IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

A random minimax

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2011 Russ. Math. Surv. 66 421

(http://iopscience.iop.org/0036-0279/66/2/R05)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 141.211.173.82
The article was downloaded on 06/04/2012 at 16:24

Please note that terms and conditions apply.

Russian Math. Surveys **66**:2 421–426
© 2011 RAS(DoM) and LMS
Uspekhi Mat. Nauk **66**:2 217–222
DOI 10.1070/RM2011v066n02ABEH004743

Dedicated to the memory of
Anatolii Alekseevich Karatsuba

A random minimax

H. L. Montgomery

Abstract. Stones are placed randomly on an $m \times n$ board, one at a time, with no more than one stone per unit cell. By means of a two-dimensional cross-classification, a formula is derived for the probability that a row becomes full (with n stones) at a time when there is another row that is still entirely empty.

Bibliography: 2 titles.

Keywords: random matrix, minimax, inclusion-exclusion.

1. Introduction and formulation of results. Let A be an $m \times n$ matrix whose entries are the integers $1, \dots, mn$, each one exactly once. We suppose that each of the $(mn)!$ matrices of this type are equally likely to occur, and on this probability space we consider the random variables

$$X = \min_{1 \leqslant i \leqslant m} \max_{1 \leqslant j \leqslant n} a_{ij}, \tag{1}$$

$$Y = \max_{1 \leqslant i \leqslant m} \min_{1 \leqslant j \leqslant n} a_{ij}. \tag{2}$$

Our object is to determine the distribution and expectation of X and of Y, and to derive a formula for $\mathsf{P}(X < Y)$.

Theorem 1. *Let X be defined as in (1). Then the mass function is*

$$p_X(k) = \mathsf{P}(X - k) = \sum_{r=1}^{m} (-1)^{r-1} \frac{\binom{m}{r}\binom{k-1}{rn-1}}{\binom{mn}{rn}}$$

for $1 \leqslant k \leqslant m$, and

$$\mathsf{E}[X] = \frac{m!\, n^m}{\prod_{i=1}^{m-1}(in+1)} = (mn+1) \prod_{i=1}^{m} \frac{in}{in+1}\,.$$

If π is a permutation of the numbers $1, \dots, mn$, let π' denote the associated permutation $\pi'(i) = mn + 1 - \pi(i)$. It is easy to see that $Y(\pi) = mn + 1 - X(\pi')$. Hence the distribution and expectation of Y follow immediately from those of X, which is to say we have the following theorem.

Theorem 2. *Let X and Y be defined as in (1) and (2). Then*

$$p_Y(k) = p_X(mn + 1 - k)$$

for $1 \leqslant k \leqslant mn$, and $\mathsf{E}[Y] = mn + 1 - \mathsf{E}[X]$.

AMS 2010 *Mathematics Subject Classification.* Primary 05A05; Secondary 33C05.

Theorem 3. *Let X and Y be defined as in (1) and (2). Then*

$$\mathsf{P}(X<Y) = 1 + \sum_{k=0}^{m}(-1)^k \binom{m}{k} \sum_{r=0}^{k} \frac{\binom{k}{r}}{\binom{kn}{rn}}\,.$$

According to Doron Zeilberger (oral communication), the formula for p_X in Theorem 1 cannot be written in closed form. Also, the formula in Theorem 3 can be written neither in closed form nor as a one-dimensional sum.

Suppose that a deck of 52 playing cards is to be placed face up in 13 rows and 4 columns, with aces in the first row, dueces in the second, and so on, and spades in the first column, hearts in the second, diamonds in the third, and clubs in the fourth. The deck is in random order, and the cards are placed one at a time. (This was done for the purpose of stacking the deck in preparation for performing a magic trick invented by J. H. Conway.) With $m = 13$ and $n = 4$ the variable X is the index of the card that for the first time completes a row of the array. When the card with index Y is placed, we have for the first time at least one card in each row of the array. When this procedure was executed, it was noted empirically that often one row is completed while there is still a row that is empty, which is to say that $X < Y$. From our theorems we find that

$$\begin{aligned}
\mathsf{E}[X] &= \frac{68\,719\,476\,736}{2\,748\,462\,675} = \frac{2^{36}}{3\cdot 5^2\cdot 7^2\cdot 17\cdot 29\cdot 37\cdot 41} = 25.00287792,\\
\mathsf{E}[Y] &= \frac{76\,949\,045\,039}{2\,748\,462\,675} = \frac{317\cdot 8221\cdot 29527}{3\cdot 5^2\cdot 7^2\cdot 17\cdot 29\cdot 37\cdot 41} = 27.99712208,\\
\mathsf{P}(X<Y) &= \frac{1\,309\,858\,309\,850\,791}{2\,149\,964\,274\,784\,935} = \frac{709\cdot 250708771\cdot 7369}{3\cdot 5\cdot 7\cdot 17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47}\\
&= 0.6092465467.
\end{aligned}$$

It would be tempting to conjecture that the inequality $\mathsf{E}[X] < \mathsf{E}[Y]$ holds if and only if $\mathsf{P}(X<Y) > 1/2$, but in fact this fails when $m = 5,\ n = 21$ since then

$$\begin{aligned}
\mathsf{E}[X] &= 52.64173261,\\
\mathsf{E}[Y] &= 53.35826739,\\
\mathsf{P}(X<Y) &= 0.4977715896.
\end{aligned}$$

2. Proof of Theorem 1. Suppose that k is given. For $1 \leqslant i \leqslant m$ let A_i denote the event $\{\max_{1\leqslant j\leqslant n} a_{ij} \leqslant k\}$. Thus the event $\{X \leqslant k\}$ is $\bigcup_{i=1}^m A_i$. By the inclusion-exclusion principle it follows that

$$\mathsf{P}(X>k) = 1 - \mathsf{P}\Big(\bigcup A_i\Big) = \sum_{r=0}^{m}(-1)^r \sum_{1\leqslant i_1<\cdots<i_r\leqslant m} \mathsf{P}(A_{i_1}\cdots A_{i_r}). \tag{3}$$

If $rn > k$, then $A_{i_1}\cdots A_{i_r} = \varnothing$. If $rn \leqslant k$, then we construct the event $A_{i_1}\cdots A_{i_r}$ by first choosing the rn numbers not exceeding k that are to fall in the rows $i_1,\dots,i_r$ of the matrix A. This can be done in $\binom{k}{rn}$ ways. Next we choose where each of these rn numbers is positioned in the rn places. This can be done in $(rn)!$ ways. The remaining $mn - rn$ numbers can be assigned to the remaining $mn - rn$ locations in $(mn - rn)!$ ways. Hence

$$\mathsf{P}(A_{i_1} \cdots A_{i_r}) = \frac{\binom{k}{rn}(rn)!\,(mn-rn)!}{(mn)!} = \frac{\binom{k}{rn}}{\binom{mn}{rn}}\,.$$

We note that this formula holds also when $rn > k$. On inserting this in (3), we deduce that

$$\mathsf{P}(X > k) = \sum_{r=0}^{m} (-1)^r \frac{\binom{m}{r}\binom{k}{rn}}{\binom{mn}{rn}}\,.$$

Thus

$$p_X(k) = \mathsf{P}(X > k-1) - \mathsf{P}(X > k) = \sum_{r=0}^{m} (-1)^r \frac{\binom{m}{r}}{\binom{mn}{rn}} \left[\binom{k-1}{rn} - \binom{k}{rn}\right].$$

The difference inside the square brackets on the right vanishes when $r = 0$, so the term $r = 0$ can be omitted. For $r > 0$, this difference is $-\binom{k-1}{rn-1}$. Thus we have the stated value for $p_X(k)$.

To determine the expectation of X it suffices to note that

$$\begin{aligned}
\mathsf{E}[X] = \sum_{k=1}^{mn} k p_X(k) &= mn \sum_{k=1}^{mn} \sum_{r=1}^{m} (-1)^{r-1} \frac{\binom{m-1}{r-1}\binom{k}{rn}}{\binom{mn}{rn}} \\
&= mn \sum_{r=1}^{m} (-1)^{r-1} \frac{\binom{m-1}{r-1}}{\binom{mn}{rn}} \sum_{k=1}^{mn} \binom{k}{rn} \\
&= mn \sum_{r=1}^{m} (-1)^{r-1} \frac{\binom{m-1}{r-1}\binom{mn+1}{rn+1}}{\binom{mn}{rn}} \\
&= mn(mn+1) \sum_{r=0}^{m-1} (-1)^r \frac{\binom{m-1}{r}}{rn+n+1}\,.
\end{aligned}$$

To complete the proof it suffices to show that

$$\sum_{r=0}^{m-1} (-1)^r \frac{\binom{m-1}{r}}{rn+n+1} = \frac{1}{mn} \prod_{i=1}^{m} \frac{in}{in+1}\,. \tag{4}$$

By using the binomial theorem and integrating term-by-term it is clear that the left-hand side above is

$$\int_0^1 (1-x^n)^{m-1} x^n\, dx.$$

On making the change of variable $u = x^n$ we see that this is

$$\frac{1}{n} \int_0^1 (1-u)^{m-1} u^{1/n}\, du.$$

Here the integral is the Beta function, for which there is a simple formula (see Theorem 1.1.4 of [1]):

$$\mathrm{B}(\alpha,\beta)=\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\,.$$

By the recurrence $z\Gamma(z)=\Gamma(z+1)$, it follows that the expression we wish to evaluate is

$$\frac{1}{n}\,\mathrm{B}\Big(1+\frac{1}{n}\,,\,m\Big)=\frac{\Gamma(1+1/n)\Gamma(m)}{n\Gamma(m+1+1/n)}=\frac{m!\,\Gamma\big(\frac{1}{n}\big)\frac{1}{n}}{mn\Gamma\big(\frac{1}{n}\big)\frac{1}{n}\frac{n+1}{n}\cdots\frac{mn+1}{n}}\,,$$

which gives (4).

3. Proof of Theorem 3. Fix k and ℓ with $1\leqslant k<\ell\leqslant mn$. For $1\leqslant i\leqslant m$, let A_i denote the event $\{\max_{1\leqslant j\leqslant n}a_{ij}\leqslant k\}$, and similarly for $1\leqslant h\leqslant m$, let B_h denote the event $\{\min_{1\leqslant j\leqslant n}a_{hj}\geqslant \ell\}$. Then $A=\bigcup A_i$ is the event $\{X\leqslant k\}$, and $B=\bigcup B_i$ is the event $\{Y\geqslant \ell\}$. Since $A^cB^c=(A\cup B)^c$, we see that

$$\begin{aligned}\mathsf{P}(X>k,Y<\ell)&=1-\mathsf{P}\Big(\bigcup_{i=1}^m A_i\cup\bigcup_{h=1}^m B_h\Big)\\&=\sum_{r=0}^m\sum_{s=0}^m(-1)^{r+s}\sum_{\substack{1\leqslant i_1<\cdots<i_r\leqslant m\\1\leqslant h_1<\cdots<h_s\leqslant m}}\mathsf{P}(A_{i_1}\cdots A_{i_r}B_{h_1}\cdots B_{h_s}).\end{aligned}\tag{5}$$

To calculate these probabilities, we first note that $A_iB_i=\varnothing$, since

$$\max_j a_{ij}\leqslant k<\ell\leqslant\min_j a_{ij}$$

is impossible. Thus we may assume that $r+s\leqslant m$. The intersection $A_{i_1}\cdots A_{i_r}$ is empty if $rn>k$. Thus we may restrict our attention to $rn\leqslant k$. Similarly, we may assume that $sn\leqslant mn+1-\ell$. The rn integers that are to fall in rows $i_1,\dots,i_r$ can be chosen in $\binom{k}{rn}$ ways, and there are $(rn)!$ ways of positioning these numbers. Similarly, the sn integers that are to fall in rows $h_1,\dots,h_s$ can be chosen in $\binom{mn+1-\ell}{sn}$ ways, and these sn numbers can be positioned in $(sn)!$ ways. Finally, the remaining $(m-r-s)n$ numbers can be positioned in $\big((m-r-s)n\big)!$ ways. Thus

$$\begin{aligned}\mathsf{P}(A_{i_1}\cdots A_{i_r}B_{h_1}\cdots B_{h_s})&=\frac{\binom{k}{rn}(rn)!\,\binom{mn+1-\ell}{sn}(sn)!\,\big((m-r-s)n\big)!}{(mn)!}\\&=\frac{\binom{k}{rn}\binom{mn+1-\ell}{sn}}{\binom{mn}{(r+s)n}\binom{(r+s)n}{rn}}\,.\end{aligned}$$

We note that this last formula also holds when $rn > k$ or $sn > mn + 1 - \ell$. The r rows i_μ and the s rows h_ν can be chosen in $\binom{m}{r\ s\ m-r-s}$ ways. Thus

$$\begin{aligned} P(X > k, Y < \ell) &= \sum_{r=0}^{m} \sum_{s=0}^{m-r} (-1)^{r+s} \binom{m}{r\ \ s\ \ m-r-s} \frac{\binom{k}{rn}\binom{mn+1-\ell}{sn}}{\binom{mn}{(r+s)n}\binom{(r+s)n}{rn}} \\ &= \sum_{t=0}^{m} \frac{(-1)^t \binom{m}{t}}{\binom{mn}{tn}} \sum_{r=0}^{t} \frac{\binom{t}{r}\binom{k}{rn}\binom{mn+1-\ell}{(t-r)n}}{\binom{tn}{rn}}. \end{aligned} \tag{6}$$

We note that

$$\begin{aligned} \mathsf{P}(X < Y) &= 1 - \mathsf{P}(X \geqslant Y) = 1 - \sum_{k=1}^{mn} \mathsf{P}(X = k,\ Y \leqslant k) \\ &= 1 - \sum_{k=1}^{mn} \mathsf{P}(X = k,\ Y < k+1). \end{aligned}$$

Since the event $\{X = k\}$ is the event $\{X > k-1\} \setminus \{X > k\}$, the above is

$$= 1 - \sum_{k=1}^{mn} \big(\mathsf{P}(X > k-1,\ Y < k+1) - \mathsf{P}(X > k,\ Y < k+1)\big).$$

By (6) this is

$$\begin{aligned} &1 - \sum_{k=1}^{mn} \sum_{t=0}^{m} \frac{(-1)^t \binom{m}{t}}{\binom{mn}{tn}} \sum_{r=0}^{t} \frac{\binom{t}{r}\binom{mn-k}{(t-r)n}}{\binom{tn}{rn}} \left(\binom{k-1}{rn} - \binom{k}{rn}\right) \\ &\quad = 1 + \sum_{t=0}^{m} \frac{(-1)^t \binom{m}{t}}{\binom{mn}{tn}} \sum_{r=0}^{t} \frac{\binom{t}{r}}{\binom{tn}{rn}} \sum_{k=1}^{mn} \binom{k-1}{rn-1}\binom{mn-k}{(t-r)n}. \end{aligned}$$

To complete the proof it suffices to show that

$$\sum_{k=1}^{mn} \binom{k-1}{rn-1}\binom{mn-k}{(t-r)n} = \binom{mn}{tn}. \tag{7}$$

To this end we recall by the 'negative' binomial theorem that

$$\begin{aligned} \frac{1}{(1-z)^{rn}} &= \sum_{i=0}^{\infty} \binom{rn-1+i}{rn-1} z^i, \\ \frac{1}{(1-z)^{(t-r)n+1}} &= \sum_{j=0}^{\infty} \binom{(t-r)n+j}{(t-r)n} z^j, \\ \frac{1}{(1-z)^{tn+1}} &= \sum_{k=0}^{\infty} \binom{tn+k}{tn} z^k, \end{aligned}$$

for $|z| < 1$. Here the third function is the product of the first two, and hence its power series coefficients can be written as the Cauchy convolution of the first two sets of coefficients, which is to say that

$$\sum_{i=0}^{k} \binom{rn-1+i}{rn-1} \binom{(t-r)n+k-i}{(t-r)n} = \binom{tn+k}{tn}.$$

In particular, when $k = (m-t)n$, we find that

$$\sum_{i=0}^{(m-t)n} \binom{rn-1+i}{rn-1} \binom{(m-r)n-i}{(t-r)n} = \binom{mn}{tn}.$$

This is (7) apart from a change in the indexing, so the proof is complete.

It is worth noting that the identities (4) and (7), for which we provided simple proofs, have been known for many years, perhaps for more than seven centuries, and indeed they are both special cases of a much more general identity, which we now describe. For $|z| < 1$, we define the hypergeometric function

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!\,(c)_k}\, z^k,$$

where $(x)_k = x(x+1)\cdots(x+k-1)$. If it happens that b is a negative integer, then the sum terminates, and the Chu–Vandermonde identity (which is Corollary 2.2.3 in [1]) asserts that

$${}_2F_1\left(\begin{matrix} a, -N \\ c \end{matrix}; 1\right) = \sum_{k=0}^{N} \frac{(a)_k (-N)_k}{k!\,(c)_k} = \frac{(c-a)_N}{(c)_N}\,.$$

For a discussion of the history of this identity, see Chap. 7 of [2]. Our identity (4) is obtained by taking $N = m-1$, $a = 1+1/n$, and $c = 2+1/n$, and (7) follows on taking $N = (m-t)n$, $a = rn$, $c = -(m-r)n$.

The author is grateful to Richard Askey and George E. Andrews for assistance in locating this antecedent.

Bibliography

[1] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia Math. Appl., vol. 71, Cambridge Univ. Press, Cambridge 1999, xii+664 pp.

[2] R. Askey, *Orthogonal polynomials and special functions*, Society for Industrial and Applied Mathematics, Philadelphia, PA 1975, vii+110 pp.

H. L. Montgomery
University of Michigan, USA
E-mail: hlm@umich.edu

Received 03/OCT/10