Philosophy Compass 7/8 (2012): 559–570, 10.1111/j.1747-9991.2011.00388.x

Philosophical Aspects of Quantum Field Theory: I

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Abstract

This is the first of a two-part introduction to some interpretive questions that arise in connection with quantum field theories (QFTs). Some of these questions are continuous with those familiar from the discussion of ordinary non-relativistic quantum mechanics (QM). For example, questions about locality can be rigorously posed and fruitfully pursued within the framework of QFT. A stark disanalogy between QFTs and ordinary QM – the former, but not the latter, typically admit infinitely many putatively physically inequivalent realizations – prompts relatively novel questions, questions about how to understand and adjudicate different strategies for equipping quantum theories with content. Part I sketches the fate of locality and related notions in QFT, then documents the non-uniqueness unprecedented in ordinary QM but rampant in QFT. Part II presents foundations issues raised by non-uniqueness.

1. *Introduction*

Philosophical discussion of non-relativistic quantum mechanics largely centers on two issues that can be motivated by appeal to systems possessing a very few degrees of freedom. The first issue, *quantum non-locality*, is exemplified by the distant correlations imposed by the spin singlet state, a state of a pair of electrons, each modeled as bivalent (spin up or down). The second, the *measurement problem*, is exemplified by Schrödinger's cat, a bivalent (alive or dead) system coupled to a similarly bivalent (undecayed or decayed) atom. A Hilbert space of four dimensions (spanned, in the spin singlet case, by the simultaneous spin eigenvectors $|+\rangle|-\rangle, |+\rangle|+\rangle, |-\rangle|-\rangle, |-\rangle|+\rangle$) suffices for the orthodox quantum treatment of each of these exemplary settings.

By contrast, quantum field theory (QFT), and the thermodynamic limit of quantum statistical mechanics (QSM), address systems of infinitely many degrees of freedom. QFTs arise by quantizing classical field theories, which assign real or complex numbers (the field amplitudes) to every point of spacetime; to take the thermodynamic limit of QSM is to let the number of microsystems (constituting a macrosystem with interesting bulk thermodynamic behavior) and the volume they occupy go to infinity while their density remains finite. Grouping these theories under the heading of QM_∞, I aim in this two-part contribution to chronicle some of the ways their foundational investigation rewards attention, both for deepening and extending themes already familiar from the philosophy of ordinary QM and for introducing new questions that dramatize themes applicable to all physical theories. Section 2 discusses a theme extended from ordinary QM to QFT: locality and entanglement. Section 3 offers one account of the project of interpreting physical theories. Sections 4–5, and Part II, survey a set of questions, questions this account would brand interpretive, which are largely unprecedented in the foundations of ordinary QM. These questions pertain to theories of QM_∞ in virtue of their admitting (by the lights of ordinary QM) infinitely many *physically inequivalent* realizations. Section

4 explains why this non-uniqueness does not arise in ordinary QM; Section 5 illustrates how it can arise in theories that aren't field theories. Part II introduces properly field theoretic examples of non-uniqueness and the interpretive questions they prompt.

These essays will be mathematically informal – I will presuppose only some familiarity with the rudiments of ordinary non-relativistic QM (see Redhead (1988) or Hughes (1989) for introductions) – and limited in scope. Halvorson and Müger (2007) (for QFT) and Sewell (2002) (for QSM) correct both these failings. Discussions of the renormalization of interacting QFTs, along with important interpretive approaches neglected here due to constraints of space, can be found in Huggett (2000), Teller (1995), Sklar (2002), and Auyang (1995).

2. Locality and Entanglement

QM_∞ can supply a new vantage from which to consider interpretive problems familiar from ordinary QM. This section surveys some of QM_∞'s contributions to the discussion of quantum non-locality.[1] Phenomena accurately described by ordinary QM violate the Bell's inequalities, signalling that one of the assumptions generating those inequalities must be false. Some these assumptions can be cast in a form strongly reminiscent of the special theory of relativity (STR)'s folkloric ban on superluminal causal influence. This has lent the question of whether QM and the STR can 'peacefully coexist' (Redhead 1983) urgency.

As urgent as the question is, as stated it lacks precision. STR is not readily understood as a theory about causes and their admissible configurations. STR rather requires of spacetime theories formulated in Minkowski spacetime that they be Lorentz covariant. Non-relativistic QM, which is not a spacetime theory, is not subject to STR's requirements. So the question of whether STR and QM are capable of peaceful coexistence can be posed only after extensive precarious and heroic interpretive work – work addressing exceedingly non-trivial questions such as how to understand relativistic constraints on causal action in a stochastic setting – has been completed.

Another question – can there be Lorentz-covariant quantum theories? – is much more tractable. Axiomatic algebraic approaches to QFT set up an association between open bounded regions of a spacetime and algebras of observables. Section 4 will make the notion of algebra more precise. For now, think of the algebra $\mathfrak{A}(O)$ associated with a spacetime region O as a collection of magnitudes measurable in that regions. Axiomatic algebraic QFT subjects such associations to axioms expressing natural desiderata. For instance, an *Isotony* axiom requires algebras associated with spacetime regions to reduplicate the inclusion relations between those regions. Haag and Kastler's (1964) axioms were formulated with Minkowski spacetime in mind; Dimock's (1980) generalize these. The Haag–Kastler axioms include an axiom of Lorentz covariance. The existence of QFTs satisfying the Haag–Kastler axioms (including models of *interacting* quantum fields in two or three spacetime dimensions (Glimm and Jaffe 1972)) positively settles the tractable version of the peaceful coexistence question.

It is significant that the axiom of Lorentz covariance is *logically independent* of Haag and Kastler's *Microcausality* axiom, which requires algebras associated with space-like separated regions – that is, regions none of whose points are connectable by signals travelling at or below the speed of light – to commute with one another. *This* axiom, not the one demanding compliance with STR in the form of Lorentz covariance, is generally taken to express, and be motivated by, a ban on superluminal signal propagation. The intuition underlying the motivation is that operators associated with spacelike separated regions correspond to measurements whose temporal order is different for different inertial observers. Lest the quantum probabilities assigned outcomes of those measurements also vary from observer to observer, it must not matter what order those operators act in; that is, they must commute. But not even the microcausality axiom succeeds in guarding QFTs against provocative non-local entanglements. The Reeh–Schlieder theorem, a consequence of the standard axioms, is a striking example of this. The theorem concerns an algebra $\mathfrak{A}(O)$ of observables associated with an open bounded region O of spacetime, e.g. a physics laboratory over the course of a fall afternoon. The Minkowski vacuum state $|0\rangle$ is a state on the algebra associated with the entirety of spacetime, a state axiomatically characterized in terms of the symmetries of Minkowski spacetime. The theorem states that the set of states obtained by acting on $|0\rangle$ with elements of $\mathfrak{A}(O)$ is *dense* in the set of possible states for all of spacetime. In other words, any state of an axiom-satisfying QFT on Minkowski spacetime can be approximated arbitrarily closely by acting on the vacuum state by polynomial combinations of observables in $\mathfrak{A}(O)$. If it were appropriate to model events in the region O as applications of elements of $\mathfrak{A}(O)$ to the global vacuum state, this would mean that machinations in local regions, such as physics laboratories on fall afternoons, could produce arbitrary approximations of arbitrary global states!

The Reeh–Schleider theorem also implies that $|0\rangle$ is an eigenstate of no observable associated with a finite spacetime region: this makes vacuum correlations ubiquitous, in the sense that there is no local region with which the vacuum state associates a pure state. Redhead (1995) illuminates the Reeh–Schleider theorem by explicating analogies between the Minkowski vacuum state and how the spin-singlet state stands to algebras of spin observables pertaining to its component systems. Summers and Buchholz (2005) is a more recent, and more technical, account of the state of play. Thus work on non-locality and holism initiated in the context of ordinary QM continues, and continues with a vengeance, in the context of QFT.

3. *The Content of Physical Theories*

QM_∞ prompts questions without notable counterpart in the foundations of ordinary QM. Consider a system of finitely many particles. Standardly, to quantize such a system is to find what I'll call a *Hilbert space representation* of its quantum physics. That physics is encapsulated by a set of relation between physical magnitudes, aka *observables*, pertaining to the system: *canonical commutation relations* (CCRs), interrelating the positions and momenta of mechanical systems, or *canonical anticommutation relations* (CARs), interrelating different components of spin for spin systems. A Hilbert space representation of the quantum physics of a given system, then, is a set of operators acting on a Hilbert space and obeying the relations characteristic of that system at hand.[2] These operators correspond to fundamental physical observables. Other observables pertaining to the system correspond to polynomials of, and limits of sequences of polynomials of, the representation-bearing operators. A state of the system is a well-behaved expectation value assignment to these observables. In this way, a Hilbert space representation of the CCRs or CARs constituting a quantum theory supplies a *kinematics* – an account of the possible states and the magnitudes in their scope – for that theory.

Most interpretations of ordinary QM take quantum kinematics, in the form determined by a Hilbert space representation, as their point of departure. From there, in pursuit of a solution to the measurement problem, they typically proceed in different directions. Some supplement the basic quantum observables with hidden variables; others append schizophrenic dynamics riddled by measurement collapse; still others maintain that a quantum system can exhibit a determinate observable value its state cannot predict with certainty. But interpreters of ordinary QM generally don't worry that the basic Hilbert space kinematics framing their interpretive dispute is *the wrong kinematics*. They don't worry, that is, that the instruction, 'quantize this system', could be carried out in multiple, incompatible ways. They don't worry about this because *the Stone–von Neumann uniqueness theorem* tells them not to. The theorem states that all Hilbert space representations of the CCRs for a particular classical Hamiltonian theory of finitely many particles stand to one another in a mathematical relation called *unitary equivalence* (see Summers 2001). Because unitarily equivalent Hilbert space representations agree, and agree systematically, in what expectation values they assign, and because expectation value assignments are supposed to be the vehicle of the empirical content of quantum theories, unitary equivalence is widely accepted as a standard of physical equivalence for Hilbert space representations. It follows that variant representations of the CCRs for a given classical Hamiltonian theory are simply and unalarmingly different ways of expressing the same quantum kinematics. For finitely many spin systems subject to CARs, the Jordan–Wigner theorem likewise guarantees uniqueness (see Emch 1972, 269–75). For systems of finitely many particles, there is essentially only one way to follow the instruction: 'quantize!'.

This brings us to novel interpretive matters dramatized by QM_∞. QFT and the thermodynamic limit of QSM fall outside the scope of these uniqueness theorems. According to very same criterion of physical equivalence by whose lights Hilbert space representations for ordinary quantum theories are reassuringly unique, a QM_∞ theory can admit infinitely many presumptively physically inequivalent Hilbert space representations. This raises a host of interpretive questions, some of which are sketched in the following sections. Section 4 offers a brief account of quantization and its uniqueness for ordinary QM. Section 5 develops simple but provocative examples of the non-uniqueness of Hilbert space representations. Part II continues the discussion with examples of non-uniqueness drawn from QFT and the philosophical responses they elicit.

4. Quantization and Uniqueness

4.1 REPRESENTING THE CCRS/CARS

In classical Hamiltonian mechanics, the state of a simple mechanical system is given by its position and momentum. The position and momentum variables q_i and p_j therefore serve as coordinates for the phase space M of possible states of the system. System observables are functions from M to the real numbers $\mathbb{R}$. The position and momentum *observables* are examples: they map points in phase space to their q_i and p_j coordinate values, respectively. All other observables can be expressed as functions of these observables. Foremost among these is the Hamiltonian observable H, which usually coincides with the sum of the system's kinetic and potential energies. The Hamiltonian helps identify dynamically possible trajectories $\mathrm{q}(t)$, $\mathrm{p}(t)$ through phase space as those obedient to Hamilton's equations of motion, which are equivalent to Newton's second law.

To motivate the canonical Hamiltonian quantization recipe, remark that the collection of classical observables exhibits an *algebraic structure*, where this is understood as follows: as smooth functions on phase space, classical observables form a set that is also a vector space over the real numbers. To first approximation, an algebra is a collection of elements equipped with a way of forming linear combinations and products of those elements. More formally, an algebra is a linear vector space V endowed with a (not necessarily associative) multiplicative structure. The vector space of classical observables becomes a

Lie algebra upon being equipped with a multiplicative structure supplied by the *Poisson bracket*. The Poisson bracket $\{f, g\}$ of classical observables f: $M \rightarrow \mathbb{R}$ and g: $M \rightarrow \mathbb{R}$ is

$$\{f,g\} := \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{1}$$

Notice that for observables p and q

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \quad \{p_i, q_j\} = -\delta_{ij} \tag{2}$$

The canonical Hamiltonian quantization recipe enjoins us to quantize a classical theory cast in Hamiltonian form by promoting its canonical observables to symmetric operators $\hat{q}_i$, $\hat{p}_i$ acting on a separable Hilbert space $\mathcal{H}$ and obeying commutation relations corresponding to the Poisson brackets of the classical theory. In the case of a classical theory with phase space $\mathbb{R}^{2n}$ and canonical observables q_i and p_i, these CCRs are (where $[\hat{A}, \hat{B}] := \hat{A}\hat{B} - \hat{B}\hat{A}$, $\hat{I}$ is the identity operator, and $\hbar$ is set to one)

$$[\hat{p}_i, \hat{p}_j] = [\hat{q}_i, \hat{q}_j] = 0, [\hat{p}_i, \hat{q}_j] = -i\delta_{ij}\hat{I} \tag{3}$$

The standard execution of the recipe is *Schrödinger's wave function representation*, set in the Hilbert space $L^2(\mathbb{R}^n)$ of square integrable complex-valued functions of $\mathbb{R}^n$. For $n = 1$, Schrödinger's wave function representation defines $\hat{q}\psi(x) = x\psi(x)$ and $\hat{p}\psi(x) = -i\frac{d\psi(x)}{dx}$.

To build a quantum theory for a single spin system, one finds symmetric operators $\{\hat{\sigma}(x), \hat{\sigma}(y), \hat{\sigma}(z)\}$ acting on a Hilbert space $\mathcal{H}$ to satisfy the *Pauli Relations (CARs)*, which include

$$[\hat{\sigma}(x), \hat{\sigma}(y)] = i\hat{\sigma}(z), \ [\hat{\sigma}(y), \hat{\sigma}(z)] = i\hat{\sigma}(x), \ [\hat{\sigma}(z), \hat{\sigma}(x)] = i\hat{\sigma}(y) \tag{4}$$

Call these $\{\hat{\sigma}(i)\}$ the *Pauli spin observables*. The generalization to n spin systems is straightforward. To quantize such a system, one finds for each spin system k a Pauli spin $\hat{\sigma}^k = (\hat{\sigma}^k(x), \hat{\sigma}^k(y), \hat{\sigma}^k(z))$ satisfying the Pauli Relations, expanded to include the requirement that spin observables for different systems commute.

Having crafted a Hilbert space representation of the CARs/CCRs, one prosecutes QM as usual. One enriches ones set of physical magnitudes by taking polynomials, and limits of sequences of polynomials, of the canonical magnitudes representing the CARs/CCRs. Here an important subtlety arises: to what criterion of convergence ought we appeal when deciding which sequence have limits? Adopting the criterion furnished by the *weak operator topology*, the result is $\mathfrak{B}(\mathcal{H})$, the set of bounded operators on $\mathcal{H}$, the Hilbert space carrying the representation.[3] Observables of the quantum theory are identified with the self-adjoint elements of $\mathfrak{B}(\mathcal{H})$. The question before us is: can Erwin and Werner, diligently following the Hamiltonian recipe starting from the same classical theory, cook up physically distinct quantizations?

4.2. UNIQUENESS THEOREMS

Think of a physical theory as sorting logically possible worlds into two piles. One pile contains logically possible worlds that are also, according to the theory, physically possible; the other pile contains the worlds the theory deems physically impossible. And suppose (along with philosophers as various as David Lewis and Ludwig Wittgenstein) that the content of a theory, so understood, consists of the set of worlds possible according to it. Then two theories are physically equivalent – that is, they have identical content – just in case they put the same worlds into the 'physically possible' pile. This content coincidence criterion of physical equivalence is one component in the construction of the Stone–von Neumann and Jordan–Wigner theorems as demonstrations of the physical equivalence of quantizations in their scope.

Other components are substantive assumptions about the details of those quantizations, the sets of physical magnitudes they deploy, and the sets of states they entertain. It is in terms of these sets that the quantizations describe and individuate physical possibilities. In particular, it is assumed that a quantization identifies physical observables with the self adjoint elements of $\mathfrak{B}(\mathcal{H})$ for some separable $\mathcal{H}$. *Supposing* that states should be normalized, linear, positive, and countably additive, it follows that a quantization identifies possible states with the set $\mathfrak{W}$ of density operators on $\mathcal{H}$. Each $\hat{W} \in \mathfrak{W}$ determines a state via the trace prescription, which assigns each self-adjoint $\hat{A} \in \mathfrak{B}(\mathcal{H})$ the expectation value $\mathrm{Tr}(\hat{W}\hat{A})$.

The pair $(\mathfrak{B}(\mathcal{H}), \mathfrak{W})$ encapsulates what worlds are possible according to a quantization of the sort under discussion. We want a criterion of physical equivalence that requires the sets of worlds possible according to equivalent quantizations $(\mathfrak{B}(\mathcal{H}), \mathfrak{W})$ and $(\mathfrak{B}(\mathcal{H}'), \mathfrak{W}')$ to coincide. But we should demand more than a one-to-one correspondence between those sets of possible worlds. Part of the content of a theory is the functional relationships it posits between the physical magnitudes it recognizes. After all, these functional relationships are entangled in its laws: $\hat{H} = \hat{p}^2/2m$ makes the energy of a free system of mass m a function of its momentum; the Schrödinger equation uses the energy of an isolated system to build a family $\hat{U}(t) = \mathrm{e}^{-i\hat{H}t}$ of operators describing that system's time evolution, which implies (roughly speaking) that the operator $\hat{H}$ is a limit of a sequence of functions of the operators $\hat{U}(t)$. Confining attention to relations between magnitudes that are uncontentiously kinematic, the *algebraic structure* of a collection of magnitudes – which in some sense encapsulates the genealogy of their descent from a collection of canonical magnitudes generating them – is part of the kinematic content of a pair $(\mathfrak{B}(\mathcal{H}), \mathfrak{W})$. So let us adopt the following account of physical equivalence for Hilbert space representations (cf. Clifton and Halvorson 2001):

> $(\mathfrak{B}(\mathcal{H}), \mathfrak{W})$ and $(\mathfrak{B}(\mathcal{H}'), \mathfrak{W}')$ are physically equivalent exactly when there are bijections i_{obs}: $\mathfrak{B}(\mathcal{H}) \rightarrow \mathfrak{B}(\mathcal{H}')$ and i_{state}: $\mathfrak{W} \rightarrow \mathfrak{W}'$ such that i_{obs} preserves relevant algebraic structure and
>
> $$\mathrm{Tr}\left(i_{state}(\hat{W})i_{obs}(\hat{A})\right) = \mathrm{Tr}(\hat{W}\hat{A}) \tag{5}$$

for all $\hat{W} \in \mathfrak{W}$ and all $\hat{A} \in \mathfrak{B}(\mathcal{H})$.

Equation (5) guarantees that each state $\hat{W}$ in one representation has a counterpart $i_{state}(\hat{W})$ in the other representation such that $i_{state}(\hat{W})$'s assignment of expectation values to observables $i_{obs}(\hat{A})$ exactly duplicates $\hat{W}$ assignment of expectation values to observables $\hat{A}$. It guarantees further a relevant isomorphism of algebraic structure between the observables wielded by each quantization. With the tools at hand at present, the notion of 'relevant isomorphism' must remain vague, but its core idea is that $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{B}(\mathcal{H}')$, *considered as algebras generated by representations of the same CCRs*, instantiate the same algebraic structure.

Now quantizations $(\mathfrak{B}(\mathcal{H}), \mathfrak{W})$ and $(\mathfrak{B}(\mathcal{H}'), \mathfrak{W}')$ are physically equivalent (in the content coincidence sense), if and only if the collections of operators bearing their representations of the CCRs are unitarily equivalent, where this is understood as follows.

> A Hilbert space $\mathcal{H}$, and a collection of operators $\{\hat{O}_i'\}$ is *unitarily equivalent* to $(\mathcal{H}', \{\hat{O}_i'\})$ if and only if there exists a one-to-one, invertible, linear, norm-preserving transformation ('unitary map') U: $\mathcal{H} \rightarrow \mathcal{H}'$ such that $U^{-1}\hat{O}_i'U = \hat{O}_i$ for all i.

Think of the sets of observables mapped to one another by a such a U as the observables giving a primed and an unprimed representation of the CCRs, respectively. The U effecting the unitary equivalence of the primed and the unprimed representation of the CCRs furnishes both the bijection i_{state} from the first theory's state space to the second's and the bijection i_{obs} from the first theory's observable set to the second's; it also ensures the isomorphism of their observable sets considered as algebras generated by representations of the CCRs.

Having attributed quantum theories a Hilbert space structure, von Neumann demonstrated in 1931 what had been conjectured the previous year by Stone: the *unitary equivalence* (up to multiplicity) of any pair of Hilbert space quantizations of a classical system with configuration space $\mathbb{R}^n$.[4] Unitary equivalence is the relation, the Jordan–Wigner theorem assures us, all representations of the CARs for n degrees of freedom (n finite) enjoy with one another. According to the content coincidence criterion of physical equivalence, unitarily equivalent quantizations are physically equivalent. Whatever else interpreters of ordinary QM have to worry about, they can rest assured that the Hilbert space representations constituting the quantizations they set out to interpret are essentially unique.

5. *Non-uniqueness in QM_∞*

5.1. THE INFINITE SPIN CHAIN

Anyone even minimally acquainted with philosophical treatments of QM has on hand the resources to describe a quantum system admitting unitarily inequivalent representations: a chain of infinitely many spin $\frac{1}{2}$ systems.[5] Let us work our way up to this system starting with a finite number n of spin $\frac{1}{2}$ systems, arranged in a one-dimensional lattice. To construct a quantization for this system, one equips each location k with a Pauli spin $\hat{\sigma}^k = (\hat{\sigma}^k(x), \hat{\sigma}^k(y), \hat{\sigma}^k(z))$ satisfying the Pauli Relations.

Here's one way to do this: use a vector space $\mathcal{H}$ spanned by a basis consisting of sequences s_k, where each entry takes one of the values ± 1, and k ranges from 1 to n. (NB there are finitely many distinct such sequences–finitely many ways to map a set of finite cardinality into the set $\{+1,-1\}$.) Operators $\hat{\sigma}^j(z), j = 1$ to n are introduced in such a way that sequences s_k whose jth entry is ± 1 correspond to $\hat{\sigma}^j(z)$ eigenvectors associated with the eigenvalue ± 1. Operators $\hat{\sigma}^j(y), \hat{\sigma}^j(x)$ conspiring with these to satisfy the Pauli Relations can then be introduced.

There are many other ways to do this. But because we're considering only finitely many (indeed, n) spin systems, the Jordan–Wigner theorem guarantees that other ways are only notational variants on our way. At the risk of pedantry, let's spell out what that means. Suppose Werner and Erwin each find a representation of the Pauli Relations for a spin chain of length n. Let $\sigma^k(i)^W$ be the operator on $\mathcal{H}_W$ by which Werner represents the ith component of spin for the kth particle; let $\sigma^k(i)^E$ be the operator on $\mathcal{H}_E$ by which Werner represents the ith component of spin for the kth particle.[6] To say that Werner's representation and Erwin's are unitarily equivalent is to say that there exists a unitary map U: $\mathcal{H}_E \rightarrow \mathcal{H}_W$ such that

$$U\sigma^k(i)^E U^{-1} = \sigma^k(i)^W \text{ for all } i \in \{x, y, z\}, k \in \{1, 2, \ldots, n\} \quad (6)$$

Because unitary maps are linear and norm preserving, this unitary map, which establishes a correspondence between the Pauli spin operators figuring in Erwin's representation and the Pauli spin operators figuring in Werner's representation, extends *in a way that preserves that correspondence* to the full set of bounded operators on each quantizer's Hilbert space. Supposing that each is prosecuting an ordinary quantum theory, the ordinary quantum theories they prosecute are (presumptively) physically equivalent.

The *polarization* of a system is described by a vector whose magnitude ($\in [0,1]$) gives the strength and whose orientation gives the direction of the system's magnetization. On a single electron, it is represented by an observable $\hat{\mathrm{m}}$ whose three components correspond to three orthogonal components of quantum spin. For example, in the +1 eigenstate $|+\rangle$ of $\hat{\sigma}(z)$ (understood as the z-component of spin), the polarization has an expectation value of +1 along the z-axis. For the finite spin chain, the polarization observable has components $\hat{m}_i$ that are just the average over the chain of the corresponding component of spin: $\hat{m}_i = \frac{1}{n}\sum_{k=1}^{n} \hat{\sigma}^k(i)$. Let $[s_k]_j \in \{\pm 1\}$ stand for the j^{th} entry of the sequence s_k. In the basis sequence s_k, the z-component of polarization $\hat{m}_z$ takes an expectation value of magnitude $\frac{1}{n}\sum_{j=1}^{n}[s_k]_j$. This quantity attains extreme values (of ± 1) for those sequences every term of which is the same.

Let $\hat{W}$ be a state in Werner's representation assigning $\hat{m}_z$ the expectation value +1. The Jordan–Wigner theorem ensures that any other representation of the Pauli Relations will be unitarily equivalent to Werner's. Any other representation of those relations is thus guaranteed to contain a state $\hat{W}'$ (the image of $\hat{W}$ under the unitary map implementing the equivalence of the representations) and an observable $\hat{m}'_z$ (the image of $\hat{m}_z$ under that map) such that the expectation value of $\hat{m}'_z$ in the state $\hat{W}'$ is +1.

Now, consider a doubly infinite chain, labelled by the positive and negative integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, of spin $\frac{1}{2}$ systems. As before, to construct a quantum theory of this system is to associate with each site k a Pauli spin satisfying the Pauli Relations. But if we follow the strategy adopted for the finite spin chain, and attempt to construct our Hilbert space from a basis consisting all possible maps from $\mathbb{Z}$ to $\{\pm 1\}$, we are foiled. The set of such maps is non-denumerable, thus the Hilbert space we'd construct would be non-separable, breaking the tradition of using separable Hilbert spaces (that is, those whose bases are countable) for physics.

Here's one way to build a separable Hilbert space representation of the Pauli relations for an infinite chain of spins. Start with the sequence $[s_k]_j = +1$ for $j \in \mathbb{Z}$, and add all sequences obtainable therefrom by finitely many local modifications. The resulting basis consists of all sequences for which all but a finite number of sites take the value +1. Continue to follow the model of the finite spin chain to introduce operators $\hat{\boldsymbol{\sigma}}^k(i)^+$ satisfying the Pauli Relations (Sewell 2002, section 2.3 has details). I will call this *the* $\mathcal{H}^+$ *representation* – but please keep in mind that it matters to the algebraic structure of this representation *which* elements of $\mathfrak{B}(\mathcal{H}^+)$ play the role of *which* Pauli spins.

Notice how the polarization observable $\hat{\mathbf{m}}^+$ behaves on the $\mathcal{H}^+$ representation. For each state s_k in the basis, the polarization will be oriented along the z-axis and take the value $\lim_{n\to\infty}\frac{1}{2n+1}\sum_{j=-n}^{n}[s_k]_j$. Because for each basis element, all but a finite number of its entries take the value +1, this limit will be 1. Every ordinary quantum state on the $\mathcal{H}^+$ representation will inherit this feature from the basis vectors in terms of which it is expressed: every state in the representation will have unit polarization in the positive z direction.

Because the chain is infinite, the Jordan–Wigner theorem does not imply that the representation just constructed is unique up to unitary equivalence. And it is not. Consider, for contrast, a representation set in a Hilbert space whose basis elements correspond to the sequence $[s_k]_j = -1$ for $j \in \mathbb{Z}$, along with all sequences obtainable from this one by finitely many local modifications. The basis consists, then, of sequences for which all but a finite number of sites take the value −1. Operators $\hat{\sigma}^k(i)^-$ satisfying the Pauli relations are introduced in such a way that $[s_k]_j$, the j^{th} entry in the basis sequence s_k, gives the expectation value of $\hat{\sigma}^j(z)^-$ (Sewell 2002, section 2.3 has details). Call this *the* $\mathcal{H}^-$ *representation*. By parity of reasoning, the polarization observable $\hat{\mathbf{m}}^-$ in this quantization is assigned the expectation value −1 by each of the representation's states.

We can expose the $\mathcal{H}^-$ and $\mathcal{H}^+$ representations as unitarily *in* equivalent. The exposition will be informal. (Formalities may be found in Sewell (2002), section 2.3.3) Suppose, for contradiction, that these representations of the CARs *were* unitarily equivalent. Then there'd be a unitary map establishing a correspondence between the Pauli spins of the $\mathcal{H}^+$ representation and those of the $\mathcal{H}^-$ representation. That is, there'd be a U: $\mathcal{H}^+ \to \mathcal{H}^-$ such that $U\hat{\boldsymbol{\sigma}}^j(z)^+U^{-1} = \hat{\boldsymbol{\sigma}}^j(z)^-$ for all j. Where $\hat{m}(z)^{\pm}_N := \frac{1}{2N+1}\sum_{k=-N}^{N}\hat{\boldsymbol{\sigma}}^k(z)^{\pm}$, this implies that $\hat{m}(z)^-_N = U\hat{m}(z)^+_N U^{-1}$. For $|\psi^+\rangle$ and $|\psi^-\rangle$, unit vectors in $\mathcal{H}^+$ and $\mathcal{H}^-$ related by $|\psi^-\rangle = U|\psi^+\rangle$, it follows that

$$\langle\psi^+|\hat{m}(z)^+_N|\psi^+\rangle = \langle\psi^-|\hat{m}(z)^-_N|\psi^-\rangle \tag{7}$$

But in the limit $N \to \infty$, Equation (7) breaks down: the RHS (which gives the expectation value the state $|\psi^-\rangle$ assigns the polarization observable $\hat{\mathbf{m}}(z)^-$) and the LHS (which gives the expectation value the state $|\psi^+\rangle$ assigns the polarization observable $\hat{\mathbf{m}}(z)^+$) go to +1 and −1, respectively. This establishes the failure of the $\mathcal{H}^+$ and $\mathcal{H}^-$ representations of the infinite spin chain to be unitarily equivalent.

Unitarily inequivalent Hilbert space quantizations are supposed to rival physical theories. The rivalry of the quantizations under discussion consists in this: embracing the Hilbert space theory circumscribed by the $\mathcal{H}^+$ representation, we deny that polarizations different from +1 in the z direction are possible; embracing the Hilbert space theory circumscribed by the $\mathcal{H}^-$, we assert that such polarizations are the only ones possible.

In the sections which follow, this toy example will illustrate the phenomena of phase structure and broken symmetry, as well as the selection pressures those phenomena exert on interpretations of QM_∞.

5.2. THE POSITION AND MOMENTUM REPRESENTATIONS

This section presents an example of unitarily inequivalent representations lurking in what seems like the most ordinary of quantum theories: a single system on the real line. The aim is to illustrate how the apparatus of algebra and representation, essential to framing and addressing questions about QM_∞, also fosters insight into provocative aspects of more familiar quantum theories.

It is well known that in ordinary quantum mechanics, there are no *exact position eigenstates*, that is, eigenstates of the position operator associated with punctual eigenvalues $\lambda \in \mathbb{R}$. This is plain from the standard spectral measure for self adjoint Hilbert space operators with continuous spectra (see Prugovecki 1971, III.5.5) The spectral measure for the operator $\hat{A}$ acting on $\mathcal{H}$ is a map from measurable subsets Δ of $\mathbb{R}$ to projection operators on $\mathcal{H}$. Let $\hat{P}^A_\Delta$ be the image of this map for the set Δ. $\hat{P}^A_\Delta$ serves as a device for assigning probabilities: in a density operator state $\hat{\rho}$, the probability that a measurement of $\hat{A}$ yields an outcome in the set Δ is given by $\mathrm{Tr}(\hat{\rho}\hat{P}^A_\Delta)$. The standard spectral measure for the position operator maps degenerate intervals $\Delta = [a, a]$ to the zero operator 0. Thus in any density operator state $\hat{\rho}$, the probability that position assumes a value in $[a, a]$ – a point value – is $\mathrm{Tr}(\hat{\rho}0) = 0$. In other words, no density operator state is an exact position eigenstate. In ordinary QM, every state is a density operator state. Ergo in ordinary QM, no state is an exact position eigenstate. The continuous momentum observable likewise lacks exact eigenstates.

Ordinary QM is conducted in a separable Hilbert space. But a non-separable Hilbert space is exactly what we need to sustain a full set of exact position eigenstates. For should an eigenstate of position exist for every possible position $\lambda \in \mathbb{R}$, there would be as many pairwise orthogonal position eigenvectors in the Hilbert space as there are real numbers (and mutatis mutandis for momentum). Lacking a countable basis, the Hilbert space would fail to be separable.

The *position representation,*[7] for example, is set in the non-separable vector space $\ell_2(\mathbb{R})$ of square summable functions f: $\mathbb{R} \to \mathbb{C}$. These are functions supported on a countable subset S_f of $\mathbb{R}$ and such that $\sum_{x \in S_f} |f(x)|^2 < \infty$. The functions

$$\varphi_\lambda(x) = 1 \quad \text{if } \lambda = x \varphi_\lambda(x) = 0 \quad \text{if } \lambda \neq x \tag{8}$$

for $\lambda \in \mathbb{R}$ furnish an (uncountable!) orthonormal basis for $\ell_2(\mathbb{R})$.

The Weyl relations are obtained by 'exponentiating' the CCRs, a maneuver motivated by mathematical niceties: the position and momentum operators $\hat{q}$ and $\hat{p}$ are unbounded and so not everywhere defined; the unitary operators $\hat{U}(a) = e^{ia\hat{q}}$ and $\hat{V}(b) = e^{ib\hat{p}}$ involved in the Weyl relations are bounded and everywhere defined (see Summers 2001 for details). On $\ell_2(\mathbb{R})$, the Weyl relations are satisfied by operators $\hat{U}(a)$ and $\hat{V}(b)$ defined by the following actions on basis elements φ_λ:

$$\hat{U}(a)\varphi_\lambda(x) = e^{ia\lambda}\varphi_\lambda(x) \tag{9}$$

$$\hat{V}(b)\varphi_\lambda(x) = \varphi_{\lambda-b}(x) \tag{10}$$

The family of unitaries $\hat{U}(a)$ is weakly continuous. Stone's theorem tells us that a weakly continuous family of unitary operators $U(t)$ has a self-adjoint generator $\hat{A}$ such that $U(t) = e^{it\hat{A}}$. It tells us, then, there is some self-adjoint $\hat{Q}$ on $\ell_2(\mathbb{R})$ such that $\hat{U}(a) = e^{ia\hat{Q}}$. From Equation (9) and a Taylor series expansion of $\hat{U}(a)$ in terms of $\hat{Q}$, it follows that

$$\hat{Q}\varphi_\lambda(x) = \lambda\varphi_\lambda(x) \tag{11}$$

In other words, $\varphi_\lambda(x)$ is an eigenvector of $\hat{Q}$ associated with eigenvalue λ. Interpreting $\hat{Q}$ as the position operator, $\varphi_\lambda(x)$ is an exact position eigenstate. By contrast, the family $V(b)$ lacks weak continuity, and has no a self-adjoint generator. Thus the position representation lacks a presumptive momentum operator

There is a completely analogous non-regular representation, the *momentum representation*, with exact eigenstates of momentum. The uncountable orthonormal basis of the Hilbert space bearing the momentum representation is furnished by plane waves e^{ikx}, eigenstates of the momentum operator for each real number k. Owing to the family $\hat{U}(a)$'s failure to be weakly continuous, the momentum representation lacks a position operator – a fact for which Emch (1972, 231–2) offers the quasi-physical explanation that it makes no sense to speak of the position of a plane wave.

The official statement of the Stone–von Neumann theorem concerns representations of the Weyl relations and requires the unitary operators conveying them to be weakly continuous. The position and momentum representations flout this requirement. With the assumption of weak continuity suspended, the Stone–von Neumann theorem ceases to apply. The position and momentum representations have uncountable bases; the standard Schrödinger wave function representation has a countable one. Each of the former is unitarily inequivalent to the latter. Less obviously, each of the former is unitarily inequivalent to the other (see Halvorson 2004, 51 (Theorem 1), for a simple argument).

The position and momentum representations are important not only for challenging received notions of what physical possibilities can be entertained by quantum theories of single non-relativistic particles, but also because of striking features of *generic* states in theories of QM_∞ they illuminate. To put the analogy roughly, just as the 'pure' states of exact position inhabiting the position representation are 'orthogonal' to every state of the standard, separable, Schrödinger wave function representation, so too *every* pure state of the observable algebras that typically crop up in QM_∞ is 'orthogonal' to most physically significant states on those algebras. Given the propensity of many interpretations of ordinary QM to code the condition a quantum system is 'really' in with a pure state (consider, for example, the value states of modal interpretations), this complicates the extension of strategies for interpreting ordinary QM to QM_∞. Clifton (2000) offers, on behalf of modal interpretations, a way through the complication; Earman and Ruetsche (2005) criticize the result for saying too little about too many systems of interest.

6. Conclusion and Preview

Without even venturing into QFT, we've seen how the uniqueness promised by the Stone–von Neumann theorem breaks down outside the scope of its assumptions. The non-uniqueness calls for interpretive work: if we retain unitary equivalence as a criterion of physical equivalence, we face prima facie embarrassments of the following sort: infinitely many physically inequivalent quantum theories vie for the title 'quantum theory of the infinite spin chain'. It seems we must either eliminate all but one unitary equivalence class as viable contenders, or rethink what it is to be a physical theory in a way that reconciles us to such flamboyant underdetermination. If we depose unitary equivalence as a criterion of physical equivalence, we're beholden to articulate – and motivate – an account of physical equivalence for quantum theories to take its place.

Part II begins with a brief catalog of quantum field theoretic examples of non-uniqueness. It then presents the basics of the algebraic approach to quantum theories, which discloses a structure even unitarily inequivalent Hilbert space representations can share. Finally, it introduces and evaluates a handful of strategies for interpreting QM_∞ in the face of the non-uniqueness of Hilbert space representations.

Acknowledgement

I am grateful to the editors for their patience and to Paul Teller and other readers at UC-Davis for terrific feedback on an earlier draft.

Short Biography

Laura Ruetsche has been a Professor of Philosophy at the University of Michigan since 2008. She has also held tenure track appointments at the University of Pittsburgh (where she got her PhD under the direction of John Earman in 1995) and Middlebury College, as well as visiting appointments at Rutgers and Cornell. Her *Interpreting Quantum Theories* will be published this spring by Oxford University Press.

Notes

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[1] I should acknowledge, although I do not explicitly discuss, a problem. QM_∞ inherits from ordinary QM: the measurement problem.
[2] 'Representation' has an official sense which Part II will explain. For now, we will understand a representation to be a realization, by means of Hilbert space operators, of theoretically central relations between physical magnitudes.
[3] For an account of different topologies, consult Kadison and Ringrose 1983, Ch. 3.
[4] This needs to be qualified slightly. See Summers (2001) for details.
[5] Here I follow Sewell (2002, section 2.3), to which I refer the reader for details.
[6] For the duration of this explication, I'm dropping hats over operators to minimize notational clutter.
[7] Halvorson (2004), which I follow here, gives a lucid exposition of position and momentum representations. Clifton and Halvorson (2001) use them to formulate Bohrian complementarity.

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