AIAA Guidance, Navigation, and Control Conference
08 - 11 August 2011, Portland, Oregon

AIAA 2011-6315

Adaptive State Estimation for Nonminimum-Phase Systems with Uncertain Harmonic Inputs

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We develop a method for obtaining state estimates for a possibly nonminimum-phase system in the presence of an unknown harmonic input. We construct a state estimator based on the system model, and then introduce an estimator input provided by an adaptive feedback model whose goal is to drive the estimated output to the measured output despite the presence of the unknown harmonic input. Using input reconstruction based on a retrospective surrogate cost, we reconstruct the unknown harmonic input. Using the reconstructed input we update the parameters of the adaptive model using recursive least squares identification. We then extend the method to nonlinear systems. The performance of this method is compared with the Kalman filter for linear examples, as well as with the extended and unscented Kalman filters for nonlinear examples.

I. Introduction

The classical Kalman filter is the optimal state estimator for linear systems under white process and sensor noise with zero mean and finite second moments. Implementation of the optimal estimator under these idealized conditions depends on knowledge of the linear dynamics and noise covariances. When these assumptions are not satisfied, the accuracy of the Kalman filter can be degraded.[13,14,16]

If the transfer function from the process noise to the measurements is minimum phase, the number of outputs equals the number of disturbances, and there is no sensor noise, then the minimum achievable estimation error is zero.[10] On the other hand, the presence of nonminimum-phase zeros increases the minimum achievable estimation error and thus, for harmonic disturbances, the Kalman filter does not give perfect state estimates.[8,9]

A more proactive approach is to implement an adaptive state estimator, where the goal is to identify the dynamics and noise statistics during system operation and use this information to tune the estimator on-line.[15]

In addition to compensating for white process noise, the Kalman filter accommodates the presence of a known, deterministic input. By injecting this signal into the estimator, the estimator experiences no loss of estimation accuracy relative to the case in which no deterministic input is present. This feature is essential when the Kalman filter is used in conjunction with the linear-quadratic regulator for constructing the full-order dynamic LQG controller.

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In practice, however, the deterministic input may not be known exactly, and this error can viewed as a component of the process noise. However, this approach may be conservative and can lead to bias when the unknown input has a nonzero "mean" value. Consequently, a more direct approach is to extend the estimator to include an estimate of the unknown input.[21–25] Yet another approach is to constrain the gains of the estimator in order to guarantee that the state estimates are unbiased.[17–20]

In the present paper we consider state estimation for minimum- or nonminimum-phase systems in the presence of an unknown harmonic input. To address this problem we consider the estimator structure shown in Figure 1 with an auxiliary input $\hat{u}$, which is the output of an adaptive feedback system that is updated on-line. The signal $\hat{u}$ is estimated using a retrospective-cost-based input-reconstruction technique. In this way, the adaptive feedback system uses knowledge of the estimator residual to improve the accuracy of the state estimator by reconstructing the harmonic disturbance, thereby achieving perfect estimates in the minimum and nonminimum-phase cases. A related technique is used in.[26]

The contents of this paper are as follows. In Section II we describe the state estimation problem and construct a state estimator that uses an auxiliary input from an adaptive subsystem. In Section III, we describe an input-reconstruction technique that constructs the auxiliary input by minimizing the residual error, that is, the difference between the measured output and the output of the estimator system. In Section IV we use least squares to estimate the adaptive subsystem parameters from the reconstructed auxiliary input and current state estimates.

Next, we numerically examine the adaptive state estimation error in comparison to the optimal state estimator. In Section V we demonstrate the adaptive state estimator on linear numerical examples, and compare the results to the Kalman filter. In Section VI we extend the method to nonlinear state estimation, and in Section VII we demonstrate the method on nonlinear examples with comparisons to the extended and unscented Kalman filters.

II. Problem Formulation

Consider the linear-time-invariant system

$$x(k+1) = Ax(k) + Bu(k) + Bw(k), \tag{1}$$
$$y(k) = Cx(k), \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is the unknown state, $u(k) \in \mathbb{R}^m$ is an unknown input, $w(k) \in \mathbb{R}^m$ is unknown zero-mean Guassian white noise, and $y(k) \in \mathbb{R}^p$ is the measured output, which is assumed to be bounded. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are known, and (A, C) is observable. Furthermore we assume that $u(k)$ is the output of a Lyapunov-stable, linear system.

In order to obtain an estimate $\hat{x}(k) \in \mathbb{R}^n$ of the state $x(k)$, we construct an adaptive state estimator (ASE) of the form

$$\hat{x}(k+1) = A\hat{x}(k) + B\hat{u}(k), \tag{3}$$
$$\hat{y}(k) = C\hat{x}(k), \tag{4}$$
$$z(k) = y(k) - \hat{y}(k), \tag{5}$$

where $\hat{y}(k) \in \mathbb{R}^p$ is the estimated output, $\hat{u}(k) \in \mathbb{R}^m$ is the estimator input, and $z(k) \in \mathbb{R}^p$ is the measured output error. Furthermore, $\hat{u}(k)$ is the output of the strictly proper adaptive feedback system of order n_c, with input $z(k)$, given by

$$\hat{u}(k) = \sum_{i=1}^{n_{\mathrm{c}}} M(k)_i \hat{u}(k-i) + \sum_{i=0}^{n_{\mathrm{c}}} N(k)_i z(k-i), \tag{6}$$

where $M_{k,i} \in \mathbb{R}^{m\times m}$, $i = 1, \ldots, n_{\mathrm{c}}$, and $N_{k,i} \in \mathbb{R}^{m\times p}$, $i = 0, \ldots, n_{\mathrm{c}}$. The goal is to update $M_{i,k}$ and $N_{i,k}$ using the measured output error $z(k)$. Figure 1 shows the adaptive estimator structure.

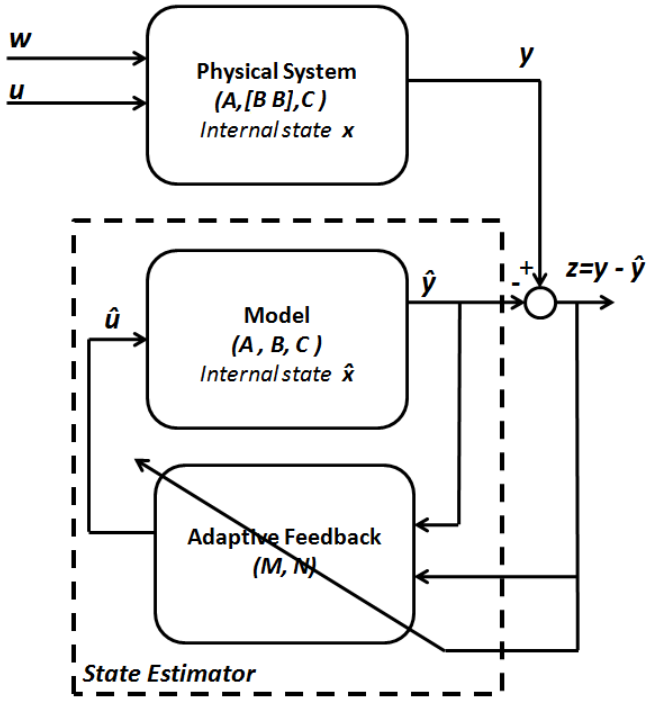

Figure 1. Adaptive State Estimator Architecture

III. State Estimation Using a Retrospective Surrogate Cost

For $i \geq 1$, define the Markov parameter H_i of (A, B, C) given by

$$H_i \triangleq CA^{i-1}B. \tag{7}$$

For example, $H_1 = CB$ and $H_2 = CAB$. Let r be a positive integer. Then, for all $k \geq r$,

$$\hat{x}(k) = A^r\hat{x}(k-r) + \sum_{i=1}^{r} A^{i-1}B\hat{u}(k-i), \tag{8}$$

and thus

$$z(k) = CA^r\hat{x}(k-r) - y(k) + \bar{H}\hat{\bar{U}}(k-1), \tag{9}$$

where

$$\bar{H} \triangleq \begin{bmatrix} H_1 & \cdots & H_r \end{bmatrix} \in \mathbb{R}^{p\times rm}$$

and

$$\bar{U}(k-1) \triangleq \begin{bmatrix} \hat{u}(k-1) \\ \vdots \\ \hat{u}(k-r) \end{bmatrix}.$$

Next, we rearrange the columns of $\bar{H}$ and the components of $\hat{\bar{U}}(k-1)$ and partition the resulting matrix and vector so that

$$\bar{H}\hat{\bar{U}}(k-1) = \mathcal{H}'\hat{U}'(k-1) + \mathcal{H}\hat{U}(k-1), \tag{10}$$

where $\mathcal{H}' \in \mathbb{R}^{p\times(rm-l_{\hat{U}})}$, $\mathcal{H} \in \mathbb{R}^{p\times l_{\hat{U}}}$, $\hat{U}'(k-1) \in \mathbb{R}^{rm-l_{\hat{U}}}$, and $\hat{U}(k-1) \in \mathbb{R}^{l_{\hat{U}}}$. Then, we can rewrite (9) as

$$z(k) = \mathcal{S}(k) + \mathcal{H}\hat{U}(k-1), \tag{11}$$

where

$$\mathcal{S}(k) \triangleq CA^r\hat{x}(k-r) - y(k) + \mathcal{H}'\hat{U}'(k-1). \tag{12}$$

For example, $\bar{H} = \begin{bmatrix} H_1 & H_2 & H_3 & H_4 & H_5 \end{bmatrix}$,

$$\mathcal{H}' = \begin{bmatrix} H_1 & H_2 & H_4 \end{bmatrix}, \quad U'(k-1) = \begin{bmatrix} \hat{u}(k-1) \\ \hat{u}(k-2) \\ \hat{u}(k-4) \end{bmatrix},$$

and

$$\mathcal{H} = \begin{bmatrix} H_3 & H_5 \end{bmatrix}, \quad \hat{U}(k-1) = \begin{bmatrix} \hat{u}(k-3) \\ \hat{u}(k-5) \end{bmatrix}.$$

Next, for $j = 1, \ldots, s$, we rewrite (11) with a delay of k_j time steps, where $0 \le k_1 \le k_2 \le \cdots \le k_s$, in the form

$$z(k-k_j) = \mathcal{S}_j(k-k_j) + \mathcal{H}_j\hat{U}_j(k-k_j-1), \tag{13}$$

where (12) becomes

$$\mathcal{S}_j(k-k_j) \triangleq CA^r x(k-k_j-r) - y(k-k_j) + \mathcal{H}_j'\hat{U}_j'(k-k_j-1)$$

and (10) becomes

$$\bar{H}\hat{\bar{U}}(k-k_j-1) = \mathcal{H}_j'\hat{U}_j'(k-k_j-1) + \mathcal{H}_j\hat{U}_j(k-k_j-1), \tag{14}$$

where $\mathcal{H}_j' \in \mathbb{R}^{p\times(rm-l_{\hat{U}_j})}$, $\mathcal{H}_j \in \mathbb{R}^{p\times l_{\hat{U}_j}}$, $\hat{U}_j'(k-k_j-1) \in \mathbb{R}^{rm-l_{\hat{U}_j}}$, and $\hat{U}_j(k-k_j-1) \in \mathbb{R}^{l_{\hat{U}_j}}$. Now, by stacking $z(k-k_1), \ldots, z(k-k_s)$, we define the *extended performance*

$$Z(k) \triangleq \begin{bmatrix} z(k-k_1) \\ \vdots \\ z(k-k_s) \end{bmatrix} \in \mathbb{R}^{sp}. \tag{15}$$

Therefore,

$$Z(k) \triangleq \tilde{\mathcal{S}}(k) + \tilde{\mathcal{H}}\hat{\bar{U}}(k-1), \tag{16}$$

where

$$\tilde{\mathcal{S}}(k) \triangleq \begin{bmatrix} \mathcal{S}_1(k-k_1) \\ \vdots \\ \mathcal{S}_s(k-k_s) \end{bmatrix} \in \mathbb{R}^{sp}, \tag{17}$$

$\hat{\tilde{U}}(k-1)$ has the form

$$\hat{\tilde{U}}(k-1) \triangleq \begin{bmatrix} \hat{u}(k-q_1) \\ \vdots \\ \hat{u}(k-q_{l_{\hat{U}}}) \end{bmatrix} \in \mathbb{R}^{l_{\hat{U}}}, \tag{18}$$

where, for $i = 1, \ldots, l_{\hat{U}}$, $k_1 \le q_i \le k_s + r$, and $\tilde{\mathcal{H}} \in \mathbb{R}^{sp \times l_{\hat{U}}}$ is constructed according to the structure of $\hat{\tilde{U}}(k-1)$. The vector $\hat{\tilde{U}}(k-1)$ is formed by stacking $\hat{U}_1(k-k_1-1), \ldots, \hat{U}_s(k-k_s-1)$ and removing copies of repeated components.

For example, with $k_1 = 0$ and $k_2 = 1$, stacking $\hat{U}_1(k-1) = \begin{bmatrix} \hat{u}(k-1) \\ \hat{u}(k-2) \end{bmatrix}$ and $\hat{U}_2(k-2) = \hat{u}(k-2)$ results in $\hat{\tilde{U}}(k-1) = \begin{bmatrix} \hat{u}(k-1) \\ \hat{u}(k-2) \end{bmatrix}$. The coefficient matrix $\tilde{\mathcal{H}}$ consists of the entries of $\mathcal{H}_1, \ldots, \mathcal{H}_s$ arranged according to the structure of $\hat{\tilde{U}}(k-1)$.

Next, we define the *surrogate performance*

$$\hat{z}(k-k_j) \triangleq \mathcal{S}_j(k-k_j) + \mathcal{H}_j U_j^*(k-k_j-1), \tag{19}$$

where the past controls $\hat{U}_j(k-k_j-1)$ in (13) are replaced by the surrogate controls $U_j^*(k-k_j-1)$. In analogy with (15), the *extended surrogate performance* for (19) is defined as

$$\hat{Z}(k) \triangleq \begin{bmatrix} \hat{z}(k-k_1) \\ \vdots \\ \hat{z}(k-k_s) \end{bmatrix} \in \mathbb{R}^{sp} \tag{20}$$

and thus is given by

$$\hat{Z}(k) = \tilde{\mathcal{S}}(k) + \tilde{\mathcal{H}}\tilde{U}^*(k-1), \tag{21}$$

where the components of $\tilde{U}^*(k-1) \in \mathbb{R}^{l_{\hat{U}}}$ are the components of $U_1^*(k-k_1-1), \ldots, \hat{U}_s^*(k-k_s-1)$ ordered in the same way as the components of $\hat{\tilde{U}}(k-1)$. Subtracting (16) from (21) yields

$$\hat{Z}(k) = Z(k) - \tilde{\mathcal{H}}\hat{\tilde{U}}(k-1) + \tilde{\mathcal{H}}\tilde{U}^*(k-1). \tag{22}$$

Finally, we define the *retrospective cost function*

$$J(\tilde{U}^*(k-1), k) \triangleq \hat{Z}^{\mathrm{T}}(k) R(k) \hat{Z}(k), \tag{23}$$

where $R(k) \in \mathbb{R}^{ps \times ps}$ is a positive-definite performance weighting. The goal is to determine refined controls $\hat{\tilde{U}}(k-1)$ that would have provided better performance than the controls $U(k)$ that were applied to the system. The refined control values $\tilde{U}^*(k-1)$ are subsequently used to update the controller.

IV. Cost Function Optimization with Adaptive Regularization

To ensure that (23) has a global minimizer, we consider the regularized cost

$$\bar{J}(\tilde{U}^*(k-1), k) \triangleq \hat{Z}^{\mathrm{T}}(k) R(k) \hat{Z}(k) + \eta(k) \tilde{U}^{*\mathrm{T}}(k-1) \tilde{U}^*(k-1), \tag{24}$$

where $\eta(k) \geq 0$. Substituting (22) into (24) yields

$$\bar{J}(\tilde{U}^*(k-1), k) = \tilde{U}^*(k-1)^{\mathrm{T}} \mathcal{A}(k) \tilde{U}^*(k-1) + \mathcal{B}(k) \tilde{U}^*(k-1) + \mathcal{C}(k), \tag{25}$$

where

$$\mathcal{A}(k) \triangleq \tilde{\mathcal{H}}^{\mathrm{T}} R(k) \tilde{\mathcal{H}} + \eta(k) I_{l_{\hat{U}}}, \tag{26}$$

$$\mathcal{B}(k) \triangleq 2\tilde{\mathcal{H}}^{\mathrm{T}} R(k)[Z(k) - \tilde{\mathcal{H}}\hat{\tilde{U}}(k-1)], \tag{27}$$

$$\mathcal{C}(k) \triangleq Z^{\mathrm{T}}(k) R(k) Z(k) - 2Z^{\mathrm{T}}(k) R(k) \tilde{\mathcal{H}}\hat{\tilde{U}}(k-1) + \hat{\tilde{U}}^{\mathrm{T}}(k-1)\tilde{\mathcal{H}}^{\mathrm{T}} R(k) \tilde{\mathcal{H}}\hat{\tilde{U}}(k-1). \tag{28}$$

If either $\tilde{\mathcal{H}}$ has full column rank or $\eta(k) > 0$, then $\mathcal{A}(k)$ is positive definite. In this case, $\bar{J}(\hat{\tilde{U}}(k-1), k)$ has the unique global minimizer

$$\tilde{U}^*(k-1) = -\frac{1}{2}\mathcal{A}^{-1}(k)\mathcal{B}(k). \tag{29}$$

IV.A. Adaptive Feedback Update

The control $\hat{u}(k)$ is given by the strictly proper time-series controller of order n_{c} given by

$$\hat{u}(k) = \sum_{i=1}^{n_{\mathrm{c}}} M_i(k)\hat{u}(k-i) + \sum_{i=1}^{n_{\mathrm{c}}} N_i(k) z(k-i), \tag{30}$$

where, for all $i = 1, \dots, n_{\mathrm{c}}$, $M_i(k) \in \mathbb{R}^{m \times m}$ and $N_i(k) \in \mathbb{R}^{m \times p}$. The control (30) can be expressed as

$$\hat{u}(k) = \theta(k)\phi(k-1), \tag{31}$$

where

$$\theta(k) \triangleq [M_1(k) \ \cdots \ M_{n_{\mathrm{c}}}(k) N_1(k) \ \cdots \ N_{n_{\mathrm{c}}}(k)] \in \mathbb{R}^{m \times n_{\mathrm{c}}(m+p)} \tag{32}$$

and

$$\phi(k-1) \triangleq \begin{bmatrix} \hat{u}(k-1) \\ \vdots \\ \hat{u}(k-n_{\mathrm{c}}) \\ z(k-1) \\ \vdots \\ z(k-n_{\mathrm{c}}) \end{bmatrix} \in \mathbb{R}^{n_{\mathrm{c}}(m+p)}. \tag{33}$$

Let d be a positive integer such that $\tilde{U}(k-1)$ contains $u(k-d)$. Next, we define the cumulative cost function

$$J_{\mathrm{R}}(\theta(k)) \triangleq \sum_{i=1}^{k} \lambda^{k-i} \|\phi^{\mathrm{T}}(k-d-1)\theta^{\mathrm{T}}(k-1) - \hat{u}^{\mathrm{T}}(k-d)\|^2, \tag{34}$$

where $\|\cdot\|$ is the Euclidean norm, and $\lambda(k) \in (0,1]$ is the forgetting factor. Minimizing (34) yields

$$\begin{aligned} \theta^{\mathrm{T}}(k) = \theta^{\mathrm{T}}(k-1) &+ P(k-1)\phi(k-d-1)[\phi^{\mathrm{T}}(k-d)P(k-1)\phi(k-d-1) + \lambda(k)]^{-1} \\ &\times [\phi^{\mathrm{T}}(k-d-1)\theta^{\mathrm{T}}(k-1) - \hat{u}^{\mathrm{T}}(k-d)]. \end{aligned} \tag{35}$$

The error covariance is updated by

$$\begin{aligned} P(k) = \lambda^{-1}(k)P(k-1) &- \lambda^{-1}(k)P(k-1)\phi(k-d-1) \cdot [\phi^{\mathrm{T}}(k-d-1)P(k-1)\phi(k-d) + \lambda(k)]^{-1} \\ &\times \phi^{\mathrm{T}}(k-d-1)P(k-1). \end{aligned} \tag{36}$$

We initialize the error covariance matrix as $P(0) = \gamma I$, where $\gamma > 0$.

V. Linear Examples

In this section, we apply the adaptive state estimator to several linear examples and compare its performance with the Kalman filter (KF). Define the error metric

$$\varepsilon_k = \frac{1}{\ell} \sum_{i=k}^{k+l-1} ||e_i||, \tag{37}$$

where ℓ is the window size. For all examples in this section, $\ell = 2000$.

V.A. Example 1: Dual Spring-Mass-Damper System, Minimum-Phase

Consider the dual spring-mass-damper system shown in Figure 2. For $i = 1, 2$, let q_i be the position of i^{th} mass, let m_i be the mass of the i^{th} block, let k_i be the stiffness of the i^{th} spring, and let c_i be the damping coefficient of the i^{th} damper. Finally, let u be the force applied to the first block.

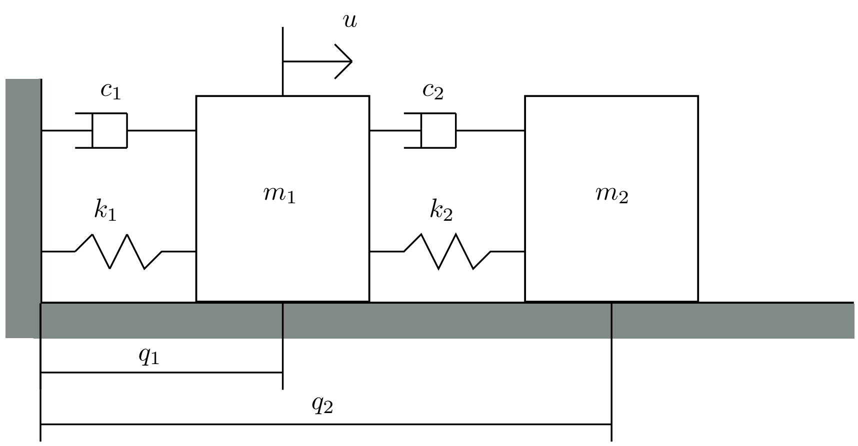

Figure 2. Dual spring-mass-damper system

The equations of motion of this system are

$$\dot{x} = A_c x + B_c u,$$

where

$$x = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}, A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix}, \ B_c = \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix}. \tag{38}$$

We choose $m_1 = 5$, $m_2 = 4$, $k_1 = k_2 = 0.01$, $c_1 = 0.5$, and $c_2 = 0.05$. We discretize the system using

$$A = e^{A_c T_s} \ , \ B = A_c^{-1}(A - I)B_c, \tag{39}$$

where $T_s = 1$ is the sampling time. The output matrix is

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \tag{40}$$

which represents the position of the first mass. The zeros of the discretized system are -0.9680 and $0.9852 \pm 0.0687\jmath$. For the ASE, let $\eta(k) = 0$, $n_c = 3$, $P(0) = 1 \times 10^{15} I_{6\times6}$, and $\tilde{\mathcal{H}} = CB$. For the Kalman filter, the noise covariance matrix is $Q = BB^{\mathrm{T}}$, and the initial error covariance is $I_{4\times4}$. Finally, $u(k) = 20\sin(k)$. Figure 3 shows that ε_k converges to zero for both the KF and ASE.

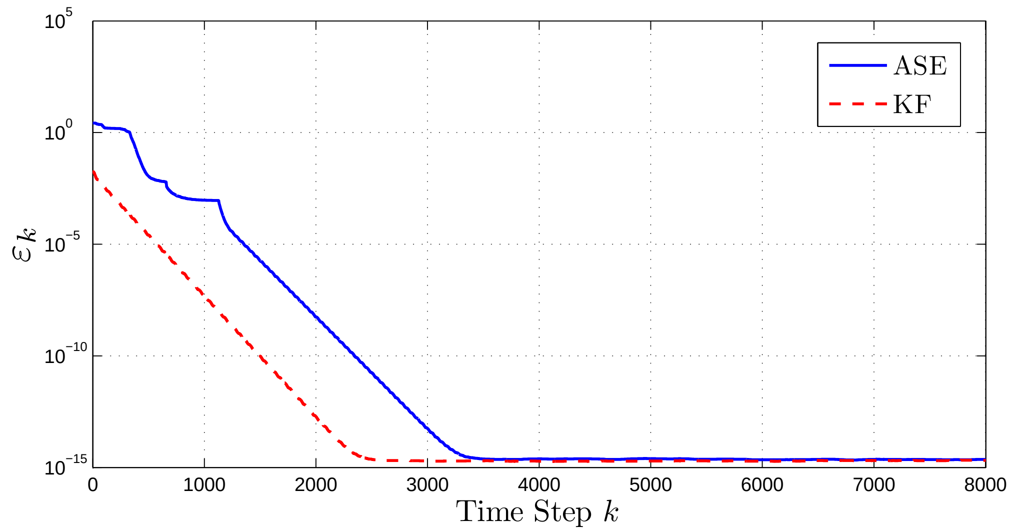

Figure 3. Comparison of the performance ε_k of the ASE and KF for the minimum-phase spring-mass-damper system.

V.B. Example 2: Dual Spring-Mass-Damper system, Nonminimum-Phase

For this example, we consider the system defined by (39) with the output matrix

$$C = \begin{bmatrix} -10 & 0 & 1 & 0 \end{bmatrix}, \tag{41}$$

which represents a difference in the positions of the two masses. The zeros of the discretized system are -0.9632, 1.1293, and 0.8767, and thus (A, B, C) is nonminimum phase. For the ASE, let $\eta(k) = 5 \times 10^{-5}$, $n_c = 5$, $P(0) = 10I_{10\times10}$, and $\tilde{\mathcal{H}} = CAB$. For the Kalman filter, the noise covariance matrix is $Q = BB^{\mathrm{T}}$, and the initial error covariance is $I_{4\times4}$. Finally, $u(k) = 20\sin(k)$. Figure 4 shows the performance ε_k of the ASE and KF.

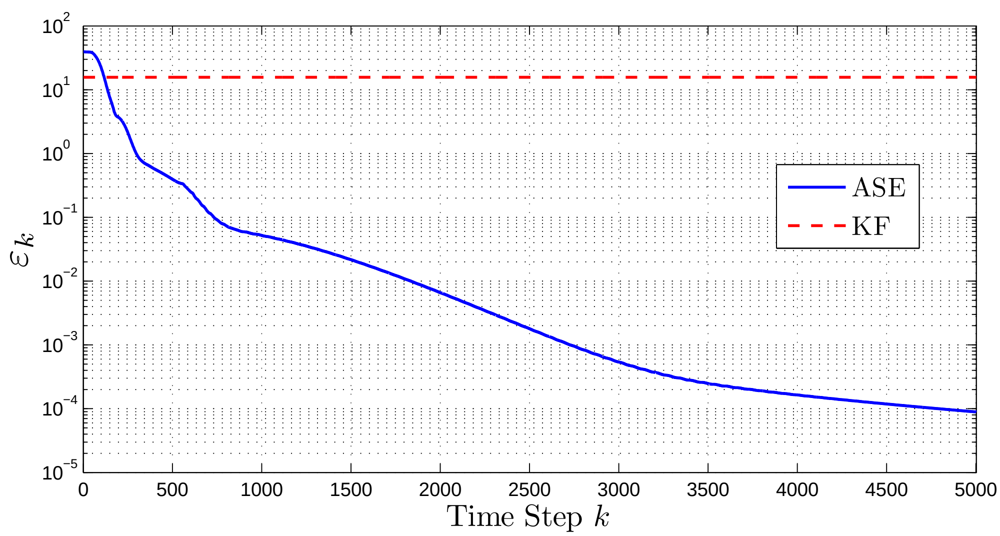

Figure 4. Comparison of the performance ε_k of the ASE and KF for the nonminimum-phase spring-mass-damper system.

V.C. Example 3: Nonminimum-Phase Linearized Planar Linkage

We consider the planar linkage system shown in Figure 5. Let p_1 be the point where the first link is attached to the horizontal plane, and let p_2 be the point where the two links are connected. Furthermore, for $i = 1, 2$, let q_i be the center of mass of the i^{th} link, let m_i be the mass of the i^{th} link, let c_i be the damping at the joint p_i, and let k_i be the stiffness of the joint p_i.

Next, let $\mathrm{F_A}$ be an inertial frame with the orthogonal unit vectors $(\hat{\imath}_\mathrm{A}, \hat{\jmath}_\mathrm{A}, \hat{k}_\mathrm{A})$, where $\hat{\imath}_\mathrm{A}$ and $\hat{\jmath}_\mathrm{A}$ lie in the plane of motion of the planar linkage system. For simplicity, we assume that the origin of $\mathrm{F_A}$ is located at p_1. In addition, for $i = 1, 2$, let $\mathrm{F}_{\mathrm{B}_i}$ be a body-fixed frame attached to the i^{th} link. More specifically, $\mathrm{F}_{\mathrm{B}_i}$ is a body-fixed frame that rotates as the i^{th} link rotates. For $i = 1, 2$, let $\mathrm{F}_{\mathrm{B}_i}$ have orthogonal unit vectors $(\hat{\imath}_{\mathrm{B}_i}, \hat{\jmath}_{\mathrm{B}_i}, \hat{k}_{\mathrm{B}_i})$, where $\hat{\imath}_{\mathrm{B}_i}$ is in the direction from p_i to q_i, and $\hat{\jmath}_{\mathrm{B}_i}$ is orthogonal to $\hat{\imath}_{\mathrm{B}_i}$ in the plane of motion. Furthermore, u_1 is an external torque applied at p_1. Finally, for $i = 1, 2$, let θ_i be the angle from $\hat{\imath}_\mathrm{A}$ to $\hat{\imath}_{\mathrm{B}_i}$. All frames are right handed.

The equations of motion of the planar linkage system[6] are given by

$$\begin{aligned} u_1 &= (\frac{1}{3}m_1 l_1^2 + m_2 l_1^2)\ddot{\theta}_1 + \frac{1}{2}m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + \frac{1}{2}m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 \\ &\quad + (k_1 + k_2)\theta_1 - k_2\theta_2 + (c_1 + c_2)\dot{\theta}_1 - c_2\dot{\theta}_2, \\ 0 &= (\frac{1}{3}m_2 l_2^2)\ddot{\theta}_2 - \frac{1}{2}m_2 l_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 + \frac{1}{2}m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 \\ &\quad - k_2\theta_1 + k_2\theta_2 - c_2\dot{\theta}_1 + c_2\dot{\theta}_2, \end{aligned} \tag{42}$$

where $m_1 = 2$, $m_2 = 1$, $l_1 = 3$, $l_2 = 2$, $k_1 = 7$, $k_2 = 5$, $c_1 = 10$, and $c_2 = 1$. The output of the system

$$y = \theta_2, \tag{43}$$

is the angle θ_2, which represents the angle from $\hat{\imath}_\mathrm{A}$ to $\hat{\imath}_{\mathrm{B}_2}$. Linearizing and discretizing (42) with sampling time $T_s = 1$ yields

$$x(k+1) = Ax(k) + Bu(k),$$

where

$$x(k) = \begin{bmatrix} \theta_1(k) \\ \theta_2(k) \\ \dot{\theta}_1(k) \\ \dot{\theta}_2(k) \end{bmatrix}, A = \begin{bmatrix} 0.661 & 0.174 & 0.5517 & 0.1408 \\ 1.0993 & -0.0223 & 0.928 & 0.26 \\ -0.134 & -0.065 & 0.39 & 0.161 \\ 0.009 & -0.3 & 0.744 & -0.083 \end{bmatrix}, B = \begin{bmatrix} 0.024 \\ -0.011 \\ 0.028 \\ 0.042 \end{bmatrix}^T. \tag{44}$$

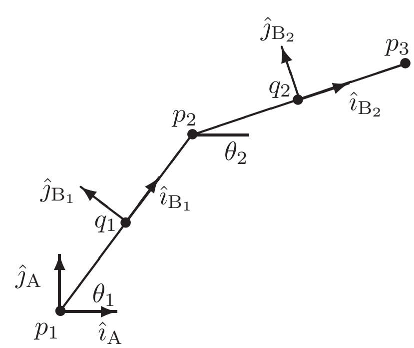

Figure 5. Planar linkage system. All motion is in the horizontal plane.

The zeros of the discretized system are 6.6598, 0.3219, and −0.2619, and thus (A, B, C) is nonminimum phase. For the ASE, let $\eta(k) = 5 \times 10^{-4}$, $n_c = 10$, $P(0) = 1 \times 10^6 I_{20\times20}$, and $\tilde{\mathcal{H}} = CB$. For the Kalman filter, the noise covariance matrix is chosen to be $Q = BB^{\mathrm{T}}$, and the initial error covariance is $I_{4\times4}$. Finally, $u(k) = 20\sin(k)$. Figure 6 shows the performance ε_k of the ASE and KF.

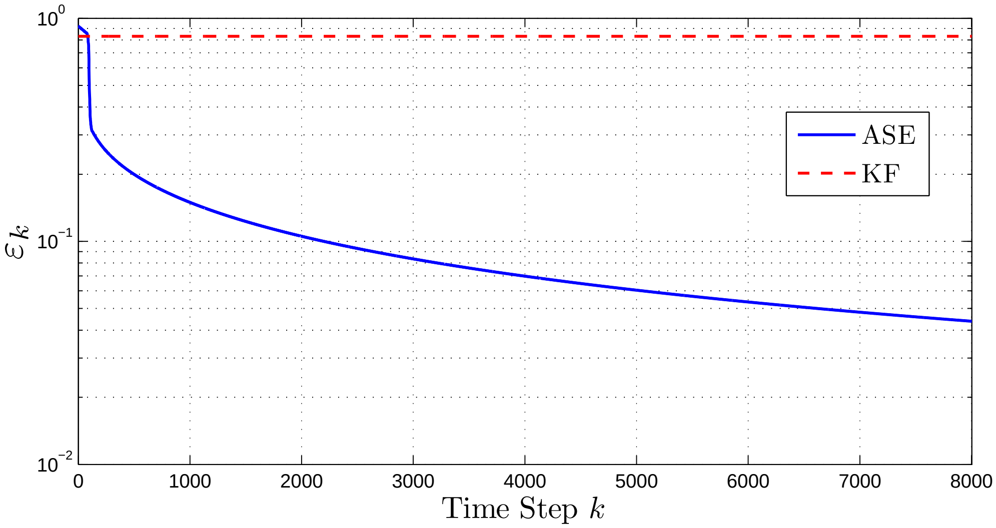

Figure 6. Comparison of the performance ε_k of the ASE and KF for the nonminimum-phase linearized planar linkage.

Next, consider the linearized planar linkage system in Example 3 with the input $u(k) = 20\sin(k) + 5\sin(0.3k) + 80\sin(0.01k)$. For the ASE, $\eta(k) = 1\times10^{-5}$, $n_c = 7$, $P(0) = 1\times10^4 I_{14\times14}$, and $\tilde{\mathcal{H}} = CAB$. For the Kalman filter, the noise covariance matrix is chosen to be $Q = BB^{\mathrm{T}}$, and the initial error covariance is $I_{4\times4}$. For this example, Figure 7 shows the performance ε_k of the ASE and KF.

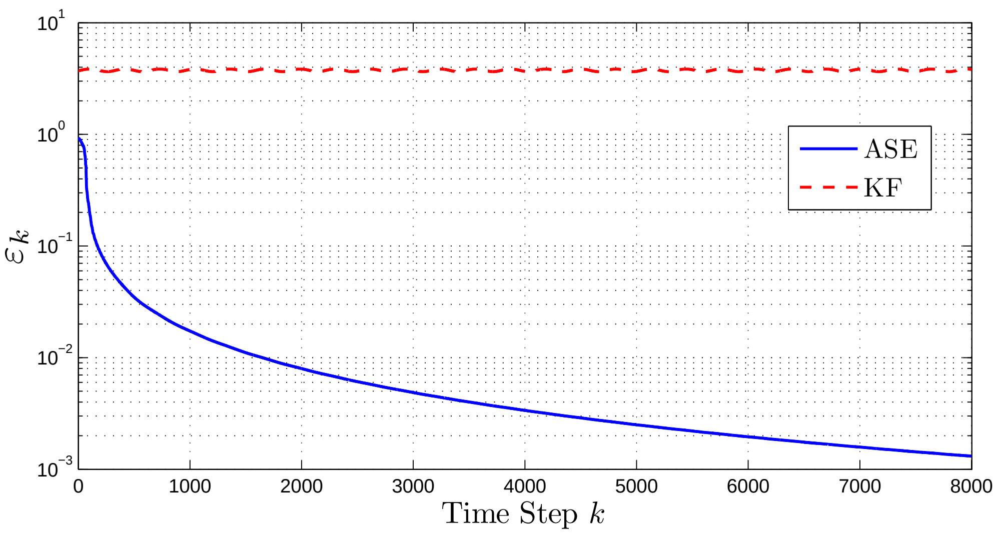

Figure 7. Comparison of the performance ε_k of the ASE and KF for the nonminimum-phase linearized planar linkage with a multi-harmonic input.

Next, consider the linearized planar linkage system in Example 3 with the input $u(k) = 10$. For the ASE, $\eta(k) = 2\times10^{-3}$, $n_c = 1$, $P(0) = 1\times10^3 I_{1\times1}$, and $\tilde{\mathcal{H}} = CAB$. For the Kalman filter, the noise covariance matrix is chosen to be $Q = BB^{\mathrm{T}}$, and the initial error covariance is $I_{4\times4}$. For this example, Figure 8 shows the performance ε_k of the ASE and KF.

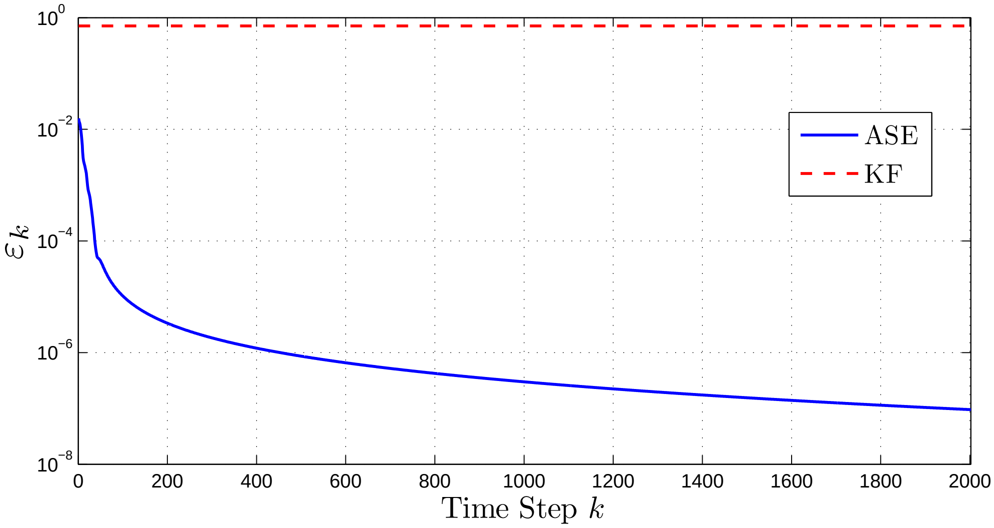

Figure 8. Comparison of the performance ε_k of the ASE and KF for the nonminimum-phase linearized planar linkage with a constant input.

V.D. Example 5: Linearized Planar Linkage with Process Noise

Consider the system

$$x(k+1) = Ax(k) + Bu(k) + \alpha Bw(k),$$

where $x(k)$, A, and B are given by (44). The output of this system is given by (43). We test the ASE and KF for $\alpha = 0$, $\alpha = 10^{-6}$, $\alpha = 10^{-4}$, and $\alpha = 10^{-2}$. For the ASE, $\eta(k) = 1 \times 10^{-5}$, $n_c = 5$, $P(0) = 1 \times 10^3 I_{10\times10}$, and $\tilde{\mathcal{H}} = CA^2B$. For the KF, the noise covariance matrix is $Q = \alpha BB^{\mathrm{T}}$, and the initial error covariance is $I_{4\times4}$. Finally, $u(k) = 20\sin(k)$. Figure 9 shows the performance ε_k of the ASE and KF. For the KF, the effect of α is negligible and thus we show ε_k of the KF for only $\alpha = 0$.

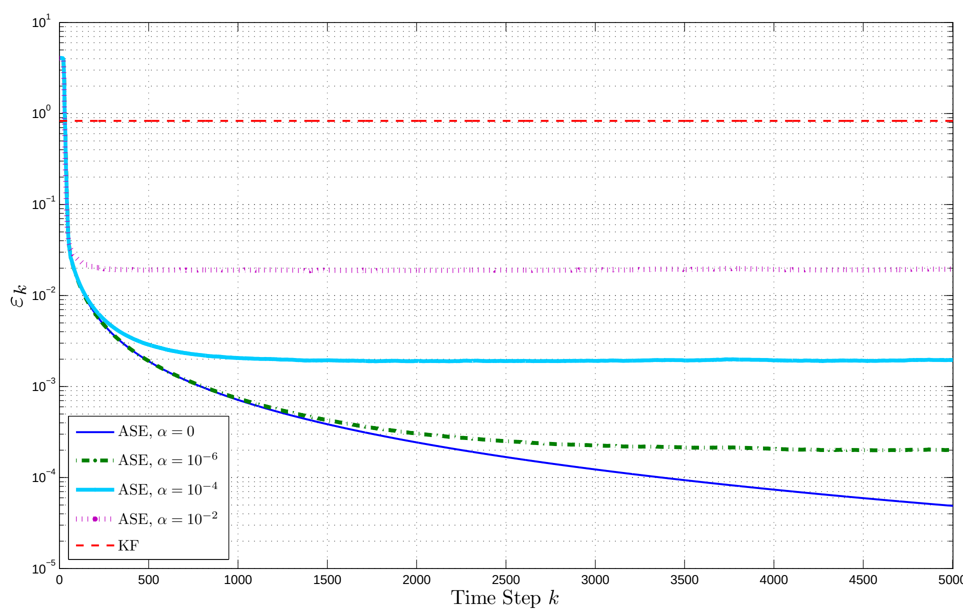

Figure 9. Comparison of the performance ε_k of the ASE and KF for the nonminimum-phase linearized planar linkage with process noise.

VI. Nonlinear State Estimation

Consider the MIMO nonlinear time-invariant system,

$$x(k+1) = f(x(k)) + g(u(k), w(k)), \tag{45}$$
$$y(k) = h(x(k)), \tag{46}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^p$ is an unknown input, $w(k) \in \mathbb{R}^m$ is unknown zero-mean Gaussian white noise, $y(k) \in \mathbb{R}^p$ is the measured output, which is assumed to be bounded, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^m \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}^p$ are known functions. Furthermore, we assume that $u(k)$ is the output of a Lyapunov-stable linear system.

In order to obtain estimates $\hat{x}(k) \in \mathbb{R}^n$ of the state $x(k)$, we construct a state estimator of the form

$$\hat{x}(k+1) = f(\hat{x}(k)) + g(\hat{u}(k)), \tag{47}$$
$$\hat{y}(k) = h(\hat{x}(k)), \tag{48}$$

where the estimated output is $\hat{y}(k) \in \mathbb{R}^p$, and $\hat{u}(k) \in \mathbb{R}^m$ is the estimator input given by (6). M_k and N_k are updated as in the linear case, where H_i is redefined as

$$H_i \triangleq HF^{i-1}G, \tag{49}$$

where x_{eq} is an equilibrium point and

$$F \triangleq \left.\frac{\partial f}{\partial \hat{x}}\right|_{x_{\mathrm{eq}}}, \quad H \triangleq \left.\frac{\partial h}{\partial \hat{x}}\right|_{x_{\mathrm{eq}}}, \quad G \triangleq \left.\frac{\partial g}{\partial \hat{x}}\right|_{x_{\mathrm{eq}}}. \tag{50}$$

Unlike the extended Kalman filter, the adaptive state estimator does not require a linearization of f, g, h, at each step k, which is used by the extended Kalman filter to propagate the error covariance.

VII. Nonlinear Examples

In this section, we compare the adaptive state estimator (ASE) with the extended Kalman filter (EKF) and the unscented Kalman filter (UKF). We consider the nonlinear planar linkage system and the Van der Pol oscillator. For all examples, the error metric is given by (37) with $\ell = 2000$.

VII.A. Example 5: Nonlinear Planar Linkage

In this example, we consider the nonlinear planar linkage given by discretizing (42) with $T_s = 1$, and with the output matrix (43). For the ASE, let $\eta(k) = 5 \times 10^{-4}$, $n_c = 10$, $P(0) = 10I_{20\times 20}$, and $\tilde{\mathcal{H}} = HG$, where H and G are obtained by linearizing and discretizing the system about the origin. For the EKF and UKF, we set $Q = 100B_kB_k^{\mathrm{T}}$, where B_k is the input vector obtained by linearizing and discretizing the system about the current state estimate $\hat{x}(k)$. The initial error covariance matrix for the EKF and UKF is $I_{4\times 4}$. Furthermore, for the UKF, we use nine sigma points, and we set $\kappa = 0$, $\beta = 2$, and $\alpha = 0.1$. Finally, $u(k) = 20\sin(k)$. Figure 10 shows the state estimates of the ASE and UKF for the last fifty steps. Figure 11 shows the performance ε_k of the ASE, UKF, and EKF.

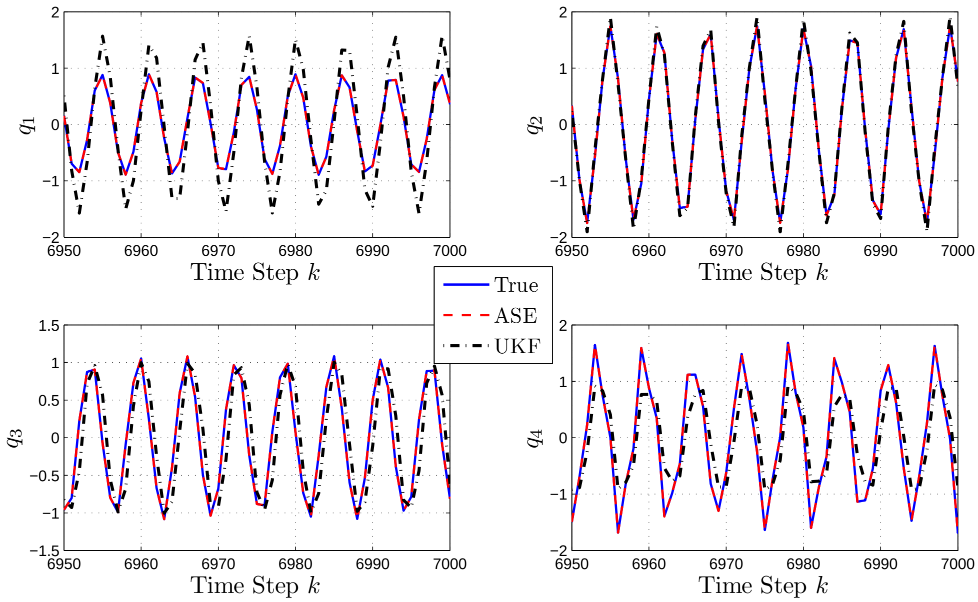

Figure 10. State estimates for the nonlinear planar linkage.

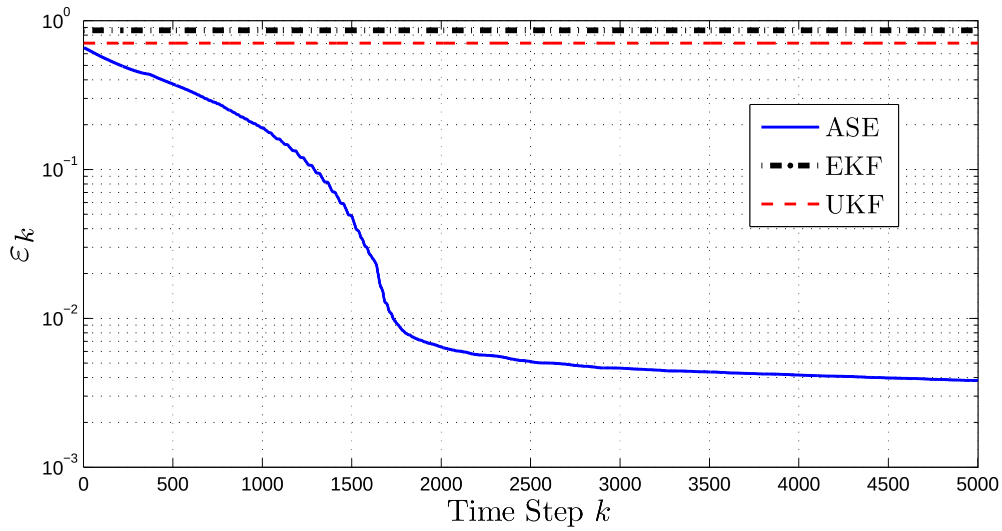

Figure 11. Comparison of the performance ε_k of the ASE, EKF, and UKF for the nonlinear planar linkage.

VII.B. Example 6: Van der Pol Oscillator

We consider the Van der Pol oscillator

$$\ddot{q} - \mu(1 - q^2)\dot{q} + q = u,$$

where $\mu = 1$. The output of this system is $y = q$. We discretize this system with $T_s = 0.1$ to obtain

$$x_1(k+1) = T_s x_2(k) + x_1(k),$$
$$x_2(k+1) = T_s \mu(1 - x_1(k)^2)x_2(k) - T_s x_1(k) + T_s u(k) + x_2(k),$$

where $x_1 = q$ and $x_2 = \dot{q}$. For the ASE, let $\eta(k) = 5 \times 10^{-4}$, $n_c = 2$, $P(0) = 100I_{4\times 4}$, and $\tilde{\mathcal{H}} = HG$, where H and G are obtained by linearizing the discretized system about the origin. For the EKF and UKF, the noise covariance matrix is $Q = [0\ 10]^{\mathrm{T}}[0\ 10]$, and the initial error covariance matrix is $I_{2\times 2}$. Furthermore, for the UKF, we use five sigma points and we set $\kappa = 0$, $\alpha = 0.01$, and $\beta = 1$. Finally, $u(k) = 10\sin(k)$. Figure 12 shows the phase portrait for this example. Figure 13 shows the state estimates for the ASE and UKF. Figure 14 shows the performance ε_k of the ASE, EKF, and UKF.

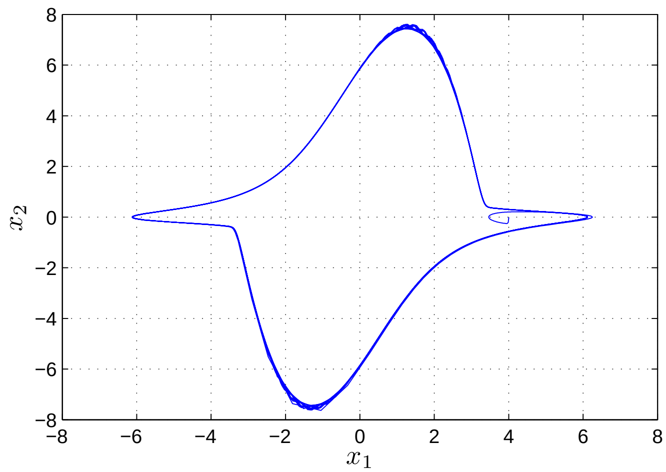

Figure 12. Phase portrait of the Van der Pol oscillator.

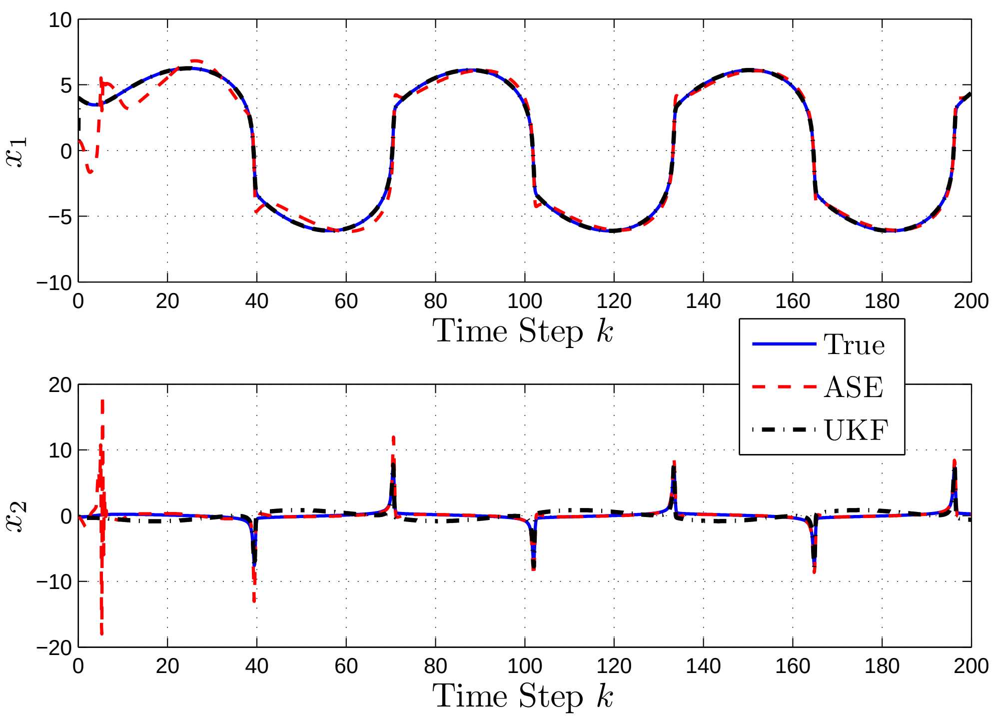

Figure 13. State estimates for the ASE and UKF.

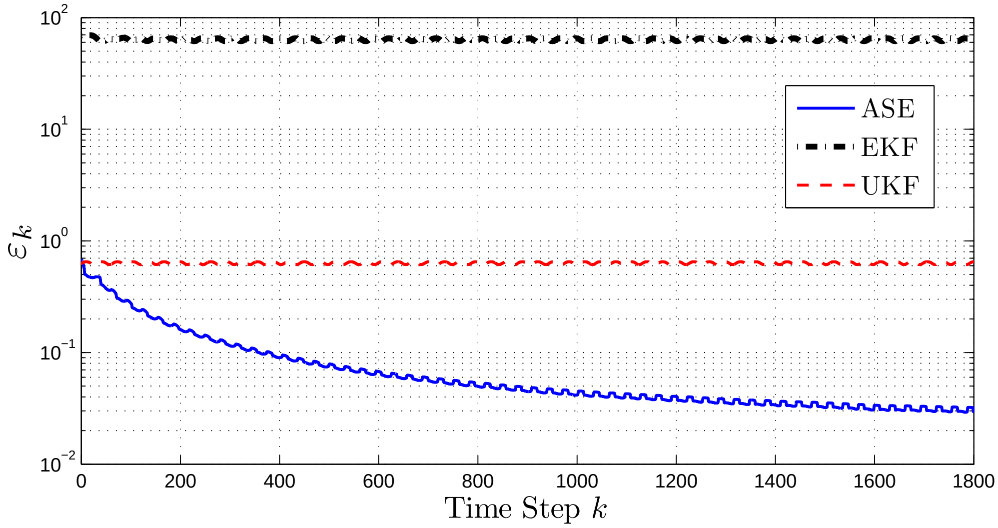

Figure 14. Comparison of the performance ε_k of the ASE, EKF, and UKF for the Van der Pol oscillator.

Next, consider the Van der Pol oscillator in Example 6 with the alternative output

$$y = \dot{q}. \tag{51}$$

For the ASE, let $\eta(k) = 1 \times 10^{-4}$, $n_c = 3$, $P(0) = 100 I_{6\times 6}$, and $\tilde{\mathcal{H}} = HFG$, where H, F and G are obtained by linearizing the discretized system about the origin. For the EKF and UKF, the noise covariance matrix is $Q = [0\ 10]^{\mathrm{T}}[0\ 10]$, and the initial error covariance matrix is $I_{2\times 2}$. Furthermore, for the UKF, we use five sigma points and we set $\kappa = 0$, $\alpha = 0.01$, and $\beta = 1$. Finally, $u(k) = 10\sin(k)$. Figure 15 shows the state estimates for the ASE and UKF. Figure 14 shows the performance ε_k of the ASE, EKF, and UKF.

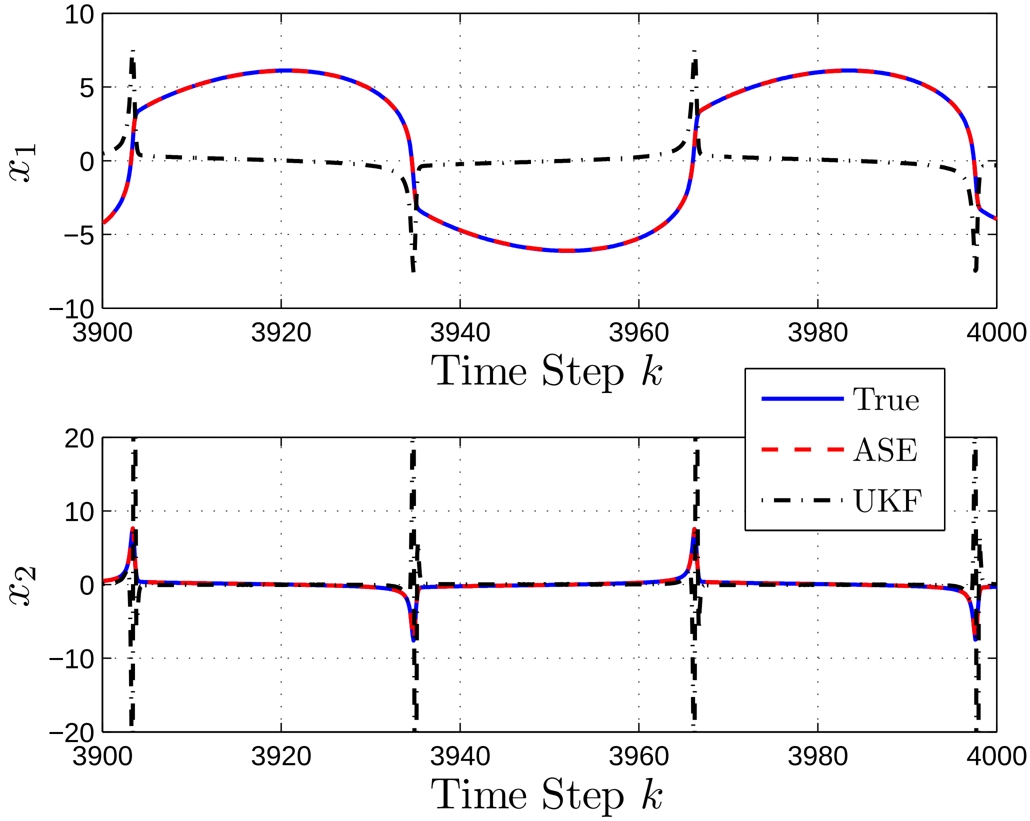

Figure 15. State estimates for the ASE and UKF.

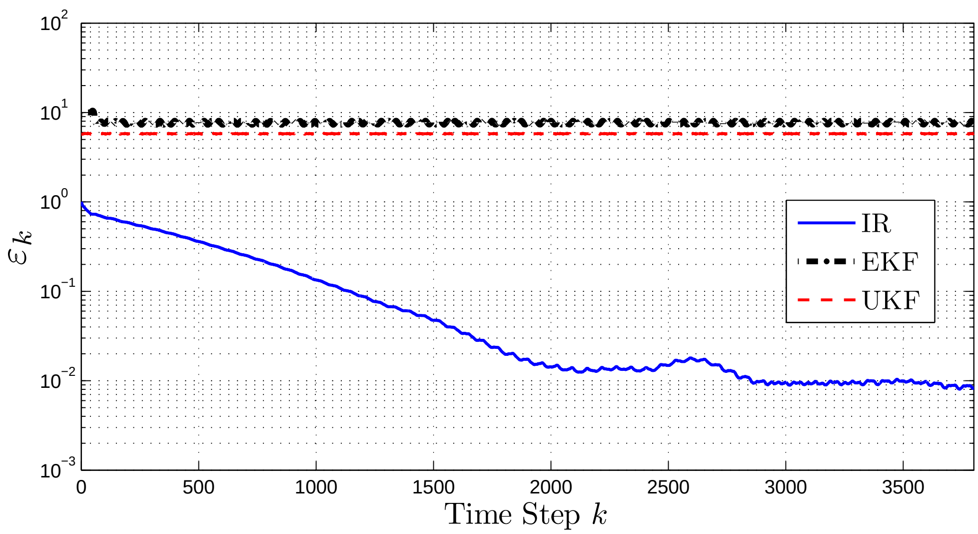

Figure 16. Comparison of the performance ε_k of the ASE, EKF, and UKF for the Van der Pol oscillator.

VII.C. Example 7: Van der Pol Oscillator with Process Noise

Consider the discretized nonlinear Van der Pol oscillator

$$x_1(k+1) = T_s x_2(k) + x_1(k),$$
$$x_2(k+1) = T_s \mu (1 - x_1(k)^2) x_2(k) - T_s x_1(k) + x_2(k) + T_s u(k) + \gamma T_s w(k),$$

where $\mu = 1$ and $T_s = 0.1$. The output of this system is $y = x_1$. Furthermore, $u(k) = 20\sin(k)$ and $w(k)$ is the realization of a zero mean Gaussian white noise process with unit variance. We test the ASE for $\gamma = 0$, $\gamma = 10^{-2}$, and $\gamma = 1$. We let $\eta(k) = 5 \times 10^{-4}$, $n_c = 2$, $P(0) = 100 I_{4\times 4}$, $\tilde{\mathcal{H}} = HG$, where H and G are obtained by linearizing the discretized system about the origin. For the UKF, we choose the same parameters as in Example 6. Figure 17 shows the performance ε_k of the ASE for several values of γ. For the UKF, the effect of γ is negligible and thus we show ε_k of the KF only for $\gamma = 0$.

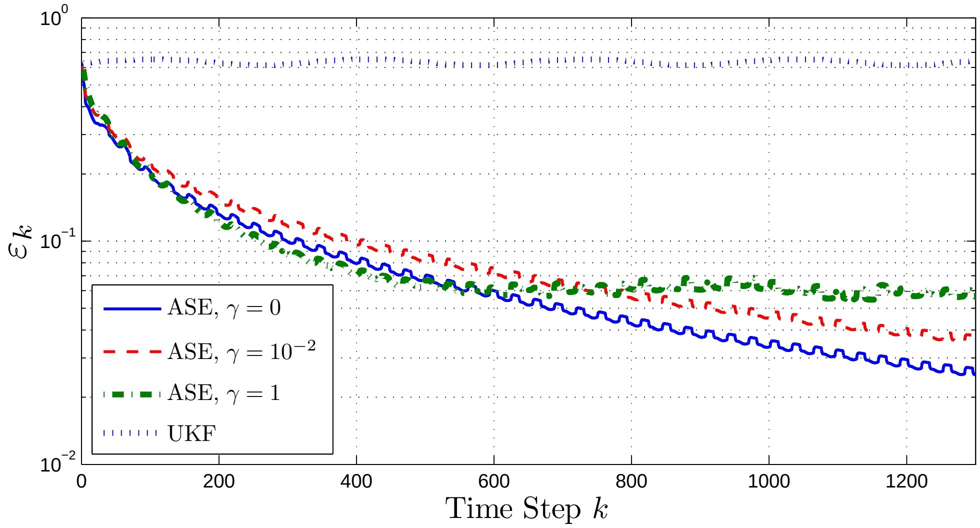

Figure 17. Comparison of the performance ε_k of the ASE for the Van der Pol oscillator with process noise.

VIII. Conclusions

In this paper we demonstrated a method for obtaining state estimates for minimum- and nonminimum-phase systems in the presence of harmonic process noise. First we constructed an estimator based on the known system model. At each step k we reconstruct the signal $u(k)$, called $\hat{u}^*(k)$, which minimizes the residual error $y(k) - \hat{y}(k)$. We then estimate a feedback system with input $z(k)$ and output $\hat{u}(k)$. Using the signal $\hat{u}(k)$ as the input to the estimator, we obtain estimates $\hat{x}$ of the system state $x(k)$.

We demonstrated the method on several linear examples including minimum and nonminimum-phase systems. In the minimum-phase case, the adaptive input reconstruction filter and the Kalman filter asymptotically reach zero state-estimation-error. In the nonminimum-phase case, the Kalman filter reaches a finite lower bound, on the state-estimation-error. The adaptive input reconstruction filter outperforms the Kalman filter in this case.

Finally, we extended the method to nonlinear state estimation. We compare the adaptive input reconstruction filter to the extended Kalman filter and the unscented Kalman filter. We note that the adaptive input reconstruction filter does not require knowledge of the process noise covariance or linearizations of the model about each state estimate.

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