) is false, and Prior seems to think that it depends on the order in which you and I attempt to have the relevant attitudes. Consider another case that Prior discusses, the idea for which he attributes to Jean Buridan:

Suppose there are four people who on a certain occasion say one thing each. A says that 1 and 1 are 2—a truth. B says that 2 and 2 are 4—a truth. C says that 2 and 2 are 5—a falsehood. And D says that exactly as many truths as falsehoods are uttered on this occasion. [Pri61, p. 20]

As with the other paradoxes, we can prove that what D has said is both true and false. Prior says that in the situation described by the first four sentences of this passage, the fifth and final sentence simply cannot be true—D simply cannot say *that exactly as many truths as falsehoods are uttered on this occasion*. Similarly, concerning the paradox involving Tarski that began Section 2.2, Prior writes,

[S]ince *snow is white*, Tarski cannot say that *snow is white* ... if he says nothing else immediately after I have said that either what I am then saying or what he will say immediately after, but not both of these things, is false. ... [Pri61, p. 29]

In both of these cases Prior seems to rely on a first-come, first-served principle: D cannot say what she tried to say because A, B, and C got there first; Tarski cannot say *that snow is white* because Prior got there first. The problem with this should be abundantly clear in light of the complex Prior paradox: Prior is committed to saying that for the past few years I have been the only thinking, hoping, fearing, asserting, knowing, etc. being (unless somebody beat me to the punch!), and surely this is about as unacceptable a result as one could ask for from a resolution of these paradoxes.

In Section 5.5, I propose to use truth-value gaps, as I do for the quantifier domain restriction resolutions, to identify the attitudes that must be blocked, so that, for instance, (11) is false once (12) (or something similar) is true. Even this is not perfect, though, as one is left to wonder what happens in the interim: Do I succeed in thinking [(13)] until you say [(14)] and then suddenly cease to have thought it? That will not do without some elaboration, since we will otherwise be able to construct new paradoxes in which I think about it being true at a particular time that I thought something or other. But it is also strange to say that it is never true that I thought [(13)], since before anybody else bore an attitude to anything, there was no paradox. Perhaps if determinism is true, whether a thought succeeds or fails could depend on how things will turn out, but if determinism is false, even that will not work.

Once more, I want to set these questions aside until Section 7.3. Before leaving our three approaches to the paradoxes, though, we should consider how Prior can handle the expression *Liar*. The proposition *Liar* is, of course, out of his reach, because it involves no propositional attitudes, and I suspect that any proponent of this sort of resolution will have to follow the domain-restriction theorists in prohibiting propositions that are necessarily viciously circular. But there are no propositional attitudes involved in the expression *Liar* either. Still, insofar as attitudes can be blocked because they put ourselves in relation to the rest world, perhaps meaning (and even meaning-in-normal-circumstances, as discussed above) can be blocked because it puts language in relation to the rest of the world. If so, then Prior can handle the expression *Liar* paradox as well as he can handle the complex Prior and Epimenides paradoxes.

CHAPTER 4

Logic

I use a slightly simplified form of Church’s Russellian Simple Type Theory [Chu74]. It is an intensional logic in the loose sense that the formulas we are ultimately interested in are of a type p of propositions rather than a type t of truth values: I translate every English sentence with a formula of type p , but I also translate explicit references to propositions, such as that-clauses, with formulas of type p . The same formulas can thus appear both as arguments to connectives and as arguments to predicates representing propositional attitudes. For instance, I translate (9) and (10) as

$$\forall x[Hx \rightarrow \neg x] \text{ and} \tag{16}$$

$$\forall x[Fx \rightarrow x] \tag{17}$$

respectively.

One could, following [Tho80], use different symbols for this logic, such as \rightsquigarrow alongside \rightarrow . This would be especially helpful if one wished to include an extensional logic along with the intensional part—if one wished to have both a type p and a type t . Since I need only an intensional logic, though, I do not bother with such a notational distinction; following Church, I use the familiar symbols with the understanding that they do not have their familiar truth-functional interpretations.

One consequence of this is that some of the constructions involving truth values are slightly long-winded. Notions of satisfaction and consistency, for instance, must derive from restrictions placed on \mathcal{T} and \mathcal{F} , the sets of true and false propositions in the models.

Finally, the logic I use does not include an explicit truth or falsity predicate. I thus make no distinction between, e.g., the proposition *that snow is white* and the proposition *that the proposition that snow is white is true*—I used the bare ‘ x ’ in

(16) and (17) rather than something like ‘ Tx ’. This treatment of truth is not novel: it is the one Arthur Prior uses [Pri61] and it has been defended as the appropriate treatment of (at least many instances of) the English ‘true’ [GCB74]. But I think that my decision to translate English sentences in this way is not a substantial one; the constructions in Chapter 5 should work just as well with an explicit truth predicate.

4.1 Language

The set TS of *type symbols* of our language \mathcal{L} is the smallest set such that $p \in TS$ and for all $\tau, \sigma \in TS$, $\langle \tau, \sigma \rangle \in TS$. Intuitively, p is the type of propositions, $\langle p, p \rangle$ is the type of functions from propositions to propositions, and so on.¹⁷ The *proper symbols* are the constants $\neg^{\langle p, p \rangle}$, $\wedge^{\langle p, \langle p, p \rangle \rangle}$, $\vee^{\langle p, \langle p, p \rangle \rangle}$, $\rightarrow^{\langle p, \langle p, p \rangle \rangle}$, $\leftrightarrow^{\langle p, \langle p, p \rangle \rangle}$; for each $\tau \in TS$, the constants $=^{\langle \tau, \langle \tau, p \rangle \rangle}$ and an infinite alphabet of variables x^τ , y^τ , etc.; and several additional constants with superscript $\tau \in TS$ that I introduce as needed to formalize the paradoxes. The *improper symbols* are \forall , \exists , $[$, and $]$.¹⁸ I omit superscripts when no ambiguity thereby arises.

Any proper symbol with superscript τ is a *well-formed formula* of type τ . If \mathbf{P} is a well-formed formula of type $\langle \tau, \sigma \rangle$ and \mathbf{Q} is a well-formed formula of type τ , then $\mathbf{P}\mathbf{Q}$ (often written $\mathbf{P}(\mathbf{Q})$) is a well-formed formula of type σ . If \mathbf{P} is a well-formed formula of type p and \mathbf{x} is a variable, $\forall \mathbf{x}[\mathbf{P}]$ and $\exists \mathbf{x}[\mathbf{P}]$ are well-formed formulas of type p . Φ is the set of well-formed formulas of \mathcal{L} . I employ standard abbreviations, using $\mathbf{P}(\mathbf{Q}, \mathbf{R})$ for $\mathbf{P}^{\langle \tau, \langle \sigma, \rho \rangle \rangle} \mathbf{Q}^\tau \mathbf{R}^\rho$ and $\mathbf{P} \mathcal{C} \mathbf{Q}$ for $\mathcal{C}(\mathbf{P}, \mathbf{Q})$ when \mathcal{C} is a binary connective or relation symbol. I sometimes insert brackets to disambiguate scope; such disambiguations are necessary only because of my abbreviations. When I omit brackets and parentheses, I assume that juxtaposition has the narrowest scope possible, followed by relation symbols like $=$. Thus, for instance, given constants $a^{\langle p, p \rangle}$, b^p , and c^p we should read $ab = ac \wedge c$ as $[a(b) = a(c)] \wedge c$, not something like $a(b = a(c \wedge c))$.

I have already used ‘ \mathbf{P} ’, ‘ \mathbf{Q} ’, etc. as metavariables over well-formed formulas and ‘ \mathbf{x} ’ as a metavariable over variables, and I continue to do so, sometimes with superscripts to restrict their ranges. Also as I already have, I allow symbols and formulas

¹⁷One could, of course, easily include other types, such as a type i of individuals, but we will not need them.

¹⁸One could make \mathcal{L} more general by including λ as an improper symbol and replacing \forall and \exists with, for each $\tau \in TS$, constants $\forall^{\langle \langle \tau, p \rangle, p \rangle}$ and $\exists^{\langle \langle \tau, p \rangle, p \rangle}$. But we will not need λ -abstraction, so I omit it for simplicity’s sake. I have taken both quantifiers and all the connectives as primitive, rather than defining some in terms of others, because we are dealing with propositions, not truth values. I do not want to assume that, for instance, conjunctive propositions are identical to negations of certain disjunctive ones, although I do not rule out such identities.

to name themselves, omitting corner quotes. But I avoid using formulas to name their denotations; I speak of not the proposition $\mathbf{P} \wedge \mathbf{Q}$ but the proposition denoted by $\mathbf{P} \wedge \mathbf{Q}$ or the proposition $\llbracket \mathbf{P} \wedge \mathbf{Q} \rrbracket$, where $\llbracket \cdot \rrbracket$ is the interpretation function introduced in Section 4.2.

4.2 Models

A *model* \mathfrak{M} is a quadruple $\langle \mathcal{D}, \mathcal{T}, \mathcal{F}, \llbracket \cdot \rrbracket \rangle$ where

- \mathcal{D} is a set of sets (*domains*) \mathcal{D}_τ , one for each $\tau \in TS$;
- \mathcal{T} and \mathcal{F} are disjoint subsets of \mathcal{D}_p , whose purpose I explain below; and
- $\llbracket \cdot \rrbracket$ is an interpretation function, a function $\Phi \rightarrow \bigcup \mathcal{D}$ (Φ , recall, is the set of well-formed formulas of our language \mathfrak{L}) taking each well-formed formula \mathbf{P}^τ to an element of \mathcal{D}_τ .

When f is a partial function $\mathcal{D}_p \rightarrow \{0, 1\}$, I sometimes write ‘ \mathfrak{M}_f ’ for a model in which $\mathcal{T} = \{x \in \mathcal{D}_p : f(x) = 1\}$ and $\mathcal{F} = \{x \in \mathcal{D}_p : f(x) = 0\}$. (I use ‘ \mathbf{x} ’, ‘ \mathbf{y} ’, etc. throughout as variables over elements of $\bigcup \mathcal{D}$.)

As I have said, I make no assumptions about the nature of propositions; this translates into placing no restrictions on \mathcal{D}_p . The other \mathcal{D}_τ , as well as $\llbracket \cdot \rrbracket$, are entirely standard. For any $\langle \tau, \sigma \rangle \in TS$, $\mathcal{D}_{\langle \tau, \sigma \rangle} = \mathcal{D}_\sigma^{\mathcal{D}_\tau}$, the set of functions from \mathcal{D}_τ to \mathcal{D}_σ . When \mathbf{P} is a lone proper symbol, $\llbracket \mathbf{P} \rrbracket$ is unrestricted. When \mathbf{P} has the form $\forall \mathbf{x}[\mathbf{Q}]$ or $\exists \mathbf{x}[\mathbf{Q}]$, $\llbracket \mathbf{P} \rrbracket$ is mostly unrestricted: we need to ensure only that identity is preserved under change of bound variables and substitution of identicals, so that, e.g., we have $\llbracket \forall x[Ax] \rrbracket = \llbracket \forall y[By] \rrbracket$ if we also have $\llbracket A \rrbracket = \llbracket B \rrbracket$. I discuss one way to achieve this in Section 5.1. Of course, we must have $\llbracket \mathbf{P}(\mathbf{Q}) \rrbracket = \llbracket \mathbf{P} \rrbracket(\llbracket \mathbf{Q} \rrbracket)$; $\llbracket \cdot \rrbracket$ is thus entirely determined by its (mostly arbitrary) values for proper symbols and quantified formulas.

\mathcal{T} and \mathcal{F} can be thought of as containing the true and false propositions respectively. In assuming that \mathcal{T} and \mathcal{F} are disjoint, I assume that there are no truth-value gluts; this requirement could be relaxed if one wanted to pursue paraconsistent resolutions. In not requiring \mathcal{T} and \mathcal{F} to jointly exhaust \mathcal{D}_p , I allow truth-value gaps. I do however require that identity propositions never be gappy and always have the correct truth values—I require that for any $\tau \in TS$ and $\mathbf{x}, \mathbf{y} \in \mathcal{D}_\tau$ we have $\llbracket =^{\langle \tau, \langle \tau, p \rangle} \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}$ if $\mathbf{x} = \mathbf{y}$ and $\llbracket =^{\langle \tau, \langle \tau, p \rangle} \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{F}$ otherwise. This requirement does not conflict with any of the resolutions I develop, because I am not considering resolutions that involve making propositional identities contingent. It is slightly risky,

because we invite unresolved paradoxes every time we prohibit truth-value gaps, but I have not discovered any paradoxes that rely on propositional identities being always determinate.

Finally, we must place some restrictions on our models to ensure that our truth values and truth-value gaps are well-behaved. As explained in Section 3.1, I use the strong Kleene scheme (extended to quantification following [Kri75]). In fact, I require only that \mathcal{T} and \mathcal{F} follow one direction of the strong Kleene scheme; the other direction will be established during the construction of the intended models. Thus, I insist only that if a conjunction is in \mathcal{T} , then both conjuncts must be in \mathcal{T} , and if it is in \mathcal{F} , then one of the conjuncts must be in \mathcal{F} ; that if a universal quantification is in \mathcal{T} , then every instance must be in \mathcal{T} , and if it is in \mathcal{F} , then an instance must be in \mathcal{F} ; and so on. Since this goes in only one direction, it does not require that, e.g., a conjunction be in \mathcal{T} if both its conjuncts are.

We can state this restriction explicitly but much more tediously as follows. For any variable \mathbf{x}^τ and $z \in \mathcal{D}_\tau$, let $\llbracket \cdot \rrbracket^{\mathbf{x}/z}$ be that interpretation function just like $\llbracket \cdot \rrbracket$ except that $\llbracket \mathbf{x} \rrbracket^{\mathbf{x}/z} = z$.¹⁹ Then for all $\mathbf{x}, \mathbf{y} \in \mathcal{D}_p$, I require that

- (a) if $\llbracket \neg \rrbracket(\mathbf{x}) \in \mathcal{T}$, then $\mathbf{x} \in \mathcal{F}$;
- (b) if $\llbracket \neg \rrbracket(\mathbf{x}) \in \mathcal{F}$, then $\mathbf{x} \in \mathcal{T}$;
- (c) if $\llbracket \vee \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}$, then $\mathbf{x} \in \mathcal{T}$ or $\mathbf{y} \in \mathcal{T}$;
- (d) if $\llbracket \vee \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{F}$, then $\mathbf{x}, \mathbf{y} \in \mathcal{F}$;
- (e) if $\llbracket \wedge \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}$, then $\mathbf{x}, \mathbf{y} \in \mathcal{T}$;
- (f) if $\llbracket \wedge \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{F}$, then $\mathbf{x} \in \mathcal{F}$ or $\mathbf{y} \in \mathcal{F}$;
- (g) if $\llbracket \rightarrow \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}$, then $\mathbf{x} \in \mathcal{F}$ or $\mathbf{y} \in \mathcal{T}$;
- (h) if $\llbracket \rightarrow \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{F}$, then $\mathbf{x} \in \mathcal{T}$ and $\mathbf{y} \in \mathcal{F}$;
- (i) if $\llbracket \leftrightarrow \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{T}$, then $\mathbf{x}, \mathbf{y} \in \mathcal{T}$ or $\mathbf{x}, \mathbf{y} \in \mathcal{F}$;
- (j) if $\llbracket \leftrightarrow \rrbracket(\mathbf{x})(\mathbf{y}) \in \mathcal{F}$, then either $\mathbf{x} \in \mathcal{T}$ and $\mathbf{y} \in \mathcal{F}$ or $\mathbf{x} \in \mathcal{F}$ and $\mathbf{y} \in \mathcal{T}$;
- (k) if $\llbracket \forall \mathbf{x}^\tau \rrbracket[\mathbf{P}] \in \mathcal{T}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for all $z \in \mathcal{D}_\tau$;
- (l) if $\llbracket \forall \mathbf{x}^\tau \rrbracket[\mathbf{P}] \in \mathcal{F}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for some $z \in \mathcal{D}_\tau$;
- (m) if $\llbracket \exists \mathbf{x}^\tau \rrbracket[\mathbf{P}] \in \mathcal{T}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for some $z \in \mathcal{D}_\tau$; and
- (n) if $\llbracket \exists \mathbf{x}^\tau \rrbracket[\mathbf{P}] \in \mathcal{F}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for all $z \in \mathcal{D}_\tau$.

¹⁹We are sure to have a unique such function because an interpretation function is entirely determined by the arbitrary values it assigns to the proper symbols and quantificational formulas.

4.3 Formalizing the paradoxes

4.3.1 The complex Epimenides paradox

At this point, we have the resources to reconstruct the paradoxes in our system. We can capture the essence of (5)–(8) with (18) and (19), letting (7) and (8) be vacuously satisfied. (16) and (17) are reproduced from above (with a change in variable).

$$\forall y[Hy \rightarrow \neg y] \tag{16}$$

$$\forall y[Fy \rightarrow y] \tag{17}$$

$$\forall x[Fx \leftrightarrow x = \forall y[Hy \rightarrow \neg y]] \tag{18}$$

$$\forall x[Hx \leftrightarrow x = \forall y[Fy \rightarrow y]] \tag{19}$$

The paradox is that if we suppose $\llbracket(18)\rrbracket, \llbracket(19)\rrbracket \in \mathcal{T}$ and eliminate truth-value gaps, requiring $\mathcal{T} \cup \mathcal{F} = \mathcal{D}_p$, then we can prove that $\llbracket(16)\rrbracket$ and $\llbracket(17)\rrbracket$ are in both \mathcal{T} and \mathcal{F} . Given that \mathcal{T} and \mathcal{F} must be disjoint, this amounts to saying that there are no gapless models with $\llbracket(18)\rrbracket, \llbracket(19)\rrbracket \in \mathcal{T}$.

The proof parallels the informal derivation in Section 2.1. Given $\llbracket(18)\rrbracket, \llbracket(19)\rrbracket \in \mathcal{T}$, suppose $\llbracket(16)\rrbracket \in \mathcal{T}$. Then by clauses (k), (g), and (b) of the restriction above, we know that for all $z \in \mathcal{D}_p$, if $\llbracket H \rrbracket(z) \in \mathcal{T}$, then $z \in \mathcal{F}$. Since $\llbracket(19)\rrbracket \in \mathcal{T}$, we know by clauses (k) and (i) and our insistence that identity statements be well-behaved that we have $\llbracket H \rrbracket(\llbracket(17)\rrbracket) \in \mathcal{T}$, and so we have $\llbracket(17)\rrbracket \in \mathcal{F}$. By clauses (l) and (h), we thus know that for some $z \in \mathcal{D}_p$ we have $\llbracket F \rrbracket(z) \in \mathcal{T}$ and $z \in \mathcal{F}$. Since $\llbracket(18)\rrbracket \in \mathcal{T}$, we know by clauses (k) and (i) and the well-behavedness of identity that $\llbracket(16)\rrbracket$ is the only $z \in \mathcal{D}_p$ for which we have $\llbracket F \rrbracket(z) \in \mathcal{T}$, and so we have $\llbracket(16)\rrbracket \in \mathcal{F}$, which contradicts our initial supposition, given that \mathcal{T} and \mathcal{F} are disjoint.

$\llbracket(16)\rrbracket$, then, cannot be in \mathcal{T} , and so must be in \mathcal{F} if we do not allow truth-value gaps. But from here, by similar reasoning, we can prove that it is in \mathcal{T} , contra the disjointness of \mathcal{T} and \mathcal{F} but this time with no suppositions.

Truth-value gaps, of course, block this proof by prohibiting the inference from $\llbracket(16)\rrbracket \notin \mathcal{T}$ to $\llbracket(16)\rrbracket \in \mathcal{F}$. Ramification blocks the proof by restricting clauses (k)–(n), as we see in Section 5.3.

4.3.2 The other paradoxes

As before, I omit the derivations of contradictions for the remaining paradoxes because they parallel the proof for the complex Epimenides.

Replacing “someone else bears a propositional attitude towards” with “you say” in (11), we can roughly capture (11)–(14) with (20)–(23) respectively.

$$\forall x \left[Tx \leftrightarrow x = [\forall y [Ty \rightarrow \neg y] \leftrightarrow \neg \exists y [Sy]] \right] \quad (20)$$

$$Ss \quad (21)$$

$$\forall y [Ty \rightarrow \neg y] \leftrightarrow \neg \exists y [Sy] \quad (22)$$

$$s \quad (23)$$

Then the trouble is that when we have $\llbracket(20)\rrbracket, \llbracket(21)\rrbracket \in \mathcal{T}$ and $\mathcal{T} \cup \mathcal{F} = \mathcal{D}_p$ we can prove both $\llbracket(22)\rrbracket \in \mathcal{T}$ and $\llbracket(22)\rrbracket \in \mathcal{F}$.

As I have said, I set the expression Liar aside until Section 7.1.

The proposition Liar is perhaps the simplest to capture with our logic: we simply introduce one more constant a^p and assume $\llbracket(24)\rrbracket \in \mathcal{T}$.

$$a = \neg a \quad (24)$$

CHAPTER 5

Four resolutions

5.1 A toy theory of propositions

In Section 4.2, I said that we were assuming that denotations were conserved under change of variables and substitution of identicals. To prove the results in the remainder of this chapter, we need to also ensure that we do not have any bizarre identities, such as $\llbracket(18)\rrbracket = \llbracket a \rrbracket$, which would make it impossible to get the complex Epimenides off the ground.

One simple way to satisfy these assumptions is to prohibit nearly all propositional identities. (I discuss a way to substantially relax this in Section 5.6.) As an illustration, we can take our cue from Anderson’s models of Church’s Logic of Sense and Denotation [And80].²⁰ Let \mathcal{D}_p be the set of equivalence classes of closed formulas of type p for a particular equivalence relation \mathcal{R} .²¹ For the most restrictive account of identity that is of interest for our purposes, \mathcal{R} is (the reflexive, transitive closure of) the relation that holds between all and only (i) formulas that vary only in their bound variables (suitably and familiarly restricted to avoid relating problematic pairs like $\forall x[x = \forall y[x = y]]$ and $\forall y[y = \forall x[x = y]]$); (ii) formulas \mathbf{P} and \mathbf{P}' where (a) \mathbf{P}' is the result of replacing a part \mathbf{Q} of \mathbf{P} with \mathbf{R} and (b) either $\mathcal{R}(\mathbf{Q}, \mathbf{R})$ or $\mathcal{R}(\mathbf{R}, \mathbf{Q})$; and (iii) a and $\neg a$. Our predicates then denote functions from sets of formulas to sets of formulas. $\llbracket F \rrbracket$, for instance, is the function that takes a set of formulas and returns the set of formulas constructed by prepending F to the elements of the argument. That is (momentarily insisting on corner quotes and recalling that officially there are

²⁰Though its spirit is similar, there are differences between my construction and Anderson’s. He prohibits circular synonymies, while I do not, and my construction is simpler, as I am not trying to capture a Fregean theory of sense and denotation. At the same time, though, my construction is much more fragile, as I explain in note 22.

²¹I have not defined ‘closed’, but the familiar definition suffices. Of course I do not mean to suggest that propositions are actually sets of formulas of \mathcal{L} .

no parentheses in \mathfrak{L}), for a set of formulas \mathfrak{x} , $\llbracket F \rrbracket(\mathfrak{x}) = \{\ulcorner FP \urcorner : \mathbf{P} \in \mathfrak{x}\}$.²²

5.2 Truth-value gaps

Call a model \mathfrak{M}' an *extension* of a model \mathfrak{M} iff $\mathfrak{D} = \mathfrak{D}'$, $\llbracket \cdot \rrbracket = \llbracket \cdot \rrbracket'$, $\mathcal{T} \subseteq \mathcal{T}'$, and $\mathcal{F} \subseteq \mathcal{F}'$, and call an extension \mathfrak{M}' of \mathfrak{M} *proper* iff either $\mathcal{T} \subset \mathcal{T}'$ or $\mathcal{F} \subset \mathcal{F}'$. Recalling from Section 4.2 that \mathfrak{M}_f is a model in which \mathcal{T} and \mathcal{F} are defined by a partial function f , \mathfrak{M}_g is then a proper extension of \mathfrak{M}_f iff $\mathfrak{D}_f = \mathfrak{D}_g$, $\llbracket \cdot \rrbracket_f = \llbracket \cdot \rrbracket_g$, and $f \subset g$. Call \mathfrak{M}_f *maximal* iff it has no proper extensions—iff for any $g \supset f$, \mathfrak{M}_g would violate one of clauses (a)–(n) above. In effect, a model is maximal iff no more propositions can be made true or false without violating our restrictions on truth-value assignments. According to a truth-value gap resolution, each maximal model wholly determines one possible way the world can be.

That maximal models exist, if any models do, is almost immediate. (This should not be a surprise, as they are closely analogous to maximal consistent sets of sentences.) Let F be a set of partial functions $f : \mathcal{D}_p \rightarrow \{0, 1\}$ such that for all $f \in F$, \mathfrak{M}_f is a model. It suffices to show that if F is totally ordered by \subset (which guarantees that $\bigcup F$ is a function), then $\mathfrak{M}_{\bigcup F}$, abbreviated \mathfrak{M}_F , is also a model—also treats truth values as clauses (a)–(n) require.

This is straightforward. Suppose, for instance, that for some $\mathfrak{x}, \mathfrak{y} \in \mathcal{D}_p$ we have $\llbracket \wedge \rrbracket(\mathfrak{x})(\mathfrak{y}) \in \mathcal{T}_F$. We need to show $\mathfrak{x}, \mathfrak{y} \in \mathcal{T}_F$ in order to show that \mathfrak{M}_F satisfies clause (e). But we have this: if $\llbracket \wedge \rrbracket(\mathfrak{x})(\mathfrak{y}) \in \mathcal{T}_F$, then for some $f \in F$ we have $\llbracket \wedge \rrbracket(\mathfrak{x})(\mathfrak{y}) \in \mathcal{T}_f$, whence, since \mathfrak{M}_f satisfies clause (e) by supposition, we have $\mathfrak{x}, \mathfrak{y} \in \mathcal{T}_f$, whence we have $\mathfrak{x}, \mathfrak{y} \in \mathcal{T}_F$ immediately. Every other case proceeds in exactly the same fashion: if something is in \mathcal{T}_F or \mathcal{F}_F , then it is in some \mathcal{T}_f or \mathcal{F}_f respectively, whence whatever the relevant clauses require of \mathfrak{M}_F holds of \mathfrak{M}_f , whence it holds of \mathfrak{M}_F as well.

Knowing that maximal models exist is a step in the right direction, but it does not quite get us where we want to be. Ultimately, we need maximal models in which our paradoxical assumptions are true—models in which $\llbracket (18) \rrbracket, \llbracket (19) \rrbracket, \llbracket (20) \rrbracket, \llbracket (21) \rrbracket \in \mathcal{T}$. Truth values are preserved in extensions—taking extensions is monotonic—so this amounts to needing just one such model, maximal or otherwise.

As I said above, we cannot in general be certain that there are models in which $\llbracket (18) \rrbracket, \llbracket (19) \rrbracket, \llbracket (20) \rrbracket, \llbracket (21) \rrbracket \in \mathcal{T}$ because we cannot in general be certain that we do

²²Such a fine-grained treatment of propositions opens the door for a violation of Cantor's theorem like the one in the Appendix B paradox: if we have a unique formula of type p for every $\mathfrak{x} \in \mathcal{D}_{\langle p, p \rangle}$, then this construction falls apart. But our current language \mathfrak{L} is safe, because it can express only four of the (infinitely many) elements of $\mathcal{D}_{\langle p, p \rangle}$ — $\llbracket F \rrbracket$, $\llbracket H \rrbracket$, $\llbracket T \rrbracket$, and $\llbracket S \rrbracket$.

not have bizarre propositional identities. But we can be certain of this given the theory of propositions from Section 5.1. Working with that theory, all $\llbracket(18)\rrbracket \in \mathcal{T}$ requires, for instance, is that propositions of the form $\llbracket F \rrbracket(\mathbf{x})$ be true or false. Of course, clauses (a)–(n) say nothing about propositions of this form, and so we can be certain that there will be models with $\llbracket(18)\rrbracket \in \mathcal{T}$ —the model in which \mathcal{F} is empty and $\mathcal{T} = \{\llbracket F \rrbracket(\mathbf{x}) : \mathbf{x} \in \mathcal{D}_p\}$ is one example. Similar considerations apply to the other three paradoxical assumptions, and so we can be certain that there are maximal models in which all four are true.

It remains to show that in such a model none of $\llbracket(16)\rrbracket$, $\llbracket(17)\rrbracket$, $\llbracket(22)\rrbracket$, and $\llbracket a \rrbracket$ will have a truth value. But this follows immediately from the observations in Section 4.3.

5.3 Compressed ramification

The idea is to begin with a maximal, gappy model from the previous section; restrict the domain of quantification to just those propositions that have been assigned truth values; and push the other propositions up to a higher order, where we can (eventually) assign them truth values. The truth-value gaps disappear at the end of the day, but along the way they help us get exactly the results, both symmetrical and asymmetrical, described in Section 3.2.2. Consider, for instance, the desired asymmetrical result for the complex Prior paradox—option (vi) from Section 3.2.2. Nothing can stop $\llbracket s \rrbracket$ from having a truth value, given our toy model, because clauses (a)–(n) place no restrictions on it. Thus, $\llbracket s \rrbracket$ will always be in the domain of quantification, and so the only way we will ever resolve the paradox is by keeping $\llbracket(22)\rrbracket$ out of the domain until we are at a sufficiently high order.

To begin, we must enrich our language and models to make room for infinitely many domains of propositional quantification. Let \mathcal{I} be a non-empty set of ordinals with no greatest element. The natural numbers will do, although transfinite orders should pose no problems. The only change to our language is to replace each variable \mathbf{x}^p with the variables \mathbf{x}^i , $i \in \mathcal{I}$. These new variables are treated as having superscript p for syntactic purposes.

We now need to add orders to the variables in (16)–(19), (20), and (22). In the interest of not (further) overworking italic Latin letters, I use ‘ α ’, ‘ β ’, etc. as variables over elements of \mathcal{I} when we will need to refer to the orders later.

$$\forall y^\alpha [Hy \rightarrow \neg y] \tag{16'}$$

$$\forall y^\beta [Fy \rightarrow y] \quad (17')$$

$$\forall x^\gamma [Fx \leftrightarrow x = \forall y^\alpha [Hy \rightarrow \neg y]] \quad (18')$$

$$\forall x^\delta [Hx \leftrightarrow x = \forall y^\beta [Fy \rightarrow y]] \quad (19')$$

$$\forall x^\eta [Tx \leftrightarrow x = [\forall y^\epsilon [Ty \rightarrow \neg y] \leftrightarrow \neg \exists y^\zeta [Sy]]] \quad (20')$$

$$\forall y^\epsilon [Ty \rightarrow \neg y] \leftrightarrow \neg \exists y^\zeta [Sy] \quad (22')$$

(18') says that I fear (and fear only, through order γ) that everything you hope of order α is false (i.e., I fear $\llbracket(16')\rrbracket$), and (19') says that you hope (and hope only, through order δ) that everything I fear of order β is true (i.e., you hope $\llbracket(17')\rrbracket$). Similarly, (20') says that I think (and think only, through order η) that everything of order ϵ is false iff you do not say anything of order ζ (i.e., I think $\llbracket(22')\rrbracket$). For simplicity, suppose $\alpha < \beta < \gamma < \delta$ and $\epsilon < \zeta < \eta$. The goal of compressed ramification is then to have $\llbracket(16')\rrbracket$ and $\llbracket(17')\rrbracket$ in order $\alpha + 1$, $\llbracket(22')\rrbracket$ in order $\epsilon + 1$, and $\llbracket s \rrbracket$ in order 0; β , γ , δ , ζ , and η can mostly drop out of the picture.

Notice in particular that I do not impose orders on arguments or outputs of functions— $\llbracket F \rrbracket$, for instance, can still take and return propositions of any order. This is different from traditional ramification. But it is natural if one does not want to assume in advance that propositions have particular orders.

The changes to the models are only slightly more involved. A model \mathfrak{M} is now a quintuple $\langle \mathcal{D}, \mathcal{Q}, \mathcal{T}, \mathcal{F}, \llbracket \cdot \rrbracket \rangle$, where \mathcal{Q} is a set of domains of propositional quantification \mathcal{Q}_i , $i \in \mathcal{I}$.²³ To make the \mathcal{Q}_i actually function as domains of quantification, we need to change clauses (k)–(n) of our restriction on \mathcal{T} and \mathcal{F} when the superscript τ on \mathbf{x} is some $i \in \mathcal{I}$ (when $\tau \notin \mathcal{I}$, the original clauses suffice).

(kⁱ) If $\llbracket \forall \mathbf{x}^i [\mathbf{P}] \rrbracket \in \mathcal{T}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for all $z \in \mathcal{Q}_i$;

(lⁱ) if $\llbracket \forall \mathbf{x}^i [\mathbf{P}] \rrbracket \in \mathcal{F}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for some $z \in \mathcal{Q}_i$;

(mⁱ) if $\llbracket \exists \mathbf{x}^i [\mathbf{P}] \rrbracket \in \mathcal{T}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for some $z \in \mathcal{Q}_i$; and

(nⁱ) if $\llbracket \exists \mathbf{x}^i [\mathbf{P}] \rrbracket \in \mathcal{F}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for all $z \in \mathcal{Q}_i$.

5.3.1 First attempt

It is tempting to think that constructing the orders is almost trivial. Suppose we have a maximal model \mathfrak{M} from Section 5.2. In effect, this is a model in which

²³We need not require that we have $\mathcal{Q}_i \subseteq \mathcal{D}_p$ or $\llbracket \mathbf{x}^i \rrbracket \in \mathcal{Q}_i$. The former is covered by the construction of the models, and the latter is not necessary: the order of a variable only matters when it is bound, and then its initial value is unimportant.

$\mathcal{Q}_i = \mathcal{D}_p$ for all $i \in \mathcal{I}$. Thus, we know we can assign truth values to the propositions in \mathcal{T} or \mathcal{F} even when everything quantifies over them, and so they can be in order 0 without trouble.

Why, then, can we not simply let $\mathcal{T} \cup \mathcal{F}$ be \mathcal{Q}_0 ? We could repeat the process for all the subsequent orders: keeping our new \mathcal{Q}_0 fixed, we simply find another maximal model, with orders 1 and up unrestricted; make the new $\mathcal{T} \cup \mathcal{F}$ our \mathcal{Q}_1 ; and so on.

The trouble is with the first step. When we cut \mathcal{Q}_0 down to $\mathcal{T} \cup \mathcal{F}$ from \mathcal{D}_p , we cannot retain the original truth-value assignment. To see this, consider the proposition $\llbracket \exists x[Fx] \rrbracket$. When $\mathcal{Q}_0 = \mathcal{D}_p$, we have $\llbracket \exists x[Fx] \rrbracket \in \mathcal{T}$. That is, it is true that I fear something. But when we move to $\mathcal{Q}_0 = \mathcal{T} \cup \mathcal{F}$, this proposition must be false, because the only thing I fear, $\llbracket (16') \rrbracket$, is now outside the domain of quantification.

This would not be troublesome on its own—we could simply adjust the truth-value assignment and then move on to constructing \mathcal{Q}_1 . But there may be propositions that can be assigned truth values only when other propositions are in the domain of quantification. That is, moving to $\mathcal{Q}_0 = \mathcal{T} \cup \mathcal{F}$ might introduce new truth-value gaps.

We can see this by considering another paradox inspired by Prior. Prior's original paradox was problematic only when snow was white. My thought was problematic only in situations in which somebody else bore an attitude towards a proposition. And indeed we can make a paradox contingent on any fact, such as whether I fear something. More formally, consider the following two formulas. Glossing over the orders, (26) says that the only A -proposition is the proposition *that every A-proposition is true iff I fear something*, and (25) denotes that proposition.

$$\forall y^{\alpha'}[Ay \rightarrow y] \leftrightarrow \exists y^0[Fy] \tag{25}$$

$$\forall x^{\beta'}[Ax \leftrightarrow x = [\forall y^{\alpha'}[Ay \rightarrow y] \leftrightarrow \exists y^0[Fy]]] \tag{26}$$

Let a model $\mathfrak{M}_{f,\mathcal{Q}}$ be the model like \mathfrak{M}_f with $\mathcal{Q}_0 = \mathcal{Q}$. With this notation, our initial maximal, gappy model is $\mathfrak{M}_{f,\mathcal{D}_p}$, and the model after we have cut \mathcal{Q}_0 down to $\mathcal{T} \cup \mathcal{F}$ would be $\mathfrak{M}_{f,\text{Dom}(f)}$, where $\text{Dom}(f)$ is the domain of f . The trouble is that $\mathfrak{M}_{f,\text{Dom}(f)}$ is not guaranteed to be a model. If we have $\llbracket (26) \rrbracket \in \mathcal{T}$, then $\llbracket (25) \rrbracket$ is truth valueless whenever we have $\llbracket \exists y^0[Fy] \rrbracket \in \mathcal{F}$ —whenever we have $\llbracket (16') \rrbracket \notin \mathcal{Q}_0$. And we have this in $\mathfrak{M}_{f,\text{Dom}(f)}$ so long as we have $\llbracket (18') \rrbracket, \llbracket (19') \rrbracket \in \mathcal{T}$ —so long as the complex Epimenides paradox gets off the ground. Thus, $\mathfrak{M}_{f,\text{Dom}(f)}$ will not do if $\llbracket (26) \rrbracket \in \mathcal{T}$; we must instead find some maximal $\mathfrak{M}_{g,\text{Dom}(f)}$, which does not assign a truth value to $\llbracket (25) \rrbracket$. This means we need to cut \mathcal{Q}_0 down further to $\text{Dom}(g)$, so that it does not include $\llbracket (25) \rrbracket$. And, of course, once we have done so, we may need to cut it down

still further due to other propositions, which were assigned truth values in $\mathfrak{M}_{f, \mathcal{D}_p}$ and $\mathfrak{M}_{g, \text{Dom}(f)}$ but cannot be assigned truth values in any $\mathfrak{M}_{h, \text{Dom}(g)}$. And so on.²⁴

5.3.2 Second attempt

Luckily, following through on the “and so on” at the end of the last paragraph can be monotonic in the sense that the domain of quantification never grows, and so we can be guaranteed to have fixed points that can serve as our order 0. We must, however, change our approach slightly: $\mathfrak{M}_{f, \mathcal{Q}}$ must be the model like \mathfrak{M}_f with $\mathcal{Q}_i = \mathcal{Q}$ for all $i \in \mathcal{I}$. The plan is thus to cut every domain down, opening the later ones back up to \mathcal{D}_p only after we have fixed \mathcal{Q}_0 .

To see why this is necessary, imagine that the superscript 0 in (25) and (26) were a 1, and suppose that $\min(\alpha + \beta) \geq 1$. This ensures that $\llbracket (26) \rrbracket \in \mathcal{T}$ is paradoxical during the construction of \mathcal{Q}_1 , and so we must have $\llbracket (25) \rrbracket \notin \mathcal{Q}_1$. Now, if we were not cutting every subsequent order down while constructing \mathcal{Q}_0 , we would have $\llbracket \exists y^1[Fy] \rrbracket \in \mathcal{T}$ during that construction, whence we would be able to assign a truth value to $\llbracket (25) \rrbracket$, whence we would have $\llbracket (25) \rrbracket \in \mathcal{Q}_0$. But the orders are supposed to be cumulative, so that is a contradiction. Ultimately, the idea behind this change is that we cannot know in advance where a proposition will first enter the hierarchy of orders, and so if we discover that a proposition must be kept out of the order we are constructing, the only safe approach (in light of these Prior-inspired paradoxes) is to assume, for the duration of that construction, that it cannot be in any subsequent order either.²⁵

With that change made, call a model $\mathfrak{M}_{f, \mathcal{Q}}$ *intermediate* if $\text{Dom}(f) \subseteq \mathcal{Q}$: intermediate models are models in which every proposition that has a truth value is also in the domain of quantification.²⁶ The argument for the existence of maximal models from Section 5.2 carries over without alteration: we can be certain that if there are any intermediate models for some \mathcal{Q} , then there is a maximal intermediate $\mathfrak{M}_{f, \mathcal{Q}}$.

Now let \mathcal{G} be a function on subsets of \mathcal{D}_p which, given some set $\mathcal{Q} \subseteq \mathcal{D}_p$, returns $\text{Dom}(f)$, where $\mathfrak{M}_{f, \mathcal{Q}}$ is a maximal intermediate model (and returns \emptyset if there are no

²⁴There may also be propositions that can be assigned truth values only after the domain has been cut down. If we prepend a \neg to $\exists y^0[Fy]$ in (26) and (25), then $\llbracket (25) \rrbracket$ will be such a proposition when $\llbracket (26) \rrbracket \in \mathcal{T}$. In the interest of simplicity, I do not try to expand \mathcal{Q}_0 to include such propositions.

²⁵Notice also that in light of this, truth values will change as the orders grow: for any $i \in \mathcal{I}$ and $x \in \mathcal{Q}_{i+1} \setminus \mathcal{Q}_i$, we must have $\llbracket \exists y[y = x] \rrbracket^{x/x} \in \mathcal{F}$ during the construction of \mathcal{Q}_i but $\in \mathcal{T}$ during the construction of \mathcal{Q}_{i+1} .

²⁶I think that focusing on models of this sort is not necessary for avoiding the paradoxes, but it is simplifying. It guarantees, for instance, that there is no expansion of the sort described in note 24.

intermediate models at all with domain of quantification \mathcal{Q}).²⁷ Thus, for instance, working with the f and g from the end of Section 5.3.1, we could have $\mathcal{G}(\mathcal{D}_p) = \text{Dom}(f)$ and $\mathcal{G}(\text{Dom}(f)) = \text{Dom}(g)$.

\mathcal{G} is guaranteed to have fixed points—we always have $\mathcal{G}(\mathcal{Q}) \subseteq \mathcal{Q}$ —but the most common is bound to be \emptyset . It thus remains to show that if we begin with \mathcal{D}_p and require only that we have $\llbracket(18')\rrbracket, \llbracket(19')\rrbracket, \llbracket(20')\rrbracket, \llbracket(21)\rrbracket \in \mathcal{T}$, then \mathcal{G} will always return a \mathcal{Q} for which there are intermediate models. The challenge here is analogous to that of showing that there are models with $\llbracket(18)\rrbracket, \llbracket(19)\rrbracket, \llbracket(20)\rrbracket, \llbracket(21)\rrbracket \in \mathcal{T}$ in Section 5.2. As with that case, we cannot guarantee that, given arbitrary true propositions (or arbitrary propositional identities), \mathcal{G} never returns \emptyset . For instance, if we insist on looking at models with $\llbracket\exists x[Ax]\rrbracket \in \mathcal{T}$, then we will have $\mathcal{G}(\mathcal{G}(\mathcal{D}_p)) = \emptyset$. But also as in Section 5.2, a simple examination of $\llbracket(18')\rrbracket, \llbracket(19')\rrbracket, \llbracket(20')\rrbracket, \llbracket(21)\rrbracket$ should put these fears to rest: these assumptions require only that certain propositions of the form $\llbracket F \rrbracket(\mathbf{x})$, $\llbracket H \rrbracket(\mathbf{x})$, $\llbracket T \rrbracket(\mathbf{x})$, and $\llbracket S \rrbracket(\mathbf{x})$ be assigned particular truth values, and truth-value assignments to such propositions are never jeopardized as the domain of quantification shrinks, because (once again barring bizarre identities with the model from Section 5.1) our clauses (a)–(n) place no substantive restrictions on them.

The proposal is to use fixed points of \mathcal{G} , starting from $\mathcal{G}(\mathcal{D}_p)$, for our orders. A (non-empty) fixed point of \mathcal{G} is a domain of quantification \mathcal{Q} such that there is a maximal intermediate model $\mathfrak{M}_{f,\mathcal{Q}}$ with $\text{Dom}(f) = \mathcal{Q}$. We can take such a \mathcal{Q} as our order 0—it is a set of propositions whose presence in every domain of quantification never causes problems. Then we begin anew, working on order 1. The notation quickly becomes unwieldy, but let $\mathfrak{M}_{f,\mathcal{Q},\mathcal{Q}'}$ be a model with $\mathcal{Q}_0 = \mathcal{Q}$ and $\mathcal{Q}_i = \mathcal{Q}'$ for all $i > 0$. The idea is that, given our fixed point \mathcal{Q} from above, we start over with a maximal $\mathfrak{M}_{f,\mathcal{Q},\mathcal{D}_p}$, in which all the orders aside from 0 are again unrestricted. We then cut each \mathcal{Q}_i , $i > 0$, down to $\text{Dom}(f)$, and then to $\text{Dom}(g)$ for some maximal intermediate $\mathfrak{M}_{g,\mathcal{Q},\text{Dom}(f)}$, and so on until we reach another fixed point.²⁸ That gives us order 1. And then we do it again for every subsequent order, letting any transfinite order be the union of every lower order as usual.

As we proceed through the orders, we have progressively fewer truth-value gaps. If the only paradoxical assumptions that we care about are (18'), (19'), (20'), and (21), then we could very well have orders $\min(\alpha, \epsilon)$ and lower be identical and orders

²⁷It turns out that maximal models can disagree about the location of truth-value gaps—we can have maximal models $\mathfrak{M}_{f,\mathcal{Q}}$ and $\mathfrak{M}_{g,\mathcal{Q}}$ with $\text{Dom}(f) \neq \text{Dom}(g)$. Constructing a suitable \mathcal{G} thus requires the axiom of choice. Since we are ultimately interested in fixed points, it is probably best to always choose \mathcal{Q} if possible.

²⁸Technically, we need a new \mathcal{G} , which holds order 0 fixed and varies only subsequent orders.

$\max(\alpha, \epsilon) + 1$ and up be identical, so long as our truth-value assignment didn't happen to make any other paradoxical assumptions true accidentally (recall that we assumed $\alpha < \beta < \gamma < \delta$ and $\epsilon < \zeta < \eta$). In any event, as far as the complex Epimenides paradox is concerned, as we build up through order α , $\llbracket(16')\rrbracket$ and $\llbracket(17')\rrbracket$ will lack truth values and thus be kept outside the orders. Once we hit $\alpha + 1$, however, we will be able to assign them both truth values without trouble: $\llbracket(16')\rrbracket$ can be vacuously true, because $\llbracket(17')\rrbracket$ is the only proposition you hope (at least through order δ) and it cannot be in order α . And then $\llbracket(17')\rrbracket$ can be true, because $\llbracket(16')\rrbracket$ is the only proposition I fear (at least through order γ , and thus certainly through β) and we just saw that it can be true. Similarly, for the complex Prior, we will be able to assign $\llbracket(22')\rrbracket$ a truth value only after we have constructed order ϵ .

5.3.3 The proposition Yablo paradox

I want to quickly make good on my promise in Section 2.6 that we can construct the proposition Yablo paradox without propositional quantification by instead quantifying over the (non-propositional) arguments to a propositional function. Add a type o of ordinals, insisting that \mathcal{D}_o be non-empty and have no greatest element, and a constant $>^{(o, \langle o, p \rangle)}$, with accompanying restrictions on \mathcal{T} and \mathcal{F} to parallel those for propositions of the form $\llbracket = \rrbracket(x)(y)$.²⁹ The paradox then comes from introducing a constant $Y^{(o, p)}$ and requiring that for every $x \in \mathcal{D}_o$, $\llbracket Y \rrbracket(x) = \llbracket \forall y [y > x \rightarrow \neg Y y] \rrbracket^{x/x}$. That is, we have a problem so long as we have an infinite hierarchy of propositions $\llbracket Y \rrbracket(x)$, each of which says *that $\llbracket Y \rrbracket(y)$ is false for every y greater than x* .

With no propositional quantification to restrict, each $\llbracket Y \rrbracket(x)$ will forever lack a truth value in the above construction (as well as in the construction in Section 5.4), as will $\llbracket a \rrbracket$ from the proposition Liar. This reinforces my claim in Section 3.2.4 that the best option for a quantifier domain restriction approach is to prohibit things like $\llbracket a \rrbracket$ and $\llbracket Y \rrbracket$ from the outset: if we are willing to accept truth-value gaps in the proposition Liar and Yablo paradoxes, we seem to have no reason to move beyond Section 5.2 in the first place.

5.4 Two domains

If we return to the unramified system, we can also avoid the paradoxes while filling in all the truth-value gaps by beginning with a maximal, gappy \mathfrak{M} from Section 5.2;

²⁹In light of our toy model, we must also add a unique name \mathbf{n} for every element of \mathcal{D}_o to ensure that we have enough propositions.

setting $\mathcal{Q} = \text{Dom}(f)$; and then amending our clauses as follows.

- (k^{alt}) If $\llbracket \forall \mathbf{x}^p[\mathbf{P}] \rrbracket \in \mathcal{T}$ and \mathcal{Q} , then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for all $z \in \mathcal{D}_p$;
- (l^{alt}) if $\llbracket \forall \mathbf{x}^p[\mathbf{P}] \rrbracket \in \mathcal{T}$ but $\notin \mathcal{Q}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for all $z \in \mathcal{Q}$;
- (m^{alt}) if $\llbracket \forall \mathbf{x}^p[\mathbf{P}] \rrbracket \in \mathcal{F}$ and \mathcal{Q} , then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for some $z \in \mathcal{D}_p$;
- (n^{alt}) if $\llbracket \forall \mathbf{x}^p[\mathbf{P}] \rrbracket \in \mathcal{F}$ but $\notin \mathcal{Q}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for some $z \in \mathcal{Q}$;
- (o^{alt}) if $\llbracket \exists \mathbf{x}^p[\mathbf{P}] \rrbracket \in \mathcal{T}$ and \mathcal{Q} , then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for some $z \in \mathcal{D}_p$;
- (p^{alt}) if $\llbracket \exists \mathbf{x}^p[\mathbf{P}] \rrbracket \in \mathcal{T}$ but $\notin \mathcal{Q}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for some $z \in \mathcal{Q}$;
- (q^{alt}) if $\llbracket \exists \mathbf{x}^p[\mathbf{P}] \rrbracket \in \mathcal{F}$ and \mathcal{Q} , then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for all $z \in \mathcal{D}_p$; and
- (r^{alt}) if $\llbracket \exists \mathbf{x}^p[\mathbf{P}] \rrbracket \in \mathcal{F}$ but $\notin \mathcal{Q}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for all $z \in \mathcal{Q}$.

Now propositions within \mathcal{Q} look at every proposition, but propositions outside \mathcal{Q} care about only those inside it. This obviates the need for the fixed-point construction of Section 5.3.2: we can skip directly from the maximal, gappy $\mathfrak{M}_{f, \mathcal{D}_p}$ to a final, gapless (modulo the proposition Liar and Yablo paradoxes), restricted-quantification model with $\mathcal{Q} = \text{Dom}(f)$. In this final model, $\llbracket \forall x[Ax \rightarrow \neg x] \rrbracket$, for example, is vacuously true, but our problematic $\llbracket b \rrbracket$ from above is still either true or false, because $\llbracket \exists x[Ax] \rrbracket$ is in \mathcal{Q} and is thus still unrestricted in its quantification.

5.5 Restrictions on attitudes

There is no way to be as systematic in this case as we have been with the previous resolutions, because we are not here trying to develop models. Instead, we already have the models; what we have to do is explain where our judgments about possibility have gone wrong. That is, all we need is the ability to take a set of inconsistent assumptions about propositional attitudes and identify the one that is false. Prior's method was, again, to proceed chronologically through the attitudes, making as many true as possible without backtracking.

Now that we have our truth-value gaps we can adopt a different method: begin with the paradoxical assumptions, construct a gappy model, and block any attitudes that have as their object a proposition that lacks a truth value in that model. For each gappy model there will be one theory about which assumptions are false—which propositional attitudes were blocked. In the complex Epimenides paradox, for instance, both my fear and your hope will always be blocked, because neither receives a truth value in any model. Likewise, in the complex Prior paradox, my thought will always be blocked, but your speech never will be, because the proposition that snow is black will always receive a truth value.

This revision of Prior’s resolution does not address all the problems. In particular, it has nothing to say about the temporal concerns. But it does address what I think is the most pressing problem with Prior’s resolution: no longer can we preempt another’s propositional attitudes without some measure of cooperation.

5.6 Other theories of propositions

Using our toy model from Section 5.1, propositions are about as fine-grained as they can be, but we can easily construct models at the other extreme, identifying propositions (such as $\llbracket \mathbf{P} \rrbracket$ and $\llbracket \mathbf{P} \wedge \forall x[x = x] \rrbracket$) whenever they necessarily have the same truth value: given a particular resolution, add $\langle \mathbf{P}, \mathbf{Q} \rangle$ to \mathcal{R} if for all models \mathfrak{M} of that resolution we have $\llbracket \mathbf{P} \rrbracket \in \mathcal{T}$ iff $\llbracket \mathbf{Q} \rrbracket \in \mathcal{T}$ and $\llbracket \mathbf{P} \rrbracket \in \mathcal{F}$ iff $\llbracket \mathbf{Q} \rrbracket \in \mathcal{F}$. Unfortunately, we cannot interpret the old models with the new \mathcal{R} , because even if $\llbracket \mathbf{P} \rrbracket$ and $\llbracket \mathbf{Q} \rrbracket$ always have the same truth value, there is no guarantee that $\llbracket \mathbf{R}(\mathbf{P}) \rrbracket$ and $\llbracket \mathbf{R}(\mathbf{Q}) \rrbracket$ will for arbitrary \mathbf{R} s. Thus we will have to repeat the process. But the process is a monotonic function of \mathcal{R} , and so it will reach a fixed point, which will be an \mathcal{R} that holds among every pair of necessarily equivalent formulas—formulas whose denotations are assigned the same truth values in every model.

CHAPTER 6

The Appendix B paradox

I begin with a general version of the Appendix B paradox involving propositional functions. Suppose that we have a constant $m^{\langle(p,p),p\rangle}$ for which the following is true.

$$\forall x^{\langle p,p\rangle} \forall y^{\langle p,p\rangle} [mx = my \rightarrow x = y] \quad (27)$$

As with the original Appendix B paradox, such a supposition should be worrying, because if it is correct, then there is a unique proposition for every function from propositions to propositions, in violation of Cantor's theorem. Indeed, if we also suppose

$$\forall x^p [wx \leftrightarrow \exists y [x = my \wedge \neg yx]], \quad (28)$$

we can prove that $\llbracket w \rrbracket(\llbracket mw \rrbracket)$ is both true and false, assuming it must be one or the other. Suppose that it is true—suppose $\llbracket w \rrbracket(\llbracket mw \rrbracket) \in \mathcal{T}$. Then by (28) we know that for some \mathbf{y} , $\llbracket mw \rrbracket = \llbracket m \rrbracket(\mathbf{y})$ and $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{F}$. By the former and (27) we have $\mathbf{y} = \llbracket w \rrbracket$, whence by the latter we have $\llbracket w \rrbracket(\llbracket mw \rrbracket) \in \mathcal{F}$, contra our assumption. Thus $\llbracket w \rrbracket(\llbracket mw \rrbracket)$ must be false—we must have $\llbracket w \rrbracket(\llbracket mw \rrbracket) \in \mathcal{F}$. Then by (28) we know that for no \mathbf{y} do we have both $\llbracket mw \rrbracket = \llbracket m \rrbracket(\mathbf{y})$ and $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{F}$. In particular, we do not have those for $\mathbf{y} = \llbracket w \rrbracket$. Of course, $\llbracket mw \rrbracket = \llbracket m \rrbracket(\llbracket w \rrbracket)$, so we must not have $\llbracket w \rrbracket(\llbracket mw \rrbracket) \in \mathcal{F}$, a contradiction.

Nevertheless, according to some intuitions, (27) seems plausible. If, for instance, $\llbracket m \rrbracket(x)$ is the proposition *that x is my favorite function from propositions to propositions*, it is difficult to see how (27) could be false—surely that proposition is unique to x . One can then think of $\llbracket w \rrbracket$ as that function which is true of a proposition x (more carefully: which returns a true proposition when given a proposition x) just in case for some \mathbf{y} , (i) x is the proposition *that y is my favorite function from propositions to propositions* and (ii) \mathbf{y} is not true of x — $\mathbf{y}(x) \in \mathcal{F}$.

If one can argue independently that one of (27) and (28) is false—if, for instance, one’s preferred theory of propositions ensures that (27) is false, as many do—then of course the paradox dissolves. In fact, it is impossible to make (27) true given the models I have been discussing; this is why I put off discussing this paradox until now. The domains of those models are sets, and so $\mathcal{D}_{\langle p,p \rangle}$, which is the set of all functions from \mathcal{D}_p to itself, must be larger than \mathcal{D}_p .³⁰ But then it is not possible to have a one-to-one function from $\mathcal{D}_{\langle p,p \rangle}$ to \mathcal{D}_p , as $\llbracket m \rrbracket$ must be according to (27). Nevertheless, one can imagine more flexible models which do not immediately make (27) false, and it is instructive to study the paradox more carefully, if only because it highlights an important difference between uncompressed and compressed ramification. The other resolutions of the Appendix B paradox are relatively straightforward.

6.1 Truth-value gaps

The construction from Section 5.2 immediately resolves this paradox by ensuring that $\llbracket w \rrbracket(\llbracket mw \rrbracket)$ will never have a truth value because, as we have seen, any attempt to assign it a truth value results in a contradiction.

6.2 Uncompressed ramification

Traditional ramification resolves this paradox by placing restrictions on quantification over propositional functions. One can think of the original implementation of ramification as replacing every p in a type symbol with a numeral, as we did before for the variables of type p . That is, for every function, we restrict both the possible arguments and the possible values to a particular order. Thus, for instance, for $\alpha, \beta, \gamma \in \mathcal{I}$ with $\gamma \leq \alpha$, (27) becomes

$$\forall x^{\langle \alpha, \beta \rangle} \forall y^{\langle \alpha, \beta \rangle} [m^{\langle \langle \alpha, \beta \rangle, \gamma \rangle} x = my \rightarrow x = y]. \quad (27')$$

(We must have $\gamma \leq \alpha$ so that the output of $\llbracket m \rrbracket$ can be an argument to its input—so that $w(mw)$ is well-formed. We do not need to have $\gamma = \alpha$ because orders are cumulative.)

We can define the order of a function recursively as the sum of the orders of its input and output. Thus, for instance, $\llbracket m \rrbracket$ ³¹ is of order $\alpha + \beta + \gamma$.³² It is then easiest to

³⁰Here, I assume that there is more than one proposition—that $|\mathcal{D}_p| \geq 2$.

³¹Here and throughout I depart slightly from Church: he uses comprehension schemas, rather than talking directly about the denotations of formulas, but this shift is harmless for my purposes.

³²This is not quite the way orders are defined in [Chu76]. There, functions of arbitrarily many

see how the contradiction is blocked by introducing Church’s comprehension schema, which I have translated into the present system. For every \mathbf{P}^p we are guaranteed to have the following true, so long as \mathbf{z} (does not appear in \mathbf{P} and) is (i) of higher order than every bound variable in \mathbf{P} and (ii) of at least as high an order as every free variable and constant in \mathbf{P} [Chu76, p. 750].³³

$$\exists \mathbf{z} [\forall \mathbf{x} [\mathbf{z}\mathbf{x} \leftrightarrow \mathbf{P}]] \tag{29}$$

We need the following in order to be certain we have the right sort of $\llbracket w \rrbracket$, and thus to derive a contradiction from (27’).

$$\exists z^{\langle \alpha, \beta \rangle} \left[\forall x^\gamma [zx \leftrightarrow \exists y^{\langle \alpha, \beta \rangle} [x = my \wedge \neg yx]] \right]$$

However, this is not an instance of (29) for two reasons. First, z is of the same order as y , in violation of clause (i) above. Second, z is of lower order than m , in violation of clause (ii). (Here it is important that the lowest order in Church’s system is 1.) The only true instance is

$$\exists z^{\langle \alpha, \delta \rangle} \left[\forall x^\gamma [zx \leftrightarrow \exists y^{\langle \alpha, \beta \rangle} [x = my \wedge \neg yx]] \right] \tag{28’}$$

for some $\delta \geq \beta + \gamma$ (and thus $> \beta$, because, again, Church’s lowest order is 1). Thus, although we can be certain from (28’) that there is a function very much like the $\llbracket w \rrbracket$ that the paradox requires, there are two reasons that we cannot take it as $\llbracket w \rrbracket$ and still prove that $\llbracket w(mw) \rrbracket$ is both true and false. First, $\llbracket w \rrbracket$ would then be outside the domain of the quantifiers in (27’). Second, $w(mw)$ would then be ill-formed because mw would be ill-formed— w would be not the right type to appear as an argument to m .

6.3 Compressed ramification

I think it is telling that in both crucial steps above, it was overdetermined that the paradox could not go through. We had two reasons that $\llbracket w \rrbracket$ could not be of type $\langle \alpha, \beta \rangle$, which gave us two reasons that $\llbracket w(mw) \rrbracket$ was not both true and false.

arguments are considered, so that we can have types like $\langle \sigma_1, \sigma_2, \dots, \sigma_n, \tau \rangle$, and the order of such a type is the sum of the order of τ and the highest of the orders of σ_1 – σ_n . Luckily, we do not need to address functions of more than one argument, so the simple definition in the text suffices.

³³The schema is actually more general, allowing for arbitrarily many free \mathbf{x} in \mathbf{P} , but we can make do with the schema for a single \mathbf{x} .

Compressed ramification sets out to retain just the first of each pair of reasons—to retain the one reason in each case that involves quantification. Once we see that those reasons are enough to resolve the paradox, we can greatly simplify our orders. We no longer need to care about the orders of the inputs to a function: we can look exclusively at the orders of its outputs, which can of course vary in a theory of compressed ramified types. The story then remains that when we quantify over propositional functions, we really quantify over only functions of a particular order. The difference is that the order of a function is now simply the highest of the orders of its outputs.

6.3.1 Machinery

Recall that \mathcal{I} is a set of indices, intuitively corresponding to our orders. Before, when we restricted propositional quantification, we replaced every variable \mathbf{x}^p with a collection of variables \mathbf{x}^i . Now we want to restrict quantification over functions as well, so for every variable \mathbf{x} , if the superscript on \mathbf{x} ends in a p , we replace that variable with a collection of variables on which that last p has been replaced with some $i \in \mathcal{I}$. Thus, for instance, $x^{\langle(p,p),p\rangle}$ is replaced with a collection of variables $x^{\langle(p,p),i\rangle}$, $y^{\langle p,\langle p,p\rangle\rangle}$ with $y^{\langle p,\langle p,i\rangle\rangle}$, and so on. is are still treated as ps for syntactic purposes; as before, they play a role only in restrictions (k)–(n) on truth-values assignments.

Intuitively, $x^{\langle p,i\rangle}$ ranges over only functions all of whose outputs are of order i or less. We can capture this more formally by defining a domain of quantification \mathcal{Q}_τ for every τ that is a variable superscript. It is easiest to recursively define \mathcal{Q}_τ more generally for every τ that is either a type symbol or a variable superscript. We already have \mathcal{Q}_i for $i \in \mathcal{I}$ from our models. Let $\mathcal{Q}_p = \mathcal{D}_p$, and then let $\mathcal{Q}_{\langle\tau,\sigma\rangle} = \mathcal{Q}_\sigma^{\mathcal{Q}_\tau}$, where $\mathcal{Q}_\sigma^{\mathcal{Q}_\tau}$, recall, is the set of functions from \mathcal{Q}_τ to \mathcal{Q}_σ . Indices appear only at the ends of variable superscripts, so this amounts to having $\mathcal{Q}_{\langle\tau,\sigma\rangle} = \mathcal{Q}_\sigma^{\mathcal{D}_\tau}$. This makes $\mathcal{Q}_{\langle\tau,\sigma\rangle}$ the set of functions in $\mathcal{D}_{\langle\tau,\sigma\rangle}$ whose ranges are subsets of \mathcal{Q}_σ , as intended.

Now we can return to nearly the original forms of the last four restrictions on truth-value assignments, changing the \mathcal{D} s to \mathcal{Q} s and, of course, widening the range of τ from TS to include all our new variable superscripts:

- (k') if $\llbracket \forall \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{T}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for all $z \in \mathcal{Q}_\tau$;
- (l') if $\llbracket \forall \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{F}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for some $z \in \mathcal{Q}_\tau$;
- (m') if $\llbracket \exists \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{T}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for some $z \in \mathcal{Q}_\tau$; and
- (n') if $\llbracket \exists \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{F}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for all $z \in \mathcal{Q}_\tau$.

An unintended consequence of this construction is that $\llbracket \wedge \rrbracket$, $\llbracket \neg \rrbracket$, etc. never appear in a domain of quantification unless we have a domain of universal propositional quantification—unless for some $i \in \mathcal{I}$ we have $\mathcal{Q}_i = \mathcal{D}_p$. This is because they have outputs of every order. This is not an ideal consequence, but I think that it is also not fatal.

6.3.2 Resolution

At this point we have enough to explain how compressed orders resolve the paradox. In fact, the construction from Section 5.3 can go through unchanged as long as we use our new $(k')\text{--}(n')$. Neither (27) nor (27') is a formula of our language; we must instead have

$$\forall x^{\langle p, \alpha \rangle} \forall y^{\langle p, \alpha \rangle} [m^{\langle \langle p, p \rangle, p \rangle} x = my \rightarrow x = y] \quad (27'')$$

for some $\alpha \in \mathcal{I}$. We must also rewrite (28). We can assume that there is a value for w , rather than going through a comprehension schema, because our compressed orders take care of the contradiction.

$$\forall x^\beta [w^{\langle p, p \rangle} x \leftrightarrow \exists y^{\langle p, \gamma \rangle} [x = my \wedge \neg yx]] \quad (28'')$$

To get the paradox off the ground, suppose that we have $\llbracket (27'') \rrbracket, \llbracket (28'') \rrbracket \in \mathcal{T}$. For simplicity, I suppose that for all $x \neq \llbracket mw \rrbracket$, $\llbracket w \rrbracket(x)$ can be assigned a truth value unproblematically.³⁴ β matters only insofar as there is no paradox at all if we have some reason to exclude $\llbracket mw \rrbracket$ from \mathcal{Q}_β , so also suppose that we have $\llbracket mw \rrbracket \in \mathcal{Q}_\beta$.³⁵ In what follows, **let ‘y’ range over only those functions for which we have $\mathbf{y} \neq \llbracket w \rrbracket$ and $\llbracket mw \rrbracket = \llbracket m \rrbracket(\mathbf{y})$** . Of course, there may very well be no such \mathbf{y} . Whether there is such a \mathbf{y} can be determined before we begin constructing the orders, as it is purely a matter of identities, and identities are fixed from the outset.

First, notice that if we have $\llbracket w(mw) \rrbracket \in \mathcal{T}$ at an order δ (more precisely, if we have it during the construction of \mathcal{Q}_δ), then both $\llbracket w \rrbracket$ and some \mathbf{y} such that $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{F}$ are in $\mathcal{Q}_{\langle p, \delta \rangle}$: $\llbracket w \rrbracket$ because of our simplifying assumption that every other proposition of the form $\llbracket w \rrbracket(x)$ has a truth value and such a \mathbf{y} because we need a witness to the right-hand side of (28''). Thus we cannot make $\llbracket w(mw) \rrbracket$ true at an order $\delta \leq \alpha$ (again, more precisely, during the construction of \mathcal{Q}_δ , $\delta \leq \alpha$), on pain of contradiction

³⁴This is entirely a simplifying assumption. I discuss it below.

³⁵It is unlikely that this assumption will be false very often. Orders depend entirely on truth-value assignments and there are no restrictions on the truth values of propositions of the form $\llbracket m \rrbracket(x)$, so we have no reason to expect that $\llbracket mw \rrbracket$ will ever be gappy, and thus no reason to expect to have even $\llbracket mw \rrbracket \notin \mathcal{Q}_0$, let alone $\llbracket mw \rrbracket \notin \mathcal{Q}_\beta$.

from (27'') (recall the bold stipulation above).

Now, there are two exhaustive but not exclusive possibilities.

- (i) **If for some \mathbf{y} we can consistently have $\mathbf{y} \in \mathcal{Q}_{\langle p, \gamma \rangle}$ and $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{F}$, then that \mathbf{y} can witness the truth of the right-hand side of (28''). For this \mathbf{y} , there is a least $\gamma' \leq \gamma$ such that $\mathbf{y} \in \mathcal{Q}_{\langle p, \gamma' \rangle}$,³⁶ and we can have $\llbracket w(mw) \rrbracket \in \mathcal{T}$ at order $\min(\gamma', \alpha + 1)$ but no earlier.³⁷ That is, we have models in which $\llbracket w(mw) \rrbracket \in \mathcal{T}$ and $\notin \mathcal{Q}_{\min(\gamma', \alpha + 1) - 1}$ but $\in \mathcal{Q}_{\min(\gamma', \alpha + 1)}$. We need to be at least as high as γ' to make sure that we have our witness, and we need to be beyond order α for the reasons in the preceding paragraph.**
- (ii) **If we can consistently have $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{T}$ for every $\mathbf{y} \in \mathcal{Q}_{\langle p, \gamma \rangle}$, then we can have $\llbracket w(mw) \rrbracket \in \mathcal{F}$ at order $\gamma + 1$ but no earlier. That is, we have models in which $\llbracket w(mw) \rrbracket \in \mathcal{F}$ and $\notin \mathcal{Q}_\gamma$ but $\in \mathcal{Q}_{\gamma + 1}$. This covers the case in which there are no \mathbf{y} —in which we do not have a $\mathbf{y} \neq \llbracket w \rrbracket$ such that $\llbracket mw \rrbracket = \llbracket m \rrbracket(\mathbf{y})$. We cannot make $\llbracket w(mw) \rrbracket$ false before order $\gamma + 1$, given our simplifying assumption, because then we would have $\llbracket w \rrbracket \in \mathcal{Q}_{\langle p, \gamma \rangle}$, and $\llbracket w \rrbracket$ itself would witness the truth of the right-hand side of (28''), contra the falsity of $\llbracket w(mw) \rrbracket$.**

If the antecedent of only (whence exactly, since they are exhaustive) one of (i) and (ii) holds, then the corresponding consequent tells us what the models look like. If both antecedents hold, then we have some of each type of model.³⁸

If we do away with the simplifying assumption that $\llbracket w(mw) \rrbracket$ is the only proposition of the form $\llbracket w \rrbracket(\mathbf{y})$ whose truth is problematic, then we add a further layer to our cases: we could make $\llbracket w(mw) \rrbracket$ false earlier than order $\gamma + 1$ if doing so didn't force $\llbracket w \rrbracket$ into $\mathcal{Q}_{\langle p, \gamma \rangle}$, and we could make $\llbracket w(mw) \rrbracket$ true earlier than order $\alpha + 1$ if doing so didn't force $\llbracket w \rrbracket$ into $\mathcal{Q}_{\langle p, \alpha \rangle}$. But this only multiplies (in relatively uninteresting directions) the ways we can resolve the paradox, so the only result of doing away with this assumption would be more clauses in the already complex possibilities described above; it really is playing only a simplifying role.

³⁶If no $\mathbf{y}(x)$ is potentially paradoxical, then we should have $\gamma' = 0$, but we can harmlessly work with the more general case.

³⁷Barring bizarre identities as always.

³⁸One might think that the possible models depend on how $\min(\gamma', \alpha + 1)$ and γ compare: that if $\min(\gamma', \alpha + 1) \leq \gamma + 1$, then we have models of the sort described in (i); if $\gamma + 1 \leq \min(\gamma', \alpha + 1)$, then we have models of the sort described in (ii); and only if $\min(\gamma', \alpha + 1) = \gamma + 1$ do we have models of both sorts. But this assumes that once a proposition has been assigned a truth value during the construction of an order, it retains that truth value through the construction of each subsequent order. This is not only not required, but actually guaranteed to be false as long as the orders grow at all; see note 25. Thus, even if, for instance, both antecedents are true and $\min(\gamma', \alpha + 1) < \gamma + 1$, we will eventually see models of both sorts, and we will be restricted to models of the first sort only until we have begun constructing order $\gamma + 1$.

6.3.3 Summary

This is a reiteration of the bold text above. Let ‘ \mathbf{y} ’ range over only those functions for which we have $\mathbf{y} \neq \llbracket w \rrbracket$ and $\llbracket mw \rrbracket = \llbracket m \rrbracket(\mathbf{y})$. If for some \mathbf{y} we can consistently have $\mathbf{y} \in \mathcal{Q}_{\langle p, \gamma \rangle}$ and $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{F}$, then there is a least $\gamma' \leq \gamma$ such that $\mathbf{y} \in \mathcal{Q}_{\langle p, \gamma' \rangle}$, and we can have $\llbracket w(mw) \rrbracket \in \mathcal{T}$ at order $\min(\gamma', \alpha + 1)$. If we can consistently have $\mathbf{y}(\llbracket mw \rrbracket) \in \mathcal{T}$ for every $\mathbf{y} \in \mathcal{Q}_{\langle p, \gamma \rangle}$, then we can have $\llbracket w(mw) \rrbracket \in \mathcal{F}$ at order $\gamma + 1$.

6.4 Two domains

As before, the resolution is an adaptation of the idea behind compressed ramification. Given a maximal, gappy model \mathfrak{M} , let \mathcal{Q}_p be $\mathcal{T} \cup \mathcal{F}$ and $\mathcal{Q}_{\langle \tau, \sigma \rangle}$ be $\mathcal{Q}_\sigma^{\mathcal{D}_\tau}$ —the latter ensures that a function is in $\mathcal{Q}_{\langle \tau, \sigma \rangle}$ iff all its outputs are in \mathcal{Q}_σ . Since $\llbracket w \rrbracket(\llbracket mw \rrbracket)$ lacks a truth value, it will not be in \mathcal{Q}_p , and so $\llbracket w \rrbracket$ will not be in $\mathcal{Q}_{\langle p, p \rangle}$. This means that with the following updated clauses we will have extended our two-domain resolution to propositional functions, and $\llbracket w \rrbracket(\llbracket mw \rrbracket)$ will be false. These clauses are a bit strange, because they restrict quantifiers over types other than p on the basis of their appearing at the beginning of a formula of type p that lacks a truth-value, but I have not been able to discover any new paradoxes that this introduces.

- (k^{alt'}) If $\llbracket \forall \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{T}$ and \mathcal{Q} , then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for all $z \in \mathcal{D}_\tau$;
- (l^{alt'}) if $\llbracket \forall \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{T}$ but $\notin \mathcal{Q}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for all $z \in \mathcal{Q}_\tau$;
- (m^{alt'}) if $\llbracket \forall \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{F}$ and \mathcal{Q} , then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for some $z \in \mathcal{D}_\tau$;
- (n^{alt'}) if $\llbracket \forall \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{F}$ but $\notin \mathcal{Q}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for some $z \in \mathcal{Q}_\tau$;
- (o^{alt'}) if $\llbracket \exists \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{T}$ and \mathcal{Q} , then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for some $z \in \mathcal{D}_\tau$;
- (p^{alt'}) if $\llbracket \exists \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{T}$ but $\notin \mathcal{Q}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{T}$ for some $z \in \mathcal{Q}_\tau$;
- (q^{alt'}) if $\llbracket \exists \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{F}$ and \mathcal{Q} , then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for all $z \in \mathcal{D}_\tau$; and
- (r^{alt'}) if $\llbracket \exists \mathbf{x}^\tau [\mathbf{P}] \rrbracket \in \mathcal{F}$ but $\notin \mathcal{Q}$, then $\llbracket \mathbf{P} \rrbracket^{\mathbf{x}/z} \in \mathcal{F}$ for all $z \in \mathcal{Q}_\tau$.

To see that $\llbracket w \rrbracket(\llbracket mw \rrbracket)$ will be false, recall the assumption about w :

$$\forall x^p [wx \leftrightarrow \exists y[x = my \wedge \neg yx]]. \quad (28)$$

Given this assumption, as I said in Section 6.1, $\llbracket w \rrbracket(\llbracket mw \rrbracket)$ will lack a truth value. This means that the right-hand side of the biconditional in (28) lacks a truth value when $\llbracket x \rrbracket = \llbracket mw \rrbracket$, because by (27), which ensures the uniqueness of $\llbracket m \rrbracket(\mathbf{x})$ to \mathbf{x} , the only value of y for which $\llbracket \exists y[x = my \wedge \neg yx] \rrbracket$ could be true is $\llbracket w \rrbracket$, and we

already know that we do not have $\llbracket \neg w(mw) \rrbracket$. Thus, when $\llbracket x \rrbracket = \llbracket mw \rrbracket$, we have $\llbracket \exists y[x = my \wedge \neg yx] \rrbracket \in \mathcal{Q}$, and so when we go to evaluate it in our new two-domain theory (for instance when we are checking to see whether $\llbracket w \rrbracket(\llbracket mw \rrbracket)$ can be assigned a truth value), we will ignore the variable assignment $\llbracket y \rrbracket = \llbracket w \rrbracket$, because $\llbracket w \rrbracket \notin \mathcal{Q}_{(p,p)}$. Again by the uniqueness of $\llbracket m \rrbracket(\mathbf{x})$ to \mathbf{x} , this ensures that the left conjunct in $\exists y[x = my \wedge \neg yx]$ will never be true when $\llbracket x \rrbracket = \llbracket mw \rrbracket$, and so $\llbracket w \rrbracket(\llbracket mw \rrbracket)$ will be false.

6.5 Variations

The goal of compressed ramification and the two-domain resolution is to retain traditional ramification's ability to resolve the paradoxes while allowing for more flexible domains of quantification. I think their treatment of the above version of the Appendix B paradox satisfies this goal. But there are other paradoxes that highlight important differences between uncompressed ramification on the one hand and these two quantifier domain restriction resolutions on the other. For simplicity, I talk about compressed ramification, but all the following points apply equally to the two-domain resolution. As usual, the truth-value gap approach resolves these paradoxes without trouble.

6.5.1 The original Appendix B paradox

In the original version of the Appendix B paradox, $\llbracket m \rrbracket(\mathbf{x})$ is the proposition *that every proposition of which \mathbf{x} is true is true*, i.e., the proposition $\llbracket \forall x[yx \rightarrow x] \rrbracket^{y/\mathbf{x}}$. With this interpretation, the problematical assumption is not (27) but

$$\forall x^{(p,p)} \forall y^{(p,p)} [\forall z[xz \rightarrow z] = \forall z[yz \rightarrow z] \rightarrow x = y]. \quad (30)$$

Since each $\llbracket m \rrbracket(\mathbf{x})$ now involves propositional quantification, traditional ramification provides a different resolution of the paradox. This comes from Church's first comprehension schema [Chu76, p. 750]: for every \mathbf{P} of type p we are guaranteed to have the following true, so long as \mathbf{x} (does not appear in \mathbf{P} and) is (i) of higher order than every bound variable in \mathbf{P} and (ii) of at least as high an order as every free variable and constant in \mathbf{P} .

$$\exists \mathbf{x}[\mathbf{x} \leftrightarrow \mathbf{P}] \quad (31)$$

This comprehension schema ensures that $\llbracket \forall z[wz \rightarrow z] \rrbracket$ is of at least as high an order as $\llbracket w \rrbracket$. But then, since the order of $\llbracket w \rrbracket$ is the sum of the orders of its inputs and outputs, $\llbracket \forall z[wz \rightarrow z] \rrbracket$ cannot be an argument to $\llbracket w \rrbracket$. (Again, Church's lowest

propositional order is 1.) It is thus wrong to even ask whether $\llbracket w(\forall z[wz \rightarrow z]) \rrbracket$ is true or false.

Compressed ramification does away with restrictions on the arguments to functions, so it cannot provide this resolution. I think, however, that this is not a great shortcoming. In the general case, when $\llbracket mw \rrbracket$ does not involve quantification, even uncompressed ramification must fall back on restrictions on quantification over propositional functions; compressed ramification merely extends that reliance to the original Appendix B paradox.

6.5.2 Sets, properties, pluralities, etc.

A greater shortcoming of compressed ramification is that it has no easy answer to paradoxes that arise if we extend the logic to cover, for example, sets, properties, or pluralities.

- (i) For every set x , there seems to be the proposition $m(x)$ *that x is my favorite set*. But what about the set w of all and only propositions of the form $m(x)$ such that $m(x) \notin x$ —do we have $m(w) \in w$?
- (ii) For every property x , there seems to be the proposition $m(x)$ *that x is my favorite property*. But what about the property w of being a proposition of the form $m(x)$ that does not have the property x —does $m(w)$ have w ?
- (iii) For any propositions xx , there is the proposition $m(xx)$ *that xx are my favorites*. But what about those propositions ww that are of the form $m(xx)$ but not one of xx —is $m(ww)$ one of ww ?³⁹

None of these paradoxes can be constructed in the system as it stands, but we can imagine introducing, say, a type s of sets of propositions and a constant $\in^{(p, \langle s, p \rangle)}$ in order to capture the paradox in (i).

It would be natural for traditional ramification to insist that each set be capped at some order or other: this parallels the requirement that propositional functions take input of only a certain order, and it blocks the paradox. But compressed ramification does away with the requirement that such an insistence parallels—functions in a theory of compressed ramified types can take inputs of any order—and so it cannot block the paradox in the same manner. All compressed ramification allows is ordering functions by the orders of their outputs. The parallel of this is ordering a set x by the orders of propositions of the form $\llbracket \mathbf{P} \in x \rrbracket^{x/x}$ —the orders of propositions about

³⁹This paradox is the subject of [MR00].

membership in x . This strikes me as a truly bizarre method of assigning orders to sets.

Similar contortions are required in cases (ii) and (iii), and I think the story in the case of plural quantification is even less plausible than it is in the cases of sets and properties—surely pluralities should be ordered by the orders of their parts, and not by the orders of propositions about being among them. These paradoxes, then, show that while it might make some sense to determine the order of a function by the orders of its outputs, this does not naturally extend to some of the purposes to which functions are often put. For some purposes, for instance, we can use functions in place of sets and a membership relation, but extending the orders from Section 6.3 to sets in order to block the paradox in (i) seems hopelessly ad hoc.

6.5.3 A return to the proposition Liar?

One response here is to insist on behalf of the quantifier domain restriction approach that none of the paradoxes in this section (and perhaps even in this chapter) can even get off the ground. I observed at the beginning of this chapter that (27) cannot be true in the models we have been working with, and certainly it is plausible for a theory of functions, sets, properties, or pluralities to prohibit having a unique proposition for every function, set, property, or plurality of propositions. We have already seen that these resolutions must appeal to a theory of propositions to block the proposition Liar paradox; perhaps we can find good independent reasons, coming from our theories of functions, sets, etc., to block these paradoxes as well.

Of course, such a response leaves something to be desired: we still want an explanation of where our intuitions have gone wrong. Why must there be two distinct functions of type $\langle p, p \rangle$ x and y such that the proposition *that x is my favorite function* is identical to the proposition *that y is my favorite function*? What functions are they? Why can't anyone even *believe* that one is my favorite without thereby believing that the other is my favorite? Taking this route amounts to buying a simple theory of sets at the cost of intuitive stories about propositional identity and differentiation.

CHAPTER 7

Comparing the resolutions

7.1 The expression Liar paradox

To capture the expression Liar, we must extend our language to be able to talk about formulas and expression. This paradox, recall, involves the sentence

$$(15) \text{ does not express a true proposition.} \tag{15}$$

One easy way to make these extensions is to introduce a new primitive type symbol f and a new constant $E^{(f, \langle p, p \rangle)}$. Intuitively, f is the type of formulas— \mathcal{D}_f is Φ —and $\llbracket E(\mathbf{P}, \mathbf{Q}) \rrbracket$ is the proposition *that* $\llbracket \mathbf{P} \rrbracket$ expresses $\llbracket \mathbf{Q} \rrbracket$. As we did with $=$, we suppose that $\llbracket E(\mathbf{P}, \mathbf{Q}) \rrbracket$ always has the correct truth value. That is, we suppose that we have $\llbracket E(\mathbf{P}, \mathbf{Q}) \rrbracket \in \mathcal{T}$ iff $\llbracket \mathbf{P} \rrbracket$ is a formula that really does denote $\llbracket \mathbf{Q} \rrbracket$ and $\llbracket E(\mathbf{P}, \mathbf{Q}) \rrbracket \in \mathcal{F}$ otherwise. (As a consequence, $\llbracket E(\mathbf{P}, \mathbf{Q}) \rrbracket$ is always in either \mathcal{T} or \mathcal{F} —it is always either true or false, and is never gappy.) We can put this somewhat awkwardly by saying that we have $\llbracket E(\mathbf{P}, \mathbf{Q}) \rrbracket \in \mathcal{T}$ iff $\llbracket \llbracket \mathbf{P} \rrbracket \rrbracket = \llbracket \mathbf{Q} \rrbracket$, since $\llbracket \mathbf{P} \rrbracket$ is a formula that can itself be interpreted by our interpretation function.

We also need a constant l^f , which denotes the formula $\neg \exists x [E(l, x) \wedge x]$, which is a rough translation of (15) into \mathfrak{L} . That is, $\langle l^f \rangle$ (momentarily dropping the convention of allowing formulas to name themselves) denotes a formula of type p , whose denotation in turn is the proposition *that the formula denoted by* $\langle l^f \rangle$ *does not denote a true proposition*. Put yet another way, $\llbracket l \rrbracket$ is the translation of (15) into \mathfrak{L} . This amounts to a restriction on $\llbracket \cdot \rrbracket$: we must have $\llbracket l \rrbracket = \neg \exists x [E(l, x) \wedge x]$ (reinstating our omission of mention quotes).

We do not need anything further; these assumptions about $\llbracket \cdot \rrbracket$, \mathcal{T} , and \mathcal{F} allow us to prove both $\llbracket \neg \exists x [E(l, x) \wedge x] \rrbracket \in \mathcal{T}$ and $\llbracket \neg \exists x [E(l, x) \wedge x] \rrbracket \in \mathcal{F}$ when we have $\mathcal{T} \cup \mathcal{F} = \mathcal{D}_p$. As with the other intensional paradoxes, this is easily handled by

the truth-value gap approach: $\llbracket \neg \exists x [E(l, x) \wedge x] \rrbracket$ is simply neither true nor false, as there is no way to assign it a truth value in any model, maximal or otherwise. Thus, according to compressed ramification, it will be kept out of whatever order x is restricted to; according to the two-domain theory, it will be kept out of \mathcal{Q} and x will be restricted to \mathcal{Q} ; and if we extend Prior’s restriction on propositional attitudes to expression, we will see that since $\llbracket \neg \exists x [E(l, x) \wedge x] \rrbracket$ lacks a truth value, it must be that our assumption that it is expressed by $\llbracket l \rrbracket$ is false (clearly, this approach will require that we relax our assumption about $\llbracket E(\mathbf{P}, \mathbf{Q}) \rrbracket$ always having a truth value). This is why I did not bother to address the paradox from the start—its formulation requires complications, such as our new type f , and its resolution requires nothing beyond the constructions in Chapter 5.

7.2 Strengthened paradoxes

One might be dissatisfied with this resolution of the expression Liar on the grounds that it does violence to the intended meaning of (15)—on the grounds that the formula denoted by l is a poor translation of the original English sentence. This is, after all, a *strengthened* form of the expression Liar; at least according to one intuition, the proposition expressed by (15) should be *true* if it is neither true nor false. Similarly, if $\llbracket \neg \exists x [E(l, x) \wedge x] \rrbracket$ is vacuously true, as it is according to every quantifier domain restriction resolution, then, quantifier domains aside, it is intuitively clear that (15) denotes a true proposition, and so that it (the sentence, and thus the proposition it denotes) ought to be false.

It should not be surprising that strengthened forms of the paradoxes have intuitively unsatisfactory resolutions; this problem is not limited to the resolutions I have developed, and it is not limited to the intensional paradoxes. Most resolutions of either type of paradox require some expressive limitation or other, whether it is an inability to talk about truth-value gaps (be they gaps in \mathcal{T} and \mathcal{F} or gaps in satisfaction) or an inability to talk about a domain of quantification of one form or another. The situation semantics of [BE87], Glanzberg’s contextualist machinery in [Gla04], and Burge’s contextualist suggestions in [Bur79, Bur84] can all, to some extent, address the worries posed by strengthened intensional paradoxes. The idea, very roughly, is that when we start talking about the treatment of some paradox according to some particular resolution, observing that the resolution seems to force us to say something different about the paradoxical proposition or sentence, we move to a new situation or context.

What is important for my purposes is that all these approaches to the strengthened paradoxes require an initial, non-contextualist resolution. Contextualism (or situation semantics; for simplicity I use only contextualist examples) can explain the reasoning at the beginning of this section by saying that we begin in a context in which (15) either doesn't express a proposition in the domain of quantification (Glanzberg) or expresses a proposition that lacks a truth value (Burge, roughly) and then move to a context in which it expresses a proposition in the domain/expresses one that has a truth value. But we still need to be able to identify the initial, pre-shift context, in which (15) fails to denote a proposition/denotes a proposition that lacks a truth value. This is what I have done in Sections 5.3 (and 5.4) and 5.2 respectively.

This is not to say that strengthened paradoxes have nothing to say about resolutions at the general level at which I have been working. One of the reasons Glanzberg favors his quantifier domain restriction approach is that we have independent reason to think that domains of quantification are context-sensitive, and that is certainly a benefit of that approach over the truth-value gap approach. And there are problems facing quantifier domain restriction approaches that might be addressed by turning to situations, as Jon Barwise, John Etchemendy, and Willem Groeneveld [BE87, Gro94] have. But we cannot be held hostage to pre-existing work: all the context- or situation-based machinery in the world will not help if we cannot get a satisfactory pre-shift, paradox-free context or situation off the ground. By working at a very general level, I think we can identify (and have identified) problems and choices that face at least the majority of resolutions, no matter how they incorporate ideas from Glanzberg, Burge, etc.

I also do not mean to say that one can easily tack a theory of contexts or situations onto the resolutions I have developed; it is not at all clear what one could rely on in the logic I have been using to shift the context to one in which our problematic paradoxes are assigned truth values/are within the domain of quantification,⁴⁰ let alone what those new contexts would look like. As I said in Section 3.2.1, there is a real question about how domains of quantification can vary with the context. If we want to take a contextualist approach to the strengthened paradoxes, this issue arises again with more force, and applies equally to truth-value gap approaches: what triggers the context shift in the reasoning at the beginning of this section? Any complete resolution of the paradoxes must have an answer.

⁴⁰Perhaps this is a reason to introduce explicit truth predicates, contra [GCB74]. As I said at the beginning of Chapter 4, I do not think that such an addition would affect my constructions.

7.3 Ramsey’s division

In [Ram25], Frank Ramsey argued that we ought to split the extensional paradoxes into two now-familiar categories. In modern terms, these are the semantical paradoxes, such as the satisfaction Liar, and the set-theoretical paradoxes, such as Russell’s paradox of the set that contains only all sets that do not contain themselves. Despite some objections, such as Graham Priest’s [Pri94], this division has guided much of the literature on the extensional paradoxes. For the most part, resolutions of the semantical paradoxes try to find a way to block the derivation of a contradiction from sentences like “this sentence is false,” while the set-theoretical paradoxes are mostly resolved by insisting that they cannot even get off the ground because they assume a naïve theory of sets, theories of non-wellfounded sets notwithstanding.

By taking this divide-and-conquer approach, most existing resolutions have purchased a simpler theory of truth and meaning with their set theory. Priest’s call for a unified solution is not ignored, but his own paraconsistent approach involves admitting truth-value gluts—admitting that some sentences are both true and false—and this seems to be too big a pill to swallow for most theorists.

We can see a similar tension arising with the four resolutions I have developed. The truth-value gap resolution is a unified resolution in the sense that it blocks every contradiction. The cost of this unification, though, is allowing propositions, the fundamental bearers of truth, to lack truth values. And notice that this lack of truth values extends beyond the initially gappy propositions. If x lacks a truth value, then so does the proposition *that x lacks a truth value*, on pain of a strengthened paradox. But then the propositions denoted by sentences like “the proposition *that x lacks a truth value* must lack a truth value” must also lack truth values. Indeed, the propositions we use to capture the core tenets of the truth-value gap theory must lack truth values. And so must the previous and current sentences!

This is troubling. If, for instance, the goal of assertion is to assert true propositions, we have failed. But we cannot even say that we have failed—the previous sentence is neither true nor false, and neither is this one. We can all understand the problem here, but that understanding cannot involve grasping true propositions. And, once again, the second conjunct of the previous sentence, along with this sentence, lack truth values. These problems are more serious in this intensional case than in the extensional analogue: this is not merely an expressive limitation in a language, but rather a lack of truths to even study.

The truth-value gap resolution, then, purchases a unified resolution of the para-

doxes and a simple theory of propositions and propositional attitudes at the cost of our theory of truth.

Similarly, the two-domain resolution retains a simple theory of truth and a relatively simple theory of propositional attitudes, but must purchase this with a theory of propositions that can (i) allow the ad hoc shifts in quantifier domains that the theory requires and (ii) explain why the propositions required for proposition Liar and Yablo paradoxes and the variations of the Appendix B paradox cannot exist.

Compressed ramification is simultaneously better and worse off. It must rely on a theory of propositions for (ii), but orders eliminate the need for (i). But now we must ask where the orders come from, and I think that any answer to this will have at least some important consequences for our theory of mental states, since we presumably must be able to retrieve the orders from such states.

Compressed ramification also suffers from a problem similar to the one facing the truth-value gap resolution. We might try to state one of the core tenets of ramification by saying, “Every proposition has an order.” But we know that this must actually be equivalent to “Every proposition of order n has an order,” and we have no guarantee that there is an order that contains every proposition. (For traditional ramification, such an order is impossible; for compressed ramification, it is merely improbable.) Where the truth-value gap resolution cannot be true because any proposition that corresponds to it cannot be true (and neither can this sentence, which attempts to state the problem, be true), a ramified theory of types cannot be true because there is no proposition that corresponds to it in the first place. And notice that even that last claim does not really mean what I want it to mean, because all it actually means is “there is no proposition of order n that corresponds to it.”

One might at this point turn to the machinery discussed in Section 7.2 for help. These concerns are, after all, similar to those raised by strengthened paradoxes. But such machinery has a lot of work to do. We can, it seems, understand these issues. But understanding is a propositional attitude. So what true propositions are we grasping when we understand these problems? The problem is one of lacking not a universal metalanguage, but rather a universal way the world is, and there seems to me to be no way around this consequence for either the truth-value gap resolution or ramification. All we can hope to do, I think, is search for an explanation of why this is not so great a cost to pay. The situation semantics of [BE87] is, I think, particularly promising as such an explanation, because it does not rely on sentences at all. But as it does not even begin to address propositional quantification, it is only a promise at this point.

Finally, we can see one of the major benefits of Prior's approach. By being conservative about the nature of propositions and truth—indeed, by conserving *all* the intuitions that permit the derivation of a contradiction from paradoxical assumptions—it avoids these problems. But it pays for this by being even less unifying than the quantifier domain restriction resolutions, and by giving up some of what we can believe, what sentences can express, and so on. Prior's approach gives us a story to tell about the paradoxes, but it is one on which, for instance, thoughts can be blocked by future states of affairs, which is at least very strange.

I do not pretend to have solved these issues. My goal from the start was only to develop and explore the space of resolutions of these paradoxes (and even this I have not completely done, as paraconsistent approaches are notably absent). I hope, at least, that we now (i) have a better idea of what the available options are, and in particular realize that the quantifier domain restriction approach and Prior's approach need not be as rigid as they have historically been; and (ii) better understand the costs and most fundamental commitments of each approach.

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