# Currents and equidistribution in holomorphic dynamics

by

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to Sarah

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# CHAPTER I

# Introduction

## 1.1 Historical development and known results

Let $\mathbb{P}^k$ be the complex projective space of dimension $k$ and let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of algebraic degree $d \geq 2$. It is well known (see [FS94], [HP94]) that there exists a positive closed (1,1)-current $T_f$, the *Green current*, such that for every smooth (1,1)-form $\alpha$ in the cohomology class of the Fubini-Study form $\omega$, the sequence of smooth (1,1)-forms $d^{-n}(f^n)^*\alpha$ converges to $T_f$ in the sense of currents. A natural question to ask is if such behavior also occurs when we replace smooth forms by currents. More precisely, if $S$ is a positive closed (1,1)-current of mass 1, when does the convergence

$$d^{-n}(f^n)^*S \to T_f \tag{1.1}$$

hold? This last convergence is what we refer to as *equidistribution*. The answer: Not always. Assume for example that there exists a *totally invariant* irreducible hypersurface $X \subset \mathbb{P}^k$, i.e. $f^{-s}(X) = X$ for some $s \in \mathbb{N}$ (which for simplicity we take to be $s = 1$); then its current of integration $[X]$ satisfies $f^*[X] = d[X]$ giving us that

$$d^{-n}(f^n)^*[X] = [X] \nrightarrow T_f$$

since the current $T_f$ has no mass on any algebraic subsets of $\mathbb{P}^k$ (in particular, it cannot be the current of integration of an algebraic variety. See [Sib99] for more details). Therefore, it seems that the appearance of totally invariant algebraic sets restricts the possibility of having equidistribution. A very important feature of holomorphic maps is that the collection of all totally invariant algebraic subsets of $\mathbb{P}^k$ is finite (see [FS94], [DS08]).

In dimension $k = 1$ (i.e. $\mathbb{P}^1$ is the Riemann sphere and $T_f$ is an invariant probability measure), a famous result by Brolin [Bro65] (for the case of a polynomial self-maps of $\mathbb{C}$) and by Lyubich [Lju83], Freire-Lopes-Mañé [FLM83] (for the case of rational self-maps of $\mathbb{P}^1$) states that there exists a collection $\mathcal{E}_f$ of totally invariant points (also called *exceptional points*), with cardinality of $\mathcal{E}_f \leq 2$ with the following property:

Given any probability measure $\nu$ on $\mathbb{P}^1$, $d^{-n}(f^n)^*\nu$ converges to $T_f$ if and only if $\nu(\{p\}) = 0$ for all $\{p\} \in \mathcal{E}_f$. In particular, for every $x \in \mathbb{P}^1$ which is not exceptional

$$d^{-n}(f^n)^*\delta_x = \frac{1}{d^n} \sum_{f^n(y)=x} \delta_y \to T_f \tag{1.2}$$

as $n \to +\infty$, where $\delta_x$ denotes the Dirac mass at $x$. The equation (1.2) also shows that the sequence of preimages of points outside $\mathcal{E}_f$ accumulate along the Julia set of $f$.

The situation for $k = 2$ is already highly more involved. Some partial results for equidistribution in $\mathbb{P}^2$ for holomorphic (and meromorphic) maps were obtained by J. E. Fornæss and N. Sibony [FS95], A. Russakovskii. and B. Shiffman [RS97] and others. C. Favre and M. Jonsson finished the characterization for the two-dimensional case in [FJ03] (see also [FJ07]) proving the following: There exists a family $\mathcal{E}_f$ of totally invariant irreducible algebraic subsets of $\mathbb{P}^2$, containing at most 3 lines and a finite number of points, with the following property: given any positive closed (1,1)-current $S$ of mass 1, $d^{-n}(f^n)^*S$ converges to $T_f$ if and only if $S$ has no mass on any element of $\mathcal{E}_f$. The elements of $\mathcal{E}_f$ are attracting in nature and this collection can be strictly smaller than the collection of all totally invariant irreducible algebraic subsets of $\mathbb{P}^k$.

In higher dimensions the situation is not as well understood, particularly since we do not have any satisfactory classification of totally invariant algebraic subsets of $\mathbb{P}^k$. The equidistribution problem in higher dimensions was studied already in [FS95], [RS97] and [Sib99]. In [Gue03] V. Guedj showed that for a given positive closed (1,1)-current $S$ with Lelong numbers zero everywhere we have

$$d^{-n}(f^n)^*S \to T_f$$

as $n \to +\infty$ (his result also holds for $f$ meromorphic). In [DS08] T.-C. Dinh and N. Sibony established the following: There exists a finite collection $\mathcal{E}_{\mathrm{DS}}$ of totally invariant irreducible algebraic subsets of $\mathbb{P}^k$ with the following property: given any positive closed (1,1)-current $S$ of mass 1 whose local potentials are not identically $-\infty$ on any element of $\mathcal{E}_{\mathrm{DS}}$, $d^{-n}(f^n)^*S$ converges to $T_f$ (their result is uniform in $S$ in certain sense). The collection $\mathcal{E}_{\mathrm{DS}}$ obtained by Dinh and Sibony is constructed inductively by studying the induced dynamics on totally invariant sets. Note that neither Guedj's result nor Dinh and Sibony's result imply each other.

### 1.2 Presentation of results

As we have mentioned already, the main difficulty for proving equidistribution arises from the lack of a satisfactory classification of totally invariant algebraic subsets of $\mathbb{P}^k$. In order to tackle this, we make an assumption on the singular locus of totally invariant algebraic subsets which will allow us to develop our methods.

We present our main result

**Theorem I.1** (Main Theorem)**.** *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of algebraic degree $d \geq 2$ and assume that all totally invariant algebraic subsets have normalizations with at worst isolated quotient singularities. Then there exists a finite collection $\mathcal{E}_f$ of irreducible totally invariant algebraic sets with the following property: given any positive closed (1,1)-current $S$ of mass 1 with no mass on any element of $\mathcal{E}_f$ we have*

$$d^{-n}(f^n)^*S \to T_f$$

*as $n \to +\infty$ in the sense of currents.*

The finite family $\mathcal{E}_f$ coincides with the ones already obtained for $k = 1$ and $k = 2$. It is constructed following the ideas of Favre and Jonsson in [FJ03], but we push the methods further.

Our assumption on the singularities of the totally invariant algebraic subsets (namely, to have normalizations with at worst isolated quotient singularities) holds for $k = 3$, as it can be derived from the work of J. Wahl [Wah90], N. Nakayama [Nak99], D.Q. Zhang [Zha00] and C. Favre [Fav10]. From this we obtain as a corollary a sharper equidistribution result in dimension 3

**Corollary I.2.** *Let $f : \mathbb{P}^3 \to \mathbb{P}^3$ be a holomorphic map of degree $d \geq 2$. There exists a finite collection $\mathcal{E}_f$ of irreducible totally invariant algebraic sets with the following property: given any positive closed (1,1)-current $S$ with no mass on any element of $\mathcal{E}_f$, the sequence $d^{-n}(f^n)^*S$ converges to $T_f$ in the sense of currents.*

We conjecture that the converse implication is also true: if the sequence $d^{-n}(f^n)^*S$ converges to $T_f$ then $S$ has no mass on any element of $\mathcal{E}_f$; this would extend the results already known in dimensions one and two.

It is important to notice that the families $\mathcal{E}_f$ and $\mathcal{E}_{\mathrm{DS}}$ are different and they satisfy the relation

$$\mathcal{E}_f \subset \mathcal{E}_{\mathrm{DS}} \subset \left\{\text{All totally invariant alg subsets of } \mathbb{P}^k\right\}$$

where the first and/or the second inclusion can be strict.

Our techniques will also allow us to generalize what has been so far obtained by Dinh-Sibony and Guedj and provide the sharpest results known by the author for dimensions $k \geq 3$ in the holomorphic setting. We extend Dinh-Sibony's result

**Theorem I.3** (Dinh-Sibony improved)**.** *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of degree $d \geq 2$. There exists a finite collection $\mathcal{E}_{DS}$ of totally invariant algebraic subsets of $\mathbb{P}^k$ with the following property: given any positive closed (1,1)-current $S$ with no mass on any element of $\mathcal{E}_{DS}$ we have*

$$d^{-n}(f^n)^*S \to T_f$$

*as $n \to +\infty$ in the sense of currents.*

As opposed to Theorem I.1, we make no assumptions on the singularities of the totally invariant algebraic subsets. The collection $\mathcal{E}_{\mathrm{DS}}$ is the same one constructed by Dinh and Sibony in [DS08] (see also [Din09] for a different construction) and our improvement is based on approximating the current $S$ by currents that satisfy the conditions imposed by Dinh and Sibony in their theorem.

As an immediate consequence of the theorem above, we can extend Guedj's result

**Corollary I.4** (Guedj improved)**.** *Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of degree $d \geq 2$. There exist a totally invariant proper algebraic subset $E \subset \mathbb{P}^k$ with the following property: given any positive closed (1,1)-current $S$ such that $\nu(S, x) = 0$ for all $x \in E$ we have*

$$d^{-n}(f^n)^*S \to T_f$$

*as $n \to +\infty$ in the sense of currents, where $\nu(S,x)$ denotes the Lelong number of $S$ at $x$.*

The proof of this corollary follows by taking $E$ as the union of the elements of $\mathcal{E}_{\rm DS}$ (which is algebraic). The tools introduced here will allow us to present an independent proof of the result above.

In order to verify equidistribution, we will use a characterization due to V. Guedj (see [Gue03], Theorem 1.4) which states that it is enough to test the asymptotic behavior of Lelong numbers. More precisely, he proved the equivalence

$$d^{-n}(f^n)^*S \to T_f \iff \sup_{x\in\mathbb{P}^k} \nu\left(d^{-n}(f^n)^*S, x\right) \to 0. \tag{1.3}$$

Our approach uses a mixture of analytic tools in order to control the asymptotic behavior of Lelong numbers and verify Guedj's condition (1.3). This work can be found in [Par11].

In order to do this, we use a technique due to J. P. Demailly of approximation of currents by currents with analytic singularities. As a consequence, we are able to reduce the general problem for positive closed (1,1)-currents to the case of the currents of integration of a suitable hypersurface. Applying Dinh-Sibony's result to this hypersurface allows us to obtain Theorem I.3.

For the proof of Theorem I.1, we proceed as in [FJ03] and study the Lelong numbers of $d^{-n}[\mathcal{C}_{f^n}]$, where $\mathcal{C}_{f^n}$ denotes the critical set of $f^n$. We prove that the (totally invariant) set

$$E := \bigcup_{\delta>0}\bigcap_{n\in\mathbb{N}} \left\{x\in\mathbb{P}^k \mid \operatorname{ord}_x(\mathcal{C}_{f^n}) \geq \delta d^n\right\}$$

is algebraic. In order to do this, we prove a refined version of certain self-intersection inequalities à la Demailly (Theorem III.5, proved in [Par10]), used in [DS08].

We then proceed inductively on the irreducible components $X \subset E$. The main problem that arises, is that $X$ might be too singular, hence, making sense of the critical set of $f|_X$ is hard. Our assumptions on the singular locus of $X$ will let us get around this problem.

## CHAPTER II

# Background

## 2.1 Currents

In this section we introduce the main results concerning positive closed currents and Lelong numbers. The basic reference for this section will be the book [Dem09], chapter III unless otherwise stated.

### 2.1.1 Lelong numbers

The main tool we have in order to 'measure' the size of the singular locus of a current are the Lelong numbers. Let $X$ be a compact Kähler manifold with Kähler form $\omega$ and let $S$ be a positive closed $(p,p)$-current on $X$. By definition, the $(k,k)$-form

$$\sigma_S := S \wedge \omega^{k-p}$$

is a finite positive measure on $X$. We refer to this measure as the *trace measure* of $S$.

If $x \in X$ and $B(x,r)$ is an Euclidean ball with center $x$ and radius $r > 0$, then the function

$$r \mapsto \nu(S,x,r) := \frac{\sigma_S\left(B(x,r)\right)}{\pi^p r^{2p}}$$

is increasing in $r > 0$. We define the *Lelong number* $\nu(S,x)$ of $S$ at $x$ as the limit

$$\nu(S,x) := \lim_{r \to 0^+} \nu(S,x,r).$$

This limit always exists and $\nu(S,x)$ does not depend on neither the chosen local chart nor $\omega$. The quantity defined above can be seen as a generalization of the multiplicity $\mathrm{mult}_x(Z)$ of a variety $Z$ at $x$. More precisely, if $Z$ is an irreducible analytic subvariety of $X$, then

$$\nu([Z],x) = \mathrm{mult}_x(Z),$$

where $[Z]$ denotes the current of integration along $Z$.

A very important feature of Lelong numbers is the upper semicontinuity in both variables, which can be obtained from its definition. But the upper semicontinuity of $\nu(S,\cdot)$ is remarkably stronger since it is not only true in the standard topology but also in the Zariski topology: For every positive closed $(p,p)$-current $S$ and every $c > 0$ we denote by $E_c(S)$ the *Lelong upper level set*

$$E_c(S) := \{x \in X \mid \nu(S,x) \geq c\}.$$

A fundamental theorem proved by Siu [Siu74] states that $E_c(S)$ is always an analytic subset of $X$, hence $\nu(S,\cdot)$ is Zariski upper semicontinuous.

Note that by Siu's theorem, given any irreducible analytic subset $V$ of $X$, the quantity

$$\nu(S,V) := \min_{x \in V} \nu(S,x)$$

is equal to $\nu(S,x)$ for $x$ generic, i.e. for $x$ outside a proper analytic subset of $V$. We define the *Lelong number of $S$ along $V$* as $\nu(S,V)$.

As a consequence of Siu's theorem, it is possible to prove the following decomposition formula: If $S$ is a positive closed $(p,p)$-current, then there is a unique decomposition of $S$ as a (possibly infinite) weakly convergent series

$$S = \sum_{j\geq 1} \lambda_j [A_j] + R$$

where $[A_j]$ is the current of integration over an irreducible analytic variety $A_j \subset X$ of codimension $p$, $\lambda_j > 0$ the generic Lelong numbers of $S$ along $A_j$ and $R$ is a positive closed current such that for every $c > 0$, the level set $E_c(R)$ has dimension strictly less than $\dim(X) - p$.

This formula (known as Siu's decomposition theorem) states that the singular locus of a positive closed current can be decomposed into a union of analytic subsets plus a residual part with small size.

#### 2.1.2 Extension and intersection of currents

We state here some known results on positive closed currents that we will need in this thesis.

A subset $P \subset X$ is said to be *complete pluripolar* if for every $x \in P$ there exist an open neighborhood $U \ni x$ and a plurisubharmonic function $u$ not identically $-\infty$ such that

$$P \cap U = \{z \in U \mid u(z) = -\infty\}.$$

In particular all analytic subsets of $X$ are closed complete pluripolar sets.

**Theorem II.1** (El Mir)**.** *Let $P \subset X$ be a closed complete pluripolar subset and let $S$ be a positive closed current on $X \setminus P$ with bounded mass on a neighborhood of every point of $P$. Then, the trivial extension by zero of $S$ on $X$ is a positive closed current.*

It is well known that for any irreducible analytic subset $A \subset X$, the current of integration $[A_{\text{reg}}]$ has finite mass in a neighborhood of every point of $A_{\text{sing}}$, hence the current of integration $[A]$, meaning its extension by zero through $A_{\text{sing}}$, is a well defined positive closed current on $X$.

We finally discuss intersection of currents. Given an open set $\Omega \subset \mathbb{C}^k$, a plurisubharmonic function $\varphi$ and a positive closed current $S$ in $\Omega$, we would like to have a notion of intersection $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi \wedge S$ on $\Omega$. More precisely, we would like to define

$$\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi \wedge S := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(\varphi S).$$

The equation above does not always make sense but it is well defined as long as the sizes of the singular sets involved are not too big; note in particular that it is well defined if $\varphi$ is locally integrable with respect to the trace measure of $S$. We proceed to introduce a more general result concerning intersection of currents.

Let $S_1, \ldots, S_q$ be positive closed (1,1)-currents with local potentials $\varphi_1, \ldots, \varphi_q$ respectively. We denote by $L(\varphi_j)$ the *unbounded locus* of $\varphi_j$, namely, the set

$$L(\varphi_j) := \{x \in X \mid \varphi_j \text{ is not bounded near } x\}.$$

**Theorem II.2.** *Let $\Theta$ be a positive, closed $(k-p, k-p)$-current. Assume that for any choice of indices $j_1 < \cdots < j_m$ in $\{1, \ldots, q\}$ the set*

$$L(\varphi_{j_1}) \cap \cdots \cap L(\varphi_{j_{m,}}) \cap \mathrm{Supp}(\Theta)$$

*has $(2p-2m+1)$-Hausdorff measure zero. Then the wedge product $S_1 \wedge \cdots \wedge S_q \wedge \Theta$ is well defined. Moreover, the product is weakly continuous with respect to monotone decreasing sequences of plurisubharmonic functions.*

We end this subsection with a useful comparison of Lelong numbers of products of currents: If $S_1$ is a positive closed (1,1)-current and $S_2$ is a positive closed $(p,p)$-current such that the product $S_1 \wedge S_2$ (which is given locally by the local potentials of $S_1$) is well defined, then

$$\nu(S_1 \wedge S_2, x) \geq \nu(S_1, x)\nu(S_2, x) \tag{2.1}$$

for every $x \in X$.

#### 2.1.3 Currents on singular projective varieties

We will need to deal with positive closed currents defined on singular varieties. If $X$ is a projective variety and $\iota : X \hookrightarrow \mathbb{P}^N$ an embedding, we will say that $\omega$ is a Fubini-Study form on $X$ if $\omega = \iota^*\omega_{\mathbb{P}^N}|_X$, where $\omega_{\mathbb{P}^N}$ is the Fubini-Study form on $\mathbb{P}^N$. Note that $\omega$ is a positive smooth differential form on $X_{\mathrm{reg}}$.

**Definition II.3.** If $X$ is a (possibly singular) projective irreducible variety and $S$ is a positive closed $(p,p)$-current defined on $X_{\mathrm{reg}}$, we will say that $S$ has *bounded mass around $X_{sing}$* if there exist an open neighborhood $U$ of $X_{\mathrm{sing}}$ such that

$$\int_{U\cap X_{\mathrm{reg}}} S \wedge \omega^{\dim(X)-p} < +\infty.$$

In a complex manifold, for any given two hermitian forms $\omega$ and $\omega'$ there is always a positive constant $A$ such that $A^{-1}\omega \leq \omega' \leq A\omega$, in particular it is easy to see that above definition does not depend of the embedding of $X$.

### 2.2 Holomorphic Dynamics on the Projective Space

Throughout this thesis $f : \mathbb{P}^k \to \mathbb{P}^k$ will denote a holomorphic map of degree $d \geq 2$, i.e. the map $f$ is given by a $(k+1)$-tuple of homogeneous polynomials of degree $d \geq 2$ in $k+1$ variables, having $(0,\ldots,0) \in \mathbb{C}^{k+1}$ as the only common root.

#### 2.2.1 The Green current

Denoting by $\omega$ the Fubini-Study metric on $\mathbb{P}^k$ and using that the De Rham cohomology group $H^2_{\mathrm{DR}}(\mathbb{P}^k;\mathbb{R})$ is generated by $\{\omega\}$, we see that there exists a smooth function $u : \mathbb{P}^k \to \mathbb{R}$ such that

$$d^{-1}f^*\omega = \omega + dd^c u,$$

where $dd^c = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$. In particular, for every $n \in \mathbb{N}$ we observe that

$$d^{-n}(f^n)^*\omega = \omega + dd^c u_n,$$

where $u_n := \sum_{i=0}^{n-1} d^{-i} u \circ f^i$ is a sequence of smooth functions on $\mathbb{P}^k$. Since $u$ is bounded, it follows that the series defining $u_n$ converges uniformly to a (continuous) function $g_f$ on $\mathbb{P}^k$. Hence, the sequence of smooth forms $\omega + dd^c u_n$ converges in the sense of currents to the positive closed (1,1)-current

$$T_f := \omega + dd^c g_f.$$

The current $T_f$ is called the *Green current* associated to $f$ and it plays a central role for the understanding of the dynamics given by the map $f$. It is invariant (i.e. $f^* T_f = dT_f$), it has Lelong numbers zero everywhere and its support equals the Julia set of $f$ ([FS94], [HP94]).

As we stated at the introduction, given a positive closed (1,1)-current $S$ we can verify equidistribution (i.e. $d^{-n}(f^n)^* S$ converges to $T_f$ in the sense of currents) if the sequence $\sup_{x \in \mathbb{P}^k} \nu \left( d^{-n}(f^n)^* S, x \right)$ converges to 0 [Gue03]. We will show that there is a link between equidistribution and the appearence of certain exceptional sets.

It will be convenient to replace iterates $f^s$ by $f$, for this we need the following

**Lemma II.4.** *Let $S$ be a positive closed (1,1)-current of mass 1. Then the following are equivalent*

*(i)* $d^{-n}(f^n)^* S \to T_f$*;*

*(ii)* $d^{-ns}(f^{ns})^* S \to T_f$ *for some* $s \geq 1$*.*

Note that (i) $\Rightarrow$ (ii) follows immediately and that (ii) implies

$$d^{-n}(f^n)^* S \to d^{-l} \left( f^l \right)^* T_f$$

for some $l \in \{0, 1, \ldots, s-1\}$. The $f$-invariance of $T_f$ finally implies that $d^{-l} \left( f^l \right)^* T_f = T_f$.

### 2.2.2 Totally invariant algebraic sets

A subset $X \subset \mathbb{P}^k$ is said to be *totally invariant* if $f^{-s}(X) \subset X$ for some $s \geq 1$. If $X \subset \mathbb{P}^k$ is an irreducible algebraic totally invariant set, then it follows that $f^{-s}(X) = X$. Moreover, if $X$ is a totally invariant algebraic set of codimension $p$ in $\mathbb{P}^k$ then the holomorphic map

$$g := f^s|_X : X \to X$$

has topological degree $d^{sp}$ and $(f^s)^*[X] = d^{sp}[X]$, where $[X]$ denotes the current of integration of $X$. For a detailed discussion on the properties of holomorphic (and meromorphic) dynamics on projective spaces we refer the reader to [Sib99].

A crucial property of totally invariant algebraic subsets of $\mathbb{P}^k$ is the following (non-trivial) well known fact

**Theorem II.5.** *The collection of all proper totally invariant algebraic subsets of* $\mathbb{P}^k$ *is finite.*

A proof of Theorem II.5 can be found in [DS08] (see also [FS94] for the case $k = 2$). For the more general situation $g : X \to X$ where $g$ is a regular map and $X$ is a projective variety, the same conclusion can be derived from the work of Dinh-Sibony in [DS08], giving us the more useful result

**Theorem II.6.** *Let $g : X \to X$ be a regular self-map of a projective variety $X$. Then, the collection of all proper totally invariant subsets of $X$ is finite.*

## 2.3 Characterization of equidistribution

As we mentioned earlier in the introduction, we can test equidistribution looking at the asymptotic behavior of Lelong numbers. More concretely

**Theorem II.7.** *Let $S$ be a positive closed (1,1)-current of mass 1 on $\mathbb{P}^k$. The following are equivalent*

*(i)* $d^{-n}(f^n)^*S \to T_f$ *weakly;*

*(ii)* $\sup_{x\in\mathbb{P}^k} \nu(d^{-n}(f^n)^*S, x) \to 0$.

*Proof.* (*i*) $\Rightarrow$ (*ii*)

This implication follows from the usc of the map $(S,x) \mapsto \nu(S,x)$ and the fact that $\nu(T_f, x) = 0$ for all $x \in \mathbb{P}^k$. Assume that there exists a sequence $\{x_n\} \subset \mathbb{P}^k$ and a constant $A > 0$ such that

$$\nu\left(d^{-n}(f^n)^*S, x_n\right) \geq A$$

for all $n \in \mathbb{N}$. Up to passing to a subsequence, there exists a point $\bar{x} \in \mathbb{P}^k$ such that $x_n \to \bar{x}$, hence

$$0 < A \leq \limsup_{n\to+\infty} \nu\left(d^{-n}(f^n)^*S, x_n\right) \leq \nu(T_f, \bar{x}) = 0$$

which is a contradiction.

(*ii*) $\Rightarrow$ (*i*)

Before starting the proof of this implication we recall two results by Kiselman [Kis00] and Guedj [Gue04] respectively.

**Theorem II.8** (Kiselman)**.** *Let $U \subset \mathbb{C}^2$ be an open set, $K$ a compact subset of $U$, and $u$ a plurisubharmonic function on $U$. For any real number $\alpha < 2\left(\sup_{p\in K}\nu(u,p)\right)^{-1}$, there exist a constant $C_\alpha$ such that for any $t \geq 0$, the estimate*

$$\mathrm{Vol}(\{K \cap \{u \leq -t\}) \leq C_\alpha \exp(-\alpha t)$$

*holds.*

**Theorem II.9** (Guedj)**.** *There exist a positive constant $C$ independent of $n \geq 0$ and any ball $B \subset \mathbb{P}^k$ such that*

$$\mathrm{Vol}(f^n(B)) \geq \exp\left(-\frac{C}{\mathrm{Vol}(B)}d^n\right),$$

*for every $n \geq 0$.*

Since $S$ is a positive closed (1,1)-current of mass 1 we can write $S$ as $S = \omega + dd^c u$ where $\omega$ is the Fubini-Study form on $\mathbb{P}^k$. It follows that for every $n \geq 0$,

$$d^{-n}(f^n)^* S = d^{-n}(f^n)^*\omega + d^{-n}dd^c(u \circ f^n),$$

where $d^{-n}(f^n)^*\omega \to T_f$ in the sense of currents. We will proceed by contradiction. If $d^{-n}(f^n)^* S \nrightarrow T$, we have that $d^{-n}dd^c(u \circ f^n) \nrightarrow 0$ which is equivalent to $d^{-n}(u \circ f^n) \nrightarrow 0$ in $L^1_{\mathrm{loc}}$. Therefore, by Hartog's Lemma [Hör90, Theorem 1.6.13] we can find a ball $B \subset \mathbb{P}^k$, a subsequence $n_j \to \infty$ and a positive constant $\alpha$ such that

$$f^{n_j}(B) \subset \{u < -\alpha d^{n_j}\}.$$

Since

$$\sup_{p \in \mathbb{P}^k} \nu(d^{-n} u \circ f^n, p) \to 0,$$

for every $\epsilon > 0$ there exist $N > 0$ such that $\sup_{p \in \mathbb{P}^k} \nu(u \circ f^n, p) < 2/\epsilon$ for $n \geq N$. Using Theorem II.8, in our case $K = \mathbb{P}^k$ and $u$ the potential of $S$ we get that

$$\mathrm{Vol}\left(\{u < -\alpha d^n\}\right) \leq C_\epsilon \exp(-\epsilon\alpha d^n).$$

On the other hand, if $f^{n_j}(B) \subset \{u < -\alpha d^{n_j}\}$, from Theorem II.9 we deduce that

$$\exp\left(-\frac{C}{\mathrm{Vol}(B)}d^{n_j}\right) \leq C_\epsilon \exp(-\epsilon\alpha d^{n_j}),$$

which is a contradiction since $\epsilon > 0$ is very small and independent of $B$, $n_j$, $C$ and $\alpha$.

$\square$

# CHAPTER III

# Intersection Inequality

## 3.1 Approximation of currents

In this section we will discuss the approximation of (1,1)-currents by currents with analytic singularities. The entire section is based on the work of J. P. Demailly (particularly [Dem93], [Dem92]). However, we add some details of the proof of Theorems III.2 and III.4 since these techniques are crucial for this work and the author believes that they are not very well known.

The main ingredients of the approximation are the mean value inequality and the Ohsawa-Takegoshi $L^2$-extension theorem

**Theorem III.1** (Ohsawa-Takegoshi's $L^2$-Extension Theorem)**.** *Let $X$ be a projective manifold. Then there is a positive line bundle $A \to X$ over $X$ with smooth hermitian metric $h_A$ and a constant $C > 0$ such that for every line bundle $L \to X$ provided with a singular hermitian metric $h_L$ and for every $x \in X$ such that $h_L(x) \neq 0$, there exist a section $\sigma$ of $L + A$ such that*

$$\|\sigma\|_{h_L \otimes h_A} \leq C|\sigma(x)|.$$

The original version of Theorem III.1 can be found in [OT87]. The more general form used here can be found in [Man93].

#### 3.1.1 Approximation by divisors

Let $X$ be a projective manifold and let $S$ be a positive closed current representing the first Chern class $c_1(L)$ of a hermitian line bundle $L \to X$. More precisely, we can endow $L \to X$ with a singular hermitian metric $h_L$ and curvature form $\Theta(h_L)$ where

$$S \in c_1(L) = \{\Theta(h_L)\}\,.$$

Now, let $A \to X$ be an ample line bundle with smooth hermitian metric $h_A = e^{-\varphi_A}$. Its positive curvature form $\omega := \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_A$ endows $X$ with a Kähler metric. We can fix a smooth hermitian metric $h$ on $L$, hence we can write $h_L = he^{-2\varphi}$ and $S = \Theta(h_L) = \Theta(h) + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$. We endow $mL+A$ with the (singular) metric $h_L^{\otimes m} \otimes h_A$ and we define the (finite dimensional) Hilbert space $\mathcal{H}_m \subset H^0(X; \mathcal{O}_X(mL+A))$ as

$$\mathcal{H}_m := \left\{\sigma \in H^0(X; \mathcal{O}_X(mL+A)) \mid \|\sigma\|_m^2 < +\infty\right\}$$

where the norm $\|\cdot\|_m^2$ is given by

$$\|\sigma\|_m^2 := \int_X h_L^{\otimes m} \otimes h_A(\sigma) dV_\omega.$$

We present the following theorem which can be found in [Dem93] (see also [Bou02]).

**Theorem III.2.** *Let $X$, $S$, $L \to X$, $\varphi$ and $A \to X$ be as before. Let $\{\sigma_{m,j}\}_{j=1}^{N_m}$ be an orthonormal basis of $\mathcal{H}_m$ and define*

$$\varphi_m(x) := \frac{1}{2m}\log\left(\sum_{j=1}^{N_m} h^{\otimes m} \otimes h_A(\sigma_{m,j}(x))\right).$$

*Then there exist positive constants $C_1$ and $C_2$ independent of $m$ such that, for every $x \in X$ we have:*

$$\varphi(x) - \frac{C_1}{m} \le \varphi_m(x) \le \sup_{z \in B(x,r)} \varphi(z) + \frac{C_2}{m} + C(x,r),$$

*where $C(x,r)$ tends to 0 as $r \to 0$.*

*Proof of Theorem III.2.* First we cover $X$ by finitely many small open balls $\{B\}$ giving local trivializations for both line bundles $A$ and $L$. On $L$, $A|_B \simeq B \times \mathbb{C} \subset \mathbb{C}^k \times \mathbb{C}$ we pick smooth metrics $\psi$, $\psi_A$ for $L$ and $A$ respectively, i.e. for all $(x, v) \in B \times \mathbb{C}$

$$h(x, v) = |v|^2 e^{-2\psi(x)}, \quad h_A(x, v) = |v|^2 e^{-2\psi_A(x)},$$

hence if $\sigma \in \mathcal{H}_m$ is a section supported on $B$ we have that

$$h^{\otimes m} \otimes h_A(\sigma(x)) = |\sigma(x)|^2 e^{-2m\psi(x)-2\psi_A(x)}.$$

Since $\sigma : B \to \mathbb{C}$ is holomorphic, by the mean value inequality for all $x \in B$ and $r < \operatorname{dist}(x, \partial B)$ we have

$$|\sigma(x)|^2 \leq \frac{k!}{\pi^k r^{2k}} \int_{B(x,r)} |\sigma(z)|^2 dV(z),$$

implying

$$\begin{aligned}(3.1)\quad h^{\otimes m} \otimes h_A(\sigma(x)) &\leq \frac{C}{r^{2k}} e^{-2m\psi(x)-2\psi_A(x)} \int_{B(x,r)} |\sigma(z)|^2 dV(z) \leq \\ &\leq \frac{C}{r^{2k}} e^{2m[\sup_{B(x,r)} \psi - \psi(x)] + 2[\sup_{B(x,r)} \psi_A - \psi_A(x)]} \int_{B(x,r)} h^{\otimes m} \otimes h_A(\sigma(z)) dV(z).\end{aligned}$$

Denote by $c(x, r) := \sup_{B(x,r)} \psi - \psi(x)$, $c_A(x, r) := \sup_{B(x,r)} \psi_A - \psi_A(x)$ and note that

$$\int_{B(x,r)} h^{\otimes m} \otimes h_A(\sigma(z)) dV(z) \leq \left( \sup_{B(x,r)} e^{2m\varphi} \right) \|\sigma\|_m^2,$$

therefore

$$(3.2)\qquad h^{\otimes m} \otimes h_A(\sigma(x)) \leq \frac{C}{r^{2k}} e^{mc(x,r)+c_A(x,r)} \left( \sup_{B(x,r)} e^{2m\varphi} \right) \|\sigma\|_m^2.$$

We can write $\varphi_m$ as

$$e^{2m\varphi_m(x)} = \sup_{\|\sigma\|=1} h^{\otimes m} \otimes h_A(\sigma(x));$$

therefore taking log of (3.2) and the supremum over $\|\sigma\|_m = 1$ we obtain

$$(3.3) \quad 2m\varphi_m(x) \leq \log\left(\frac{C}{r^{2k}}\right) + mc(x,r) + c_A(x,r) + 2m \sup_{B(x,r)} \varphi \Longrightarrow$$

$$\Longrightarrow \varphi_m(x) \leq \sup_{B(x,r)} \varphi + C(x,r) + \frac{1}{m} \log\left(\frac{C'}{r^k}\right),$$

where $C' > 0$ and $C(x,r) \to 0$ as $r \to 0$.

For the other inequality we use Ohsawa-Takegoshi's $L^2$ Extension Theorem: Let $x \in X$ such that $h_L(x) \neq 0$. Since $h_L = he^{-2\varphi}$ we can find a section $\sigma \in \mathcal{H}_m$ such that

$$\|\sigma\|_m^2 \leq C^2 |\sigma(x)|^2 = C^2 h^{\otimes m} \otimes h_A(\sigma(x)) e^{-2m\varphi(x)},$$

for some $C > 0$. Using (again) that

$$e^{2m\varphi_m(x)} = \sup_{\|\sigma\|_m = 1} h^{\otimes m} \otimes h_A(\sigma(x)),$$

we take log of the inequality and the supremum over $\|\sigma\|_m = 1$ obtaining

$$\varphi(x) \leq \varphi_m(x) + \frac{C'}{m}.$$

This concludes the proof. □

The theorem above can be reformulated as

**Theorem III.3.** *Let $X$ be a projective complex manifold and let $S$ be a positive closed (1,1)-current in the cohomology class of a line bundle. Then there exist a sequence of closed (1,1)-currents $S_m$ in the cohomology class of $S$ such that*

*(i) $S_m \geq -\frac{1}{m}\omega$;*

*(ii) The sequence $S_m$ converges weakly to $S$;*

*(iii) For every $x \in X$ the Lelong numbers at $x$ satisfy*

$$\nu(S,x) - \frac{C}{m} \leq \nu(S_m, x) \leq \nu(S,x),$$

*for some $C > 0$. In particular, the Lelong numbers $\nu(S_m, x)$ converge uniformly to $\nu(S, x)$.*

*Proof.* Let $L \to X$ be a positive hermitian line bundle with singular hermitian metric $h_L$ such that $S \in \{\Theta(h_L)\}$. We can take a smooth metric $h$ on $L$ such that $h_L$ can be written as $h_L = he^{-2\varphi}$ and therefore we can define

$$S_m := \Theta(h) + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_m,$$

with $\varphi_m$ as in the theorem above.

For any $x \in X$, pick a trivialization $\Omega$ of $L$ and $A$ around $x$. On $\Omega$, we have that

$$h^{\otimes m} \otimes h_A(\sigma)(x) = |\sigma(x)|^2 e^{-2m\varphi(x) - 2\varphi_A(x)}$$

giving us

$$\varphi_m(x) = \frac{1}{2m}\log\left(\sum_{j=1}^{N_m} |\sigma_{m,j}(x)|^2\right) - \varphi(x) - \frac{1}{m}\varphi_A(x).$$

Therefore

$$\underbrace{\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_m + \Theta(h)}_{S_m} + \frac{1}{m}\underbrace{\Theta(h_A)}_{\omega} \geq 0$$

giving us (i). It is routine to check that the sequence $S_m$ converges to $S = \Theta(h) + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ for (ii) and part (iii) follows immediately. $\square$

#### 3.1.2 Attenuation of Lelong numbers

We finish this section with a refined version of the theorem of the subsection above which will allow us to approximate positive closed currents by currents with analytic singularities and attenuated Lelong numbers. We state the main theorem of this section proved in [Dem93].

**Theorem III.4.** *Let $X$ be a projective manifold and let $S$ be a positive closed (1,1)-current representing the class $c_1(L)$ of some hermitian line bundle $L \to X$. Fix a sufficiently positive line bundle $G$ over $X$ such that $TX \otimes G$ is nef. Then for every $c > 0$ there exist a sequence of closed (1,1)-currents $S_{c,m}$ converging weakly to $S$ over $X$ such that*

- $S_{c,m} \geq -\frac{2}{m}\omega - cu$*, where $u$ is the curvature form of $G$ and;*
- $\max\left(\nu(S,x) - c - \dim(X)/m, 0\right) \leq \nu(S_{c,m}, x) \leq \max\left(\nu(S,x) - c, 0\right).$

The proof of the above theorem in a more general case, namely $X$ is a compact Kähler manifold and $S$ is any almost positive closed (1,1)-current can be found in [Dem92]; the proof involves a very technical gluing procedure which is beyond the scope of what we want to present here. For the case $X$ projective and $S$ the curvature current of a positive line bundle, the proof is simpler and can be obtained in a more direct way; we present the proof given in [Dem93] with some details added.

*Proof.* As in Theorem III.2 it is possible to construct sections $\sigma_{m,j} \in H^0(X; mL+A)$, $1 \leq j \leq N_m$ such that

$$\nu(S,x) - \frac{\dim X}{m} \leq \frac{1}{m} \min_{j=1,\ldots,N_m} \operatorname{ord}_x(\sigma_{m,j}) \leq \nu(S,x).$$

We consider the $l$-jet sections $J^l\sigma_{m,j}$ with values in the vector bundle $J^l\mathcal{O}_X(mL+A)$. We have the exact sequence

$$0 \to S^lT^*X \otimes \mathcal{O}_X(mL+A) \to J^l\mathcal{O}_X(mL+A) \to J^{l-1}\mathcal{O}_X(mL+A) \to 0.$$

Dualizing the above sequence we obtain the short exact sequence

$$0 \to (J^{l-1}\mathcal{O}_X(mL+A))^* \to (J^l\mathcal{O}_X(mL+A))^* \to (S^lT^*X \otimes \mathcal{O}_X(mL+A))^* \to 0$$

which can be rewritten as

$$0 \to (J^{l-1}\mathcal{O}_X(mL+A))^* \to (J^l\mathcal{O}_X(mL+A))^* \to S^lTX \otimes \mathcal{O}_X(-mL-A) \to 0.$$

Twisting this exact sequence with $\mathcal{O}_X(mL+2A+lG)$ we obtain that

$$\begin{aligned}(3.4)\quad & 0 \to (J^{l-1}\mathcal{O}_X(mL+A))^* \otimes \mathcal{O}_X(mL+2A+lG) \to \\ & \to (J^l\mathcal{O}_X(mL+A))^* \otimes \mathcal{O}_X(mL+2A+lG) \to S^lTX \otimes \mathcal{O}_X(lG+A) \to 0\end{aligned}$$

is exact. By hypothesis, the vector bundle $TX \otimes G$ is nef and therefore $S^l(TX \otimes \mathcal{O}_X(G)) = S^lTX \otimes \mathcal{O}_X(lG)$ is nef for all symmetric powers of order $l$, hence

$$S^lTX \otimes \mathcal{O}_X(lG+A) = \underbrace{\left(S^lTX \otimes \mathcal{O}_X(lG)\right)}_{\text{nef}} \otimes \underbrace{\mathcal{O}_X(A)}_{\text{ample}}$$

is ample. Since hte extremes of the exact sequence (3.4) are ample, we use induction on $l \geq 1$ to conclude that the middle term

$$(J^l\mathcal{O}_X(mL+A))^* \otimes \mathcal{O}_X(mL+2A+lG)$$

is also ample.

By definition of amplitude of vector bundles there exist $q \geq 1$ such that

$$S^q\left(J^l\mathcal{O}_X(mL+A)\right)^* \otimes \mathcal{O}_X(qmL+2qA+qlG)$$

is generated by holomorphic sections $g_{m,i}$. Using this together with the pairing of $(J^l\mathcal{O}_X(mL+A))^*$ and $J^l\mathcal{O}_X(mL+A)$ we obtain sections

$$S^q(J^l\sigma_{m,j})g_{m,i} \in H^0(X; \mathcal{O}_X(qmL+2qA+qlG))$$

which in a trivialization give us the metric

$$\varphi_{m,l} := \frac{1}{qm}\log\sum_{i,j}|S^q(J^l\sigma_{m,j})g_{m,i}| - \frac{2}{m}\psi_A - \frac{l}{m}\psi_G.$$

Note that $\psi_A$ and $\psi_G$ are smooth; therefore we have

$$\nu(\varphi_{m,l}, x) = \frac{1}{m} \min_j \operatorname{ord}_x(J^l \sigma_{m,j}) = \frac{1}{m} \left( \min_j \operatorname{ord}_x(\sigma_{m,j}) - l \right).$$

This gives us the inequality

$$\max \left( \nu(S, x) - \frac{l + \dim(X)}{m}, 0 \right) \leq \nu(\varphi_{m,l}, x) \leq \max \left( \nu(S, x) - \frac{l}{m}, 0 \right).$$

Finally, for every $c > 0$ and every $m \gg 0$ it is possible to find $l > 0$ such that $c < l/m < c + 1/m$, hence

$$\frac{\sqrt{-1}}{\pi} \partial\bar{\partial}\varphi_{c,m} \geq -\frac{2}{m}\omega - cu,$$

where $\varphi_{c,m} := \varphi_{m,l}$ for this choice of $m$, $l$, and $\omega$ and $u$ are the curvature forms of $A$ and $G$ respectively.

Now, for any smooth metric $h$ on $L$, the sequence of currents

$$S_{c,m} := \Theta(h) + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}\varphi_{c,m},$$

converges weakly to $S$ and satisfies the desired properties. □

### 3.2 The intersection theorem

let $X$ be a (possibly singular) projective variety and let $S$ be a positive closed (1,1)-current on $X_{\text{reg}}$ with bounded mass near $X_{\text{sing}}$ (see Definition II.3) and $Y \subset X$ an irreducible algebraic subset of codimension $l$ in $X$. We want to study the locus inside $Y$ where the Lelong numbers of $S$ are larger than the generic Lelong number of $S$ along $Y$. For every $c > 0$ we denote by $E_c(S)$ the Lelong upper level sets of $S$ defined as the analytic subset

$$E_c(S) := \overline{\{x \in X_{\text{reg}} \mid \nu(S, x) \geq c\}}$$

and by $E_c^Y(S) = E_c(S) \cap Y$ the Lelong upper level sets of $S$ at $Y$. Let

$$0 \leq \beta_1 \leq \beta_2 \ldots \leq \beta_{\dim(X)-l+1}$$

be the *jumping numbers of* $E_c^Y(S)$, i.e. for every $c \in ]\beta_p, \beta_{p+1}]$ the algebraic set $E_c^Y(S)$ has codimension $p$ in $Y$ with at least one component of codimension exactly $p$. Let $\{Z_{p,r}\}_{r\geq 1}$ be the countable collection of irreducible components of $\bigcup_{c\in]\beta_p,\beta_{p+1}]} E_c^Y(S)$ of codimension exactly $p$ in $Y$ and denote by $\nu_{p,r}$ the generic Lelong number of $S$ at $Z_{p,r}$. Note that $\beta := \nu(S,Y)$ the generic Lelong number of $S$ along $Y$ corresponds to $\beta_1$. Then we obtain the main result of this section

**Theorem III.5.** *With the same notation as above, there exist a positive constant* $C$*, depending only on the geometry of* $X$ *and* $Y$*, such that*

$$\sum_{r\geq 1} (\nu_{p,r} - \beta)^p \int_{Z_{p,r}} \omega^{k-l-p} \leq C \int_{Xreg} S \wedge \omega^{\dim(X)-1},$$

*for all* $p = 1, \ldots, \dim(X) - l + 1$*, where* $\omega$ *is the Fubini-Study metric of* $X$*.*

We have proved this result in [Par10].

### 3.2.1 Examples

We provide a few examples showing the value of our result.

**Example III.6.** Let's start with the trivial case where $X = Y$ is a projective curve. In this case, the current $S$ is a positive finite measure on $X$, giving us

$$\int_X S \geq \sum_{x\in X} \nu(S,x) = \sum_{r\geq 1} \nu_{1,r}.$$

So Theorem III.5 follows immediately.

**Example III.7.** If $X$ is a projective manifold (i.e. smooth) and $Y = X$, hence $l = 0$, $\beta = 0$ and $E_c^Y(S) = E_c(S)$, then the inequality of Theorem III.5 can be written as

$$\sum_{r\geq 1} \nu_{p,r}^p \int_{Z_{p,r}} \omega^p \leq C.$$

In [Dem92] Theorem 7.1, J.P. Demailly proved that under the same assumptions as above, we have that there is a positive constant $C' > 0$ such that

$$\sum_{r\geq 1}(\nu_{p,r}-\beta_1)\cdots(\nu_{p,r}-\beta_p)\int_{Z_{p,r}}\omega^p \leq C',$$

where $\beta_1 \leq \ldots \leq \beta_{k+1}$ are the jumping numbers of $S$. Observing that

$$\nu_{p,r} \geq \nu_{p,r} - \beta_j \quad \forall\ j = 1,\ldots,p$$

Theorem III.5 shows that

$$\sum_{r\geq 1}(\nu_{p,r}-\beta_1)\cdots(\nu_{p,r}-\beta_p)\int_{Z_{p,r}}\omega^p \leq \sum_{r\geq 1}\nu_{p,r}^p\int_{Z_{p,r}}\omega^p \leq C$$

implying Demailly's result.

In the same setting, it is also interesting to observe the two extreme cases $p = 1$ and $p = k$:

a) The case $p = 1$ follows immediately from Siu's decomposition theorem, since

$$S = \sum_{j\geq 1}\lambda_j[A_j] + R$$

with $R \geq 0$ and $\{Z_{1,r}\}_r \subset \{A_j\}_j$, therefore

$$S \geq \sum_{r\geq 1}\nu_{1,r}[Z_{1,r}]$$

and the result follows after integrating this inequality with $\int \cdot \wedge \omega^{k-1}$.

b) The case $p = k$ is especially interesting when $E_c(S)$ is countable for all $c > 0$. A remarkable result proved in Corollary 6.4 of [Dem92] is that if $E_c(S)$ countable then the class $\{S\}$ is nef. Moreover, Demailly gave the more refined inequality

$$\sum_{r\geq 1}\nu_{k,r}^k + \int_X S_{\mathrm{ac}}^k \leq \int_X \{S\}^k,$$

where $S_{\mathrm{ac}}$ is the absolutely continuous part in the Lebesgue decomposition of the coefficients of $S$.

**Example III.8.** Let $X$ be the projective plane $\mathbb{P}^2$, let $Y \subset \mathbb{P}^2$ be an irreducible curve and let $S$ be the current of integration defined by $S := (\deg(D))^{-1}[D]$ (hence $\|S\| = 1$), where $D \neq Y$ is another irreducible curve. It is not hard to see from the proof of our theorem (see Subsection 3.2.2) that the constant $C > 0$ satisfies $C = \deg(Y)$ and that

$$\sum_{r \geq 1} \nu_{1,r} \int_{Z_{1,r}} \omega = \#(Y \cap D) \cdot (\deg(D))^{-1}.$$

Therefore Theorem III.5 is nothing but Bézout's theorem for these two curves.

**Example III.9.** If $X$ is a projective manifold (i.e. smooth) and $Y \subset X$ an irreducible smooth hypersurface, hence $l = 1$, then using Siu's decomposition theorem it is easy to see that the closed (1,1)-current $S - \beta[Y]$ is positive. Assume that $S - \beta[Y]$ admits local potentials not identically $-\infty$ along $Y$. Then we can restrict $S - \beta[Y]$ to $Y$ and the statement is reduced to Example III.7.

It is important to remark that it is not always possible to restrict positive close currents. Part of the idea in the proof of Theorem III.5 is to restrict an approximation of the current.

*Remark* III.10. If $Y \subset X$ has codimension $l > 1$ in $X$, then it is not even possible to subtract $[Y]$ from $S$ since the dimensions do not match. Therefore there is no direct method for studying the Lelong numbers of $S$ inside $Y$, so our theorem proves to be useful in the general case.

#### 3.2.2 Proof of the intersection inequality

We will divide the proof of Theorem III.5 into three steps. In the first step we will assume that our projective variety $X$ is smooth and that the cohomology class of $S$ is nef. We can then find suitable smooth representatives of $\{S\}$ which, together with the sequence obtained in Theorem III.4, will allow us to approximate $S$ by a sequence of currents $S_{c,m}$ of bounded potentials and therefore we will be able to intersect such sequence with the current of integration $[Y]$; this procedure will be the key for obtaining our result in this setting. In the second step, we still assume $X$ smooth but $\{S\}$ not necessarily nef; using that $H^{1,1}(X;\mathbb{R})$ is finite dimensional and the upper semicontinuity of Lelong numbers, we will replace $S$ by a current $\hat{S}$ with nef class $\{\hat{S}\}$ but the same Lelong numbers as $S$ everywhere; then we apply our result in Step 1 to $\hat{S}$ implying the same conclusion to $S$. Finally, in Step 3 we prove the general case when $X$ is a projective variety, not necessarily smooth, by taking a resolution of singularities of $X$, and applying Step 2 to the strict transform of $S$, $Y$ and $Z_{p,r}$.

We recall the notions of *numerically effective (nef), pseudoeffective (psef) and Kähler cones* (for details see [Laz04], also [Bou02]). The space of classes of *real (1,1)-forms* $H^{1,1}(X;\mathbb{R})$ is defined as

$$H^{1,1}(X;\mathbb{R}) := H^{1,1}_{\bar\partial}(X;\mathbb{C}) \cap H^2(X;\mathbb{R}) = \left\{\alpha \in H^{1,1}_{\bar\partial}(X;\mathbb{C}) \mid \bar\alpha = \alpha\right\},$$

where $H^{1,1}_{\bar\partial}(X;\mathbb{C})$ is the Dolbeault (1,1)-cohomology of $X$.

The *Kähler cone* $\mathcal{K}(X)$, the *Psef cone* $\mathcal{P}(X)$ and the *Nef cone* $\mathcal{N}(X)$ are defined as

$$\mathcal{K}(X) := \{\alpha \in H^{1,1}(X;\mathbb{R}) \mid \alpha \text{ can be represented by a Kähler form}\},$$

$$\mathcal{P}(X) := \{\alpha \in H^{1,1}(X;\mathbb{R}) \mid \alpha \text{ can be represented by a positive closed (1,1)-current}\},$$

and

$$\mathcal{N}(X) := \{\alpha \in H^{1,1}(X;\mathbb{R}) \mid \text{ if for every } \epsilon > 0,\ \alpha \text{ can be represented by a smooth form } \alpha_\epsilon \text{ such that } \alpha_\epsilon \geq -\epsilon\omega\}$$

respectively. Note that if $X$ is Kähler (or projective) the set $\mathcal{K}(X)$ is not empty.

It follows from the definitions above that

$$\emptyset \neq \mathcal{K}(X) \subset \mathcal{N}(X) \subset \mathcal{P}(X) \quad \text{and} \quad \mathcal{K}(X) = \mathrm{Int}(\mathcal{N}(X)).$$

We prove now our Main Theorem:

**Step 1**: Assume $X$ to be a (smooth) complex projective manifold and the class $\{S\}$ to be nef. For this case, we will actually prove a slightly more general result, where we will be able to 'kill Lelong numbers' locally. More precisely, given any subset $\Xi$ of $Y$ and $p = 1, \ldots, \dim(X) - l + 1$, we denote the *jumping numbers* $b_p = b_p(S, \Xi)$ *of* $E_c^Y(S)$ *with respect to* $\Xi$ as

$$b_p := \inf\{c > 0 \mid \mathrm{codim}_x(E_c^Y(S); Y) \geq p,\ \forall\, x \in \Xi\}.$$

In our situation, the subset $\Xi$ will be a Zariski dense subset of $Y$ with a prescribed geometrical condition, namely, $\Xi$ will be the complement of all irreducible components of $E_c^Y(S)$ of codimension strictly less than $p$. Following Demailly we prove the following lemma

**Lemma III.11.** *Let* $\Xi$ *be any subset of* $Y$ *and* $0 \leq b_1 \leq b_2 \leq \ldots \leq b_{\dim(X)-l+1}$ *the jumping numbers of* $E_c^Y(S)$ *with respect to* $\Xi$*. Fix a positive line bundle with smooth curvature* $u$ *as in Theorem III.4 and assume that the class* $\{S\}$ *is nef. Then for every* $p = 1, \ldots, \dim(X) - l + 1$ *there exists a positive closed* $(l+p, l+p)$*-current* $\Theta_p$ *in* $X$ *with support on* $Y$ *such that*

$$\{\Theta_p\} = \{Y\} \cdot (\{S\} + b_1\{u\}) \cdots (\{S\} + b_p\{u\}) \in H^{l+p,l+p}(X;\mathbb{R}), \tag{3.5}$$

$$\Theta_p \geq \sum_{r \geq 1} (\nu_{p,r} - b_1) \cdots (\nu_{p,r} - b_p)[Z_{p,r}]. \tag{3.6}$$

*Proof.* Let $c > b_1$ and let $\alpha \in \{S\}$ be a smooth real (1,1)-form. Take the sequence of currents $S_{c,m} = \alpha + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_{c,m}$ as in Theorem III.4 where $\varphi_{c,m}$ is singular along $E_c(S)$ and $S_{c,m} \geq -\frac{2}{m}\omega - cu$. Since we are assuming $\{S\}$ to be nef, for every $m \in \mathbb{N}$ we can pick $\alpha_m \in \{S\}$ smooth such that $\alpha_m \geq -\frac{2}{m}\omega$ and we can write $\alpha_m$ as $\alpha_m = \alpha + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi_m$ with $\psi_m$ smooth. Set

$$\varphi_{c,m,L} := \max\{\varphi_{c,m}, \psi_m - L\},$$

for $L \gg 0$ and $S_{c,m,L} := \alpha + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_{c,m,L}$. Observe that by adding the local potentials of $\omega$ and $cu$ to the the max between $\varphi_{c,m}$ and $\psi_m - L$ we easily conclude that the closed current $S_{c,m,L}$ satisfies

$$S_{c,m,L} + \frac{2}{m}\omega + cu \geq 0.$$

The family of potentials $\{\varphi_{c,m,L}\}$ is bounded everywhere, therefore

$$\Theta_{1,c,m,L} := [Y] \wedge \left(S_{c,m,L} + \frac{2}{m}\omega + cu\right)$$

is a well defined positive closed $(l+1, l+1)$-current on $X$ with support on $Y$ by Theorem II.2. By extracting a weak limit we define

$$\Theta_{1,c,m} := \lim_{L \to +\infty} \Theta_{1,c,m,L}$$

on $X$. Since the potentials $\varphi_{c,m,L}$ decrease monotonically to $\varphi_{c,m}$ as $L \to +\infty$ we have that

$$\Theta_{1,c,m} = [Y] \wedge \left(S_{c,m} + \frac{2}{m}\omega + cu\right)$$

in a neighborhood of $\Xi$, since for every point $x \in \Xi$ we can find a neighborhood $U$ of $x$ such that the unbounded locus of $[Y]$ and $S_{c,m}$ has codimension $\geq l+1$ in $U$

(or real dimension $\leq 2\dim(X) - 2l - 2$) hence by Theorem II.2 the current $\Theta_{1,c,m}$ is well defined in a neighborhood of $\Xi$, and $\{\Theta_{1,c,m}\} = \{Y\} \cdot \big(\{S\} + \frac{2}{m}\{\omega\} + c\{u\}\big)$ for every $m \geq 1$ and every $c > b_1$ and for every $x \in X$,

$$\nu(\Theta_{1,c,m}, x) \geq \mathrm{ord}_x(Y)\nu(S_{c,m}, x) \geq \mathrm{ord}_x(Y)(\max\{\nu(S, x) - c - \dim(X)/m, 0\}).$$

Note also that the total mass of the family $\{\Theta_{1,c,m}\}$ is uniformly bounded. We extract (modulo a subsequence) a limit

$$\Theta_1 := \lim_{c \searrow b_1} \lim_{m \nearrow +\infty} \Theta_{1,c,m},$$

which satisfies $\{\Theta_1\} = \{Y\} \cdot (\{S\} + b_1\{u\})$ and by the upper semicontinuity of Lelong numbers we obtain

$$\nu(\Theta_1, x) \geq (\nu_{1,r} - b_1), \quad \forall\, x \in Z_{1,r}\ \forall\, r \geq 1.$$

By Siu's decomposition theorem, $\Theta_1$ can be written as

$$\Theta_1 = \sum_{j \geq 1} \lambda_j [V_j] + R_1,$$

where for every $j \geq 1$, $V_j$ is an irreducible variety of codimension $l + 1$ in $X$, $\lambda_j$ is the generic Lelong number of $\Theta_1$ along $V_j$ and $R_1$ is a positive closed current with upper level sets $E_c(R_1)$ of codimension strictly bigger than $l + 1$ for all $c > 0$. This in particular implies that for all $r \geq 1$ we have that $Z_{1,r} = V_{j_r}$ for some $j_r$ and for a generically chosen $x \in Z_{1,r}$ we obtain

$$\lambda_{j_r} = \nu(\Theta_1, x) \geq (\nu_{1,r} - b_1) \Longrightarrow \Theta_1 \geq \sum_{r \geq 1} (\nu_{1,r} - b_1)[Z_{1,r}].$$

Now we proceed by induction on $2 \leq p \leq \dim(X) - l + 1$. We assume we have constructed $\Theta_{p-1}$ with the desired properties and in the exact same way as before, for $c > b_p$ we define the positive closed $(l + p, l + p)$-current

$$\Theta_{p,c,m,L} := \Theta_{p-1} \wedge \left( S_{c,m,L} + \frac{2}{m}\omega + cu \right),$$

which is well defined everywhere. The current

$$\Theta_{p,c,m} := \lim_{L \to +\infty} \Theta_{p,c,m,L}$$

satisfies

- $\Theta_{p,c,m} = [Y] \wedge \left(S_{c,m} + \frac{2}{m}\omega + cu\right)$ in a neighborhood of $\Xi$,
- $\{\Theta_{p,c,m}\} = \{Y\} \cdot \left(\{S\} + \frac{2}{m}\{\omega\} + c\{u\}\right)$ for every $m \geq 1$ and every $c > b_p$ and,
- $\nu(\Theta_{p,c,m}, x) \geq \nu(\Theta_{p-1}, x) \max\{(\nu(S,x) - c - \dim(X)/m), 0\}$ for every $x \in X$.

We extract a weak limit (modulo a subsequence)

$$\Theta_p := \lim_{c \searrow b_p} \lim_{m \nearrow +\infty} \Theta_{p,c,m},$$

which (by the same arguments as above) satisfies the desired properties.

$\square$

**Step 2**: Now assume that $X$ is a complex projective manifold and let $\omega$ be any Kähler form on $X$. However, the class $\{S\}$ is not necessarily nef.

Let

$$\mathcal{P}_1 := \{\alpha \in \mathcal{P}(X) \mid \|\alpha\| = 1\} \subset H^{1,1}(X; \mathbb{R}) \tag{3.7}$$

be a slice of the pseudoeffective cone of $X$, where $\|\cdot\|$ is any norm on the finite dimensional real vector space $H^{1,1}(X;\mathbb{R})$. Since $\mathcal{P}_1$ is compact and $\mathrm{Int}(\mathcal{N}(X)) = \mathcal{K}(X) \neq \emptyset$ we can pick $A_0 = A_0(\{\omega\}) > 0$ such that $A\{\omega\} + \alpha$ is nef for every $A \geq A_0$ and every $\alpha \in \mathcal{P}_1$. Note also that the set of of positive closed currents $S$ with a fixed cohomology class is also (weakly) compact. Moreover, by the upper semicontinuity in both variables of the Lelong numbers is easy to see that there exists a constant $\tau = \tau(X)$ such that $\nu(S,x) \leq \tau$ for every $x \in X$ and every positive closed (1,1)-current $S$ so that $\{S\} \in \mathcal{P}_1$.

Now, fixing $A \geq A_0$ we define the positive closed (1,1)-current $\hat{S} := S + A\omega$. It satisfies:

- $\{\hat{S}\} \in \mathcal{N}(X)$,
- $\nu(\hat{S}, x) = \nu(S, x)$ for every $x \in X$ (in particular, the Lelong upper level sets $E_c^Y(\hat{S})$ and $E_c^Y(S)$ coincide, giving us the same decomposition in terms of jumping numbers).

Taking $\beta = \nu(S, Y) = \nu(\hat{S}, Y)$ and defining the set

$$\Xi_p := \complement \left( \cup_{c>\beta} (\text{Irreducible components of } E_c^Y(S) \text{ of codimension } < p) \right),$$

we obtain that the jumping numbers with respect to $\Xi_p$ satisfy

$$b_1(S, \Xi_p) = \ldots = b_p(S, \Xi_p) = \beta.$$

If $\{Z_{p,r}\}_{r\geq 1}$ are the irreducible components of $E_c^Y(\hat{S}) = E_c^Y(S)$ for $c \in ]\beta_p, \beta_{p+1}]$ of codimension exactly $p$ in $Y$ and $\nu_{p,r}$ the generic Lelong numbers, we apply the previous lemma to $\hat{S}$, hence we obtain a positive closed $(l+p, l+p)$-current $\Theta_p$ on $X$ with support on $Y$ such that

$$\begin{aligned} \{\Theta_p\} = \{Y\} \cdot (\{\hat{S}\} + b_1\{u\}) \cdots (\{\hat{S}\} + b_p\{u\}) = \\ = \{Y\} \cdot (\{S\} + A\{\omega\} + b_1\{u\}) \cdots (\{S\} + A\{\omega\} + b_p\{u\}) \end{aligned}$$

and

$$\sum_{r\geq 1} (\nu_{p,r} - \beta)^p [Z_{p,r}] \leq \Theta_p.$$

We apply $\int_X \cdot \wedge \omega^{\dim(X)-l-p}$ to the inequality, giving us

$$\sum_{r\geq 1}(\nu_{p,r}-\beta)^p \int_{Z_{p,r}} \omega^{\dim(X)-l-p} \leq \int_X \Theta_p \wedge \omega^{\dim(X)-l-p} =$$

$$= \int_X [Y] \wedge (S + A\omega + b_1 u) \wedge \cdots \wedge (S + A\omega + b_p u) \wedge \omega^{\dim(X)-l-p} \leq$$

$$\leq \int_X [Y] \wedge ((1+A)\omega + \tau u)^p \wedge \omega^{\dim(X)-l-p} =: C.$$

**Step 3**: We now prove the theorem in the general case.

Let $\pi : \tilde{X} \to X$ be a resolution of singularities. Since $Y$ and $Z_{p,r}$ are not contained in $X_{\text{sing}}$, we can define $\tilde{Y}$ and $\tilde{Z}_{p,r}$ the strict transforms of $Y$ and $Z_{p,r}$, respectively. Let $\tilde{S}$ be the positive closed (1,1)-current defined by

$$\tilde{S} := \pi^* S \;\text{ on }\; \pi^{-1}(X_{\text{reg}}).$$

By assumption, $\tilde{S}$ has locally bounded mass around $\pi^{-1}(X_{\text{sing}})$ hence by Theorem II.1 the extension by zero of $\tilde{S}$ is a positive closed (1,1)-current on $\tilde{X}$. On the other hand, since $\pi : \pi^{-1}(X_{\text{reg}}) \to X_{\text{reg}}$ is a biholomorphism we can conclude that $\nu(\tilde{S}, \tilde{Z}_{p,r}) = \nu_{p,r}$ and $\nu(\tilde{S}, \tilde{Y}) = \beta$.

We know by Step 2 that if $\tilde{\omega}$ is the Fubini-Study metric on $\tilde{X}$ we can find a positive constant $C$ depending only on $\tilde{X}$, $\tilde{Y}$ and $\tilde{\omega}$ such that

$$C \geq \sum_{r\geq 1}(\nu_{p,r}-\beta)^p \int_{\tilde{Z}_{p,r}} \tilde{\omega}^{\dim(X)-l-p}.$$

We prove the following lemma

**Lemma III.12.** *Let $A$ be an ample line bundle defined on $X$ and $\tilde{\omega}$ the Fubini-Study metric on $\tilde{X}$. Then, there exist $\delta > 0$ depending only on $\tilde{\omega}$ and $A$ such that for every irreducible algebraic set $Z \subset X$ not contained in $X_{sing}$ of dimension $q$ and strict transform $\tilde{Z}$ the following holds*

$$\int_{\tilde{Z}} \tilde{\omega}^q \geq \delta(A^q \cdot Z)$$

*where* $(A^q \cdot Z)$ *denotes the intersection number* $\int_Z c_1(A)^q$.

*Proof of Lemma.* First observe that

$$(A^q \cdot Z) = \int_Z c_1(A)^q = \int_{\pi_* \tilde{Z}} c_1(A)^q = \int_{\tilde{Z}} \pi^*(c_1(A)^q) = \int_{\tilde{Z}} (\pi^* c_1(A))^q.$$

Since $\tilde{\omega}$ is positive, we can find $\epsilon >$ small enough such that the class of $\{\alpha\} := \{\tilde{\omega}\} - \epsilon \pi^* c_1(A)$ is numerically effective (even ample) on $\tilde{X}$. Then

$$\int_{\tilde{Z}} \tilde{\omega}^q = \int_{\tilde{Z}} (\epsilon \pi^* c_1(A) + \alpha)^q =$$

$$= \int_{\tilde{Z}} (\epsilon \pi^* c_1(A))^q + \sum_{i=0}^{q-1} \binom{q}{i} \int_{\tilde{Z}} (\epsilon \pi^* c_1(A))^i \wedge \alpha^{q-i} \geq$$

$$\geq \epsilon^q \int_{\tilde{Z}} (\pi^* c_1(A))^q \geq \delta (A^q \cdot Z),$$

where $\delta := \epsilon^{\dim(X)}$. This proves the lemma. $\square$

Now picking $\tilde{\omega}$ on $\tilde{X}$ and $\delta > 0$ as above, and taking $A = \mathcal{O}_X(1)$ the theorem follows since

$$C' := C\delta^{-1} \geq \sum_{r \geq 1} (\nu_{p,r} - \beta)^p \delta^{-1} \int_{\tilde{Z}_{p,r}} \tilde{\omega}^{p+l} \geq \sum_{r \geq 1} (\nu_{p,r} - \beta)^p \int_{Z_{p,r}} \omega^{p+l}.$$

This completes the proof of the Theorem.

# CHAPTER IV

# Orders of vanishing and the Jacobian cocycle

## 4.1 Orders of vanishing

In this section we discuss orders of vanishing on algebraic varieties, where a major difficulty will be to deal with the singular locus. The standard references [Kol97] and [KM98] (see also [CKM88]) contain a details discussion of all the concepts discussed in this section.

Let $X$ be an irreducible projective normal variety of dimension $k$, and assume $X$ to be $\mathbb{Q}$-factorial, i.e. every Weil divisor is $\mathbb{Q}$-Cartier. If $\pi : Y \to X$ is a birational morphism, we say that $E \subset Y$ is a *divisor over* $X$ if $E$ is a smooth prime divisor in $Y$. We say that $E$ *lies over a point* $x \in X$ if $\pi(E) = \{x\}$.

If $\phi$ is a rational function on $X$ and $E$ a divisor over $X$, we write $\mathrm{ord}_E(\phi)$ for the order of vanishing of $\phi \circ \pi$ along the divisor $E$. Similarly, if $D$ is a Weil divisor on $X$, we set $\mathrm{ord}_E(D) := \frac{1}{m}\,\mathrm{ord}_E\left(\pi^*(mD)\right)$ where $m \in \mathbb{N}$ is chosen so that $mD$ is Cartier. As usual, we denote by $K_X$ the canonical divisor class of $X$.

Above, we may assume that $\pi : Y \to X$ is a *log-resolution* of $X$, i.e. the exceptional locus $\mathrm{Exc}(\pi)$ of $\pi$ has simple normal crossing. In this case, there exists a unique divisor $K_{Y/X}$ on $Y$ supported on $\mathrm{Exc}(\pi)$, the *relative canonical divisor*, which is in the divisor class of $K_Y - \pi^* K_X$. If both $X$ and $Y$ are smooth, then $K_{Y/X}$ is nothing but the effective divisor defined by the Jacobian $\mathrm{Jac}(\pi)$ of $\pi$.

Given a prime divisor $E$ over $X$, we define the *log-discrepancy* $a_E$ of $E$ by

$$a_E := \mathrm{ord}_E\left(K_{Y/X}\right) + 1. \tag{4.1}$$

We say that $X$ is klt (short for Kawamata log-terminal) if $a_E > 0$ for every prime divisor $E$ over $X$.

Let us give a simple example: Let $\pi : Y \to X$ be the blow-up of $X$ at a smooth point $0 \in X$. We can choose local coordinates $(x_1, \ldots, x_k)$ so that our map $\pi$ can be written as

$$\pi(x_1, \ldots, x_k) = (x_1x_2, \ldots, x_{k-1}x_k, x_k),$$

giving us that $\mathrm{Jac}(\pi) = x_k^{k-1}$. Hence, if $E = \pi^{-1}(0)$ is the exceptional prime divisor above 0, we have that

$$a_E = \mathrm{ord}_E\left(\mathrm{Jac}(\pi)\right) + 1 = k. \tag{4.2}$$

Recall that if $x \in X$ and $\phi \in \mathcal{O}_{X,x}$, the order of vanishing $\mathrm{ord}_x(\phi)$ of $\phi$ at $x$ is defined to be

$$\mathrm{ord}_x(\phi) := \max\left\{s \in \mathbb{N} \mid \phi \in \mathfrak{m}_x^s\right\}$$

where $\mathfrak{m}_x$ denotes the maximal ideal of the local ring $\mathcal{O}_{X,x}$. If $x$ is smooth and $E$ is an exceptional divisor of a single blowup (at $x$), then $\mathrm{ord}_x(\phi)$ and $\mathrm{ord}_E(\phi)$ coincide. However, when $x \in X$ is singular, $\mathrm{ord}_x$ may not be a valuation.

The following well known lemma (which can be found in [Tou72] p.178 Lemma 1.3) will play an important role in this thesis. We write down the proof:

**Lemma IV.1.** *Let $E$ be a prime divisor over $X$ above a smooth point $x \in X$. Then*

$$\mathrm{ord}_E(D) \leq a_E \,\mathrm{ord}_x(D)$$

*for every divisor $D \subset X$.*

*Proof.* The problem is local, hence we assume $(X, x) \equiv (\mathbb{C}^k, 0)$. Let $\phi$ be the germ of a holomorphic function at $0 \in \mathbb{C}^k$ and let $c(\phi)$ be the *complex singularity exponent* of $\phi$, i.e.

$$c(\phi) := \sup\left\{c > 0 \mid |\phi|^{-2c} \in L^1(\mathbb{C}^k, 0)\right\}.$$

It follows from a theorem of Skoda that $\frac{1}{c(\phi)} \leq \operatorname{ord}_0(\phi)$ (see also [DK01]), hence for every $c < \frac{1}{\operatorname{ord}_0(\phi)}$ we have that $|\phi|^{-2c} \in L^1(\mathbb{C}^k, 0)$.

Let $\pi : Y \to \mathbb{C}^k$ be a log-resolution over $\mathbb{C}^k$ above 0 and $E \subset \pi^{-1}(0)$ an exceptional prime divisor. For every open subset $U \subset \mathbb{C}^k$ containing 0 we have that

$$\int_{\pi^{-1}(U)} |\phi \circ \pi|^{-2c} |\operatorname{Jac}(\pi)|^2 dV = \int_U |\phi|^{-2c} dV < +\infty$$

by change of variables, where $dV$ is the standard volume form in $\mathbb{C}^k$. Therefore $|\phi \circ \pi|^{-2c} |\operatorname{Jac}(\pi)|^2 \in L^1$ along $E$ implying that

$$-2c \operatorname{ord}_E(\phi) + 2(a_E - 1) > -2 \Rightarrow \operatorname{ord}_E(\phi) < \frac{a_E}{c}.$$

Taking the limit $c \nearrow \frac{1}{\operatorname{ord}_0(\phi)}$ we obtain that

$$\operatorname{ord}_E(\phi) \leq a_E \operatorname{ord}_0(\phi).$$

□

Let $g : X \to X$ be a surjective regular map. Then there exists a unique Weil divisor $\mathcal{C}_g$ on $X$, the *critical divisor*, whose restriction to $X_{\text{reg}} \cap g^{-1}(X_{\text{reg}})$ equals the Cartier divisor

$$\left\{x \in X_{\text{reg}} \cap g^{-1}(X_{\text{reg}}) \mid \operatorname{Jac}(g) = 0\right\}.$$

Since $X$ is $\mathbb{Q}$-factorial, $\mathcal{C}_g$ is $\mathbb{Q}$-Cartier. It belongs to the divisor class of $g^*K_X - K_X$.

We need

**Lemma IV.2.** *Let $\pi : Y \to X$ be a log-resolution and let $E \subset Y$ be a prime divisor. Then, there exists a log-resolution $\pi' : Y' \to X$ and a prime divisor $E' \subset Y'$ such that the meromorphic lifting $\bar{g} : Y \dashrightarrow Y'$ satisfies $\bar{g}(E) = E'$.*

*Proof.* To any rank 1 valuation $\nu : \mathbb{C}(X) \setminus \{0\} \to \mathbb{R}$ of the function field $\mathbb{C}(X)$ we can associate two basic invariants: the *value group*

$$\Gamma_\nu := \{\nu(\phi) \mid \phi \in \mathbb{C}(X) \setminus \{0\}\} \subset \mathbb{R}$$

and the *residue field*

$$\mathbb{K}(\nu) := \{\nu \geq 0\}/\{\nu > 0\}.$$

A valuation $\nu$ is of the form $r \operatorname{ord}_E$ where $r > 0$ and $E$ is a divisor over $X$ if and only if $\Gamma_\nu = r\mathbb{Z}$ and $\mathbb{K}(\nu)$ has trascendence degree $\dim(X) - 1$ over $\mathbb{C}$ (see [Vaq00, Proposition 10.1]).

In our situation, set $\nu := \operatorname{ord}_E$ and $\nu' := g_*\nu$. Then

$$\Gamma_{\nu'} \subset \Gamma_\nu = \mathbb{Z}$$

hence $\Gamma_{\nu'} = r\mathbb{Z}$ for some $r \in \mathbb{N}$.

Furthermore, since $g$ is a finite map, $\mathbb{K}(\nu)$ is a finite extension of $\mathbb{K}(\nu')$, therefore both $\mathbb{K}(\nu)$ and $\mathbb{K}(\nu')$ have the same trascendence degree over $\mathbb{C}$, i.e. $\dim(X) - 1$. From the above description we have that $\nu' = r \operatorname{ord}_{E'}$ for some prime divisor $E'$ over $X$. ☐

As a consequence of Lemma IV.2, we know we can choose $x'' \in E$ generic, such that $\bar{g}$ is holomorphic at $x''$, the critical set $\mathcal{C}_{\bar{g}}$ of $\bar{g}$ is smooth at $x''$ and $\bar{g}(\mathcal{C}_{\bar{g}})$ smooth at $y' = \bar{g}(x'')$. Picking local coordinates $(z, z_k)$ and $(w, w_k)$ around $x''$ and $y'$ in such way that $E = \{z_k = 0\}$ and $E' = \{w_k = 0\}$, we see that

$$(w, w_k) = \bar{g}(z, z_k) = (z, z_k^r).$$

In particular, we notice that

$$\operatorname{ord}_E(\mathcal{C}_{\bar{g}}) = r - 1 \tag{4.3}$$

and

$$g_* \operatorname{ord}_E = r \operatorname{ord}_{E'} . \tag{4.4}$$

**Proposition IV.3.** *With the same notation as above, the following identity holds*

$$ra_{E'} = a_E + \operatorname{ord}_E(\mathcal{C}_g).$$

*Proof.* Let us first assume $X$ to be smooth and fix $\omega$ a meromorphic k-form on $X$, i.e. $\omega$ can be writen as

$$\omega(x) = h(x)dx_1 \wedge \cdots \wedge dx_k,$$

where $x = (x_1, \ldots, x_k)$ is some local chart and $h$ a meromorphic function on $X$.

From the commutative diagram

$$\begin{tikzcd} E \subset Y \arrow[r, dashed, "\bar{g}"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & Y' \supset E' \arrow[d, "\pi'"] \\ X \arrow[r, "g"'] & X \end{tikzcd} \tag{4.5}$$

we obtain (in local coordinates) that

$$\begin{aligned} \pi^* g^* \omega &= \pi^* g^*(h(x)dx_1 \wedge \cdots \wedge dx_k) \\ &= \pi^*(h \circ g(x) \operatorname{Jac}(g)(x) dx_1 \wedge \cdots \wedge dx_k) \\ &= h \circ g \circ \pi(y) \operatorname{Jac}(g) \circ \pi(y) \operatorname{Jac}(\pi) dy_1 \wedge \cdots \wedge dy_k. \end{aligned} \tag{4.6}$$

On the other hand

$$\begin{aligned} \bar{g}^* \pi'^* \omega &= \bar{g}^* \pi'^*(h(x)dx_1 \wedge \cdots \wedge dx_k) \\ &= \bar{g}^*(h \circ \pi'(y) \operatorname{Jac}(\pi') dy_1 \wedge \cdots \wedge dy_k) \\ &= h \circ \pi' \circ \bar{g}(y) \operatorname{Jac}(\pi') \circ \bar{g}(y) \operatorname{Jac}(\bar{g})(y) dy_1 \wedge \cdots \wedge dy_k. \end{aligned} \tag{4.7}$$

Using the identity $\pi^* g^* = \bar{g}^* \pi'^*$ from (4.5) in (4.6) and (4.7), we obtain the following identity of (Cartier) divisors

$$\pi^* \mathcal{C}_g + K_{Y/X} = \bar{g}^* K_{Y'/X} + \mathcal{C}_{\bar{g}}. \tag{4.8}$$

This in particular implies that

$$\operatorname{ord}_E(\mathcal{C}_g) + \underbrace{\operatorname{ord}_E(K_{Y/X})}_{a_E - 1} = \underbrace{\bar{g}_* \underbrace{\operatorname{ord}_E(K_{Y'/X})}_{a_{E'} - 1}}_{r(a_{E'} - 1)} + \underbrace{\operatorname{ord}_E(\mathcal{C}_{\bar{g}})}_{r-1}$$

by (4.3) and (4.4), giving us that $r a_{E'} = a_E + \operatorname{ord}_E(\mathcal{C}_g)$.

If $X$ is not smooth, we pick a meromorphic k-form $\omega$ on $X_{\text{reg}}$. Note that $\operatorname{div}(\omega)$ extends uniquely as a Weil divisor on $X$ and our assumption on $X$, namely, $X$ is $\mathbb{Q}$-factorial, allows us to obtain the identity (4.8) in the singular case. The computation then follows identically as in the smooth case. $\square$

#### 4.1.1 Comparison of orders of vanishing

By definition, $x \in X$ is an *isolated quotient singularity* if there exists a finite group $G_x \subset GL(k, \mathbb{C})$ acting freely on $\mathbb{C}^k \setminus \{0\}$ such that

$$(X, x) \cong (\mathbb{C}^k, 0)/G_x.$$

From now on, we will assume that our variety $X$ has at worst isolated quotient singularities. This in particular implies that $X$ is $\mathbb{Q}$-factorial and with klt singularities (see [KM98], Prop. 5.15 and Prop. 5.20).

**Lemma IV.4.** *For every $y \in X$ there exists a constant $C_y \geq 1$ such that*

$$\operatorname{ord}_y \phi \leq \operatorname{ord}_0 \phi \circ \varrho \leq C_y \operatorname{ord}_y \phi$$

*for every holomorphic germ $\phi \in \mathcal{O}_{X,y}$. If $y$ is smooth, we can pick $C_y = 1$.*

*Proof.* Let $\phi \in \mathcal{O}_{X,y}$ and denote by $t := \operatorname{ord}_y \phi$. We then have that

$$\phi \in \mathfrak{m}_y^t \quad \text{and} \quad \phi \notin \mathfrak{m}_y^{t+1},$$

hence

$$\phi \circ \varrho \in \varrho^* \mathfrak{m}_y^t \quad \text{and} \quad \phi \circ \varrho \notin \varrho^* \mathfrak{m}_y^{t+1}.$$

We can pick $C = C(y) \in \mathbb{N}$ such that for every $l \in \mathbb{N}$ it follows that

$$\mathfrak{m}_0^{lC} \subset \varrho^* \mathfrak{m}_y^l \subset \mathfrak{m}_0^l,$$

hence

$$\phi \circ \varrho \in \mathfrak{m}_0^t \quad \text{and} \quad \phi \circ \varrho \notin \mathfrak{m}_0^{(t+1)C}$$

$$\operatorname{ord}_y(\phi) \leq \operatorname{ord}_0(\phi \circ \varrho) \leq (\operatorname{ord}_y(\phi) + 1)C \leq 2C \operatorname{ord}_y(\phi).$$

Take $C_y = 2C$ and the inequalities follow. □

If $g : X \to X$ is a holomorphic map with $y = g(x)$ and

$$\varrho : (\mathbb{C}^k, 0) \to (X, y) \quad \text{and} \quad \varrho' : (\mathbb{C}^k, 0) \to (X, x)$$

are the quotient maps of $y$ and $x$ respectively, the map $g \circ \varrho' : (\mathbb{C}^k, 0) \to (X, y)$ can be lifted to a continuous (hence holomorphic) map $\hat{g} : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ such that the following diagram commutes

(4.9)
$$\begin{tikzcd} (\mathbb{C}^k, 0) \arrow[r, "\hat{g}"] \arrow[d, "\varrho'"'] & (\mathbb{C}^k, 0) \arrow[d, "\varrho"] \\ (X, x) \arrow[r, "g"'] & (X, y) \end{tikzcd}$$

In particular, we have that $\mathcal{C}_{\hat{g}} = \varrho^* \mathcal{C}_g$.

We state the main result of this section

**Theorem IV.5.** *There exists a positive constant $C_X \geq 1$ independent of $g$ such that the following inequality holds*

$$\operatorname{ord}_x \phi \circ g \leq C_X \left(k + \operatorname{ord}_x \mathcal{C}_g\right) \operatorname{ord}_{g(x)} \phi$$

*for every $\phi \in \mathcal{O}_{X,x}$ and every $x \in X$.*

*Proof.* **Step 1:** Assume $x$, $y = g(x) \in X$ both smooth.

Let $\pi : Y \to X$ be the blow-up of $X$ at $x$ and $E = \pi^{-1}(x)$ the exceptional divisor. By Lemma IV.2 we can find a birational morphism $\pi' : Y' \to X$ and a divisor $E' \subset Y'$ such that the lift $\bar{g} : Y \dashrightarrow Y'$ of $g$ satisfies $\bar{g}(E) = E'$ and $g_* \operatorname{ord}_E = r \operatorname{ord}_{E'}$ for some $r \in \mathbb{N}$. Since $\pi(E) = x$ we must have $\pi'(E') = y$. Hence, for every $\phi \in \mathcal{O}_{X,y}$ we have

$$\operatorname{ord}_x(\phi \circ g) = \operatorname{ord}_E(\phi \circ g) = r \operatorname{ord}_{E'}(\phi) \leq r a_{E'} \operatorname{ord}_y(\phi)$$

where the last inequality follows from Proposition IV.1.

By Proposition IV.3 we have that

$$r a_{E'} = a_E + \operatorname{ord}_E(\mathcal{C}_g),$$

where $a_E = k$ by (4.2). Therefore

$$\operatorname{ord}_x(\phi \circ g) \leq (k + \operatorname{ord}_x(\mathcal{C}_g)) \operatorname{ord}_y(\phi). \tag{4.10}$$

**Step 2:** The general case follows using diagram (4.9). Let $\phi \in \mathcal{O}_{X,y}$. For the left inequality of Theorem IV.5 we have that

$$\operatorname{ord}_x(\phi \circ g) \leq \operatorname{ord}_0(\phi \circ g \circ \varrho') = \operatorname{ord}_0(\phi \circ \varrho \circ \hat{g}).$$

Inequality (4.10) implies that

$$\operatorname{ord}_0(\phi \circ \varrho \circ \hat{g}) \leq (k + \operatorname{ord}_0(\operatorname{Jac}(\hat{g}))) \operatorname{ord}_0(\phi \circ \varrho)$$

and by Lemma IV.4 we have that

$$\operatorname{ord}_0(\phi \circ \varrho) \leq C_x \operatorname{ord}_{g(x)}(\phi)$$

for some $C_x \geq 1$. Note that

$$\operatorname{ord}_0(\operatorname{Jac}(\hat{g})) = \operatorname{ord}_0 \mathcal{C}_{\hat{g}} = \operatorname{ord}_0(\varrho'^* \mathcal{C}_g) \leq C'_x \operatorname{ord}_x \mathcal{C}_g$$

for some $C'_x \geq 1$. Taking $C_X := \max_{x \in X}\{C_x C'_x\}$ we obtain the desired inequality. □

## 4.2 The Jacobian cocycle

In this section we proceed to define one of our key tools, the Jacobian cocycle. For an extensive discussion on (analytic) cocycles we refer the reader to [Fav00], [Fav99] and [Din09].

### 4.2.1 Definition and properties of the Jacobian cocycle

Let $X$ be an irreducible normal projective variety with at worst isolated quotient singularities and let $g : X \to X$ be a surjective holomorphic self-map. Then $X$ is $\mathbb{Q}$-factorial and klt (see Section 4.1). For every $n \in \mathbb{N}$ we denote by $\mu_n^X$ the Zariski usc function

$$X \ni x \mapsto \mu_n^X(x) := C_X(\dim(X) + \operatorname{ord}_x \mathcal{C}_{g^n})$$

on $X$, with $C_X \geq 1$ as in Theorem IV.5. If $Z \subset X$ is an irreducible algebraic subset, we denote by $\mu_n^X(Z)$ the generic value of $\mu_n^X$ on $Z$ given by

$$\mu_n^X(Z) := \min_{x \in Z} \left\{\mu_n^X(x)\right\} = C_X \left(\dim(X) + \operatorname{ord}_Z(\mathcal{C}_{g^n})\right).$$

It is easy to see that the identity

$$\mathcal{C}_{g^{n+m}} = \mathcal{C}_{g^n} + (g^n)^* \mathcal{C}_{g^m} \tag{4.11}$$

follows on a suitable Zariski open subset of $X$ for every $n$, $m \in \mathbb{N}$. Since $X$ is $\mathbb{Q}$-factorial, the identity extends to all of $X$ as $\mathbb{Q}$-Cartier divisors.

**Proposition IV.6.** *The following is true for the family* $\left(\mu_n^X\right)_{n\in\mathbb{N}}$

*(i) (Comparison) Let $D$ be a divisor in $X$. Then for every $x \in X$,*

$$\operatorname{ord}_x\left((g^n)^* D\right) \leq \mu_n^X(x) \operatorname{ord}_{g^n(x)}(D),$$

*(ii) (Submultiplicativity) for every $n,\ m \in \mathbb{N}$ and for every $x \in X$ the following inequality holds*

$$\mu_{n+m}^X(x) \leq \mu_n^X(x)\mu_m^X\left(g^n(x)\right).$$

*Proof.* Part (i) follows immediately from Theorem IV.5.

For proving (ii), observe that from (4.11) we obtain

$$\mu_{n+m}^X(x) = C_X(\dim(X) + \operatorname{ord}_x(\mathcal{C}_{g^{n+m}})) =$$

$$= C_X(\dim(X) + \operatorname{ord}_x(\mathcal{C}_{g^n}) + \operatorname{ord}_x((g^n)^*\mathcal{C}_{g^m})).$$

By Theorem IV.5 we have that

$$\operatorname{ord}_x((g^n)^*\mathcal{C}_{g^m}) \leq C_X\left(\dim(X) + \operatorname{ord}_x(\mathcal{C}_{g^n})\right) \operatorname{ord}_{g^n(x)}(\mathcal{C}_{g^m})$$

implying that

$$\mu_{n+m}^X(x) \leq \left(C_X(\dim(X) + \operatorname{ord}_x(\mathcal{C}_{g^n}))\right)\left(\frac{1}{C_X} + \operatorname{ord}_{g^n(x)}(\mathcal{C}_{g^m})\right) \leq$$

$$\leq \mu_n^X(x)\mu_m^X(g^n(x)).$$

□

By the submultiplicativity property, it is easy to see that the function

$$X \ni x \mapsto \mu_\infty^X(x) := \lim_{n\to+\infty}\left(\mu_n^X(x)\right)^{\frac{1}{n}}$$

is well defined (i.e. the limit always exist for every $x \in X$). It satisfies

$$\mu_\infty^X \circ g = \mu_\infty^X.$$

### 4.2.2 Jacobian cocycles and totally invariant sets

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of algebraic degree $d \geq 2$ and let $X \subset \mathbb{P}^k$ be an irreducible algebraic set such that $f^{-1}(X) = X$. Define $g := f|_X : X \to X$ and take $\tilde{g} : \tilde{X} \to \tilde{X}$ to be the lift of $g$ to the normalization $\pi : \tilde{X} \to X$. The commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{g}"] \arrow[d, "\pi"'] & \tilde{X} \arrow[d, "\pi"] \\ X \arrow[r, "g"'] \arrow[d, "\iota"'] & X \arrow[d, "\iota"] \\ \mathbb{P}^k \arrow[r, "f"'] & \mathbb{P}^k \end{tikzcd}$$

where $\iota : X \to \mathbb{P}^k$ denotes the inclusion map, gives us the following commutative diagram of groups and homomorphisms

$$\begin{tikzcd} \mathbb{Z} \cong H^2(\mathbb{P}^k;\mathbb{Z}) \arrow[r, "f^*=d\cdot"] \arrow[d, "(\iota\pi)^*"'] & H^2(\mathbb{P}^k;\mathbb{Z}) \cong \mathbb{Z} \arrow[d, "(\iota\pi)^*"] \\ H^2(\tilde{X};\mathbb{Z}) \arrow[r, "\tilde{g}^*"'] & H^2(\tilde{X};\mathbb{Z}) \end{tikzcd}$$

where $\iota : X \hookrightarrow \mathbb{P}^k$ In particular, if $\omega$ is the Fubini-Study metric on $\mathbb{P}^k$ then $\{\omega\}$ generates $H^2(\mathbb{P}^k, \mathbb{Z})$ and $f^*\{\omega\} = d\{\omega\}$, therefore

$$\tilde{g}^*\pi^*\iota^*\{\omega\} = \pi^*g^*\iota^*\{\omega\} = \pi^*\iota^*f^*\{\omega\} = d \cdot \pi^*\iota^*\{\omega\}$$

giving us a $\tilde{g}^*$-invariant class $\{\omega_{\tilde{X}}\} := \{(\iota\pi)^*\omega\}$ in $H^2(\tilde{X};\mathbb{Z})$.The class $\{\omega\}$ represents the first Chern class of the ample line bundle $\mathcal{O}(1)$ in $\mathbb{P}^k$, which induces an ample line bundle $\mathcal{O}_{\tilde{X}}(1) := (\iota\pi)^*\mathcal{O}(1)$ on $\tilde{X}$ with

$$\tilde{g}^*c_1\left(\mathcal{O}_{\tilde{X}}(1)\right) = \tilde{g}^*\{\omega_{\tilde{X}}\} = d\{\omega_{\tilde{X}}\}.$$

The ample classes on $\tilde{X}$ form a strict open convex cone which is invariant by $\tilde{g}^*$ in the finite dimensional vector space $H^2(\tilde{X};\mathbb{R})$ (for details see [GH94], [Laz04]). Using that there exists an invariant ample class on $\tilde{X}$, namely $\{\omega_{\tilde{X}}\}$ with $(\tilde{g}^*)^n\{\omega_{\tilde{X}}\} = d^n\{\omega_{\tilde{X}}\}$ for all $n \in \mathbb{N}$, it is possible to conclude that $\|(\tilde{g}^*)^n\| \lesssim d^n$ for every $n \in \mathbb{N}$.

Moreover, if we assume $\tilde{X}$ to have at worst isolated quotient singularities then we obtain

**Lemma IV.7.** *In the same setting as above, there exists a positive constant $A$ independent of $x \in \tilde{X}$ and $n \in \mathbb{N}$ such that*

$$\mu_n^{\tilde{X}}(x) \leq Ad^n.$$

*Proof.* Note that

$$\mathcal{C}_{\tilde{g}^n} = \sum_{i=0}^{n-1} (\tilde{g}^i)^* \mathcal{C}_{\tilde{g}}$$

(see equation (4.11)).

Denote by $A' := \sup_{x\in\tilde{X}} \operatorname{ord}_x(\mathcal{C}_{\tilde{g}})$. Then we have that

$$\begin{aligned} \operatorname{ord}_x(\mathcal{C}_{\tilde{g}^n}) \leq A' \sum_{i=0}^{n-1} d^i = A' \frac{d^n - 1}{d-1} \implies& \\ \implies \mu_n^X(x) \leq C_X \left( \dim(X) + A' \frac{d^n-1}{d-1} \right) \leq& \\ \leq C_X \left( \frac{\dim(X)}{d^n} + A' \frac{1-d^{-n}}{d-1} \right) d^n \leq (\underbrace{\dim(X) + A'(d-1)^{-1}}_{A})d^n&. \end{aligned}$$

∎

**Theorem IV.8.** *The $\tilde{g}$-totally invariant set*

$$E_{\tilde{X}} := \left\{ x \in \tilde{X} \mid \mu_\infty^{\tilde{X}}(x) = d \right\} = \bigcup_{\delta>0} \bigcap_{n\in\mathbb{N}} \left\{ x \in \tilde{X} \mid \operatorname{ord}_x(\mathcal{C}_{\tilde{g}^n}) \geq \delta d^n \right\}$$

*is algebraic.*

The proof relies on the fact that the family of totally invariant algebraic subsets of $\tilde{X}$ is finite (see Section 2). The key idea is to prove that every irreducible component of $E_{\tilde{X}}$ is totally invariant for some iterate of $g$, therefore $E_{\tilde{X}}$ has only finitely many components. In order to do this we use a uniform bound from [Par10] for the orders of vanishing of $d^{-N}[\mathcal{C}_{\tilde{g}^N}]$.

*Proof.* We argue by contradiction: Let $Z \subset E_{\tilde{X}}$ be an irreducible component; define $S_N := d^{-N}[\mathcal{C}_{\tilde{g}^N}]$ for $N$ large ($N$ will be made explicit later). By definition there exists $\delta > 0$ independent of $N$ such that

$$\operatorname{ord}_x(S_N) \geq \delta, \quad \forall\, x \in Z.$$

For every $0 \leq r < N$, by the chain rule $\mathcal{C}_{\tilde{g}^N} = \mathcal{C}_{\tilde{g}^{r+N-r}} = \mathcal{C}_{\tilde{g}^r} + (\tilde{g}^r)^*\mathcal{C}_{\tilde{g}^{N-r}}$ (see equation (4.11)) we obtain that for every $x \in \tilde{g}^{-r}(Z)$

$$\operatorname{ord}_x \mathcal{C}_{\tilde{g}^N} \geq \operatorname{ord}_x \left((\tilde{g}^r)^*\mathcal{C}_{\tilde{g}^{N-r}}\right) \geq \operatorname{ord}_{\tilde{g}^r(x)} \mathcal{C}_{\tilde{g}^{N-r}} \geq \delta d^{N-r}$$

implying

$$\operatorname{ord}_x(S_N) \geq \delta d^{-r}, \quad \forall\, x \in \tilde{g}^{-r}(Z).$$

Let $Y$ be the minimal irreducible algebraic set containing $Z$ which is totally invariant by $\tilde{g}^s$ for some $s \geq 1$. For simplicity we assume $s = 1$. If $Z \neq Y$, then $Z$ has positive codimension $p > 0$ in $Y$ and since $Y \not\subset E_{\tilde{X}}$ we can find $C > 0$ and $\lambda < d$ such that $\operatorname{ord}_Y(\mathcal{C}_{\tilde{g}^n}) \leq C\lambda^n$ for all $n \geq 1$. Then it follows that

$$\operatorname{ord}_x(S_N) \leq C\left(\frac{\lambda}{d}\right)^N \ll 1$$

for every $x \in Y$. Denote by $\beta$ the generic Lelong number of $S_N$ along $Y$. Thus $0 \leq \beta \leq C(\lambda/d)^N$.

Since $Z$ is not totally invariant, following the ideas of Dinh-Sibony (see [DS08], Lemma 6.10) we know there exists a constant $\theta > 0$ (independent of $r$) and algebraic sets $Z_r \subset f^{-r}(Z)$ of degrees $d_r$ satisfying

$$d_r \geq \theta d^{rp} \quad \forall\, r \geq 1.$$

Define $Z'_0 := Z_0$ and $Z'_r := \overline{Z_r \setminus (Z_0 \cup \cdots \cup Z_{r-1})}$ for $r > 0$; note that for $r \neq s$, $Z'_r$ and $Z'_s$ have no common irreducible components. Denote by $d'_r$ the degree of $Z'_r$. It is clear by the construction that $d'_0 + \ldots + d'_r \geq d_r$ and that the generic Lelong numbers $\nu_r$ of $S_N$ at $Z'_r$ satisfy $\nu_r \geq \delta d^{-r}$.

By Theorem III.5, we know that there exists a positive constant $A_{\tilde{X}}$ independent of $r \geq 1$ and $N$ such that

$$\sum_{r=0}^{M} (\nu_r - \beta)^p d'_r \leq A_{\tilde{X}}. \tag{4.12}$$

We now fix $M < N$ (which can be made very large) such that

$$\beta d^r \leq \frac{1}{2}\delta, \quad \forall\, r = 0, \ldots, M.$$

We observe that

$$\begin{aligned}
\sum_{r=0}^{M} (\nu_r - \beta)^p d'_r \geq \sum_{r=0}^{M} (\delta d^{-r} - \beta)^p d'_r \geq& \\
\geq \left(\frac{\delta}{2}\right)^p \sum_{r=0}^{M} d^{-rp} d'_r =& \\
= \left(\frac{\delta}{2}\right)^p [d'_0(1 - d^{-p}) + (d'_0 + d'_1)(d^{-p} - d^{-2p}) + \ldots& \\
\ldots + (d'_0 + \ldots + d'_{M-1})(d^{-(M-1)p} - d^{-Mp}) + d_M d^{-Mp}].&
\end{aligned} \tag{4.13}$$

Notice that for every $r = 0, \ldots, M-1$ we have

$$d^{-rp} - d^{-(r+1)p} \geq \frac{1}{2} d^{-rp}.$$

Plugging this into (4.13) we obtain that

$$\begin{aligned}
\sum_{r=0}^{M} (\nu_r - \beta)^p d'_r \geq \frac{1}{2}\left(\frac{\delta}{2}\right)^p \sum_{r=0}^{M} d^{-rp} d_r \geq& \\
\geq \frac{1}{2}\left(\frac{\delta}{2}\right)^p \sum_{r=0}^{M} d^{-rp} \theta d^{rp} = \frac{\theta}{2}\left(\frac{\delta}{2}\right)^p (M+1),&
\end{aligned} \tag{4.14}$$

where $\delta > 0$ and $\theta > 0$ are independent of $M$ and $N$. Therefore, using equation (4.12) in the inequality (4.14) we obtain

$$\frac{\theta}{2}\left(\frac{\delta}{2}\right)^p (M+1) \leq A_{\tilde{X}}$$

which produces a contradiction if we take $N$ and $M$ sufficiently large. $\square$

**Corollary IV.9.** *The algebraic set $E_X := \pi\left(E_{\tilde{X}}\right) \subset X$ is totally invariant:*

$$g^{-1}(E_X) = E_X.$$

*Proof of Corollary IV.9.* We prove that every component of $E_X$ is totally invariant.

Let $Z \subset E_X$ be an irreducible component and write

$$\pi^{-1}(Z) = \tilde{Z} \cup \tilde{Z}',$$

where $\tilde{Z} \subset E_{\tilde{X}}$ and $\pi(\tilde{Z}) = Z$. Then there exists $l \geq 1$ such that $\tilde{g}^{-l}(\tilde{Z}) = \tilde{Z}$, this in particular implies that

$$g^l(Z) = g^l(\pi(\tilde{Z})) = \pi(\tilde{g}^l(\tilde{Z})) = \pi(\tilde{Z}) = Z \Longrightarrow Z \subset g^{-l}(Z).$$

Write $g^{-l}(Z) = Z \cup W$.

If $W \neq \emptyset$ we have that $g^l(W) = Z$. On the other hand,

$$\tilde{g}^{-l}\pi^{-1}(Z) = \tilde{g}^{-l}(\tilde{Z} \cup \tilde{Z}') = \tilde{Z} \cup \tilde{g}^{-l}(\tilde{Z}')$$

and

$$\tilde{g}^{-l}\pi^{-1}(Z) = \pi^{-1}g^{-l}(Z) = \pi^{-1}(Z \cup W) = \tilde{Z} \cup \tilde{Z}' \cup \pi^{-1}(W).$$

Putting this together we obtain that

$$\tilde{g}^{-l}(\tilde{Z}') = \tilde{Z}' \cup \pi^{-1}(W) \Rightarrow \tilde{Z}' = \tilde{g}^l(\tilde{g}^{-l}(\tilde{Z}')) = \tilde{g}^l(\tilde{Z}') \cup \tilde{g}^l(\pi^{-1}(W)).$$

Since the map $g := f|_X : X \to X$ is open, it is easy to see that

$$\tilde{g}(\pi^{-1}(W)) = \pi^{-1}(g(W)) \tag{4.15}$$

for every $W \subset X$. In particular, by identity (4.15) we have that

$$\tilde{g}^l(\pi^{-1}(W)) = \pi^{-1}(g^l(W)) = \pi^{-1}(Z) = \tilde{Z} \cup \tilde{Z}'$$

implying

$$\tilde{Z}' = \tilde{g}^l(\tilde{Z}') \cup \tilde{Z} \cup \tilde{Z}' \Longrightarrow \tilde{Z} \subset \tilde{Z}'$$

contradicting our hypothesis. Hence $W = \emptyset$ and therefore $g^{-l}(Z) = Z$. □

#### 4.2.3 The exceptional family $\mathcal{E}_f$

**Definition IV.10.** We define the *exceptional family* $\mathcal{E}_f$ *of* $f$ as the finite collection of irreducible subsets $X \subseteq \mathbb{P}^k$ such that

(i) $\mathbb{P}^k \in \mathcal{E}_f$;

(ii) $X \in \mathcal{E}_f \setminus \{\mathbb{P}^k\}$ if and only if there exist $X' \in \mathcal{E}_f$ such that $X$ is an irreducible component of $E_{X'}$. In this case we will say that $X$ is an immediate successor of $X'$.

The exceptional family $\mathcal{E}_f$ of $f$ is a partially ordered set, where $X \preceq Y$ if there exist a sequence of elements $X = X_1 \subsetneq \cdots \subsetneq X_r = Y$ in $\mathcal{E}_f$ such that $X_i$ is an immediate successor of $X_{i+1}$ for all $i = 1, \ldots, r-1$.

Note that by definition, we have that

$$\emptyset \preceq X \preceq \mathbb{P}^k \quad \forall X \in \mathcal{E}_f.$$

We will say that $X \in \mathcal{E}_f$ is an *exceptional leaf* if $\emptyset$ is the immediate succesor of $X$ (i.e. $E_X = \emptyset$).

#### 4.2.4 Asymptotic behavior

We finish this section giving a uniform estimate of the orders of vanishing outside the totally invariant algebraic subset $E_{\tilde{X}} \subset \tilde{X}$.

**Theorem IV.11.** *There exist constants $C > 0$ and $0 \leq \rho < d$ such that*

$$\sup_{x \notin E_{\tilde{X}}} \mu_n^{\tilde{X}}(x) \leq C\rho^n$$

*for all $n \in \mathbb{N}$.*

**Corollary IV.12.** *Given any hypersurface $H$ in $X$, it follows that*

$$\sup_{x \notin E_X} d^{-n} \operatorname{ord}_x \left((g^n)^* H\right) \to 0$$

*as $n \to +\infty$.*

*Proof Corollary IV.12.* Note that for every $x \in X$ we have

$$\operatorname{ord}_x \left((g^n)^* H\right) \leq \max_{y \in \pi^{-1}(x)} \operatorname{ord}_y \left((\tilde{g}^n)^* (\pi^* H)\right)$$

implying

$$\sup_{x \notin E_X} \operatorname{ord}_x \left((g^n)^* H\right) \leq \sup_{y \notin \pi^{-1}(\pi(E_{\tilde{X}})} \operatorname{ord}_y \left((\tilde{g}^n)^* (\pi^* H)\right) \leq \sup_{y \notin E_{\tilde{X}}} \operatorname{ord}_y \left((\tilde{g}^n)^* (\pi^* H)\right).$$

By Proposition IV.6 (ii), it follows that

$$\sup_{y \notin E_{\tilde{X}}} \operatorname{ord}_y \left((\tilde{g}^n)^* (\pi^* H)\right) \leq \sup_{y \notin E_{\tilde{X}}} \mu_n^{\tilde{X}}(y) \operatorname{ord}_{\tilde{g}^n(y)} H$$

which combined with Theorem IV.11 gives us

$$\sup_{x \notin E_X} d^{-n} \operatorname{ord}_x \left((g^n)^* H\right) \leq C \left(\frac{\rho}{d}\right)^n$$

for some $C > 0$ and $\rho < d$. Taking $n \to +\infty$ we obtain the desired convergence to zero. $\square$

*Proof Theorem IV.11.* Let $\tilde{X}_{0,1},\ldots,\tilde{X}_{0,m_0}$ be the irreducible components of the critical set $\mathcal{C}_{\tilde{g}}$ not contained in $E_{\tilde{X}}$. For every $i=1,\ldots,m_0$ we can pick $x_{0,i}\in\tilde{X}_{0,i}$ such that there exist $C_1>0$ and $\lambda_1<d$ satisfying

$$\max_{i=1,\ldots,m_0}\{\mu_n^{\tilde{X}}(x)\}<C_1\lambda_1^n,\quad \forall\, n\geq 1.$$

For $N>1$ large, define the algebraic set

$$\tilde{X}_1:=\{x\in\tilde{X}\mid \mu_N^{\tilde{X}}(x)\geq C_1\lambda_1^N\}.$$

We clearly have the proper inclusion of algebraic sets

$$\tilde{X}_1\subsetneq\mathcal{C}_{\tilde{g}^N},$$

where the codimension of $\tilde{X}_1$ in $\mathcal{C}_{\tilde{g}^N}$ is $\geq 1$ at every point $x\in\tilde{X}_1\setminus E_{\tilde{X}}$.

If $\tilde{X}_1\subset E_{\tilde{X}}$, for $n\gg N$ and $x\in\tilde{X}\setminus E_{\tilde{X}}$, denoting $n=tN+l$, $l\in\{0,\ldots,N-1\}$ we have

$$\mu_n^{\tilde{X}}(x)=\mu_{tN+l}^{\tilde{X}}(x)\leq\mu_l^{\tilde{X}}(x)\mu_{tN}^{\tilde{X}}(\tilde{g}^l(x))\leq\mu_N^{\tilde{X}}(x)\prod_{j=0}^{t-1}\mu_N^{\tilde{X}}(\tilde{g}^{l+jN}(x)).$$

Since $\tilde{X}\setminus E_{\tilde{X}}$ is totally invariant

$$x\notin E_{\tilde{X}}\Longrightarrow\tilde{g}^{l+jN}(x)\notin E_{\tilde{X}},$$

hence $\tilde{g}^{l+jN}(x)\notin\tilde{X}_1$ implying that

$$\mu_n^{\tilde{X}}(x)\leq(\dim(\tilde{X})+C_1\lambda_1^N)(\dim(\tilde{X})+C_1\lambda_1^N)^t\leq 2C_1^{t+1}\lambda_1^{n-l-1}.$$

Now we can find $A>0$ and $\lambda_1<\rho<d$ (independent of $x$ and $n$) such that

$$2C_1^{t+1}\lambda_1^{n-l-1}\leq A\rho^n,\quad\forall\, n\geq 1,$$

and the theorem follows.

Now assume $\tilde{X}_1 \not\subset E_{\tilde{X}}$. Let $\tilde{X}_{1,1}, \ldots, \tilde{X}_{1,m_1}$ be the irreducible components of $\tilde{X}_1$ that are not contained in $E_{\tilde{X}}$. As before, for every $i = 1, \ldots, m_1$ we can pick $x_{1,i} \in X_{1,i}$ and $C_2 \geq C_1$, $\lambda_1 < \lambda_2 < d$ such that

$$\max_{i=1,\ldots,m_1} \{\mu_n^{\tilde{X}}(x)\} < C_2 \lambda_2^n, \quad \forall\, n \geq 1.$$

Define the algebraic set

$$\tilde{X}_2 := \{x \in \tilde{X} \mid \mu_N^{\tilde{X}}(x) \geq C_2 \lambda_2^N\},$$

which has codimension $\geq 2$ in $\mathcal{C}_{\tilde{g}}$ at every point $x \in \tilde{X}_2 \setminus E_{\tilde{X}}$. Again, if $\tilde{X}_2 \subset E_{\tilde{X}}$ the theorem follows picking some $A > 0$ and $\lambda_2 < \rho < d$, so we can assume $\tilde{X}_2 \not\subset E_{\tilde{X}}$. Inductively we construct a strictly decreasing sequence of algebraic sets

$$\tilde{X}_j := \{x \in \tilde{X} \mid \mu_N^{\tilde{X}}(x) \geq C_j \lambda_j^N\},$$

for $j = 1, \ldots, \dim(\tilde{X})$, where $\lambda_1 < \lambda_2 < \cdots < \lambda_{\dim(\tilde{X})} < d$, $0 < C_1 \leq C_2 \leq \cdots \leq C_{\dim(\tilde{X})}$ and the codimension of $\tilde{X}_j$ in $\mathcal{C}_{\tilde{g}}$ is $\geq j$ at every point $x \in \tilde{X}_j \setminus E_{\tilde{X}}$. Thus, there exists $1 \leq j_0 \leq \dim(\tilde{X})$ such that $\tilde{X}_{j_0} \subset E_{\tilde{X}}$ (since $\tilde{X}_{\dim(\tilde{X})} \setminus E_{\tilde{X}} = \emptyset$), implying that there exist $A \geq C_{j_0}$ and $\lambda_{j_0} < \rho < d$ so that

$$\mu_n^{\tilde{X}}(x) \leq 2C_{j_0}^{t+1} \lambda_{j_0}^{n-l-1} \leq A\rho^n, \quad \forall\, n \geq 1,$$

for every $x \in \tilde{X}_{j_0} \setminus E_{\tilde{X}}$ as before. This concludes the proof. $\square$

# CHAPTER V

# Equidistribution

## 5.1 Reduction of the problem

We will need a version of Theorem III.3, (iii) which is preserved by the dynamical system $f : \mathbb{P}^k \to \mathbb{P}^k$, where $f$ is holomorphic of degree $d \geq 2$. We prove:

**Proposition V.1.** *For every $x \in \mathbb{P}^k$ and for every $n$, $m \in \mathbb{N}$ we have*

$$0 \leq \nu((f^n)^*S, x) - \nu((f^n)^*S_m, x) \leq (k+1)\frac{d^n}{m}.$$

*Proof.* It suffices to prove this for $n = 1$. The left inequality uses Ohsawa-Takegoshi extension theorem following the exact same argument as in Theorem III.2. We prove then the right inequality.

Let $\pi : Y \to \mathbb{P}^k$ be the blowup of $\mathbb{P}^k$ at $x \in \mathbb{P}^k$ and $E = \pi^{-1}(x)$ the exceptional divisor above $x$, hence

$$\nu(S, x) = \nu(S, E).$$

By Lemma IV.2 we can find an exceptional divisor $E'$ over $f(x) \in \mathbb{P}^k$ such that $f_* \operatorname{ord}_E = r \operatorname{ord}_{E'}$ for some $r \in \mathbb{N}$. We therefore obtain

$$\nu(f^*S, x) = \nu(f^*S, E) = r\nu(S, E')$$

implying that

$$0 \leq \nu(f^*S, x) - \nu(f^*S_m, x) = r\left(\nu(S, E') - \nu(S_m, E')\right). \tag{5.1}$$

We need the following strengthening of Theorem III.3 (iii)

**Lemma V.2.** *Let $\pi : Y \to \mathbb{P}^k$ be a modification of $\mathbb{P}^k$ at $x \in \mathbb{P}^k$ and let $E \subset \pi^{-1}(x)$ be an exceptional divisor. Then, with $S$ and $S_m$ as before, we have*

$$0 \leq \nu(S, E) - \nu(S_m, E) < \frac{a_E}{m}$$

*for every $m \in \mathbb{N}$, where $a_E$ denotes the log-discrepancy of $E$.*

The result given by Lemma V.2 can be found in [BFJ08], p.486. We sketch a proof of this

*Proof.* Pick coordinates $(x_1, \ldots, x_k)$ around a general point of $E \subset Y$ and write $E = \{x_1 = 0\}$ (locally) around this point.

If $S = \omega + dd^c\varphi$, then

$$\varphi \circ \pi \leq \nu(S, E) \log |x_1| + O(1). \tag{5.2}$$

Given a local section $\sigma$ at $x \in \mathbb{P}^k$ of some element of $\mathcal{H}_m$, we have that

$$\int_U |\sigma|^2 e^{-2m\varphi} < +\infty$$

in a neighborhood $U$ of $x$. Therefore

$$\int_{\pi^{-1}(U)} |\sigma \circ \pi|^2 e^{-2m\varphi\circ\pi} |\operatorname{Jac}(\pi)|^2 < +\infty. \tag{5.3}$$

By the definition of $a_E$ (see Section 3) we obtain that

$$|\operatorname{Jac}(\pi)|^2 \sim |x_1|^{2(a_E - 1)} \tag{5.4}$$

around $E$, and in a general point of $E$ we have

$$|\sigma \circ \pi|^2 \sim |x_1|^{2\operatorname{ord}_E(\sigma)}. \tag{5.5}$$

Putting (5.2), (5.3), (5.4) and (5.5) together we obtain that

$$\int_{\pi^{-1}(U)} |x_1|^{2\,\mathrm{ord}_E(\sigma)-2m\nu(S,E)+2(a_E-1)} < +\infty$$

which implies (by Fubini's theorem)

$$\mathrm{ord}_E(\sigma) - m\nu(S,E) + a_E > 0 \tag{5.6}$$

for all $\sigma \in \mathcal{H}_m$. Dividing equation (5.6) by $m$ and taking the maximum over $\{\sigma_1, \ldots, \sigma_{N_m}\}$ an orthonormal basis of $\mathcal{H}_m$, we finally obtain

$$\nu(S_m, E) - \nu(S,E) + \frac{a_E}{m} > 0$$

which concludes the proof. □

Now, using Lemma V.2 into equation (5.1) we obtain that

$$r\left(\nu(S,E') - \nu(S_m,E')\right) \le \frac{r a_{E'}}{m}$$

and by Proposition IV.3 we obtain that

$$r\left(\nu(S,E') - \nu(S_m,E')\right) \le \frac{r a_{E'}}{m} = \frac{k + \mathrm{ord}_x(\mathrm{Jac}(f))}{m} \le \frac{(k+1)d}{m}$$

where $a_{E'}$ is the log-discrepancy of $E'$, since $\mathrm{ord}_x(\mathrm{Jac}(f)) \le (k+1)(d-1)$. □

The following result is an immediate consequence of Proposition V.1

**Corollary V.3.** *Let $X \subset \mathbb{P}^k$ be an irreducible variety. Then,*

$$\nu(S,X) = 0 \Leftrightarrow \nu(S_m, X) = 0$$

*for $m \gg 1$. Moreover,*

$$\lim_{n\to+\infty} \sup_{x\in\mathbb{P}^k} \nu\left(d^{-n}(f^n)^* S, x\right) = 0 \Leftrightarrow \lim_{n\to+\infty} \sup_{x\in\mathbb{P}^k} \nu\left(d^{-n}(f^n)^* S_m, x\right) = 0$$

*for $m \gg 1$.*

We would like to refine our approximation and replace $S_m$ by a current of integration on a hypersurface. Observe that for every finite collection of holomorphic germs $\sigma_1, \ldots, \sigma_N \in \mathcal{O}_{\mathbb{C}^k,0}$, we can find a (Zariski) generic $\theta = (\theta_i) \in \mathbb{C}^N$ such that, if we set $\sigma_\theta := \sum_{i=1}^N \theta_i \sigma_i$ then

$$\mathrm{ord}_0(\sigma_\theta) = \min_{i=1,\ldots,N} \mathrm{ord}_0(\sigma_i).$$

In particular, fixing an orthonormal basis $\{\sigma_{m,j}\}$ of $\mathcal{H}_m$ as before and given $\theta_m = (\theta_{m,j}) \in \mathbb{C}^{N_m}$, we denote by $\varphi_{m,\theta}$ the function

$$\mathbb{P}^k \ni x \mapsto \varphi_{m,\theta}(x) := \frac{1}{2m} \log \left( h^{\otimes(m+1)} \left( \sum_{j=1}^{N_m} \theta_{m,j}\sigma_{m,j} \right)(x) \right). \tag{5.7}$$

(Note the difference with $\varphi_m$ defined in Theorem III.2). It follows immediately that

$$\nu(\varphi_{m,\theta}, x) \geq \nu(\varphi_m, x), \quad \forall\, \theta \in \mathbb{C}^{N_m}, \forall\, x \in \mathbb{P}^k.$$

On the other hand, for each $x \in \mathbb{P}^k$ we can find a Zariski open set $V_x \subset \mathbb{C}^{N_m}$ such that

$$\nu(\varphi_{m,\theta}, x) = \nu(\varphi_m, x) = \min_{j=1,\ldots,N_m} \mathrm{ord}_x(\sigma_{m,j}), \quad \forall\, \theta_m \in V_x. \tag{5.8}$$

We prove the following

**Lemma V.4.** *Let $\mathcal{E}$ be a finite family of irreducible subsets of $\mathbb{P}^k$. Then, there exists a Zariski open subset $U_m \subset \mathbb{C}^{N_m}$ such that*

$$\mathrm{ord}_X(\varphi_{m,\theta}) = \mathrm{ord}_X(\varphi_m) \quad \forall\, X \in \mathcal{E}$$

*for all $\theta \in U$.*

*Proof.* For all $j = 1, \ldots, N_m$ and all $X \in \mathcal{E}$ there exists a Zariski open subset $U_{X,j} \subset \mathbb{P}^k$ such that

$$\mathrm{ord}_X(\sigma_j) = \min_{z \in X} \mathrm{ord}_z(\sigma_j) = \mathrm{ord}_x(\sigma_j) \quad \forall\, x \in X \cap U_{X,j}.$$

Then it follows that for every $x$ in the Zariski open subset $U_{\mathcal{E}} := \bigcap_{X\in\mathcal{E},\,j} U_{X,j} \subset \mathbb{P}^k$ we have

$$\mathrm{ord}_x(\sigma_j) = \mathrm{ord}_X(\sigma_j) \quad \forall\, x \in U_{\mathcal{E}} \cap X,\ j = 1, \ldots, N_m.$$

Fixing $x \in U_{\mathcal{E}}$, we now pick $V_x \subset \mathbb{C}^{N_m}$ as in (V.4) and the conclusion follows. $\square$

Using the same notation as above, we define the closed (1,1)-current

$$S_{m,\theta} := \omega + dd^c \varphi_{m,\theta}$$

on $\mathbb{P}^k$. It also satisfies $S_{m,\theta} \geq -\frac{1}{m}\omega$ and we note that

$$S_{m,\theta} + \frac{1}{m}\omega = \frac{1}{m}[H_m]$$

where $[H_m]$ is the current of integration over the hypersurface given by

$$H_m = \mathrm{div}\left(\sum_{j=1}^{N_m} \theta_{m,j}\sigma_{m,j}\right).$$

Carrying out the same argument given in Proposition V.1, we have proved the following crucial result

**Theorem V.5.** *Let $\mathcal{E}$ be any finite collection of irreducible algebraic varieties, let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic map of degree $d \geq 2$, and let $S$ be a positive closed (1,1)-current on $\mathbb{P}^k$. Then, for every $m \geq 1$, there exists a hypersurface $H_m \subset \mathbb{P}^k$ with*

$$\mathrm{ord}_X(H_m) \leq \nu(S, X) \quad \forall\, X \in \mathcal{E},$$

*and*

$$\nu\left(d^{-n}(f^n)^*S, x\right) \leq d^{-n}\,\mathrm{ord}_x\left((f^n)^*H_m\right) + \frac{k+1}{m}$$

*for all $x \in \mathbb{P}^k$.*

As an immediate consequence, if $S$ is a positive closed (1,1)-current and

$$\sup_{x\in\mathbb{P}^k} d^{-n}\,\mathrm{ord}_x\left((f^n)^*H_m\right)\to 0$$

for every $m\geq 1$, we obtain the equidistribution of $S$ towards the Green current associated to $f$.

## 5.2 Proofs of results

In this section we prove our main results. Recall that if $X\in\mathcal{E}_f$ then there exists $s\geq 1$ (minimal) such that $f^{-s}(X)=X$. By Lemma II.4 we can assume without loss of generality that $s=1$.

### 5.2.1 Proof of Theorems I.1 and I.3

By Theorem V.5, given any positive closed (1,1)-current and any finite family $\mathcal{E}$ of irreducible varieties, we can find a sequence of hypersurfaces $H_m$ with the properties

$$\mathrm{ord}_X(H_m)\leq \nu(S,X)\quad \forall\, X\in\mathcal{E},$$

and

$$0\leq \nu\left(d^{-n}(f^n)^*S,x\right)\leq d^{-n}\,\mathrm{ord}_x\left((f^n)^*H_m\right)+\frac{k+1}{m} \tag{5.9}$$

for all $x\in\mathbb{P}^k$. In particular, if the generic Lelong number $\nu(S,X)$ of $S$ along $X$ is zero for all $X\in\mathcal{E}$, we have that $\mathrm{ord}_X(H_m)=0$ for all $X\in\mathcal{E}$ and all $m\in\mathbb{N}$.

We recall from the introduction, Guedj's characterization of equidistribution:

$$d^{-n}(f^n)^*S\to T_f \iff \lim_{n\to+\infty}\sup_{x\in\mathbb{P}^k} d^{-n}\,\mathrm{ord}_x\left((f^n)^*H_m\right)=0. \tag{5.10}$$

*Proof of Theorem I.3.* Let $\mathcal{E}_{\mathrm{DS}}$ be the collection of irreducible components of the totally invariant algebraic set constructed by Dinh-Sibony (see the introduction). By [DS08], Theorem 7.1 we have

$$\mathrm{ord}_X(H_m)=0\quad\text{for all } X\in\mathcal{E}_{\mathrm{DS}}\Rightarrow d^{-n}(f^n)^*H_m\to T_f$$

which, by the right implication in (5.10), gives us

$$\lim_{n\to+\infty} \sup_{x\in\mathbb{P}^k} d^{-n} \operatorname{ord}_x \left((f^n)^* H_m\right) = 0$$

which implies

$$\limsup_{n\to+\infty} \sup_{x\in\mathbb{P}^k} \nu\left(d^{-n}(f^n)^* S, x\right) \leq \frac{k+1}{m}$$

for all $m \in \mathbb{N}$ by the inequality (5.9). Letting $m \to +\infty$ we get

$$\lim_{n\to+\infty} \sup_{x\in\mathbb{P}^k} \nu\left(d^{-n}(f^n)^* S, x\right) = 0$$

and by the left implication in (5.10), we conclude that

$$d^{-n}(f^n)^* S \to T_f.$$

□

To prove Theorem I.1, we use the same arguments as above and reduce the problem to the case of $S$ the current of integration over some hypersurface $H \subset \mathbb{P}^k$. Here, the family $\mathcal{E}_f$ is the exceptional family defined in Section 4.3.

*Proof of Theorem I.1.* Let $H \subset \mathbb{P}^k$ be a hypersurface such that $\operatorname{ord}_X(H) = 0$ for all $X \in \mathcal{E}_f$, where $\mathcal{E}_f$ is the exceptional family. Observe that $\operatorname{ord}_X(H) = 0$ implies that $H|_X$ is a well defined Cartier divisor in $X$.

Let $X \in \mathcal{E}_f$ and define $g := f|_X : X \to X$; then

(5.11) $$d^{-n} \operatorname{ord}_x((f^n)^* H) \leq d^{-n} \operatorname{ord}_x((g^n)^* H|_X) \quad \forall\, x \in X.$$

Let $\pi : \tilde{X} \to X$ be the normalization of $X$. By assumption, $\tilde{X}$ has at worst isolated quotient singularities, hence it is $\mathbb{Q}$-factorial and klt (see Section 4.1). Moreover, there exists a regular map $\tilde{g} : \tilde{X} \to \tilde{X}$ such that the diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow[g]{} & X \end{array}$$

commutes. Note that for every $y \in \tilde{X}$ and every germ $\phi \in \mathcal{O}_{\tilde{X},\pi(x)}$ if follows that $\mathrm{ord}_{\pi(y)}(\phi) \leq \mathrm{ord}_y(\phi \circ \pi)$. This, in particular implies that

$$\mathrm{ord}_{\pi(y)}((g^n)^* H|_X) \leq \mathrm{ord}_y(\pi^*(g^n)^* H|_X) = \mathrm{ord}_y((\tilde{g}^n)^*(\pi^* H|_X))$$

giving us

$$\sup_{x \in X} d^{-n} \mathrm{ord}_x((g^n)^* H|_X) \leq \sup_{y \in \tilde{X}} d^{-n} \mathrm{ord}_y((\tilde{g}^n)^*(\pi^* H|_X)).$$

By Theorem IV.5 we have that

$$d^{-n} \mathrm{ord}_y((\tilde{g}^n)^*(\pi^* H|_X)) \leq d^{-n} \mu_n^{\tilde{X}}(y) \, \mathrm{ord}_{\tilde{g}^n(y)}(\pi^* H|_X)$$

where $\mu_n^{\tilde{X}}$ is the submultiplicative cocycle defined by

$$\mu_n^{\tilde{X}}(y) = C_{\tilde{X}} \left( \dim(\tilde{X}) + \mathrm{ord}_y(\mathcal{C}_{\tilde{g}^n}) \right) \quad \text{(see Section 4).}$$

We know from Theorem IV.11 that there exist constants $C > 0$ and $0 \leq \rho < d$ such that

$$\sup_{y \notin E_{\tilde{X}}} \mu_n^{\tilde{X}}(y) \leq C \rho^n$$

for all $n \in \mathbb{N}$, where $E_{\tilde{X}}$ is the totally invariant algebraic set

$$E_{\tilde{X}} = \left\{ x \in \tilde{X} \mid \mu_\infty^{\tilde{X}}(x) = d \right\}.$$

Recalling that the algebraic set $E_X := \pi(E_{\tilde{X}})$ is totally invariant by $g$ (Corollary IV.9), we hence obtain

$$\begin{aligned} (5.12) \quad \sup_{x \in X} d^{-n} \mathrm{ord}_x((f^n)^* H) &\leq \sup_{x \notin E_X} d^{-n} \mathrm{ord}_x((f^n)^* H) + \sup_{x \in E_X} d^{-n} \mathrm{ord}_x((f^n)^* H) \leq \\ &\leq \sup_{x \notin E_{\tilde{X}}} d^{-n} \mathrm{ord}_y((\tilde{g}^n)^*(\pi^* H|_X)) + \sup_{x \in E_X} d^{-n} \mathrm{ord}_x((f^n)^* H) \leq \\ &\leq C \left( \frac{\rho}{d} \right)^n + \sup_{x \in E_X} d^{-n} \mathrm{ord}_x((f^n)^* H). \end{aligned}$$

We now proceed by induction on the partially ordered set $\mathcal{E}_f$: If $X$ is a leaf (i.e. $E_X = \emptyset$) we obtain that

$$\sup_{x \in X} \nu(d^{-n}(f^n)^* H, x) \leq C''(\rho/d)^n \to 0$$

as $n \to +\infty$. In general, for $X \in \mathcal{E}_f$ assume that for every $X' \preceq X$ we have that

$$\sup_{x \in X'} \nu(d^{-n}(f^n)^* H, x) \to 0$$

as $n \to +\infty$.

Since every irreducible component $X'$ of $E_X$ satisfies $X' \preceq X$, we get

$$\sup_{x \in E_X} d^{-n} \operatorname{ord}_x((f^n)^* H) \to 0$$

as $n \to +\infty$, implying

$$\sup_{x \in X} \nu(d^{-n}(f^n)^* H, x) \leq C' \left(\frac{\rho}{d}\right)^n + \sup_{x \in E_X} d^{-n} \operatorname{ord}_x((f^n)^* H) \to 0$$

by inequalities (5.11) and (5.12). The desired conclusion then follows. $\square$

#### 5.2.2 Proof of Corollary I.2

If $f : \mathbb{P}^3 \to \mathbb{P}^3$ is a holomorphic map of degree $d \geq 2$ and $X \subset \mathbb{P}^3$ is an irreducible surface such that $f^{-1}(X) = X$, Corollary B would follow immediately if the normalization of every such $X$ has at worst (isolated) quotient singularities. Let $g := f|_X : X \to X$, $\pi : \tilde{X} \to X$ its normalization and $\tilde{g} : \tilde{X} \to \tilde{X}$ its holomorphic lift.

By [Fav10, Theorem B] or [Wah90]) for every $x \in \tilde{X}$, we have

(i) If $(\tilde{X}, x)$ is klt, then $(\tilde{X}, \tilde{g}(x))$ is klt;

(ii) if $x \in \mathcal{C}_{\tilde{g}}$, then $(\tilde{X}, x)$ is klt;

(iii) if $x \notin \mathcal{C}_{\tilde{g}}$, then $(\tilde{X}, x)$ is not klt (the singularity is log-canonical instead).

The case (iii) can be divided in two cases: Either $(\tilde{X}, x)$ is a cusp or not. If $(\tilde{X}, x)$ is not a cusp, Proposition 2.1 in [Fav10] implies that we can find a proper modification $\bar{\pi} : \bar{X} \to \tilde{X}$ such that $\bar{X}$ has only klt singularities and $\tilde{g}$ lifts to $\bar{X}$ as a holomorphic map. The case $(\tilde{X}, x)$ a cusp can be ruled out by Theorem 1.4 in [Nak99]. This finishes the argument of Corollary B.

In the same setting as above, in [Zha00] D.Q. Zhang found a concrete classification for $X \subset \mathbb{P}^3$. More precisely, Zhang states that either $\deg(X) = 1$ (i.e. $X$ is a plane) or $X$ is a cubic given by one of the following four defining equations

(i) $X_3^3 + X_0X_1X_2$;

(ii) $X_0^2X_3 + X_0X_1^2 + X_2^3$;

(iii) $X_0^2X_2 + X_1^2X_3$;

(iv) $X_0X_1X_2 + X_0^2X_3 + X_1^3$.

The surfaces given by (i) and (ii) are both normal with klt singularities. The singular locus of the varieties given by (iii) and (iv) is a single line which is totally invariant and their normalizations correspond to the smooth surface given by the one-point blowup of $\mathbb{P}^2$.

### 5.2.3 Proof of Corollary I.4

We provide a direct argument for this, not relying on the results given by Dinh-Sibony in [DS08].

As before, we define the Jacobian cocycle

$$\mu_n(x) := k + \mathrm{ord}_x(\mathrm{Jac}(f^n))$$

as described in Section 4.2. The totally invariant set

$$E := \bigcup_{\delta > 0} \bigcap_{n \in \mathbb{N}} \left\{ x \in \mathbb{P}^k \mid \mu_n(x) \geq \delta d^n \right\}$$

is algebraic by Theorem IV.8 and we know from Theorem IV.11 that there exist positive constants $C$ and $\rho < d$ such that

$$\sup_{x\in\mathbb{P}^k\setminus E} \mu_n(x) \leq C\rho^n, \quad \forall\, n \in \mathbb{N}. \tag{5.13}$$

Now the conclusion follows since

$$\sup_{x\in\mathbb{P}^k} \nu(d^{-n}(f^n)^*S, x) \leq \sup_{x\in\mathbb{P}^k\setminus E} \nu(d^{-n}(f^n)^*S, x) + \underbrace{\sup_{x\in E} \nu(d^{-n}(f^n)^*S, x)}_{=0}$$

and

$$\sup_{x\in\mathbb{P}^k\setminus E} \nu(d^{-n}(f^n)^*S, x) \leq \sup_{x\in\mathbb{P}^k\setminus E} d^{-n}\mu_n(x) \leq C\left(\frac{\rho}{d}\right)^n \to 0$$

by Theorem IV.5 and inequality (5.13).

## 5.3 Equidistribution in lower dimensions

We end this work providing simpler new proofs for the already known cases of dimensions 1 and 2.

### 5.3.1 Dimension 1

We observe that in dimension one, positive closed (1,1)-currents with mass 1 correspond to probability measures. We use our techniques to the provide a new proof of the famous Brolin, Lyubich, Lopes-Freire-Mañé equidistribution theorem.

**Theorem V.6** (Brolin, Lyubich, Lopes-Freire-Mañé). *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geq 2$, then there exist a finite collection of totally invariant points $\mathcal{E}_f$, with cardinality at most 2 such that: for every probability measure $\eta$ on $\mathbb{P}^1$, the following are equivalent*

*(i)* $\eta(\mathcal{E}_f) = 0$;

*(ii)* $d^{-n}(f^n)^*\eta \to T_f$ *as* $n \to +\infty$.

*Proof.* Let $\mathcal{E}_f$ be the collection of totally invariant points given by Definition IV.10. If $\#\mathcal{E}_f > 2$, since $f^{-1}(\mathcal{E}_f) = \mathcal{E}_f$, we can lift every iterate $f^n : \mathbb{P}^1 \setminus \mathcal{E}_f \to \mathbb{P}^1 \setminus \mathcal{E}_f$ to the (hyperbolic) uniformization

$$\begin{tikzcd} & \mathbb{D} \arrow[d] \\ \mathbb{P}^1 \setminus \mathcal{E}_f \arrow[ur, "\bar{f}^n"] \arrow[r, "f^n"'] & \mathbb{P}^1 \setminus \mathcal{E}_f \end{tikzcd}$$

giving us that the family $\{\bar{f}^n : \mathbb{P}^1 \setminus \mathcal{E}_f \to \mathbb{D}\}_{n\in\mathbb{N}}$ is normal by Montel's theorem, implying that the family $\{f^n\}_{n\in\mathbb{N}}$ is normal on $\mathbb{P}^1$. In particular, we can find a (uniformly) convergent subsequence $f^{n_k} \to \hat{f}$ where $\hat{f}$ is a rational self-map of $\mathbb{P}^1$. This contradicts the fact that the degrees $d^{n_k}$ of $f^{n_k}$ on $\mathbb{P}^1$ grow to $+\infty$. Hence $\#\mathcal{E}_f \leq 2$.

Observe that given $x \in \mathbb{P}^1$, the Lelong number of $\eta$ at $x$ is nothing but the point-mass $\eta(\{x\})$. Then, the implication (i)⇒(ii) follows immediately by Theorem I.1.

For the converse implication (ii)⇒(i), assume that there exists $x_0 \in \mathcal{E}_f$ with $\eta(\{x_0\}) = c > 0$. In particular, $\eta \geq c\delta_{x_0}$ where $\delta_{x_0}$ denotes the Dirac mass at $x_0$. Hence $(f^n)^*\eta \geq cd^n\delta_{x_0}$ giving us

$$\sup_{x\in\mathbb{P}^1} d^{-n}(f^n)^*\eta(\{x\}) \geq \sup_{x\in\mathbb{P}^1} d^{-n}(f^n)^*\delta_{x_0}(\{x\}) = c > 0.$$

Therefore $\sup_{x\in\mathbb{P}^1} d^{-n}(f^n)^*\eta(\{x\}) \nrightarrow 0$, hence $d^{-n}(f^n)^*\eta(\{x\}) \nrightarrow T_f$.

$\square$

#### 5.3.2 Dimension 2

The provide here a proof for the dimension 2 case.

**Theorem V.7** (Fornæss-Sibony, Favre-Jonson)**.** *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a rational map of degree $d \geq 2$, then there exists a collection of irreducible totally invariant algebraic* *sets $\mathcal{E}_f$, with with at most 3 lines and finitely many points such that: for every positive closed (1,1)-current $S$ on $\mathbb{P}^2$ with mass 1, the following are equivalent*

*(i) The current $S$ has no mass on every element of $\mathcal{E}_f$;*

*(ii) $d^{-n}(f^n)^*S \to T_f$ as $n \to +\infty$.*

*Proof.* We again take $\mathcal{E}_f$ to be the finite family of irreducible totally invariant algebraic subsets of $\mathbb{P}^2$ given in Definition IV.10. As it was proved by Fornæss and Sibony in [FS94], any irreducible totally invariant curve must be a line and there are at most 3 of them. Implication (i)⇒(ii) follows then by Theorem I.1.

For the converse implication (ii)⇒(i), let $X \in \mathcal{E}_f$ with $f^{-1}(X) = X$, and assume that the (generic) Lelong number $\nu(S, X)$ of $S$ at $X$ satisfies $\nu(S, X) = c > 0$.

If $X$ is a line, by Siu's theorem we have that $S - c[X]$ is a positive closed (1,1)-current, hence

$$d^{-n}(f^n)^*\left(S - c[X]\right) \geq 0 \Rightarrow d^{-n}(f^n)^*S \geq cd^{-n}(f^n)^*[X] = c[X]$$

implying that $d^{-n}(f^n)^*S \nrightarrow T_f$.

On the other hand, if $X$ is a point, it is not hard to see that

$$\nu\left(d^{-n}(f^n)^*S, X\right) \geq d^{-n}c_n(X)\nu(S, f^n(X)) = cd^{-n}c_n(X), \tag{5.14}$$

where $c_n(X)$ denotes the order of vanishing of $f^n$ at the point $X$. It is easy to see that

$$c_\infty(x) := \lim_{n\to+\infty} \left(c_n(x)\right)^{1/n}$$

exists for every $x \in \mathbb{P}^2$. Moreover, Favre and Jonsson proved in [FJ07, Theorem A] that there exists a uniform constant $\delta > 0$ such that $c_n(x) \geq \delta c_\infty(x)^n$ for all $x \in \mathbb{P}^2$ and in [FJ03, Proposition 3.12] that $c_\infty(X) = d$. Using this in equation (5.14), we obtain that

$$\nu\left(d^{-n}(f^n)^*S, X\right) \geq c\delta > 0$$

implying that $\sup_{x\in\mathbb{P}^2} \nu\left(d^{-n}(f^n)^*S, X\right) \nrightarrow 0$ hence $d^{-n}(f^n)^*S \nrightarrow T_f$. This concludes the proof in dimension 2.

$\square$

# BIBLIOGRAPHY

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