Nonstandard Dispersive Estimates and Linearized Water Waves

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in the University of Michigan
2013

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For my parents.

ACKNOWLEDGEMENTS

The process of raising a graduate student requires a significant village. Firstly, I need to thank my parents, my husband, and my friends, particularly Jess and Melody, for their endless long distance support during this arduous process. I also need to call attention to the Department of Mathematics staff, who over the years, have done much to abet this degree process, from organizing funding to asking about my life outside the department.

I started this work in the fall of 2009 with the support of the Institute for Mathematics and its Applications at the University of Minnesota, and I owe them a great deal for the opportunity to meet so many high-class mathematicians. My time at the IMA exposed me to a wide variety of mathematics related to fluid mechanics and ultimately shaped the mathematician I will become.

All of my professors informed a piece of this document, but I would like to particularly thank Jeffrey Rauch for all the hours spent in his office discussing my results. Those conversations were always surprising and enlightening.

Finally, I need to thank my advisor, Sijue Wu. Her work was the starting point for everything here, and her guidance and eye for detail enabled my work and these results.

TABLE OF CONTENTS

LIST OF FIGURES

Figure

CHAPTER I

Introduction

1.1 Introduction

The mathematical study of water waves is motivated in part by the need for better predictions of the behavior of the surface of the ocean for shipping and military applications. For example, there are many recorded instances of large waves in the absence of geological explanations- waves the size of tsunamis without an earthquake. These rogue waves, while uncommon, are very destructive and very unpredictable. Since as early as the 1970s ([17]), mathematicians have been attempting to identify the causes of these large waves. The question is fundamentally about the existence of solutions to a partial differential equation and, more specifically, about the persistence and size of these solutions.

The research we present here combines elements of fluid mechanics, dispersive differential equations, and harmonic analysis. We use tools from all of these areas to analyze behavior of solutions to the water wave problem in terms of the initial data. The motivation for this research is Sijue Wu's paper [20] on almost global well posedness for the full water wave problem in 2D. Wu's result roughly says that a 2D water wave with small initial height, energy, and slope in Sobolev space remains small and smooth almost globally in time. Intuitively, we expect that, for long time existence, we only need the slope of the initial wave to be small, not the height and energy.

A close examination of the techniques in [20] motivates the work. One of the main ingredients of her proof is a decay estimate on the L^∞ norm of a function in terms of the L^2 norms of specific space-time derivatives with a decay rate $1/t^{\frac{1}{2}}$. However, in practice, her estimate only provides bounds on the derivatives of certain quantities associated to our problem. The proof of this estimate [20, Proposition 3.1] uses ideas similar to those found in Klainerman's proof of the Klainerman-Sobolev type estimates for the wave equation. We will explore using different methods to control the L^∞ norms of these quantities, without the need for derivatives. The results contained in the following chapters suggest that while the data used in [20] removes some bad behavior, a broader class of data will produce the same results.

Our goal, with this research as a starting point, is to characterize completely the class of initial data which yields long time existence in the full water wave problem in two dimensions. The work so far focuses on the linear problem as the restrictions on data in the linear case will carry over to the nonlinear problem. In particular, our results identify a trajectory along which we see growth in solutions to the linear problem. In our efforts to identify the source of this surprising growing factor, we connect the size of the singularity at the origin in frequency to the spatial decay of the solution.

1.2 Outline

In Chapter II, we lay out a variety of known results, including the derivation of the water wave problem and the subsequent analysis which leads to a one dimensional dispersive equation. We also discuss results for the wave equation, specifically the work of Keel, Smith, and Sogge [5, 6, 7] which served as the inspiration for our approach.

In Chapter III, we give some preliminary results from harmonic analysis which are necessary for the proofs in the subsequent chapters. The chapter also includes a short collection of useful definitions from harmonic analysis.

In Chapter IV, we present results for a general class of one dimensional dispersive equations, including a discussion of the sharpness of the estimates. The key results of this chapter are Lemma IV.2, a new Sobolev-type decay bound, Theorem IV.3, which implies growth in the solution along certain trajectories, and Theorem IV.5, which shows that this growth factor is sharp. These theorems follow primarily from the careful analysis of several oscillatory integrals. In the case of half the water wave problem, we also include a simpler argument taking advantage of the quadratic phase.

Chapter V focuses on further analysis of the initial data of the linearized water wave problem and its effects on the regularity and decay of the solution. While Theorem IV.3 can be used to give a slow decay bound (see Corollary IV.6), by decomposing the initial data of the linearized water wave problem in frequency, we find an improved decay estimate, Theorem V.11. This proof uses an additional bound similar to those used by Klainerman for the wave equation. In the course of this proof, we identify the main impediment to our desired decay, and further analysis leads us to Theorem V.13, relating the size of the singularity at the origin in frequency to the rate of spatial decay. Finally, in Chapter VI, we make a few concluding remarks and discuss the work which will follow these results.

We also include in the form of an Appendix some results of independent interest for the two dimensional wave equation, which we derived in the course of our study of the techniques of Klainerman and Keel, Smith, and Sogge.

CHAPTER II

Motivation

2.1 Basic Framework

In order to model the behavior of waves in the deep ocean away from the effects of coastline, we considera fluid domain of infinite depth, modelled on the entirety of Euclidean space. We will consider only the case of the two dimensional water wave, that is the fluid domain is a subset of the plane. We can assume the density in the water region is constant and equal to 1, while letting the density on the air region equal zero. Let $\Omega(t) \subseteq \mathbb{R}^2$ denote the fluid domain and $\Sigma(t)$ denote the interface between the water and the air. The equations governing the velocity field $\mathbf{v}$ in the fluid domain are

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \mathbf{g} - \nabla P & \text{in } \Omega(t) \\ \nabla \cdot \mathbf{v} = 0 \qquad \nabla \times \mathbf{v} = 0 & \text{in } \Omega(t) \\ P = 0 & \text{on } \Sigma(t). \end{cases} \tag{2.1}$$

The first equation encodes that fluid particles will move with the velocity field with respect to the forces of gravity $\mathbf{g}$ and pressure P. The second set of equations enforces incompressibility and irrotationality of the vector field.

2.1.1 Rewriting the equation

The first reduction of the equation is to the interface. First, we change to Lagrangian coordinates following individual fluid packets in time. Let α denote the Lagrangian variable and $z(t,\alpha) = x(t,\alpha) + iy(t,\alpha)$ denote the equation for the interface in this variable. By construction, if we consider the complex function z as a vector (x, y), $z_t(t,\alpha) = \mathbf{v}(t, z(t,\alpha))$ and $z_{tt}(t,\alpha) = \partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}$ along the interface. If we consider $\mathbb{R}^2$ as the complex plane, i.e. $(x,y) \mapsto x + iy$, the vector $\mathbf{g}$ becomes $-i$ and the incompressibility and irrotationality conditions for $\mathbf{v} = u + iv$ can be rewritten as

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned} \tag{2.2}$$

In this framework, these equations imply that $\overline{\mathbf{v}}$ satisfies the Cauchy-Riemann equations in $\Omega(t)$. Define

$$\mathfrak{h}f(t,\alpha) = \frac{1}{\pi i}\text{p.v.}\int \frac{f(t,\beta) z_\beta(t,\beta)}{z(t,\alpha) - z(t,\beta)} d\beta,$$

the Hilbert transform along the interface $z(t,\alpha)$. Since the Hilbert transform takes the boundary values of holomorphic functions to themselves, the equation (2.2) is equivalent to $\mathfrak{h}\overline{z}_t = \overline{z}_t$. Our assumption that $P = 0$ along $\Sigma(t)$ implies that ∇P is purely in the normal direction to the interface. Denote the outward pointing unit normal vector as $\mathbf{n}$. The equation for the interface $z(t,\alpha)$ implies $\mathbf{n} = \frac{iz_\alpha}{|z_\alpha|}$, as seen in figure 2.1 . If we let $\mathfrak{a} = -\frac{\partial P}{\partial \mathbf{n}} \frac{1}{|z_\alpha|}$, we have $\nabla P = -i\mathfrak{a} z_\alpha$. Along the interface we now have the following equations, which are equivalent to (2.1):

$$\begin{aligned} &z_{tt} + i = i\mathfrak{a} z_\alpha \\ &\mathfrak{h}\overline{z}_t = \overline{z}_t. \end{aligned} \tag{2.3}$$

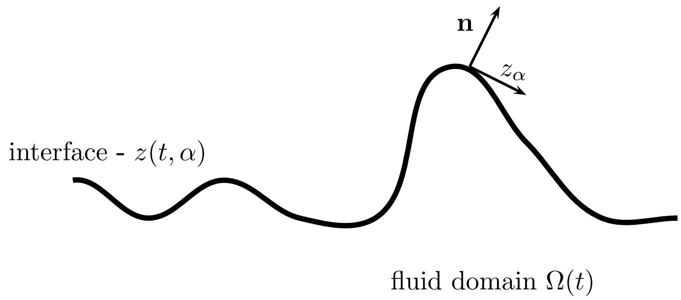

Figure 2.1: Fluid domain and $\mathfrak{a}$

The system (2.3) is fully nonlinear because of the quantity $\mathfrak{a}$ and the Hilbert transform $\mathfrak{h}$. However, we can make this equation quasilinear simply by taking a t-derivative of the first equality:

$$\begin{aligned} &z_{ttt} - i\mathfrak{a} z_{t\alpha} = i\mathfrak{a}_t z_\alpha \\ &\mathfrak{h}\overline{z}_t = \overline{z}_t. \end{aligned} \tag{2.4}$$

The first equation (2.4) displays dispersive effects in the following way: if we think of $z_t = \mathfrak{u}$, we can linearize around the free solution (which is identically 0 in this case) to the free equation

$$\partial_{tt}\mathfrak{u} + |D_\alpha|\mathfrak{u} = 0. \tag{2.5}$$

Remark II.1. We can rewrite (2.4) as $\partial_{tt}\mathfrak{u} + |D_\alpha|\mathfrak{u} = F(z, \mathfrak{u}, \mathfrak{u}_t, \mathfrak{u}_{tt}, \mathfrak{u}_\alpha)$. From now on, we will consider the linear part rather than the full equation to understand the dispersive effects.

Given initial data for $\mathfrak{u}$ and $\mathfrak{u}_t$, we can solve the linear equation explicitly:

$$\mathfrak{u}(t,x) = \frac{1}{2}\int e^{ix\xi}\left(e^{it|\xi|^{\frac{1}{2}}}\left(\widehat{\mathfrak{u}}_0 - i|\xi|^{-\frac{1}{2}}\widehat{\mathfrak{u}}_1\right) + e^{-it|\xi|^{\frac{1}{2}}}\left(\widehat{\mathfrak{u}}_0 + i|\xi|^{-\frac{1}{2}}\widehat{\mathfrak{u}}_1\right)\right) d\xi. \tag{2.6}$$

This equation shows that we have dispersive effects as different Fourier frequencies will propagate at different speeds.

The equation (2.6) is a linear combination of oscillatory inetgrals with phase $\phi(x,t,\xi) = x\xi \pm t|\xi|^{\frac{1}{2}}$. By applying the method of stationary phase, a natural control of the L^∞ bound of the solution in terms of the L^1 norm of the solutions and derivatives appears. We state this decay as the following proposition:

Proposition II.2. *If* $\mathfrak{u}(t,x) \in C^\infty(\mathbb{R}^{1+1})$ *solves equation* (2.5) *with*

$$(\mathfrak{u}(0,x), \mathfrak{u}_t(0,x)) = (\mathfrak{u}_0(x), \mathfrak{u}_1(x))$$

and $\widehat{\mathfrak{u}}_i$ *compactly supported for* $i = 0, 1$, *then for* $t > 0$,

$$\sup_x |\mathfrak{u}(t,x)| \leq \frac{1}{(1+t)^{\frac{1}{2}}} \sum_{i=0}^{1} \sum_{|k| \leq 2} \int |\partial_x^k \mathfrak{u}_i(x)| dx. \tag{2.7}$$

Remark II.3. There are several proofs for this type of bound in the literature. For example, if we apply Theorem 2.2 in [8] to each term of (2.6) we get the desired result.

This decay rate in Proposition II.2 is not ideal for our purposes for a couple of reasons. Firstly, the nonlinearities in equation (2.4) are quadratic, so a decay rate of $t^{-\frac{1}{2}}$ is not quite fast enough to give long time existence for solutions. In addition, we would prefer to control the L^∞ norm of the solution in terms of L^2 norms of the initial data, because these types of estimates are more compatible with energy methods. In [20], Wu constructed a change of variables with an accompanying change of unknowns which caused the quadratic nonlinearities to cancel leaving only cubic and higher order nonlinearities, solving the first problem. A Sobolev-type bound adapted from techniques of Klainerman for nonlinear wave equations solves the second.

2.1.2 Change of Coordinates and Unknowns

The change of variables is given as follows: let $\Phi(t,\cdot) : \Omega(t) \to P_-$ be a Riemann mapping from the fluid domain $\Omega(t)$ to the lower half plane P_- such that $\lim_{z\to\infty} \Phi(t,z) = 1$ and $\Phi(t, z(t,0)) = 2x(t,0)$, where x is the real (or horizontal) part of the interface equation $z(t,\alpha)$. Let $h(t,\alpha) = \Phi(t, z(t,\alpha))$. The change of variables used in [20] is precisely

$$k(t,\alpha) = 2x(t,\alpha) - h(t,\alpha).$$

This change of coordinates allows the problem to be recast in terms of new quantities χ, v, and λ. Let $U_k^{-1} f := f \circ k^{-1}$. Then these new quantities are given by

$$\begin{aligned} \chi &= U_k^{-1}(I - \mathfrak{h})(z - \overline{z}) \\ v &= U_k^{-1}\partial_t((I - \mathfrak{h})(z - \overline{z})) \\ \lambda &= U_k^{-1}(I - \mathfrak{h})\psi \end{aligned}$$

where $\psi(t,\alpha) = \phi(t, z(t,\alpha))$ and ϕ is the velocity potential. These quantities satisfy equations of the form

$$\partial_t^2 \Theta - i\partial_\alpha \Theta = G$$

for G cubic and higher order. Using invariant vector fields of the differential operator $\partial_t^2 - i\partial_\alpha$, Wu controls the L^∞ norm of a rapidly decaying function by a Sobolev-type bound.

2.1.3 Sobolev-type bounds

Observe that ∂_t, ∂_α, $L = \frac{t}{2}\partial_t + \alpha\partial_\alpha$ and $\Omega_0 = x\partial_t + \frac{it}{2}$ are invariant under $\partial_t^2 - i\partial_\alpha$, i.e. $[\partial_t, \partial_t^2 - i\partial_\alpha] = [\partial_\alpha, \partial_t^2 - i\partial_\alpha] = [\Omega_0, \partial_t^2 - i\partial_\alpha] = 0$ and $[L, \partial_t^2 - i\partial_\alpha] = -L$. Wu derives the following $L^2 - L^\infty$ type decay estimate with decay rate $1/t^{\frac{1}{2}}$:

Proposition II.4. *[20, Proposition 3.1] Let $u(t,x)$ be any function with sufficient decay at ∞ and let $\Gamma = \{\partial_t, \partial_x, L\}$ where $L = \frac{t}{2}\partial_t + x\partial_x$. Let $\Omega_0 = x\partial_t + \frac{it}{2}$. Then, for a multiindex $k = \{k_1, k_2, k_3\}$ and $\Gamma^k = \partial_t^{k_1}\partial_x^{k_2}L^{k_3}$, we have*

$$|u(t,x)| \leq \frac{1}{t^{\frac{1}{2}}} \left(\sum_{1 \leq |k| \leq 2} \|\Gamma^k u\|_{L^2} + \sum_{|k| \leq 1} \|\Gamma^k \Omega_0 u\|_{L^2} \right). \tag{2.8}$$

While ∂_t, ∂_α, and L are vector fields, Ω_0 is not. In fact, it is the appearance of Ω_0 in the estimates which makes Wu's decay estimates applicable only to the derivatives of the quantities associated to her problem, forcing assumptions on the initial energy and the initial height. For a careful examination of this problem in the case of the linearized water wave problem, see §5.1.2.

The focus of our work is on improving these Sobolev-type bounds, as the bounds in [20] are sufficient for proving long time existence, but they could be sharper. Since the existing proofs are complicated by the presence of Ω_0, removing all the dependence on Ω_0 is a logical way to improve the results. In the work of Keel, Smith, and Sogge [5, 6, 7], they reduce the set of vector fields used in Klainerman-type Sobolev bounds, dropping the vector fields which are inconvenient for their arguments. Their work inspired the results we present in Chapters III and IV. In the following section, we discuss Klainerman's ideas and identify some of the key points of the Keel, Smith, and Sogge papers.

2.2 Method of Invariant Vector Fields

While considering the problem of long-term existence for the nonlinear wave equation

$$(\partial_t^2 - \Delta)v = F(t, v, \dots)$$

$$(v(0,x), v_t(0,x)) = (v_0(x), v_1(x))$$

in various spatial dimensions, Klainerman developed what has become known as the method of invariant vector fields (cf. [9, 10, 11]). The details of these proofs easily fill a textbook (e.g. [18]), so we present a sketch of the ideas. The work of Klainerman is distinctive because he introduces a new class of decay estimates. Instead of using the standard Sobolev norms, which only allow spatial derivatives, Klainerman derives estimates which include a whole class of vector fields, specifically those that have favorable commutation properties with the d'Alembertian $\partial_t^2 - \Delta$, for example the vector fields for angular momentum and radiation derivatives. These new estimates produce long time existence results for a large class of nonlinear wave equations with small data in spatial dimensions greater than or equal to 2. The key to these results is the adapted Sobolev bounds.

These types of results have been generalized to other differential equations, such as the Schrödinger equation ([1]) using the appropriate class of invariant vector fields. All of these generalizations, including Proposition II.4, follow from reasoning similar to Klainerman's for the wave equation.

Other results adapted Klainerman's bounds to specialized cases of the wave equation. We are interested in the variation developed in order to handle the wave equation in three spatial dimensions with an obstacle.

2.2.1 Keel, Smith, and Sogge Bounds for the Wave equation

When trying to solve the wave equation in a domain with an obstacle, the techniques pioneered by Klainerman do not produce good results because they include the Lorentz boosts $t\partial_{x_i} + x_i\partial_t$, which grow in time in the normal direction. In their series of papers ([5, 6, 7]), Keel, Smith, and Sogge derive inequalities which use only a partial collection of vector fields for the wave equation in three space dimensions. They avoid these growing directions and get long-time existence in both the obstacle and standard Minkowski case using new inequalities. Hidano and Yokoyama showed that the key inequalities in these papers in fact hold for all space dimensions [2]. Jason Metcalfe and others increased the dimension in the obstacle case to 4 and higher ([12, 14, 15, 16]).

Remark II.5. In the course of our study of these bounds for the wave equation, we extended the results of Keel, Smith, and Sogge to the two dimensional Minkowski space. As these results are of independent interest, we include them in the appendix. Our focus here, however, is how Keel, Smith, and Sogge's techniques can inform our analysis of the linearized water wave problem.

Our primary inspiration is the following theorem from [6]:

Theorem. *[6, Theorem 2.3] Let v be solution to the homogeneous wave equation $\Box v = 0$ in $\mathbb{R}_+ \times \mathbb{R}^3$, let $v' = (\partial_t v, \nabla v)$, and let $Z = \{\partial_t, \partial_x, \Omega\}$ where $\{\Omega_{ij} = x_i\partial_j - x_j\partial_i,\ 1 \leq i < j \leq 3\}$ are the rotational vector fields. Then for any $N = 0, 1, 2, \ldots$,*

$$\sum_{|\beta|\leq N} \left(\|Z^\beta v'\|_{L^2} + (\ln(t+2))^{-\frac{1}{2}} \|(1+|x|)^{-\frac{1}{2}} Z^\beta v'\|_{L^2((s,x):0\leq s\leq t)} \right)$$
$$\leq C \sum_{|\beta|\leq N} \|(Z^\beta v)'(0,\cdot)\|_{L^2}.$$

A key ingredient in the proof of this theorem is a rotational Sobolev bound

Lemma II.6. *[6, Lemma 2.4] Suppose that $h \in C^\infty(\mathbb{R}^3)$. Then for $R > 1$,*

$$\|h\|_{L^\infty(R/2\leq|x|\leq R)} \leq CR^{-1} \sum_{|\beta|\leq 2} \|Z^\beta h\|_{L^2(R/4\leq|x|\leq 2R)}.$$

The theorem above controls weighted norms of derivatives of the time-space gradient in terms of the initial data. Notice that on both sides, the sums are for multi-indices up to N, so any estimates can be closed.

Rather than removing the Lorentz boosts in the case of the wave equation, we want to remove Ω_0. The aim is control of the L^∞ norm of solutions in space. We use an analogue to Lemma II.6 to inform our derivation of the appropriate L^2 bounds and find control in terms of homogeneous and inhomogeneous Sobolev spaces. In the next chapter, we will discuss some useful results from harmonic analysis before presenting the new decay bounds in Chapter IV.

CHAPTER III

Tools from Harmonic Analysis

In this chapter, we will outline some useful background information from harmonic analysis. There are two main results that will be used in the rest of this document. First, we will discuss the more complex one.

3.1 Basic Ideas of Harmonic Analysis

It is standard practice in harmonic analysis to define operators T on $\mathcal{S}(\mathbb{R}^n)$ in the following way:

$$Tf(x) := \int K(x,y)f(y)ds. \tag{3.1}$$

In order for this to make sense generally, we treat K as a distrubtion and call K the kernel of the operator. One of the goals of harmonic analysis is to determine the boundedness of these types of operators based on properties of the kernel. If the kernel $K(x,y)$ is in $L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, for instance, the operator T is clearly bounded from L^1 to L^∞. The discussion of these operators becomes more nuanced for kernels with singularities.

The prototypical example of this type of operator is the Hilbert transform in one dimension, defined by kernel $K(t,s) = \text{p.v.}\frac{c}{t-s}$ for a fixed constant c depending on the chose of Fourier transform. This kernel is singular along the diagonal, which would seem to be an impediment to analysis. However, the Hilbert transform is well-known to be bounded from L^p to itself for $1 < p < \infty$. Working with the Hilbert transform is straightforward because the kernel function is a convolution and has nice properties on the Fourier side. If the kernel defining a given operator is actually a convolution, that is $K(t,s) = K(t-s)$, looking at the Fourier transform is a standard technique.

3.1.1 Hardy-Littlewood-Sobolev Lemma

For example, fractional integration has a convolution type kernel. These types of kernels appear in the proofs in Chapters IV and V, and we will use the following lemma.

Lemma III.1 (Hardy-Littlewood-Sobolev Lemma). *For $n \geq 1$, $1 < p < q < \infty$, $0 < \beta < n$, and*

$$I_\beta g(x) = \int_{\mathbb{R}^n} \frac{g(z)}{|x-z|^\beta} dz, \tag{3.2}$$

$$\|I_\beta g\|_{L^q(\mathbb{R}^n)} \leq C(p,q)\|g\|_{L^p(\mathbb{R}^n)} \text{ when } \frac{1}{q} = \frac{1}{p} - \frac{n-\beta}{n}.$$

For a proof of this lemma, see [19].

Unfortunately, several operators which are not straightforward convolutions appear in our work. Instead of looking on the Fourier side, a more useful technique for our work is careful analysis of the size of the singularity and the speed of the decay of the kernel away from the singularities. There are several very powerful results along these lines, but we will use the one that fits best in the context of our problem, the $T1$ theorem. In order to state the theorem, we need some new definitions.

3.2 The $T1$ Theorem

3.2.1 Definitions

Following the work of David and Journé [4], we need to characterize precisely what sorts of kernels are allowed.

Definition III.2. Let $K(x, y)$ be a continuous function on $\mathbb{R}^n \times \mathbb{R}^n \backslash \Delta$ where $\Delta = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$, and two constants $\delta \in (0, 1]$ and $C_k > 0$. We say $K(t, s)$ is a *standard kernel* when all of the following are true:

1. $|K(x, y)| \leq C_k |x - y|^{-n}$
2. For $|x - x'| < \frac{1}{2}|x - y|$,

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \frac{C_k |x - x'|^{\delta}}{|x - y|^{n+\delta}} \tag{3.3}$$

3. For all functions f, $g \in C_0^{\infty}(\mathbb{R}^n)$ with disjoint supports,

$$\langle Tf, g\rangle = \iint K(x, y) f(y) dy\, g(x) dx.$$

Observe that the Hilbert transform mentioned in the previous section has a standard kernel. The definition above is the most general definition of a standard kernel, but observe that

$$|\nabla_x K(x, y)| + |\nabla_y K(x, y)| \leq \frac{C}{|x - y|^{n+1}} \tag{3.4}$$

implies (3.3) above. The equation (3.4) is an easier criterion check in our case, so we will take advantage of this alternate formulation.

Before we can state the $T1$ theorem, we also need to define the function space BMO, the space of functions of bounded mean oscillation:

Definition III.3. Let $f_Q := \frac{1}{|Q|}\int_Q f(x)dx$ for a cube Q. We say $f \in BMO(\mathbb{R}^n)$ when

$$\|f\|_{BMO} := \sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|\, dx < \infty \tag{3.5}$$

It is worth noting that any function in L^∞ is also in BMO and that BMO is strictly larger. For example, $\ln|x| \in BMO$ but not in L^∞. For more information on BMO and its various properties, see [19].

The last new definition needed is that of the weak boundedness property.

Definition III.4. An operator T is said to have the *weak boundedness property* if there exists an $N \geq 1$ and $C_k > 0$ such that for all ϕ, $\psi \in C_0^\infty(\mathbb{R}^n)$ with support in $B_1(0)$ and $\int \psi dx = 0$,

$$|\langle T\psi_r^y, \phi_r^x\rangle| + |\langle T\phi_r^y, \psi_r^x\rangle| \leq \frac{C_k}{r^n}\left(\sum_{|\alpha|\leq N}\|\partial^\alpha\phi\|_{L^\infty}\right)\left(\sum_{|\alpha|\leq N}\|\partial^\alpha\psi\|_{L^\infty}\right)$$

where $\phi_r^y(z) = \frac{1}{r^n}\phi(\frac{(z-y)}{r}$.

Notice that any operator bounded from L^2 to itself satisfies the weak boundedness property.

3.2.2 Statement of the $T1$ theorem

We can now state the $T1$ theorem:

Theorem III.5. *[4, Theorem 1] Let T be a continuous operator from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ associated with a standard kernel. Then, T can be extended to a bounded operator from $L^2(\mathbb{R}^n)$ to itself if and only if the three following conditions are satisfied:*

1. *$T1 \in BMO$,*

2. *$T^*1 \in BMO$,*

3. T has the weak boundedness property.

We will not prove this theorem here. See [4] for the complete proof. The key observation for our purposes is that the bounds on the operator T are a linear combination of $\|T1\|_{BMO}$ and $r^n|\langle T\psi_r^x, \varphi_r^y\rangle|$.

3.3 Useful Corollaries

In Chapter IV, we need two specific corollaries to the $T1$ theorem of David and Journé in one dimension. The proofs are very similar to that of the $T1$ theorem. We require precise constants, and so we will carefully prove Proposition III.7. Since the proof of Proposition III.6 follows from exactly the same argument, we will omit the details. We will show that our operator T in Proposition III.7 satisfies the hypotheses of the $T1$ theorem with certain constants, and those constants will lead directly to the desired bound.

Proposition III.6. *Let $a(t,s;\xi)$ be smooth function in t and s supported in ball of radius ρ in $\mathbb{R}_t \times \mathbb{R}_s$ such that $a(t,s;\xi) = a(s,t;\xi)$ and both $a(t,s;\xi)$ and $\partial_t a(t,s;\xi)$ are uniformly bounded in t, s and ξ by constants C_1 and C_2, respectively. Define the kernel $k(t,s;\xi) = a(t,s;\xi)(t-s)^{-1}$. Then $k(t,s;\xi)$ is a standard kernel uniformly bounded in ξ and the operator T associated to $k(t,s;\xi)$ is bounded from L^2 to itself independent of ξ with norm $C_1 + \rho C_2$.*

We will also have operators whose kernels are controlled by positive L^1 functions, so we will need the following proposition:

Proposition III.7. *Let $A(t,s;\xi)$ be smooth function in t and s supported in ball of radius ρ in $\mathbb{R}_t \times \mathbb{R}_s$ and $k(\xi) \in L^1(\mathbb{R})$ with $k > 0$. Assume that $A(t,s;\xi) = A(s,t;\xi)$ and $|A(t,s;\xi)| \leq C_1 k(\xi)$ and $|\partial_t A(t,s;\xi)| \leq C_2 k(\xi)$. Define the kernel $K(t,s;\xi) =$*

$A(t,s;\xi)(t-s)^{-1}$. *Then* $K(t,s;\xi)$ *is a standard kernel and the operator* T *associated to* $K(t,s;\xi)$ *is bounded from* L^2 *to itself with* $\|T\|_{2,2} \le Ck(\xi)$.

Proof. We need to show first that $K(t,s;\xi)$ is a standard kernel. Then, the bounds on the operator T follow by the $T1$ theorem.

Now, it is obvious that

$$|K(t,s;\xi)| \le \frac{C_1 k(\xi)}{|t-s|}.$$

It remains to show we have control of $|\partial_t K(t,s;\xi)| + |\partial_s K(t,s;\xi)|$. Because of the symmetry of the function $A(t,s;\xi)$, it suffices to show that $|\partial_t K(t,s;\xi)| \le Ck(\xi)|t-s|^{-2}$. This clearly follows from our assumptions on $A(t,s;\xi)$ with $C = C_1 + C_2\rho$.

In the proof of the $T1$ theorem, the operator bounds of T are less than or equal to a linear combination of the bounds on $r|\langle T\psi_r^x, \phi_r^y\rangle|$ and $\|T1\|_{BMO}$. Therefore, if we show that $r|\langle T\psi_r^x, \phi_r^y\rangle| \le Ck(\xi)$ and $\|T1\|_{BMO} \le C'k(\xi)$, we can conclude that $\|T\|_{2,2} \le Ck(\xi)$. First, we will show weak boundedness.

Take $\phi(t)$ and $\psi(t)$ as in the proof of Proposition III.6. If $|x-y| < 3r$,

$$\begin{aligned}\int\int K(t,s;\xi)\psi_r^y(s)ds\phi_r^x(t)dt &= \frac{1}{2}\int\int \left(K(t,s;\xi) - K(s,t;\xi)\right)\psi_r^y(s)ds\phi_r^x(t)dt \\ &= \frac{1}{2}\int\int K(t,s;\xi)\left(\phi_r^x(s)\psi_r^y(t) - \phi_r^x(t)\psi_r^y(s)\right)ds/,dt\end{aligned}$$

Notice that $|\phi_r^x(s)\psi_r^y(t) - \phi_r^x(t)\psi_r^y(s)| \le Cr^{-3}|t-s|$ and $|k(t,s;\xi)| \le C_1 k(\xi)|t-s|^{-1}$. In addition, $|t-y| < r$ and $|s-x| < r$, so

$$\begin{aligned}|\int\int K(t,s;\xi)\phi_r^x(s)ds\psi_r^x(t)dt| &\le \frac{1}{2}\int\int |K(t,s;\xi)|\left|\phi_r^x(s)\psi_r^y(t) - \phi_r^x(t)\psi_r^y(s)\right| ds/,dt \\ &\le \frac{1}{2}\int_{|t-y|<r}\int_{|s-x|<r} C_1 k(\xi)|t-s|^{-1}Cr^{-3}|t-s|ds/,dt \\ &\le \frac{CC_1 k(\xi)}{2r^3}(2r)^2 \le Ck(\xi)r^{-1}.\end{aligned}$$

When $|x-y|>3r$, we take advantage of the mean zero assumption on ψ to get

$$\begin{aligned}\left|\int\int K(t,s;\xi)\psi_r^y(s)ds\phi_r^x(t)dt\right| &= \left|\int\int \left(K(t,s;\xi)-K(t,y;\xi)\right)\psi_r^y(s)ds\phi_r^x(t)dt\right| \\ &\leq \int\int |K(t,s;\xi)-K(t,y;\xi)|\,|\psi_r^y(s)||\phi_r^x(t)|ds/,dt \\ \text{since } |y-s|<r, \quad &\leq \int\int |\partial_s K(t,s;\xi)||s-y||\psi_r^y(s)||\phi_r^x(t)|ds/,dt \\ &\leq \int\int \frac{Ck(\xi)|s-y|}{|t-s|^2}|\psi_r^y(s)||\phi_r^x(t)|ds/,dt.\end{aligned}$$

Now, $3|t-s|>|x-y|$ because $|x-y|<|x-t|+|t-s|+|s-y|<(2/3)|x-y|+|t-s|$, so

$$\begin{aligned}\left|\int\int K(t,s;\xi)\psi_r^y(s)ds\phi_r^x(t)dt\right| &\leq \frac{9Ck(\xi)r}{|x-y|^2}\int|\psi_r^y(s)|ds\int|\phi_r^x(t)|dt \\ &\leq Ck(\xi)r^{-1}.\end{aligned}$$

Therefore, we have precisely that $r|\langle T\psi_r^x,\phi_r^y\rangle|\leq Ck(\xi)$ It remains to show that $T1=-T^*1$ is in BMO with BMO norm controlled by $k(\xi)$. In fact, we can show that $T1\in L^\infty$:

$$\begin{aligned}\text{p.v.}\quad \int\frac{A(t,s;\xi)}{(t-s)}ds &= \lim_{\delta\to 0}\left(\int_{t+\delta}^{\infty}\frac{A(t,s;\xi)}{(t-s)}ds+\int_{-\infty}^{t-\delta}\frac{A(t,s;\xi)}{(t-s)}ds\right) \\ &= \lim_{\delta\to 0}\left(\int_{\delta}^{\infty}\frac{A(t,t+S;\xi)}{-S}dS+\int_{\delta}^{\infty}\frac{A(t,t-S;\xi)}{S}dS\right) \\ &= \lim_{\delta\to 0}\int_{\delta}^{\infty}\frac{A(t,t-S;\xi)-A(t,t+S;\xi)}{S}dS.\end{aligned}$$

Notice that the difference in the last line is close to a derivative for small S. Therefore, we consider (δ,ϵ) and (ϵ,∞) separately:

$$\begin{aligned}\sup_t\lim_{\delta\to 0}\left|\int_{\delta}^{\epsilon}\frac{A(t,t-S;\xi)-A(t,t+S;\xi)}{S}dS\right| &\leq \lim_{\delta\to 0}|\partial_s A(t,s;\xi)|_\infty\int_{\delta}^{\epsilon}|2S|\frac{dS}{S} \\ &\leq \lim_{\delta\to 0}C_2k(\xi)(\epsilon-\delta)=C_2k(\xi)\epsilon.\end{aligned}$$

For the (ϵ, ∞) case, we take a simpler bound:

$$\left|\int_\epsilon^\infty \frac{A(t,t-S;\xi)-A(t,t+S;\xi)}{S}dS\right| \leq \frac{1}{\epsilon}\int_\epsilon^\infty |A(t,t+S;\xi)|+|A(t,t-S;\xi)|dS$$

$$\text{since } |t-(t\pm S)|<C\rho, \qquad \leq \frac{4CC_1\rho k(\xi)}{\epsilon}$$

By choosing $\epsilon \leq 1$, we see that $|T1|_\infty \leq Ck(\xi)$, so $\|T1\|_{BMO} \leq 2|T1|_\infty \leq C'k(\xi)$. Therefore the operator associated to $K(t,s;\xi)$ is bounded from L^2 to itself by the $T1$ theorem and $\|T\|_{2,2} \leq C'k(\xi)$. □

CHAPTER IV

First Set of Dispersive Estimates and their Optimality

This chapter collects the new bounds for a general class dispersive equations described below. The results of primary interest for our later discussion of the linearized water wave problem are Theorem IV.3 and its related sharpness result Theorem IV.5. The precise bounds from this theorem in the case of $a = 1/2$ (found in section 4.5.2) imply that the growth factor in Theorem IV.3 exists only because of high frequency contributions in the initial data. All of the proofs rely on a careful analysis of several oscillatory integrals.

4.1 General Class of Dispersive Equations

The results in this chapter apply to a general class of one dimensional dispersive equations, for which the linearized water wave problem is a special case. We state and prove these results in their full generality, before focusing on the water wave case in Chapter V. Consider the following general class of 1D dispersive differential equations. For $0 < a$, $a \neq 1$, let $u(t, x)$ be a solution to the initial value problem

$$\begin{cases} \partial_t^2 u + |D|^{2a} u = 0 \\ u(0, x) = u_0(x) \\ u_t(0, x) = u_1(x). \end{cases} \tag{4.1}$$

We define $|D|$ by the following Fourier transform:

$$|D|f := \int e^{-ix\xi}|\xi|\widehat{f}d\xi.$$

Let $\Gamma = \{\partial_t, \partial_x, L = t\partial_t + (x/a)\partial_x\}$ and $\Omega_a = x\partial_t + at\partial_x|D|^{2a-2}$. In the case of $a = 1/2$, equation (4.1) is the linearized water wave problem. Observe that the equation above is really $(\partial_t - i|D|^a)(\partial_t + i|D|^a)u = 0$. It is sufficient to consider just half of the equation:

$$\begin{cases} \partial_t u - i|D|^a u = 0 \\ u(0,x) = u_0(x). \end{cases} \tag{4.2}$$

Remark IV.1. This decomposition is convenient for the results in the following chapters, but it is by no means the only one with merit. In the case of the linearized water wave problem, Geri Izbicki-Jennings in her thesis [3] has detailed numerical results for the so-called one way water wave operator, $\partial_t - i|D|^{1/2}\mathcal{H}$, where $\mathcal{H}$ is the Hilbert transform. The results discussed in future chapters are also applicable (after some modification) to the one-way water wave equation. We do not use the one way water wave equation simply because the equation (4.2) has some useful symmetry in our calculations.

We have the following variant on the standard Sobolev bounds, using only the vector field $L = t\partial_t + (x/a)\partial_x$

Lemma IV.2. *For any $C^1(\mathbb{R}^+, \mathbb{R})$ function $v(t,x)$ such that v decays to zero as $|x| \to \infty$ and any paramter $y \in \mathbb{R}$, we have*

$$\sup_{T \le t \le 2T} |v(t, yt^{1/a})| \le \frac{C}{T^{1/2}} \sum_{k=0}^{1} \left(\int_T^{2T} \left|L^k v(t, yt^{1/a})\right|^2 dt \right)^{\frac{1}{2}} \tag{4.3}$$

Proof. The integral curves of the vector field L are of the form $x(t) = yt^{1/a}$, where y is any real number. First, we want to consider the quantity $|v(t, yt^{1/a})|$, the restriction

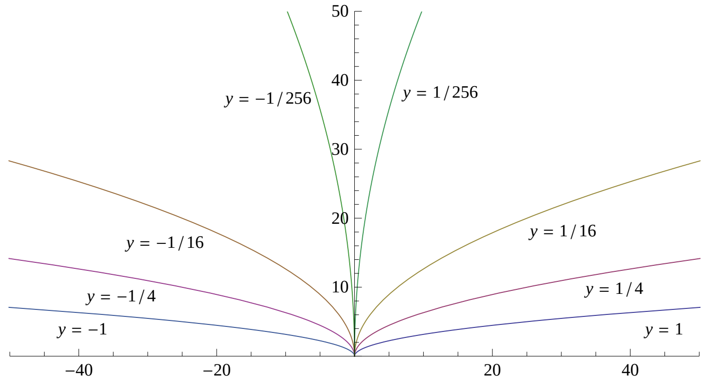

Figure 4.1: Integral curves for $a = 1/2$

of the C^1 function $v(t, x)$ to the integral curve. First, observe that:

$$\begin{aligned}
\partial_t(v(t, yt^{1/a})) =& \partial_1 v(t, yt^{1/a}) + \frac{yt^{1/a-1}}{a}\partial_2 v(t, yt^{1/a}) \\
=& \frac{1}{t}\left(t\partial_1 v(t, yt^{1/a}) + \frac{yt^{1/a}}{a}\partial_2 v(t, yt^{1/a})\right) \\
=& \frac{1}{t}\ Lv|_{(t, yt^{1/a})}
\end{aligned}$$

Using a variation on the fundamental theorem of calculus and the identity above, we find:

$$\begin{aligned}
|v(T, yT^{1/a})| \leq & \int_T^{2T} |\partial_t(v(t, yt^{1/a}))|dt + \frac{1}{T}\int_T^{2T} |v(t, yt^{1/a})|dt \\
\leq & \int_T^{2T} \frac{1}{t}|Lv(t, yt^{1/a})|dt + \frac{1}{T}\int_T^{2T} |v(t, yt^{1/a})|dt \\
\leq & \frac{1}{T}\int_T^{2T} \sum_{|k|\leq 1} \left|L^k v(t, yt^{1/a})\right| dt.
\end{aligned}$$

After an application of Cauchy-Schwarz we have the following L^2 bound:

$$|v(T, yT^{1/a})| \leq \frac{C}{T^{1/2}} \sum_{|k|\leq 1} \left(\int_T^{2T} \left|L^k v(t, yt^{1/a})\right|^2 dt\right)^{\frac{1}{2}}$$

□

We can now state the new decay estimates.

4.2 Statement of Theorems

For $0 < a$, $a \neq 1$, recall that $u(t,x)$ is a solution to the initial value problem (4.2):

$$\begin{cases} \partial_t u - i|D|^a u = 0 \\ u(0,x) = u_0(x). \end{cases} \tag{4.2}$$

Let $L = \frac{x}{a}\partial_x + t\partial_t$. Recall $Lu_0 = Lu(0,x)$.

4.2.1 New L^∞ bounds

Theorem IV.3. *Let $u(t,x) = e^{it|D|^{\frac{1}{2}}}u_0(x)$ with $L^i u_0 \in \dot{H}^{\frac{1-a}{2}}$ for $i = 0, 1$. Then, for any time $t > 0$*

$$|u(t, yt^{1/a})| \leq C(1 + |y|^{\frac{-a}{4(1-a)}})t^{-\frac{1}{2}} \left(\|u_0\|_{\dot{H}^{\frac{1-a}{2}}} + \|Lu_0\|_{\dot{H}^{\frac{1-a}{2}}} \right). \tag{4.4}$$

This theorem is a special case of the following proposition combined with Lemma IV.2:

Proposition IV.4. *Let $u(t,x) = e^{it|D|^{\frac{1}{2}}}u_0(x)$ with $L^i u_0 \in \dot{H}^{\frac{1-a}{2}}$ for $i = 0, 1$. Then we have the following restricted L^2 bounds:*

1. *For $0 < a$ and $a \neq 1$ with*

$$\sigma \in \begin{cases} \left(0, \frac{1-a}{2}\right] & \text{for } 0 < a < 1 \\ \left[\frac{1-a}{2}, 0\right) & \text{for } 1 < a, \end{cases}$$

$$\int_T^{2T} |u(t, yt^{1/a})|^2 dt \leq C|y|^{\frac{1-a-2\sigma}{a-1}} T^{\frac{a-1+2\sigma}{a}} \left(1 + y^{\frac{a}{2(a-1)}}\right) \|u_0\|^2_{\dot{H}^\sigma}. \tag{4.5}$$

2. *For $0 < a < 1$ and $\frac{1-a}{2} \leq \sigma < 1/2$,*

$$\int_T^{2T} |u(t, yt^{1/a})|^2 dt \leq C|y|^{\frac{1-a-2\sigma}{a-1}} T^{\frac{a-1+2\sigma}{a}} \left(1 + y^{\frac{a}{2(a-1)}}\right) \|u_0\|^2_{H^\sigma}. \tag{4.6}$$

In the results above, there is a growth factor in y. From the following optimality result, it is not possible to remove the growth factor without slowing down the rate of decay.

Theorem IV.5. *Choose initial data u_0 such that $\widehat{u}_0(\xi) = |\xi|^{a-1}\widehat{g\varphi}(y\xi + |\xi|^a)$ with $\|g\|_{L^2} = 1$ and $\varphi \in C_0^\infty(\mathbb{R})$ such that $0 \le \varphi$ and $\varphi^2(t) \le \chi_{[T,2T]}(t)$. Then,*

$$\left(\int_T^{2T} |u(t,yt)|^2 dt\right)^{1/2} \ge C'(a)|y|^{a/4(a-1)} C(g) \|u_0\|_{\dot{H}^{(1-a)/2}}$$

where the constant $C(g)$ is explicitly

$$C(g)^2 = \begin{cases} \displaystyle\int_0^{(a-1)|y/a|^{a/(a-1)}} |\zeta|^{-1/2} |\widehat{g\varphi}(\zeta - (a-1)|y/a|^{a/(a-1)})|^2 d\zeta & 1 < a \\ \displaystyle\int_{(a-1)|y/a|^{a/(a-1)}}^{0} |\zeta|^{-1/2} |\widehat{g\varphi}(\zeta + (1-a)|y/a|^{a/(a-1)})|^2 d\zeta & 0 < a < 1 \end{cases}.$$

We can remove the factor of y in the Theorem IV.3 above but at the cost of a slower decay rate. One such result is the following:

Proposition IV.6. *1. When $0 < a < 1$, $u(t,x) = e^{it|D|^{\frac{1}{2}}} u_0(x)$ with $L^i u_0 \in H^r$ with $r = \max \frac{1-a}{2}, \frac{2-a}{4}$ for $i = 0, 1$ satisfies the following L^∞ bound:*

$$\sup_y |u(t, yt^{1/a})| \le C(t^{-1/4} + t^{-1/2}) \sum_{|k|\le 1} \left(\|L^k u_0\|_{H^{\frac{1-a}{2}}} + \|L^k u_0\|_{H^{\frac{2-a}{4}}} \right).$$

2. When $2 \le a$, $u(t,x) = e^{it|D|^{\frac{1}{2}}} u_0(x)$ with $L^i u_0 \in \dot{H}^r$ with $r = \max\{\frac{1-a}{2}, \frac{2-a}{4}\}$ for $i = 0, 1$ satisfies the following L^∞ bound:

$$\sup_y |u(t, yt^{1/a})| \le C(t^{-1/4} + t^{-1/2}) \sum_{|k|\le 1} \left(\|L^k u_0\|_{\dot{H}^{\frac{1-a}{2}}} + \|L^k u_0\|_{\dot{H}^{\frac{2-a}{4}}} \right).$$

This proposition follows from an invariant vector field Sobolev bound (Lemma IV.2) and Proposition IV.4 for appropriate choices of σ.

The proofs of Theorem IV.3 and Proposition IV.6 are straightforward applications of Lemma IV.2 and Proposition IV.4. Most of the detailed work is in the proof of

Proposition IV.4 which has several steps. It begins by rewriting $u(t, yt^{1/a})$ as an operator on the initial data using Fourier transforms. Define the operator $\mathcal{S}_a^y$ as

$$\mathcal{S}_a^y v(t) = \int e^{i(yt^{1/a}\xi + t|\xi|^a)} \widehat{v}(\xi) d\xi.$$

Therefore, $u(t, yt^{1/a}) = \mathcal{S}_a^y u_0$. In practice, we will suppress the sub and superscript. We can rewrite the L^2 norm with respect to this new operator:

$$\begin{aligned}\left(\int_T^{2T} |u(t, yt^{1/a})|^2 dt\right)^{\frac{1}{2}} &\leq \sup_{g \in L^2, \|g\|=1} |\langle \mathcal{S}u_0(t)\varphi_T(t), g(t)\rangle| \\ &= \sup_{g \in L^2, \|g\|=1} |\langle \widehat{u_0}(\xi), \widehat{\mathcal{S}^*}(\varphi_T g)(\xi)\rangle|\end{aligned}$$

where $\varphi(t) \in C_0^\infty(\mathbb{R})$ with $\varphi = 1$ for $t \in (1, 2)$ and $\varphi = 0$ for $t \in (1/2, 5/2)^C$ and $\varphi_T(t) = \varphi(t/T)$. For simplicity, we will let $\mathcal{T} = \widehat{\mathcal{S}^*}$, so

$$\mathcal{T}h(\xi) = \int e^{-i(yt^{1/a}\xi + t|\xi|^a)} h(t) dt.$$

The proof of Proposition IV.4 relies on a careful bound for the operator $\mathcal{T}$, found in Lemma IV.7.

First, we present the proofs of Theorem IV.3 and Proposition IV.6, followed by the longer proof of Proposition IV.4.

4.2.2 Proof of Theorem IV.3 and Proposition IV.6

For these two short proofs, we will assume Proposition IV.4.

Proof of Theorem IV.3. Let $\mathcal{S}$ and $\mathcal{T}$ be the operators described above. Then,

$$\left(\int_T^{2T} \left|u(t, yt^{1/a})\right|^2 dt\right)^{\frac{1}{2}} \leq \left(\int_T^{2T} |\mathcal{S}u_0(t)\varphi_T(t)|^2 dt\right)^{\frac{1}{2}}$$

$$= \sup_{\|g\|_{L^2}=1} |\langle \mathcal{S}u_0\varphi_T, g\rangle|$$

$$= \sup_{\|g\|_{L^2}=1} |\langle \widehat{u}_0, \mathcal{T}(g\varphi_T)\rangle|$$

$$\leq \|u_0\|_{\dot{H}^{\frac{1-a}{2}}} \sup_{\|g\|_{L^2}=1} \left(\int |\mathcal{T}(g\varphi_T(t))(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

$$= \|u_0\|_{\dot{H}^{\frac{1-a}{2}}} \sup_{\|g\|_{L^2}=1} \left(\int \left|\int e^{-iy\tau^{1/a}\xi - i\tau|\xi|^a} g(\tau)\varphi_T(t)d\tau\right|^2 |\xi|^{a-1} d\xi\right)^{\frac{1}{2}}. \tag{4.7}$$

Then by Lemma IV.7,

$$\sup_{\|g\|_{L^2}=1} \left(\int \left|\int e^{-iy\tau^{1/a}\xi - i\tau|\xi|^a} g(\tau)\varphi_T(t)d\tau\right|^2 |\xi|^{a-1} d\xi\right)^{\frac{1}{2}} \leq \sup_{\|g\|_{L^2}=1} \left(1 + y^{\frac{a}{2(a-1)}}\right)^{\frac{1}{2}} \|g\|_{L^2}$$
$$\leq C(a)\left(1 + y^{\frac{a}{2(a-1)}}\right)^{\frac{1}{2}}.$$

Thus,

$$\left(\int_T^{2T} \left|u(t, yt^{1/a})\right|^2 dt\right)^{\frac{1}{2}} \leq C(a)\left(1 + y^{\frac{a}{2(a-1)}}\right)^{\frac{1}{2}} \|u_0\|_{\dot{H}^{\frac{1-a}{2}}}.$$

□

In order to get the slower decay rate, we use a similar argument, but we use a linear combination of results from Proposition IV.4 with $\sigma = \frac{1-a}{2},\ \frac{2-a}{4}$.

Proof of Proposition IV.6. Recall from Proposition IV.4, for $0 < a < 1$ and $\frac{1-a}{2} \leq \sigma < 1/2$,

$$\int_T^{2T} |u(t, yt^{1/a})|^2 dt \leq C\left(|y|^{\frac{1-a-2\sigma}{a-1}} T^{\frac{a-1+2\sigma}{a}} + |y|^{\frac{2-a-4\sigma}{2(a-1)}} T^{\frac{a-1+2\sigma}{a}}\right) \|u_0\|^2_{H^\sigma}.$$

We want to choose discrete σ so that we can control the right hand side independent of y. Notice that

$$|y|^{\frac{1}{a-1}} T^{-1/a} < 1 \Rightarrow |y|^{\frac{1}{a-1}} < T^{1/a} \Rightarrow |y|^{\frac{a}{2(a-1)}} < T^{1/2}.$$

So when $|y|^{\frac{1}{a-1}}T^{-1/a} < 1$, if we take $\sigma = \frac{1-a}{2}$, we have:

$$\int_T^{2T} |u(t, yt^{1/a})|^2 dt \leq C\left(1 + T^{1/2}\right)\|u_0\|^2_{H^{\frac{1-a}{2}}}.$$

On the other hand, if

$$|y|^{\frac{1}{a-1}}T^{-1/a} > 1 \Rightarrow |y|^{-\frac{a}{2(a-1)}}T^{1/2} < 1.$$

If we take $\sigma = \frac{2-a}{4}$, then $|y|^{\frac{1-a-2\sigma}{a-1}}T^{\frac{a-1+2\sigma}{a}} = |y|^{-\frac{a}{2(a-1)}}T^{1/2}$ and we get

$$\int_T^{2T} |u(t, yt^{1/a})|^2 dt \leq C\left(1 + T^{1/2}\right)\|u_0\|^2_{H^{\frac{2-a}{4}}}.$$

Therefore, we can conclude that

$$\int_T^{2T} |u(t, yt^{1/a})|^2 dt \leq C\left(1 + T^{1/2}\right)\left(\|u_0\|^2_{H^{\frac{1-a}{2}}} + \|u_0\|^2_{H^{\frac{2-a}{4}}}\right).$$

By combining this estimate and the Sobolev estimate from Lemma IV.2, we get the desired L^∞ bound. □

4.3 Proof of Proposition IV.4

The proofs of Theorem IV.3 and Proposition IV.6 rely on Proposition IV.4, which in turn follows from a proposition on the Fourier transform of the dual operator $\mathcal{S}^*$.

4.3.1 Reduction to Lemma IV.7

Recall the operator $\mathcal{S}$:

$$\mathcal{S}v(t) = \int e^{i(yt^{1/a}\xi + t|\xi|^a)}\widehat{v}(\xi)d\xi.$$

Then,

$$\begin{aligned} \int_T^{2T} |u(t, yt^{1/a})|^2 dt &\leq \sup_{g\in L^2, \|g\|=1} |\langle \mathcal{S}u_0(t)\varphi_T(t), g(t)\rangle| \\ &= \sup_{g\in L^2, \|g\|=1} |\langle \widehat{u_0}(\xi), \mathcal{T}(\varphi_T g)(\xi)\rangle| \qquad (4.8)\\ &= \sup_{g\in L^2, \|g\|=1} \left(\int \frac{1}{\omega(\xi)}\left|\widehat{u_0}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}} \left(\int \omega(\xi)|\mathcal{T}(\varphi_T g)(\xi)|^2 d\xi\right)^{\frac{1}{2}} \end{aligned}$$

where $\omega(\xi)$ is any weight.

Thus, we have reduced the proof of Proposition IV.4 to showing the weighted estimate contained in the following lemma.

Lemma IV.7. *Let* $g \in L^2(\mathbb{R})$ *and* $\mathcal{T}$ *and* φ_T *be as above.*

1. *For* $0 < a$ *and* $a \neq 1$ *with*

$$\sigma \in \begin{cases} \left(0, \frac{1-a}{2}\right] & \text{for } 0 < a < 1 \\ \left[\frac{1-a}{2}, 0\right) & \text{for } 1 < a, \end{cases}$$

$$\int |\xi|^{-2\sigma} |\mathcal{T}(g\varphi_T)(\xi)|^2 d\xi \leq C|y|^{\frac{1-a-2\sigma}{a-1}} T^{\frac{a-1+2\sigma}{a}} \left(1 + y^{\frac{a}{2(a-1)}}\right) \|g\|_{L^2}^2. \tag{4.9}$$

2. *For* $0 < a < 1$ *and* $\frac{1-a}{2} \leq \sigma < 1/2$*,*

$$\int (1+\xi^2)^{-\sigma} |\mathcal{T}(g\varphi_T)(\xi)|^2 d\xi \leq C|y|^{\frac{1-a-2\sigma}{a-1}} T^{\frac{a-1+2\sigma}{a}} \left(1 + y^{\frac{a}{2(a-1)}}\right) \|g\|_{L^2}^2. \tag{4.10}$$

Remark IV.8. To prove Proposition IV.4, we simply apply Lemma IV.7 to the last line of (4.8).

4.3.2 Proof of Lemma IV.7

Proof of Lemma IV.7. Assume without loss of generality that $y > 0$. The two cases in a are proved using the same techniques, but the details are subtly different. We will present the full details for the $0 < a < 1$ for (4.10) and a sketch of the ideas for (4.9). The arguments are identical for the $1 < a$ case.

The main idea of this proof is rewriting the weighted L^2 norm as an operator and analyzing the kernel of this operator. We will show that the kernel is a linear combination of standard kernels and fractional integrals. Since the kernel is an oscillatory integral, we will use a careful decomposition combined with integration by parts and the method of stationary phase to control it.

We begin by rewriting the square on the left hand side of the inequality as a product and changing the order of integration. In the proofs of (4.9) and (4.10), the only differences lie in the kernel analysis. For now, let $\omega(\xi)$ denote a general weight function. Then,

$$\int \omega(\xi)|\mathcal{T}(g\varphi_T)(\xi)|^2 d\xi = \iint g\varphi_T(t)\overline{g\varphi_T}(s) \int e^{-i\Psi(\xi)}\omega(\xi)d\xi dt ds,$$

where $\Psi(\xi) = y(t^{1/a} - s^{1/a})\xi + (t-s)|\xi|^a$ and let $\xi_0 = -\left(\frac{yf(t,s)}{a}\right)^{\frac{1}{a-1}}$ denote the critical point of this phase function $\Psi(\xi)$ with $f(t,s) := (t^{1/a} - s^{1/a})/(t-s)$. Our goal is to show kernel estimates on the ξ integral so that we can apply Hölder and produce L^2 bounds. For clarity, let $K(t,s) = \int e^{-i\Psi(\xi)}\omega(\xi)d\xi$. To get reasonable bounds on $K(t,s)$, we need to consider the integral near the critical point and away from the critical point. With different weight functions, these estimates proceed somewhat differently. First consider $\omega(\xi) = (1+|\xi|^2)^{-\sigma}$.

Case IV.8.1. $\omega(\xi) = (1+|\xi|^2)^{-\sigma}$ for (4.10)

Let $\epsilon \leq |\xi_0|/2$; then:

$$\begin{aligned} K(t,s) &= \int_{|\xi-\xi_0|<\epsilon} e^{-i\Psi(\xi)}(1+\xi^2)^{-\sigma}d\xi + \int_{|\xi-\xi_0|>\epsilon} e^{-i\Psi(\xi)}(1+\xi^2)^{-\sigma}d\xi \\ &= I + II. \end{aligned}$$

Now, $|I| \leq 2C\epsilon(1+\xi_0^2)^{-\sigma}$, and we will use integration by parts to bound the second term:

$$\begin{aligned} II &= \int_{|\xi-\xi_0|>\epsilon} \partial_\xi \left(e^{-i\Psi(\xi)}\right) \frac{1}{-i(1+\xi^2)^\sigma \Psi'(\xi)} d\xi \\ &= \left.\frac{e^{-i\Psi(\xi)}}{-i\Psi'(\xi)(1+\xi^2)^\sigma}\right|_{|\xi-\xi_0|>\epsilon} - \int_{|\xi-\xi_0|>\epsilon} e^{-i\Psi(\xi)}\partial_\xi \left(\frac{1}{-i(1+\xi^2)^\sigma\Psi'(\xi)}\right) d\xi \end{aligned}$$

Here the requirements for σ come into play. In order to have these terms be finite at zero and decay at ∞, $0 < \sigma$. We will show the bounds for the boundary term in detail, and the bounds on the remaining term are similar. When we evaluate the boundary term, the only contributions come from $\xi_0 - \epsilon$ and $\xi_0 + \epsilon$, so we have

$$BT = \frac{e^{-i\Psi(\xi_0-\epsilon)}}{-i\Psi'(\xi_0-\epsilon)(1+(\xi_0-\epsilon)^2)^\sigma} - \frac{e^{-i\Psi(\xi_0+\epsilon)}}{-i\Psi'(\xi_0+\epsilon)(1+(\xi_0+\epsilon)^2)^\sigma}.$$

First, notice that $\Psi'(\xi_0 \pm \epsilon) = \pm a(1-a)\epsilon(t-s)|\xi_0|^{a-2}$. We can neglect the higher order terms in ϵ and rewrite the boundary terms:

$$\begin{aligned}|BT| &\leq \frac{|\xi_0|^{2-a}}{a(1-a)\epsilon|t-s|}\left|\frac{e^{-i\Psi(\xi_0-\epsilon)}}{(1+(\xi_0-\epsilon)^2)^\sigma} + \frac{e^{-i\Psi(\xi_0+\epsilon)}}{(1+(\xi_0+\epsilon)^2)^\sigma}\right| \\ &\leq \frac{C|\xi_0|^{2-a}}{a(1-a)\epsilon|t-s|(1+\xi_0^2)^\sigma}.\end{aligned}$$

Now, we optimize our choice of ϵ by setting the two terms equal to each other:

$$\begin{aligned}2C\epsilon(1+\xi_0^2)^{-\sigma} =& \frac{C|\xi_0|^{2-a}}{a(1-a)\epsilon|t-s|(1+\xi_0^2)^\sigma} \\ \epsilon^2 =& \frac{C|\xi_0|^{2-a}}{a(1-a)|t-s|} \\ \epsilon =& \frac{C'|\xi_0|^{1-a/2}}{|t-s|^{1/2}}\end{aligned}$$

Case IV.8.2. $\omega(\xi) = |\xi|^{-2\sigma}$ for (4.9) For now, let $\epsilon \leq |\xi_0|/2$; then:

$$\begin{aligned}K(t,s) &= \int_{|\xi-\xi_0|<\epsilon} e^{-i\Psi(\xi)}|\xi|^{-2\sigma}d\xi + \int_{|\xi-\xi_0|>\epsilon} e^{-i\Psi(\xi)}|\xi|^{-2\sigma}d\xi \\ &= I + II.\end{aligned}$$

Now, $|I| \leq 2C\epsilon|\xi_0|^{-2\sigma}$, and we will use integration by parts to bound the second term:

$$\begin{aligned}II &= \int_{|\xi-\xi_0|>\epsilon} \partial_\xi\left(e^{-i\Psi(\xi)}\right)\frac{1}{-i|\xi|^{2\sigma}\Psi'(\xi)}d\xi \\ &= \left.\frac{e^{-i\Psi(\xi)}}{-i\Psi'(\xi)|\xi|^{2\sigma}}\right|_{|\xi-\xi_0|>\epsilon} - \int_{|\xi-\xi_0|>\epsilon} e^{-i\Psi(\xi)}\partial_\xi\left(\frac{1}{-i|\xi|^{2\sigma}\Psi'(\xi)}\right)d\xi\end{aligned}$$

Here the requirements for σ come into play. In order to have these terms be finite at zero and decay at ∞, $0 < \sigma$. We will show the bounds for the boundary term in detail, and the bounds on the remaining term are similar. When we evaluate the boundary term for $0 < \sigma < (1-a)/2$, the only contributions come from $\xi_0 - \epsilon$ and $\xi_0 + \epsilon$, so we have

$$BT = \frac{e^{-i\Psi(\xi_0-\epsilon)}}{-i\Psi'(\xi_0-\epsilon)(\xi_0-\epsilon)^{2\sigma}} - \frac{e^{-i\Psi(\xi_0+\epsilon)}}{-i\Psi'(\xi_0+\epsilon)(\xi_0+\epsilon)^{2\sigma}}.$$

First, notice that $\Psi'(\xi_0 \pm \epsilon) = \pm a(1-a)\epsilon(t-s)|\xi_0|^{a-2}$. We can neglect the higher order terms in ϵ and rewrite the boundary terms:

$$\begin{aligned}|BT| &\leq \frac{|\xi_0|^{2-a}}{a(1-a)\epsilon|t-s|}\left|\frac{e^{-i\Psi(\xi_0-\epsilon)}}{|\xi_0-\epsilon|^{2\sigma}} + \frac{e^{-i\Psi(\xi_0+\epsilon)}}{|\xi_0+\epsilon|^{2\sigma}}\right| \\ &\leq \frac{C|\xi_0|^{2-a}}{a(1-a)\epsilon|t-s||\xi_0|^{2\sigma}}.\end{aligned}$$

Now, we optimize our choice of ϵ by setting the two terms equal to each other:

$$\begin{aligned}2C\epsilon|\xi_0|^{-2\sigma} =& \frac{C|\xi_0|^{2-a}}{a(1-a)\epsilon|t-s||\xi_0|^{2\sigma}} \\ \epsilon^2 =& \frac{C|\xi_0|^{2-a}}{a(1-a)|t-s|} \\ \epsilon =& \frac{C'|\xi_0|^{1-a/2}}{|t-s|^{1/2}}\end{aligned}$$

When $\sigma = (1-a)/2$, the boundary term has an additional contribution from 0:

$$\begin{aligned}\left.\frac{e^{-i\Psi(\xi)}}{-i\Psi'(\xi)|\xi|^{1-a}}\right|_{|\xi-\xi_0|>\epsilon} &= \frac{e^{-i\Psi(\xi_0-\epsilon)}}{-i\Psi'(\xi_0-\epsilon)(\xi_0-\epsilon)^{1-a}} - \frac{e^{-i\Psi(\xi_0+\epsilon)}}{-i\Psi'(\xi_0+\epsilon)(\xi_0+\epsilon)^{1-a}} \\ &- \frac{2}{ia(t-s)} \\ |BT| &\leq \frac{2|\xi_0|^{2-a}}{a(1-a)|t-s|}\left|\frac{e^{-i\Psi(\xi_0-\epsilon)}}{(\xi_0-\epsilon)^{1-a}} - \frac{e^{-i\Psi(\xi_0+\epsilon)}}{-i\Psi'(\xi_0+\epsilon)(\xi_0+\epsilon)^{1-a}}\right| \\ &- \frac{2|\xi_0|}{a\epsilon|t-s|} \\ &\leq \frac{C(a)|\xi_0|}{\epsilon|t-s|}.\end{aligned}$$

Then, we have the same optimal choice of ϵ for $0 < \sigma \leq \frac{1-a}{2}$.

Notice that both cases give the same choice of ϵ. However, this optimal $\frac{C'|\xi_0|^{1-a/2}}{|t-s|^{1/2}}$ is not always less than $|\xi_0|/2$. Let

$$\epsilon = \min\left\{\frac{C'|\xi_0|^{1-a/2}}{|t-s|^{1/2}}, |\xi_0|/2\right\}.$$

We will do the rest of the calculations with $\omega(\xi) = (1+\xi^2)^{-\sigma}$. The homogeneous case follows by similar and slightly easier arguments.

Observe that

$$|\xi_0|/2 < \frac{C'|\xi_0|^{1-a/2}}{|t-s|^{1/2}} \Leftrightarrow y^{\frac{a}{2(1-a)}} > \frac{C|t-s|^{1/2}}{f(t,s)^{\frac{a}{2(1-a)}}}. \tag{4.11}$$

Assume $y^{\frac{a}{2(1-a)}} > \frac{C|t-s|^{1/2}}{f(t,s)^{\frac{a}{2(1-a)}}}$, and therefore $\epsilon = |\xi_0|/2$. Let $\gamma(a) = a^{\frac{1}{a-1}}$. For $0 < a < 1$, $\gamma(a) > 2$. More importantly, $\Psi(\gamma(a)\xi_0) = 0$. We adjust the decomposition of $K(t,s)$ so that one endpoint lies on this very convenient number:

$$\begin{aligned} K(t,s) &= \int_{\gamma(a)\xi_0}^{\xi_0/2} e^{-i\Psi(\xi)}(1+\xi^2)^{-\sigma}d\xi + \int_{(\gamma(a)\xi_0,\xi_0/2)^C} e^{-i\Psi(\xi)}(1+\xi^2)^{-\sigma}d\xi \\ K(t,s) &= K_1(t,s) + K_2(t,s). \end{aligned}$$

The easier term is K_1, so we will bound it first. Clearly,

$$|K_1(t,s)| < C(\gamma(a)-1/2)|\xi_0|(1+\xi_0^2)^{-\sigma}.$$

By our assumption on y, $|\xi_0| < \frac{C'|\xi_0|^{1-a/2}}{|t-s|^{1/2}}$, so we can apply the Hardy-Littlewood-Sobolev lemma, Lemma III.1, for fractional integration:

$$\begin{aligned} \left|\int g\varphi_T(t)\int K_1(t,s)\overline{g\varphi_T}(s)ds\,dt\right| &\le \int |g\varphi_T(t)| \int \frac{C'(\gamma(a)-1/2)|\xi_0|^{1-\frac{a}{2}}|g\varphi_T(s)|}{(1+\xi_0^2)^{\sigma}|t-s|^{1/2}}ds\,dt \\ &\le \frac{C'(\gamma(a)-1/2)a^{(2-a)/2(1-a)}y^{\frac{2-a}{2(a-1)}}T^{\frac{a-2}{2a}}}{(1+Ca^{\frac{2}{1-a}}y^{\frac{2}{1-a}}T^{-2/a})^{\sigma}}\|g\varphi_T\|_{L^{4/3}}^2 \\ &\le C''(\gamma(a)-1/2)a^{\frac{2-a-4\sigma}{2(1-a)}}y^{\frac{2-a-4\sigma}{2(a-1)}}T^{\frac{a-2+4\sigma}{2a}}T^{1/2}\|g\|_{L^2}^2 \\ &\le C_1(a)y^{\frac{2-a-4\sigma}{2(a-1)}}T^{\frac{a-1+2\sigma}{a}}\|g\|_{L^2}^2. \end{aligned}$$

Now we turn to $K_2(t,s)$. Recall

$$K_2(t,s) = \int_{-\infty}^{\gamma(a)\xi_0} e^{-i\Psi(\xi)}(1+\xi^2)^{-\sigma}d\xi + \int_{\xi_0/2}^{\infty} e^{-i\Psi(\xi)}(1+\xi^2)^{-\sigma}d\xi.$$

Let's begin with the first term. By integration by parts,

$$\begin{aligned}\int_{-\infty}^{\gamma(a)\xi_0} \frac{e^{-i\Psi(\xi)}}{(1+\xi^2)^{\sigma}}d\xi &= \frac{e^{-i\Psi(\xi)}}{-i\Psi'(\xi)(1+\xi^2)^{\sigma}}\Bigg|_{-\infty}^{\gamma(a)\xi_0} \\ &\qquad - \int_{-\infty}^{\gamma(a)\xi_0} e^{-i\Psi(\xi)}\partial_\xi\left(\frac{1}{-i\Psi'(\xi)(1+\xi^2)^{\sigma}}\right)d\xi \\ &= \frac{|\xi_0|^{1-a}}{-ia(1-a)(t-s)(1+a^{\frac{2}{a-1}}\xi_0^2)^{\sigma}} \\ &\qquad - \int_{-\infty}^{\gamma(a)\xi_0} e^{-i\Psi(\xi)}\partial_\xi\left(\frac{1}{-i\Psi'(\xi)(1+\xi^2)^{\sigma}}\right)d\xi \\ &= K_2'(t,s) + K_2''(t,s).\end{aligned}$$

The kernel $K_2'(t,s)$ satisfies all the conditions of Proposition III.6 in Chapter III with $C_1 = \frac{y^{-1}T^{1-1/a}}{(1-a)(1+y^{2/(a-1)}T^{-2/a})^{\sigma}}$, $C_2 = \frac{2y^{-1}T^{-1/a}}{(1-a)(1+y^{2/(a-1)}T^{-2/a})^{\sigma}}$ and $\rho = T$. Therefore,

$$\begin{aligned}|\int g\varphi_T(t)\int K_2'(t,s)\overline{g\varphi_T}(s)ds| &\le \|g\|_{L^2}\frac{Cy^{-1}T^{1-1/a}}{(1-a)(1+y^{2/(a-1)}T^{-2/a})^{\sigma}}\|g\|_{L^2} \\ &\le \frac{C}{1-a}y^{\frac{1-a-2\sigma}{a-1}}T^{\frac{2\sigma-1+a}{a}}\|g\|_{L^2}^2\end{aligned}$$

For the kernel $K_2''(t,s)$, we want to change the order of integration, so that we may integrate in t and s before integrating in ξ. Showing K_2'' is a nice kernel is complicated by the presence of the exponential; if we change the order of integration, the exponential splits into a function of norm 1 in t and s, leaving only the derivative for the kernel. In fact, $\partial_\xi\left(\frac{1}{-i\Psi'(\xi)(1+\xi^2)^{\sigma}}\right)$ satisfies the kernel conditions of Proposition III.7. It will be convenient for notation to let $\xi_2 = -(\frac{y}{a}(2T)^{(1-a)/a})^{\frac{1}{a-1}})$ and $\xi_1 = -(\frac{y}{a}T^{(1-a)/a})^{\frac{1}{a-1}})$ Since $\gamma(a)\xi_0 < \xi_2$, when we change the order of integration, we have

$$\int\int g\varphi_T(t)\overline{g\varphi_T}(s)\int_{-\infty}^{\gamma(a)\xi_0} e^{-i\Psi(\xi)}\partial_\xi\left(\frac{1}{-i\Psi'(\xi)(1+\xi^2)^\sigma}\right)d\xi$$

$$= \int_{-\infty}^{\xi_2}\iint_{\Omega(t,s)(\xi)} g(t)e^{-i(yt^{1/a}\xi+t|\xi|^a)}\frac{\varphi_T(t)\overline{\varphi_T}(s)A(t,s;\xi)}{i(t-s)}\overline{g}(s)e^{i(ys^{1/a}\xi+s|\xi|^a)}ds\,dt\,d\xi$$

where

$$A(t,s;\xi) = \varphi_T(t)\varphi_T(s)\partial_\xi\left(-i(yf(t,s)+a|\xi|^{a-1}\mathrm{sgn}\xi)(1+\xi^2)^\sigma\right)^{-1},$$

and $\Omega(t,s)(\xi)$ is the region in $\mathbb{R}^2$ from Fubini theorem. When $\xi < \xi_1$, we can apply Proposition III.7 for $(t,s)\in[T,2T]^2$, so the exact description of $\Omega(t,s)$ is not important, since its intersection with the square is clearly contained in the square. So we can decompose again:

$$\begin{aligned}
&\int_{-\infty}^{\xi_2}\iint_{\Omega(t,s)(\xi)} g(t)e^{-i(yt^{1/a}\xi+t|\xi|^a)}\frac{\varphi_T(t)\overline{\varphi_T}(s)A(t,s;\xi)}{i(t-s)}\overline{g}(s)e^{i(ys^{1/a}\xi+s|\xi|^a)}dsdtd\xi\\
&\quad= \int_{-\infty}^{\xi_1}\iint_{\Omega(t,s)(\xi)} g(t)e^{-i(yt^{1/a}\xi+t|\xi|^a)}\frac{\varphi_T(t)\overline{\varphi_T}(s)A(t,s;\xi)}{i(t-s)}\overline{g}(s)e^{i(ys^{1/a}\xi+s|\xi|^a)}dsdtd\xi\\
&\quad+ \int_{\xi_1}^{\xi_2}\iint_{\Omega(t,s)(\xi)} g(t)e^{-i(yt^{1/a}\xi+t|\xi|^a)}\frac{\varphi_T(t)\overline{\varphi_T}(s)A(t,s;\xi)}{i(t-s)}\overline{g}(s)e^{i(ys^{1/a}\xi+s|\xi|^a)}dsdtd\xi\\
&\quad= i+ii
\end{aligned}$$

We will use the following claim, combined with Proposition III.7 to bound term i:

Claim IV.9. *Let*

$$k(\xi) = \frac{1}{(1+\xi^2)^\sigma(yT^{1/a-1}/a - a|\xi|^{a-1})}.$$

Then, for $A(t,s;\xi)$ defined above and $\xi\in(-\infty,\xi_1)$, we have

$$|A(t,s;\xi)| \le \frac{2a}{1-a}\left(1+\frac{a(2^{(1-a)/a}-1)}{1-a}\right)k'(\xi)$$

and

$$|\partial_t A(t,s;\xi)| = |\partial_s A(t,s;\xi)| \le \frac{4a}{T(1-a)^2}\left(1+\frac{a(2^{(1-a)/a}-1)}{1-a}\right)k'(\xi).$$

The proof of this claim is straightforward, since by definition $yf(t,s) - a|\xi|^{a-1} > ((1-a)/a)yT^{\frac{1-a}{a}}$. Therefore,

$$
\begin{aligned}
|i| &\leq \int_{-\infty}^{\xi_1} \left| \iint_{\Omega(t,s)(\xi)} g(t) e^{-i(yt^{1/a}\xi + t|\xi|^a)} \frac{\varphi_T(t)\overline{\varphi_T}(s)A(t,s;\xi)}{i(t-s)} \overline{g}(s) e^{i(ys^{1/a}\xi + s|\xi|^a)} ds dt \right| d\xi \\
&\leq \|g\|_{L^2} \int_{-\infty}^{\xi_1} C(a) k'(\xi) d\xi \|g\|_{L^2} \\
&\leq \frac{aC(a)}{(1 + y^{2/(a-1)}T^{-2/a})^{\sigma} (1-a)yT^{(1-a)/a}} \|g\|_{L^2}^2 \\
&\leq aC(a) y^{\frac{1-a-2\sigma}{a-1}} T^{(2\sigma+a-1)/a} \|g\|_{L^2}^2
\end{aligned}
$$

where $C(a) = \frac{2a}{1-a}\left(1 + \frac{a(2^{(1-a)/a}-1)}{1-a}\right)\left(1 + \frac{2}{1-a}\right)$.

Now we consider term ii. Since we are integrating in ξ over a bounded interval, we do not need to use Proposition III.7. It suffices to show that $A(t,s;\xi)$ is bounded. With $\xi \in (\xi_1, \xi_2)$, we cannot neglect the region $\Omega(t,s)$ in favor of $[T, 2T]^2$. Observe that $\Omega(t,s) = \{|\xi|^{a-1} < yf(t,s) < \frac{y}{a}T^{(1-a)/a}\}$. Therefore, $yf(t,s) + a|\xi|^{a-1}\text{sgn}\xi > (1-a)|\xi|^{a-1}$. Using this bound and the range of ξ, we have the following claim:

Claim IV.10. *Let $A(t,s;\xi)$ be as above with $t,s \in \Omega(t,s)$ and $\xi \in (\xi_1, \xi_2)$. Then,*

$$|A| \leq C(2\sigma + a(1-a)) y^{(-a-2\sigma)/(a-1)} T^{(a+2\sigma)/a}$$

and

$$|\partial_t A| = |\partial_s A| \leq \frac{4C(2\sigma + 2a)}{T(1-a)^2} y^{(-a-2\sigma)/(a-1)} T^{(a+2\sigma)/a}.$$

Now, we use Proposition III.6:

$$
\begin{aligned}
|ii| &\leq \int_{\xi_1}^{\xi_2} \left| \iint_{\Omega(t,s)(\xi)} g(t) e^{-i(yt^{1/a}\xi + t|\xi|^a)} \frac{\varphi_T(t)\overline{\varphi_T}(s)A(t,s;\xi)}{i(t-s)} \overline{g}(s) e^{i(ys^{1/a}\xi + s|\xi|^a)} ds dt \right| d\xi \\
&\leq \|g\|_{L^2} \int_{\xi_1}^{\xi_2} \left(C(2\sigma + a(1-a)) + \frac{4C(2\sigma + 2a)}{(1-a)^2} \right) y^{(-a-2\sigma)/(a-1)} T^{(a+2\sigma)/a} \|g\|_{L^2} d\xi \\
&\leq C' \left((2\sigma + a(1-a)) + \frac{4(2\sigma + 2a)}{(1-a)^2} \right) y^{\frac{1-a-2\sigma}{a-1}} T^{(2\sigma+a-1)/a} \|g\|_{L^2}^2
\end{aligned}
$$

Finally, we address the term on $(\xi_0/2, \infty)$. The integral from $\xi_0/2$ to 0 is somewhat simpler. Since $0 < \sigma < 1/2$, we can integrate $|\xi|^{-2\sigma}$ explicitly:

$$\left| \int_{\xi_0/2}^{0} e^{-i\Psi(\xi)}(1+\xi^2)^{-\sigma} d\xi \right| \le C|\xi_0|^{1-2\sigma} \le \frac{C'|\xi_0|^{1-a/2-2\sigma}}{|t-s|^{1/2}}.$$

Therefore,

$$\left| \iint g\varphi_T(t)\overline{g\varphi_T}(s) \int_{\xi_0/2}^{0} e^{-i\Psi(\xi)}(1+\xi^2)^{-\sigma} d\xi ds\, dt \right| \le C' y^{\frac{2-a-4\sigma}{a-1}} T^{\frac{2\sigma+a-1}{a}} \|g\|_{L^2}^2.$$

When we integrate the kernel from $(0, \infty)$, we use integration by parts as before, and the boundary terms contribute nothing. For the derivative term, we need to be more precise as $\partial_\xi \left((1+\xi^2)^{-\sigma}(yf(t,s) + a|\xi|^{a-1})^{-1}\right)$ changes sign. We do an additional decomposition to preserve monotonicity. Let $\widetilde{\xi}$ be the critical point of $\partial_\xi \left((1+\xi^2)^{-\sigma}(yf(t,s) + a|\xi|^{a-1})^{-1}\right)$. Then,

$$\begin{aligned}
&\iint \frac{g\varphi_T(t)\overline{g\varphi_T}(s)}{i(t-s)} \int_0^\infty e^{-i\Psi(\xi)} \partial_\xi \left((1+\xi^2)^{-\sigma}(yf(t,s)+a|\xi|^{a-1})^{-1}\right) ds\, dt\, d\xi \\
&= \iint \frac{g\varphi_T(t)\overline{g\varphi_T}(s)}{i(t-s)} \int_0^{\widetilde{\xi}} e^{-i\Psi(\xi)} \partial_\xi \left((1+\xi^2)^{-\sigma}(yf(t,s)+a|\xi|^{a-1})^{-1}\right) ds\, dt\, d\xi \\
&\quad + \iint \frac{g\varphi_T(t)\overline{g\varphi_T}(s)}{i(t-s)} \int_{\widetilde{\xi}}^\infty e^{-i\Psi(\xi)} \partial_\xi \left((1+\xi^2)^{-\sigma}(yf(t,s)+a|\xi|^{a-1})^{-1}\right) ds\, dt\, d\xi \\
&= \iint g\varphi_T(t)\overline{g\varphi_T}(s) K_3(t,s) dt ds + \iint g\varphi_T(t)\overline{g\varphi_T}(s) K_4(t,s) dt ds
\end{aligned}$$

We will present the argument for K_3; the analysis for K_4 follows the same arguments but with slightly different constants (independent of y and T). For K_3, we apply Proposition III.7 with $k'(\xi) = \left|\partial_\xi \left((1+\xi^2)^{-\sigma}(CyT^{(1-a)/a} + a|\xi|^{a-1})^{-1}\right)\right|$. Let

ξ_3 denote the zero of $k'(\xi)$. Then,

$$\begin{aligned}
&\left|\iint \quad g\varphi_T(t)\overline{g\varphi_T}(s)K_3(t,s)dtds\right| \\
&\quad \le \int_0^\infty \left|\iint \frac{g\varphi_T(t)e^{-i\Psi(\xi)}\overline{g\varphi_T}(s)}{i(t-s)}\partial_\xi\left((1+\xi^2)^{-\sigma}(yf(t,s)+a|\xi|^{a-1})^{-1}\right)ds\,dt\right| d\xi \\
&\quad \le \|g\|_{L^2}^2\int_0^\infty \left|\partial_\xi\left(\frac{(1+\xi^2)^{-\sigma}}{CyT^{\frac{1-a}{a}}+a|\xi|^{a-1}}\right)\right| d\xi \\
&\quad \le \frac{2\|g\|_{L^2}^2}{(1+\widetilde{\xi}^2)^\sigma(CyT^{\frac{1-a}{a}}+a|\xi_3|^{a-1})}
\end{aligned}$$

In order for $\widetilde{\xi}$ to be a zero of $k'(\xi)$, it must satisfy

$$a(1-a)(1+\xi_3^2) = 2\sigma|\xi_3|^{3-a}(yT^{(1-a)/a}+a|\xi_3|^{a-1}).$$

Therefore,

$$\left|\iint \frac{g\varphi_T(t)\overline{g\varphi_T}(s)}{i(t-s)}\int_0^{\widetilde{\xi}} e^{-i\Psi(\xi)}\partial_\xi\left(\frac{(1+\xi^2)^{-\sigma}}{yf(t,s)+a|\xi|^{a-1}}\right)d\xi ds\,dt\right| \le C\|g\|_{L^2}^2|\xi_3|^{-2\sigma-a+1}.$$

Observe that $\xi_3 > C|yT^{(1-a)/a}|^{\frac{1}{a-1}}$ for a constant depending only on a and σ. When $2\sigma \ge 1-a$, that means this $\widetilde{\xi}$ term is bounded by $y^{(-2\sigma-a+1)/(a-1)}T^{(2\sigma+a-1)/a}$ and can be combined with other terms.

Now, we collect all the terms above:

Lemma IV.11. *When* $y^{\frac{a}{2(1-a)}} > \frac{C|t-s|^{1/2}}{f(t,s)^{\frac{a}{2(1-a)}}}$ *and* $1-a \le 2\sigma < 1$,

$$\int (1+\xi^2)^{-\sigma}|\mathcal{T}(g\varphi_T)(\xi)|^2 d\xi \le \left(C_1(a)y^{\frac{a}{2(a-1)}}+C_2(a)\right)y^{\frac{1-a-2\sigma}{a-1}}T^{\frac{a-1+2\sigma}{a}}\|g\|_{L^2}^2. \tag{4.12}$$

where $C_2(a) = \left(C'\left((2\sigma+a(1-a))+\frac{4(2\sigma+2a)}{(1-a)^2}\right)+aC(a)\right)$ *and* $C_1(a) = C''(\gamma(a)-1/2)a^{\frac{2-a-4\sigma}{2(1-a)}}$

It remains to show the bound for $y^{\frac{a}{2(1-a)}} < C\frac{C|t-s|^{1/2}}{f(t,s)^{\frac{a}{2(1-a)}}}$. Let $F(t,s) = \frac{C|t-s|^{1/2}}{f(t,s)^{\frac{a}{2(1-a)}}}$.

In this case, we only need to take the optimal ϵ:

$$\begin{aligned}
&\left|\iint_{y^{\frac{a}{2(1-a)}}<F(t,s)} g\varphi_T(t)\overline{g\varphi_T}(s)\int e^{-i\Psi(\xi)}(1+\xi^2)^{-\sigma}d\xi dtds\right| \\
&\qquad\leq \iint_{y^{\frac{a}{2(1-a)}}<F(t,s)} |g\varphi_T(t)||\overline{g\varphi_T}(s)|\frac{|\xi_0|^{1-a/2}}{(1+\xi_0^2)^\sigma|t-s|^{1/2}}dsdt \\
&\qquad\leq \frac{C(yT^{1/a-1})^{(2-a)/2(a-1)}}{(1+y^{2/a-1}T^{-2/a})^\sigma}\|g\varphi_T\|_{L^{4/3}}^2 \\
&\qquad\leq \frac{Cy^{(2-a)/2(a-1)}T^{-(2-a)/2a}T^{1/2}}{(1+y^{2/a-1}T^{-2/a})^\sigma}\|g\varphi_T\|_{L^2}^2 \\
&\qquad\leq Cy^{\frac{2-a-4\sigma}{2(a-1)}}T^{\frac{a-1+2\sigma}{a}}\|g\|_{L^2}^2.
\end{aligned}$$

We combine all of these terms to see that for $1-a\leq 2\sigma<1$:

$$\int(1+\xi^2)^{-\sigma}|\mathcal{T}(g\varphi_T)(\xi)|^2d\xi\leq Cy^{\frac{1-a-2\sigma}{a-1}}T^{\frac{a-1+2\sigma}{a}}\left(1+y^{\frac{a}{2(a-1)}}\right)\|g\|_{L^2}^2.$$

□

4.3.3 Remarks on the proofs of §4.2.2 and §4.3.1

The proof of Lemma IV.7 for the case $a>1$ differs in a couple key ways, but otherwise follows the same general argument. First of all, the weight in the kernel is replaced by $|\xi|^{2\sigma}$. We need the homogeneous weight here because $1/\Psi'(\xi)$ at 0 is not equal to zero, and therefore we accumulate additional powers of y and T either by evaluating the derivative at 0, or from the kernel with the exponential that we worked so hard to avoid in the case $0<a<1$. Unfortunately, the acceptable range of σ means we cannot prove Proposition IV.6 when $1<a<2$.

That said, the proof above for Proposition IV.6 is also correct when $2\leq a$. The steps in proof of Proposition IV.7 are the same, except the various bounds on the kernels are subtly different.

4.4 Alternate Proof in the case $a = \frac{1}{2}$

In the case $a = 1/2$, the phase function in Lemma IV.7 is quadratic in ξ , so we can compute those bounds precisely.

Let $u(t,x) = e^{it|D|^{\frac{1}{2}}}u_0(x)$, a solution of the following differential equation:

$$\begin{cases} \partial_t u - i|D|^{\frac{1}{2}}u = 0 \\ u(0,x) = u_0(x). \end{cases} \tag{4.13}$$

We want to control the quantity:

$$\left(\int_T^{2T} |u(t, yt^2)|^2 dt\right)^{\frac{1}{2}} = \|\mathcal{S}u_0\varphi\|_{L^2}$$

where $\mathcal{S}f(t) = \int e^{i(yt^2\xi + t|\xi|^{\frac{1}{2}})}\widehat{f}(\xi)d\xi$. Observe that

$$\begin{aligned} \left(\int_T^{2T} |u(t, yt^2|)^2 dt\right)^{\frac{1}{2}} &= \sup_{g\in L^2} |\langle \mathcal{S}u_0(t)\varphi(t), g(t)\rangle| \\ &= \sup_{g\in L^2} |\langle \widehat{u_0}(\xi), \mathcal{T}(g\varphi)(\xi)\rangle| \\ &\leq \|u_0\|_{\dot{H}^{1/4}}\left(\int |\xi|^{-\frac{1}{2}}|\mathcal{T}g\varphi(\xi)|^2 d\xi\right)^{\frac{1}{2}} \end{aligned}$$

where $\mathcal{T} = \widehat{\mathcal{S}^*}$, that is $\mathcal{T}f(\xi) = \int e^{-i(yt^2\xi + t|\xi|^{\frac{1}{2}})}f(t)dt$. Let us analyze the norm on $\mathcal{T}g\varphi$. If we rewrite the absolute value squared as a product, we can rearrange terms as

$$\int |\xi|^{-\frac{1}{2}}|\mathcal{T}g\varphi(\xi)|^2 d\xi = \iint g\varphi(t)\overline{g\varphi}(s)\int |\xi|^{-\frac{1}{2}}e^{-i(y(t^2-s^2)\xi + (t-s)|\xi|^{\frac{1}{2}})}d\xi ds\, dt.$$

We will treat the ξ integral as a kernel t and s. Without loss of generality, assume that $y > 0$. The phase function $\Psi(\xi) = y(t^2 - s^2)\xi + (t - s)|\xi|^{\frac{1}{2}}$ has strictly positive derivative, and so we can use the method of nonstationary phase to approximate the value of the integral:

$$K_1(t,s) = \int_0^\infty |\xi|^{-\frac{1}{2}} e^{-i(y(t^2-s^2)\xi+(t-s)|\xi|^{\frac{1}{2}})} d\xi$$
$$= \int_0^\infty \frac{2|\xi|^{-\frac{1}{2}}}{-iy(t^2-s^2)+(t-s)|\xi|^{-\frac{1}{2}}} \partial_\xi \left(e^{-i(y(t^2-s^2)\xi+(t-s)|\xi|^{\frac{1}{2}})}\right) d\xi$$

Straightforward calculation shows that $\varphi(t)K_1(t,s)\varphi_(s)$ is a standard kernel. Then, by a modified version of the $T1$ theorem we can conclude that

$$\left| \iint K_1(t,s)\overline{g}(s)dsg(t)dt \right| \leq C\|g\|_{L^2}^2$$

where the constant C is independent of y and T.

Now we consider the integral from $(-\infty, 0)$. First, notice that we can rewrite $\Psi(\xi)$ as a quadratic polynomial in $|\xi|^{\frac{1}{2}}$:

$$\Psi(\xi) = -y(t^2-s^2)\left(|\xi|^{\frac{1}{2}} - \frac{1}{2y(t+s)}\right)^2 + \frac{(t-s)}{4y(t+s)}.$$

This suggests a change of variables $\zeta = |\xi|^{\frac{1}{2}} - \frac{1}{2y(t+s)}$, so

$$\int_{-\infty}^0 e^{-i\Psi(\xi)}|\xi|^{-\frac{1}{2}}d\xi = e^{\frac{-i(t-s)}{4y(t+s)}} \int_{\frac{-1}{2y(t+s)}}^\infty e^{iy(t^2-s^2)\zeta^2}d\zeta$$
$$= e^{\frac{-i(t-s)}{4y(t+s)}} \left(\int_{\frac{-1}{2y(t+s)}}^0 e^{iy(t^2-s^2)\zeta^2}d\zeta + \int_0^\infty e^{iy(t^2-s^2)\zeta^2}d\zeta \right)$$

In both of these terms, we want to change variables in order to remove the factor in front of ζ^2. Let $\xi = (y(t^2-s^2))^{\frac{1}{2}}\zeta$ and change variables:

$$e^{\frac{-i(t-s)}{4y(t+s)}} \left(\int_{\frac{-1}{2y(t+s)}}^\infty e^{iy(t^2-s^2)\zeta^2}d\zeta \right) = \left(y(t^2-s^2)\right)^{-\frac{1}{2}} e^{\frac{-i(t-s)}{4y(t+s)}} \left(\int_b^0 e^{i\xi^2}d\xi + \int_0^\infty e^{i\xi^2}d\xi \right)$$

where $b = \frac{-(t-s)^{\frac{1}{2}}}{2y^{\frac{1}{2}}(t+s)^{\frac{1}{2}}}$. The second term above is bounded by a multiple of $\sqrt{\pi}$, which leaves only the first term, which is a Fresnel integral. Classical results imply that the Fresnel integral is also bounded independent of the quantity b, therefore,

$$|K_2(t,s)| \leq C|y(t^2-s^2)|^{-\frac{1}{2}}.$$

Thus,

$$\left| \iint K_1(t,s)\overline{g}(s)dsg(t)dt \right| \leq y^{-\frac{1}{2}} \iint |t^2 - s^2|^{-\frac{1}{2}} |g\varphi(s)|ds|g\varphi(t)|ds\,dt$$
$$\leq C(yT)^{-\frac{1}{2}} \|g\varphi\|_{L^{4/3}}^2$$
$$\leq Cy^{-\frac{1}{2}} \|g\|_{L^2}^2$$

4.5 Optimality and Counterexamples

In Proposition IV.6, the factor of $|y|$ acts as a barrier to our optimal time decay rate. In the following results, we explore the precise nature of this impediment. There are several different ways to consider the singularity that appears. Firstly, we will look along slightly different trajectories and find a lower bound (enforcing the optimality of our results), however this results imposes strong conditions on the initial data.

4.5.1 Lower bounds

Instead of considering $u(t, yt^{1/a})$, we will look at $u(t, yt)$ which will simplify our calculations considerably. Since

$$\left(\int_T^{2T} |u(t,zt)|^2 dt \right)^{\frac{1}{2}} \leq \sup_{[T,2T]} |u(t,zt)|$$

without the need for our Sobolev lemma, this choice makes sense. Let

$$S'f(t) = \int e^{i(yt\xi + t|\xi|^a)} \widehat{f}(\xi)d\xi$$

and

$$T'g(\xi) = \int e^{-i(yt\xi + t|\xi|^a)} g(t)\varphi(t)dt$$

where φ is a positive function with compact support such that $\varphi^2 \leq \chi_{[T,2T]}$. We will show precisely the following

Theorem IV.5. *Choose initial data* u_0 *such that* $\widehat{u}_0(\xi) = |\xi|^{a-1}\widehat{g\varphi}(y\xi + |\xi|^a)$ *with* $\|g\|_{L^2} = 1$ *and* $\varphi \in C_0^\infty(\mathbb{R})$ *such that* $0 \le \varphi \le \chi_{[T,2T]}(t)$*. Then,*

$$\left(\int_T^{2T} |u(t,yt)|^2 dt\right)^{1/2} \ge C'(a)|y|^{a/4(a-1)} C(g)\|u_0\|_{\dot{H}^{(1-a)/2}}$$

where the constant $C(g)$ *is explicitly*

$$C(g)^2 = \begin{cases} \int_0^{(a-1)|y/a|^{a/(a-1)}} |\zeta|^{-1/2}|\widehat{g\varphi}(\zeta - (a-1)|y/a|^{a/(a-1)})|^2 d\zeta & 1 < a \\ \int_{(a-1)|y/a|^{a/(a-1)}}^0 |\zeta|^{-1/2}|\widehat{g\varphi}(\zeta + (1-a)|y/a|^{a/(a-1)})|^2 d\zeta & 0 < a < 1 \end{cases}$$

This proof relies heavily on the fact that $\mathcal{T}'$ above is the Fourier transform up to a constant. Since this operator has this nice property, we will show a lower bound for $\int |\xi|^{a-1}|T'g(\xi)|^2 d\xi$ in the following proposition

Lemma IV.12. *Without loss of generality, assume that* $y > 0$*. Let* $\xi_1 = -(y/a)^{1/(a-1)}$ *and* $\Xi_1 = (a-1)|\xi_1|^a$*. For* $g \in L^2 \bigcap H^{(a-1)/2}$ *and* $\varphi \in C_0^\infty$ *with* $0 \le \varphi \le \chi_{[T,2T]}$*, we have the following inequalities:*

1. *When* $1 < a$*,*

   (4.14)
   $$\int |\xi|^{a-1}|\widehat{g\varphi}(y\xi + |\xi|^a)|^2 d\xi \ge \frac{1}{a}\int_0^\infty \Psi(\zeta)|\widehat{g}(\zeta)|^2 d\zeta + C(a)|y|^{\frac{a}{2(a-1)}} \int_0^{\Xi_1} |\zeta|^{-\frac{1}{2}}|\widehat{g\varphi}(\zeta - \Xi_1)|^2 d\zeta$$

   where $\Psi(\zeta)$ *is bounded by* $\frac{1}{a-1}$ *at 0 and tends to* $2/a$ *as* ζ *goes to infinity.*

2. *When* $0 < a < 1$*,*

   (4.15)
   $$\int |\xi|^{a-1}|\widehat{g\varphi}(y\xi + |\xi|^a)|^2 d\xi \ge \int \Phi(\zeta)|\widehat{g\varphi}(\zeta)|^2 d\zeta + C(a)|y|^{\frac{a}{2(a-1)}} \int_{\Xi_1}^0 |\zeta|^{-\frac{1}{2}}|\widehat{g\varphi}(\zeta - \Xi_1)|^2 d\zeta$$

   where $\Phi(\zeta)$ *is bounded at 0 by* $max\{1/a, 1/(1-a)\}$ *and decays like* $y^{-a}|\zeta|^{a-1}$*.*

Proof. To complete the proof of Proposition IV.5, we combine the results of Lemma IV.12 with an inner product. Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $0 \leq \varphi \leq \chi_{[T,2T]}(t)$. Therefore,

$$\left(\int_T^{2T} |u(t,yt)|^2 dt\right)^{\frac{1}{2}} \geq \left(\int |S' u_0(t)\varphi(t)|^2 dt\right)^{\frac{1}{2}}$$
$$= \sup_{\|h\|_{L^2}=1} |\langle S' u_0(t)\varphi(t), h(t)\rangle|$$
$$= \sup_{\|h\|_{L^2}=1} \left|\int \widehat{u_0}(\xi)\widehat{h\varphi}(y\xi + |\xi|^a) d\xi\right|$$
$$\geq \int |\xi|^{a-1} |\widehat{g\varphi}(y\xi + |\xi|^a)|^2 d\xi$$
$$\geq \left(\int |\xi|^{1-a} |\widehat{u_0}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(C(a) y^{\frac{a}{2(a-1)}} \int_0^{\Xi_1} |\zeta|^{-1/2} |\widehat{g\varphi}(\zeta - \Xi_1)|^2 d\zeta\right)^{\frac{1}{2}}$$
$$= C'(a) y^{a/4(a-1)} \|u_0\|_{\dot{H}^{(1-a)/2}} \left(\int_0^{\Xi_1} |\zeta|^{-1/2} |\widehat{g\varphi}(\zeta - \Xi_1)|^2 d\zeta\right)^{\frac{1}{2}}.$$

The constant in $g\varphi$ is controlled (loosely) by $\|g\|_{L^{4/3}}$. □

Lemma IV.12. We will treat the cases $a > 1$ and $0 < a < 1$ separately.

Case IV.12.1 ($a > 1$)**.** Notice that $|\xi|^{a-1} = \frac{\text{sgn}\xi}{a}\left(\frac{d}{d\xi}(y\xi + |\xi|^a) - y\right)$, so we can change variables:

$$\int |\xi|^{a-1} |\widehat{g\varphi}(y\xi + |\xi|)^a|^2 d\xi = \int \frac{\text{sgn}\xi}{a}\left(\frac{d}{d\xi}(y\xi + |\xi|^a) - y\right) |\widehat{g\varphi}(y\xi + |\xi|)^a|^2 d\xi$$
$$= \int_0^\infty \frac{\psi'(\xi)}{a} |\widehat{g\varphi}(\psi(\xi))|^2 d\xi - \int_{-\infty}^0 \frac{\psi'(\xi)}{a} |\widehat{g\varphi}(\psi(\xi))|^2 d\xi$$
$$- \frac{y}{a}\int \text{sgn}\xi |\widehat{g\varphi}(y\xi + |\xi|^a)|^2 d\xi$$
$$= \frac{2}{a}\int_0^\infty |\widehat{g\varphi}(\zeta)|^2 d\zeta - \frac{y}{a}\int \text{sgn}\xi |\widehat{g\varphi}(y\xi + |\xi|^a)|^2 d\xi.$$

On the intervals $(0,\infty)$ and $(-\infty, -y^{\frac{1}{a-1}})$, $y + a|\xi|^{a-1}\text{sgn}\xi \neq 0$, so we can change variables again as long as we keep track of the derivative factor:

$$-\frac{y}{a}\int_0^\infty |\widehat{g\varphi}(y\xi + |\xi|^a)|^2 \frac{y + a|\xi|^{a-1}\text{sgn}\xi}{y + a|\xi|^{a-1}\text{sgn}\xi} d\xi = -\int_0^\infty \frac{y}{a}\Phi_+(\zeta)|\widehat{g\varphi}(\zeta)|^2 d\zeta.$$

Observe that $\frac{y}{a}\Phi_+(\zeta)$ is bounded at 0 by $1/a$ and decays like $|\zeta|^{-(a-1)/a}$ at ∞ since $\zeta\ |\xi|^a$ for large ξ. Similarly, on $(-\infty, -y^{\frac{1}{a-1}})$, we have:

$$\frac{y}{a}\int_{-\infty}^{-y^{\frac{1}{a-1}}} |\widehat{g\varphi}(y\xi+|\xi|^a)|^2 \frac{y+a|\xi|^{a-1}\mathrm{sgn}\xi}{y+a|\xi|^{a-1}\mathrm{sgn}\xi} d\xi = \int_0^\infty \frac{y}{a}\Phi_-(\zeta)|\widehat{g\varphi}(\zeta)|^2 d\zeta$$

where $\frac{y}{a}\Phi_-(\zeta)$ is bounded as $\zeta \to 0^+$ by $1/(a(a-1))$ and decays like $|\zeta|^{-(a-1)/a}$ at positive infinity. Combining all the inequalities so far we find:

$$\int |\xi|^{a-1}|\widehat{g\varphi}(y\xi+|\xi|)^a|^2 d\xi = \int_0^\infty \left(\frac{2}{a} - \frac{y}{a}\Phi_+(\zeta) + \frac{y}{a}\Phi_-(\zeta)\right)|\widehat{g\varphi}(\zeta)|^2 d\zeta + \frac{y}{a}\int_{-y^{\frac{1}{a-1}}}^0 |\widehat{g\varphi}(y\xi+|\xi|^a)|^2 d\xi. \tag{4.16}$$

The first two terms on the right hand side are precisely the lower bounds given in the Proposition where $\Psi = \frac{2}{a} - \frac{y}{a}\Phi_+(\zeta) + \frac{y}{a}\Phi_-(\zeta)$. The growth factor in y arises from the remaining term.

In the interval $(-y^{\frac{1}{a-1}}, 0)$, the function $y\xi + |\xi|^a$ is nearly parabolic, so the natural change of variables is $y\xi + |\xi|^a + (a-1)|\xi_1|^a = (\eta - \xi_1)^2$. In particular, we will take $\eta = \eta(\xi) = \xi_1 + \mathrm{sgn}(\xi - \xi_1)\sqrt{y\xi + |\xi|^a + \Xi_1}$ so that $\dfrac{y - a|\xi|^{a-1}}{2(\eta - \xi_1)} \geq 0$. Let $J(\xi) = \dfrac{2(\eta(\xi) - \xi_1)}{y - a|\xi|^{a-1}}$. Then if we let introduce $\eta(\xi)$, we have:

$$\frac{y}{a}\int_{-y^{\frac{1}{a-1}}}^0 |\widehat{g\varphi}(y\xi+|\xi|^a)|^2 d\xi = \frac{y}{a}\int_{-y^{\frac{1}{a-1}}}^0 J(\xi)|\widehat{g\varphi}((\eta(\xi)-\xi_1)^2 - \Xi_1)|^2 \frac{d\xi}{J(\xi)}.$$

We will show that $J(\xi)$ is bounded below precisely by $C(a)y^{-1}|\xi_1|^{a/2}$ in Proposition

IV.13. Now, we use this bound and then change variables:

$$
\begin{aligned}
\frac{y}{a}\int_{-y^{\frac{1}{a-1}}}^{0} |\widehat{g\varphi}(y\xi+|\xi|^a)|^2 d\xi \geq & C(a) y^{\frac{a}{2(a-1)}} \int_{-y^{\frac{1}{a-1}}}^{0} |\widehat{g\varphi}((\eta(\xi)-\xi_1)^2-\Xi_1)|^2 \frac{d\xi}{J(\xi)} \\
= & C(a) y^{\frac{a}{2(a-1)}} \int_{\xi_1-\sqrt{\cdots}}^{\xi_1+\sqrt{\cdots}} |\widehat{g\varphi}((\eta-\xi_1)^2-\Xi_1)|^2 d\eta \\
= & C(a) y^{\frac{a}{2(a-1)}} \int_{-\sqrt{\Xi_1}}^{+\sqrt{\Xi_1}} |\widehat{g\varphi}(\eta^2-\Xi_1)|^2 d\eta \\
= & 2C(a) y^{\frac{a}{2(a-1)}} \int_{0}^{+\sqrt{\Xi_1}} \frac{1}{2\eta} |\widehat{g\varphi}(\eta^2+\Xi_1)|^2 2\eta d\eta \\
= & C(a) y^{\frac{a}{2(a-1)}} \int_{0}^{\Xi_1} \zeta^{-1/2} |\widehat{g\varphi}(\zeta+\Xi_1)|^2 d\zeta
\end{aligned}
$$

where the final step is the substitution $\zeta = \eta^2$. Therefore, we have our lower bound in the case $a > 1$.

Case IV.12.2 $(1 > a > 0)$**.** The argument in this case follows the previous case, but we omit the initial change of variable.

$$
\begin{aligned}
\int |\xi|^{a-1}|\widehat{g\varphi}(y\xi+|\xi|^a)|^2 d\xi = & \int_0^\infty |\xi|^{a-1}|\widehat{g\varphi}(y\xi+|\xi|^a)|^2 d\xi \\
& + \int_{-\infty}^{-y^{\frac{1}{a-1}}} |\xi|^{a-1}|\widehat{g\varphi}(y\xi+|\xi|^a)|^2 d\xi \\
& + \int_{-y^{\frac{1}{a-1}}}^{0} |\xi|^{a-1}|\widehat{g\varphi}(y\xi+|\xi|^a)|^2 d\xi \\
= & \int_0^\infty \frac{(y+a|\xi|^{a-1})}{y|\xi|^{1-a}+a} |\widehat{g\varphi}(y\xi+|\xi|^a)|^2 d\xi \\
& + \int_{-\infty}^{-y^{\frac{1}{a-1}}} \frac{(y-a|\xi|^{a-1})}{y|\xi|^{1-a}-a} |\widehat{g\varphi}(y\xi+|\xi|^a)|^2 d\xi \\
& + \int_{-y^{\frac{1}{a-1}}}^{0} |\xi|^{a-1}|\widehat{g\varphi}(y\xi+|\xi|^a)|^2 d\xi \\
= & \int \Phi(\zeta)|\widehat{g\varphi}(\zeta)|^2 d\eta + \int_{-y^{\frac{1}{a-1}}}^{0} |\xi|^{a-1}|\widehat{g\varphi}(y\xi+|\xi|^a)|^2 d\xi
\end{aligned}
$$

where $\Phi(\zeta)$ tends to $1/a$ as $\zeta \to 0^+$, $1/(1-a)$ for $\zeta \to 0^-$ and decays like $y^{-a}|\zeta|^{a-1}$ as $|\zeta| \to \infty$. In fact, the first term is bounded below by the $H^{(a-1)/2}$ norm (notice this is the inhomogeneous Sobolev space) and above by $\max\{1/a, 1/(1-a)\}$ times the the L^2 norm of g.

In the interval $(-y^{\frac{1}{a-1}}, 0)$, the function $y\xi + |\xi|^a$ is nearly parabolic, so the natural change of variables is $y\xi + |\xi|^a + \Xi_1 = -(\eta - \xi_1)^2$. In particular, we will take $\eta = \eta(\xi) = \xi_1 - \operatorname{sgn}(\xi - \xi_1)\sqrt{|\Xi_1| - y\xi + |\xi|^a}$ so that $\dfrac{y - a|\xi|^{a-1}}{2(\eta - \xi_1)} \geq 0$. Let $\mathcal{J}(\xi) = \dfrac{2(\eta(\xi) - \xi_1)}{y|\xi|^{1-a} - a}$. Then we have:

$$\int_{-y^{\frac{1}{a-1}}}^{0} |\xi|^{a-1} |\widehat{g\varphi}(y\xi + |\xi|^a)|^2 d\xi = \int_{-y^{\frac{1}{a-1}}}^{0} \mathcal{J}(\xi) |\widehat{g\varphi}(-(\eta(\xi) - \xi_1)^2 + |\Xi_1|)|^2 \frac{d\xi}{\mathcal{J}(\xi)}.$$

We will show that $\mathcal{J} \geq C(a) y^{\frac{a}{2(a-1)}}$ in proposition IV.13. If we change variables in the remaining term:

$$\begin{aligned}
&\frac{y}{a}\int_{-y^{\frac{1}{a-1}}}^{0} |\widehat{g\varphi}(y\xi + |\xi|^a)|^2 d\xi \\
&\quad \geq C(a) y^{\frac{a}{2(a-1)}} \int_{-y^{\frac{1}{a-1}}}^{0} |\widehat{g\varphi}(-(\eta(\xi) - \xi_1)^2 + |\Xi_1|)|^2 \frac{(y - a|\xi|^{a-1})d\xi}{2(\phi(\xi) - \xi_1)} \\
&\quad = C(a) y^{\frac{a}{2(a-1)}} \int_{\xi_1 - \sqrt{\cdots}}^{\xi_1 + \sqrt{\cdots}} |\widehat{g\varphi}(-(\eta - \xi_1)^2 + |\Xi_1|)|^2 d\eta \\
&\quad = C(a) y^{\frac{a}{2(a-1)}} \int_{-\sqrt{|\Xi_1|}}^{+\sqrt{|\Xi_1|}} |\widehat{g\varphi}(-\eta^2 + |\Xi_1|^a)|^2 d\eta \\
&\quad = 2C(a) y^{\frac{a}{2(a-1)}} \int_{0}^{\sqrt{|\Xi_1|}} \frac{1}{-2\eta} |\widehat{g\varphi}(-\eta^2 + |\Xi_1|)|^2 (-2\eta) d\eta \\
&\quad = 2C(a) y^{\frac{a}{2(a-1)}} \int_{\Xi_1}^{0} \frac{1}{2|\zeta|^{1/2}} |\widehat{g\varphi}(\zeta - \Xi_1)|^2 d\zeta
\end{aligned}$$

where the final step is the substiution $\zeta = -\eta^2$. Therefore, for the case $0 < a < 1$, we have proved our lower bounds.

□

4.5.2 Precise bounds for $a = 2$ and $a = 1/2$

It is worth noting that in the case $a = 2$ and $a = 1/2$, we can get precisely equality, rather than inequality. The Jacobian bounds that we proved in the previous section are not necessary in these two cases, as $J(\xi)$ and $\mathcal{J}(\xi)$ are constant in the case $a=2$ an $a=1/2$, respectively. In the case $a=2$ and $a=1/2$, respectively, we have:

$$\int |\xi||\widehat{g\varphi}(y\xi+\xi^2)|^2 d\xi = \int_0^\infty |\widehat{g\varphi}(\zeta)|^2 d\zeta + \frac{y}{2}\int_0^{y^2/4} |\zeta|^{-1/2}|\widehat{g\varphi}(\zeta - y^2/4)|^2 d\zeta$$

and

$$\int |\xi|^{-1/2}|\widehat{g\varphi}(y\xi+|\xi|^{1/2})|^2 d\xi = \int \frac{|\widehat{g\varphi}(\zeta)|^2 d\zeta}{(1/4+y|\zeta|)^{\frac{1}{2}}} + y^{-1/2}\int_{-(4y)^{-1}}^0 |\widehat{g\varphi}(\zeta+(4y)^{-1})|^2 \frac{d\zeta}{|\zeta|^{\frac{1}{2}}}.$$

In order to understand where precisely this undesirable decay in Theorem IV.3 is coming from in the case of the linearized water wave, we bound $u(t,zt)$ below by

$$\frac{1}{T^{\frac{1}{2}}}\left(\int_T^{2T} |u(t,zt)|^2 dt\right)^{\frac{1}{2}} \leq \sup_{[T,2T]} |u(t,zt)|.$$

By understanding the left hand side of this inequality, we can see the precise nature of the growth factor in Proposition IV.4. From the explicit bound on the analogue to operator $\mathcal{T}$, we get

$$\int_{-\infty}^{\infty} |\xi|^{-\frac{1}{2}}|\widehat{g\varphi}(z\xi+|\xi|^{\frac{1}{2}})|^2 d\xi = \int_{-\infty}^{\infty} \frac{|\widehat{g\varphi}(\zeta)|^2}{\left(\frac{1}{4}+|z|\zeta\right)^{\frac{1}{2}}} d\zeta + 2\int_0^{\frac{1}{4|z|}} \frac{|\widehat{g\varphi}(\zeta)|^2}{\left|\frac{1}{4}-|z|\zeta\right|^{\frac{1}{2}}} d\zeta \tag{4.17}$$

The singularity appears only in the second term at $1/4|z|$, suggesting that as $|z|$ (or $|y|$) gets small, the problem with the decay exists only at high frequencies.

We finish the chapter with the technical but not very deep results necessary to complete the proof of Theorem IV.5.

Precise Jacobian bounds

Recall Proposition IV.12 relied on the lower bounds of certain Jacobian bounds. The Lemma below collects these bounds.

Lemma IV.13. *1. If* $J(\xi) = \dfrac{2sgn(\xi-\xi_1)\sqrt{y\xi+|\xi|^a+(a-1)|\xi_1|^a)}}{y-a|\xi|^{a-1}}$ *with* $a>1$, *then*

$$J(\xi) \geq \begin{cases} J(0) = \dfrac{2\sqrt{(a-1)|\xi_1|^a}}{y} = 2(a-1)^{\frac{1}{2}} a^{\frac{-a}{2(a-1)}} y^{\frac{a}{2(a-1)}-1} & a > 2 \\ \\ J(-y^{\frac{1}{a-1}}) = \dfrac{2\sqrt{(a-1)|\xi_1|^a}}{(a-1)y} = \frac{2}{(a-1)^{\frac{1}{2}}} a^{\frac{-a}{2(a-1)}} y^{\frac{a}{2(a-1)}-1} & 2 > a > 1 \end{cases}$$

2. Let $\mathcal{J}(\xi) = \dfrac{-2sgn(\xi-\xi_1)\sqrt{(1-a)|\xi_1|^a - y\xi + |\xi|^a}}{y|\xi|^{1-a} - a}$ *with* $0 < a < 1$*; then,*

$$\mathcal{J}(\xi) \geq \begin{cases} \mathcal{J}(0) = \dfrac{-2\sqrt{(1-a)|\xi_1|^a}}{-a} = 2(1-a)^{\frac{1}{2}} a^{\frac{a-2}{2(a-1)}} y^{\frac{a}{2(a-1)}} & 1 > a > 1/2 \\ \\ \mathcal{J}(-y^{\frac{1}{a-1}}) = \dfrac{2\sqrt{(1-a)|\xi_1|^a}}{1-a} = \frac{2}{(1-a)^{\frac{1}{2}}} a^{\frac{-a}{2(a-1)}} y^{\frac{a}{2(a-1)}} & 1/2 > a > 0 \end{cases}$$

Proof. First, observe that $J(\xi)$ is continuous. The only possible point of discontinuity is at ξ_1, but by l'Hopital's rule,

$$\begin{aligned} \lim_{\xi\to\xi_1} J(\xi) &= \lim_{\xi\to\xi_1} \frac{2\text{sgn}(\xi-\xi_1)\sqrt{y\xi + |\xi|^a + (a-1)|\xi_1|^a}}{y - a|\xi|^{a-1}} \\ &= \lim_{\xi\to\xi_1} \frac{\text{sgn}(\xi-\xi_1)\left(y\xi + |\xi|^a + (a-1)|\xi_1|^a\right)^{-1/2}(y - a|\xi|^{a-1})}{a(a-1)|\xi|^{a-2}} \\ &= \frac{2}{a(a-1)|\xi_1|^{a-2}} \lim_{\xi\to\xi_1} \frac{1}{J(\xi)} \\ \Rightarrow \quad & \lim_{\xi\to\xi_1} J(\xi) = \sqrt{2}(a(a-1)|\xi_1|^{a-2})^{-1/2}. \end{aligned}$$

Since $J(\xi)$ is continuous, the natural way to find a lower bound is to consider the derivative of $J(\xi)$ and check for critical points. We will show that there are no critical points of $J(\xi)$ in the chosen interval, and therefore the lower bound is at one of the endpoints (which endpoint depends on the value of a). When $a = 2$, all these machinations are unnecessary as $J(\xi) = C$. From this point forward, we will assume that $a \neq 2$. First, observe that the derivative of $J(\xi)$ is

$$J'(\xi) = \frac{\text{sgn}(\xi-\xi_1)\left[-2a(a-1)|\xi|^{a-2}(y\xi + |\xi|^a + (a-1)|\xi_1|^a) + (y - a|\xi|^{a-1})^2\right]}{(y - a|\xi|^{a-1})^2(y\xi + |\xi|^a + (a-1)|\xi_1|^a)^{1/2}}.$$

Clearly, the numerator is 0 at ξ_1. Since ξ_1 is also a zero of the denominator and it is easy to check using Taylor expansions that $J'(\xi_1) \neq 0$ and is, in fact, positive and finite (implying that $J'(\xi)$ is continuous), it suffices to check if the numerator $N(\xi)$ has any additional zeroes. Since

$$N(\xi) = \operatorname{sgn}(\xi - \xi_1)\left[-2a(a-1)|\xi|^{a-2}(y\xi + |\xi|^a + (a-1)|\xi_1|^a) + (y - a|\xi|^{a-1})^2\right]$$

has a zero at ξ_1, the only way for N to have additional zeroes is if $N'(\xi)$ is zero at a point besides ξ_1. Now,

$$N'(\xi) = \operatorname{sgn}(\xi - \xi_1)2a(a-1)(a-2)|\xi|^{a-3}(y\xi + |\xi|^a + (a-1)|\xi_1|^a).$$

By construction, $y\xi + |\xi|^a + (a-1)|\xi_1|^a \geq 0$ and equal to zero only at ξ_1, so the only additional possible zero is 0 and then only when $a > 3$. Therefore, $N(\xi)$ has no additional zeroes in the open interval $(-y^{\frac{1}{a-1}}, 0)$, and $J(\xi)$ is monotone increasing when $a > 2$ and monotone decreasing when $1 < a < 2$ (since $N(\xi) \geq 0$ for $a > 2$ and $N(\xi) \leq 0$ for $1 < a < 2$). Therefore, we have

$$J(\xi) \geq \begin{cases} J(0) = \dfrac{2\sqrt{(a-1)|\xi_1|^a}}{y} = 2(a-1)^{\frac{1}{2}} a^{\frac{-a}{2(a-1)}} y^{\frac{a}{2(a-1)}} y^{-1} & a > 2 \\ \\ J(-y^{\frac{1}{a-1}}) = \dfrac{2\sqrt{(a-1)|\xi_1|^a}}{(a-1)y} = 2(a-1)^{-\frac{1}{2}} a^{\frac{-a}{2(a-1)}} y^{\frac{a}{2(a-1)}} y^{-1} & 2 > a > 1 \end{cases}$$

which completes the proof of part 1.

Now, observe that $\mathcal{J}(\xi)$ is also continuous. The only possible point of discontinuity is at ξ_1, but by l'Hopital's rule,

$$\begin{aligned}
\lim_{\xi\to\xi_1}\mathcal{J}(\xi) &= \lim_{\xi\to\xi_1}\frac{-2\mathrm{sgn}(\xi-\xi_1)\sqrt{(1-a)|\xi_1|^a-y\xi-|\xi|^a}}{y|\xi|^{1-a}-a}\\
&= \lim_{\xi\to\xi_1}\frac{-\mathrm{sgn}(\xi-\xi_1)\left((1-a)|\xi_1|^a-y\xi-|\xi|^a\right)^{-1/2}\left(-y+a|\xi|^{a-1}\right)}{-y(1-a)|\xi|^{-a}}\\
&= \frac{1}{y(1-a)}\lim_{\xi\to\xi_1}\frac{|\xi|^a(y-a|\xi|^{a-1})}{-\mathrm{sgn}(\xi-\xi_1)\left((1-a)|\xi_1|^a-y\xi-|\xi|^a\right)^{1/2}}\\
&= \frac{2|\xi_1|^{2a-1}}{a(1-a)|\xi_1|^{a-1}}\lim_{\xi\to\xi_1}\frac{1}{J(\xi)}\\
\Rightarrow\qquad & \lim_{\xi\to\xi_1}J(\xi)=\sqrt{2|\xi_1|^a}(a(a-1))^{-1/2}.
\end{aligned}$$

Since $\mathcal{J}(\xi)$ is continuous, the natural way to find a lower bound is to consider the derivative of $\mathcal{J}(\xi)$, and (if it is continuous as well) check for critical points. We will show that $\mathcal{J}'(\xi)$ is strictly postive when $0 < a < 1/2$ and strictly negative when $1/2 < a < 1$. When $a = 1/2$, $\mathcal{J}(\xi) = C$, so we will assume $a \neq 1/2$. First, observe that the derivative of $\mathcal{J}(\xi)$ is

$$\mathcal{J}'(\xi)=\frac{\mathrm{sgn}(\xi-\xi_1)\left[-2(1-a)y|\xi|^{1-a}((1-a)|\xi_1|^a-y\xi-|\xi|^a)+|\xi|^a|(y|\xi|^{1-a}-a)^2\right]}{|\xi|(y-a|\xi|^{a-1})^2(y\xi+|\xi|^a+(a-1)|\xi_1|^a)^{1/2}}.$$

Clearly, the numerator is 0 at ξ_1. Since ξ_1 is also a zero of the denominator and it is easy to check using Taylor expansions that $J'(\xi_1) \neq 0$ and is precisely $C(a)(1-2a)|\xi_1|^{5a/2-3}$, where $C(a) > 0$. The numerator also has a zero at 0, but the $|\xi|$ will force $\mathcal{J}'(\xi)$ to go to positive or negative infinity as $\xi \nearrow 0$. In order to find critical points that could be extrema of $\mathcal{J}(\xi)$, it suffices to check if the numerator $N(\xi)$ has any additional zeroes. As in the case $a > 1$, we will analyze the numerator with its derivative to check for zeroes. The numerator is slightly more complicated in this case, and it must have a critical point between ξ_1 and 0 by Rolle's theorem. However, the derivative has other properties which will allow us to draw the necessary conclusions.

Claim IV.14. *The only zeroes of $N(\xi)$ are at $\xi = \xi_1$, 0. Moreover, for $0 < a < 1/2$, $N(\xi) \leq 0$ for $\xi \in (-y^{1/(a-1)}, \xi_1)$ and $N(\xi) \geq 0$ for $\xi \in (\xi_1, 0)$. For $1/2 < a < 1$, $N(\xi) \geq 0$ for $\xi \in (-y^{1/(a-1)}, \xi_1)$ and $N(\xi) \leq 0$ for $\xi \in (\xi_1, 0)$*

Proof. Rather than draw conclusions about arbitrary a, we will discuss $0 < a < 1/2$, but exactly the same arguments will yield similar conclusions for $1/2 < a < 1$, just with opposite signs. Since $N(\xi) = -2(1-a)y|\xi|^{1-a}((1-a)|\xi_1|^a - y\xi - |\xi|^a) + |\xi|^a(y|\xi|^{1-a} - a)^2$, we can show that

$$|\xi|N'(\xi) + (1-a)N(\xi) = (1-2a)|\xi|^a(y|\xi|^{1-a} - a)^2. \tag{4.18}$$

The right hand side is always the same sign except at its zeroes, ξ_1 and 0. From this equation, we can conclude that $N'(\xi_1) = N''(\xi_1) = 0$, but $N'''(\xi_1) = 2a^2(1-a)^2(1-2a)|\xi_1|^{a-3}$, so near ξ_1, the function $N(\xi)$ is a positive cubic. The equation (4.18) also implies that at any point $x \in (-y^{-1/(a-1)}, 0)$ not equal to ξ_1 or 0 such that $N(x) = 0$ must satisfy $N'(x) > 0$. This fact means that in the subinterval $(\xi_1, 0)$, there can be no additional zeroes of $N(\xi)$. Since $N(-y^{\frac{1}{a-1}}) < 0$ and $N(\xi)$ approaches zero from below as $\xi \nearrow \xi_1$, there can only be an even number of zeroes in the subinterval $(-y^{\frac{1}{a-1}}, \xi_1)$. At one of the zeroes, $N(\xi)$ must be decreasing, but that would contradict equation (4.18). Therefore there are no zeroes of $N(\xi)$ in the subinterval $(-y^{\frac{1}{a-1}}, 0)$. Combining these two subintervals, we conclude that or $0 < a < 1/2$, $N(\xi) \leq 0$ for $\xi \in (-y^{1/(a-1)}, \xi_1)$ and $N(\xi) \geq 0$ for $\xi \in (\xi_1, 0)$. Thus the claim is proved. □

If we return to $\mathcal{J}'(\xi)$ and apply this claim, we find that when $0 < a < 1/2$, $\mathcal{J}'(\xi) > 0$ for $\xi \in (-y^{\frac{1}{a-1}}, 0)$ and when $1/2 < a < 1$, $\mathcal{J}'(\xi) < 0$ for $\xi \in (-y^{\frac{1}{a-1}}, 0)$.

Thus,

$$\mathcal{J}(\xi) \geq \begin{cases} \mathcal{J}(0) = \dfrac{-2\sqrt{(1-a)|\xi_1|^a}}{-a} = 2(1-a)^{\frac{1}{2}} a^{\frac{a-2}{2(a-1)}} y^{\frac{a}{2(a-1)}} & 1 > a > 1/2 \\ \\ \mathcal{J}(-y^{\frac{1}{a-1}}) = \dfrac{2\sqrt{(1-a)|\xi_1|^a}}{1-a} = 2(1-a)^{-\frac{1}{2}} a^{\frac{-a}{2(a-1)}} y^{\frac{a}{2(a-1)}} & 1/2 > a > 0 \end{cases}$$

□

CHAPTER V

Further Study of the Linearized Water Wave Problem

5.1 The Linearized Water Wave Problem

The results from Chapter IV suggest that the problematic regions for Theorem IV.3 are for initial data away from the origin in frequency. On the other hand, previously mentioned existing results such as [20, Proposition 3.1]) are, in some sense, only problematic for initial data with a contribution from low frequencies. We combine these results and show an improved decay rate for solutions of the linearized water wave problem by imposing further bounds on the initial data.

Recall that we have reduced the problem to the interface, and the linearized form of the initital value problems is

$$(5.1)\qquad \begin{cases} \partial_t^2 u + |D|u = 0 \\ u(0,x) = u_0(x) \\ u_t(0,x) = u_1(x). \end{cases}$$

First, consider a decay bound for the linearized water wave problem inspired by the work of Klainerman for the wave equation.

5.1.1 Analogue to [20, Proposition 3.1]

Where previous work was concerned with the removal of the troublesome quantity Ω_0, it is advantageous to reintroduce it here. Let $\Gamma = \{\partial_t, \partial_x, L = \frac{t}{2}\partial_t + x\partial_x\}$ and

$\Omega = x\partial_t + \frac{t}{2}\partial_x|D|^{-1}$. The following proposition is analogous to [20, Proposition 3.1] and is proved using techniques similar to those used by Klainerman for the wave equation. While Γ and Ω here are specific to the case of the linearized water wave problem, a similar collection of vector fields exists for (4.1) and the proposition can be generalized to this class of equations. We focus on the linearized water wave problem for the time being.

Lemma V.1. *Let $u(t,x)$ be any real-valued function which decays at infinity. Then, for a multiindex $k = \{k_1, k_2, k_3\}$ and $\Gamma^k = \partial_t^{k_1}\partial_x^{k_2}L^{k_3}$, we have:*

$$|u(t,x)| \leq \frac{C}{t^{\frac{1}{2}}}\left(\sum_{1\leq|k|\leq 2}\|\Gamma^k u(t)\|_{L^2(\mathbb{R}_x)} + \sum_{|k|\leq 1}\|\Gamma^k\Omega u(t)\|_{L^2(\mathbb{R}_x)}\right). \tag{5.2}$$

Remark V.2. The details of this proof are due to unpublished work of Sijue Wu. We duplicate the proof here for completeness.

Proof. Observe that

$$xL - \frac{t}{2}\Omega = x^2\partial_x - \frac{t^2}{4}\partial_x|D|^{-1}.$$

In some sense, $\partial_x|D|^{-1}$ is the Hilbert transform, so we will treat $\mathfrak{v} = u + iv$, where $v = \partial_x|D|^{-1}u$, or equivalently $u = -\partial_x|D|^{-1}v$. Then, we have

$$x^2\partial_x\mathfrak{v} + \frac{it^2}{4}\mathfrak{v} = xL\mathfrak{v} - \frac{t}{2}\Omega\mathfrak{v}.$$

Since $\partial_x e^{\frac{-it^2}{4x}} = \frac{it^2}{4x^2}e^{\frac{-it^2}{4x}}$, we can rewrite the above as

$$x^2\partial_x\left(e^{\frac{-it^2}{4x}}\mathfrak{v}\right) = e^{\frac{-it^2}{4x}}\left(\frac{it^2}{4}\mathfrak{v} + x^2\partial_x\mathfrak{v}\right) = e^{\frac{-it^2}{4x}}\left(xL\mathfrak{v} - \frac{t}{2}\Omega\mathfrak{v}\right).$$

Finally, we conclude that

$$\left|\partial_x\left(e^{\frac{-it^2}{4x}}\mathfrak{v}\right)\right| \leq \frac{1}{|x|}\left|L\mathfrak{v}\right| + \frac{|t|}{2x^2}\left|\Omega\mathfrak{v}\right|$$

Using a variation on the fundamental theorem of calculus, we have

$$\begin{aligned}
|\mathfrak{v}(x)| &\le \int_{|x|}^{\infty} \left|\partial_z \left(e^{\frac{-it^2}{4z}}\mathfrak{v}\right)\right| dz \\
&\le \int_{|x|}^{\infty} \frac{1}{|z|}\left|L\mathfrak{v}\right| + \frac{|t|}{2z^2}\left|\Omega\mathfrak{v}\right| dz \\
&\le \left(\int_{|x|}^{\infty} \frac{1}{|z|^2} dz\right)^{\frac{1}{2}} \|L\mathfrak{v}\|_{L^2} + \left(\int_{|x|}^{\infty} \frac{t^2}{4|z|^4} dz\right)^{\frac{1}{2}} \|\Omega\mathfrak{v}\|_{L^2}
\end{aligned}$$

Thus we can conclude that

$$|(u+iv)(x)| \le \frac{1}{|x|^{\frac{1}{2}}}\|L(u+iv)\|_{L^2} + \frac{t}{|x|^{\frac{3}{2}}}\|\Omega(u+iv)\|_{L^2} \tag{5.3}$$

Case V.2.1 ($|x| \ge t$)**.** By (5.3), we have

$$|(u+iv)(x)| \le \frac{1}{t^{\frac{1}{2}}}\|L(u+iv)\|_{L^2} + \frac{1}{t^{\frac{1}{2}}}\|\Omega(u+iv)\|_{L^2},$$

which completes the proof of this case. Since $|v(t,x)| \le |(u+iv)(x)|$ and the commutator of L and Ω with the operator $\partial_x|D|^{-1}$ are $[L, \partial_x|D|^{-1}] = 2\partial_x|D|^{-1}$ and $[\Omega, \partial_x|D|^{-1}] = 0$, we can bound $L\partial_x|D|^{-1}u$ by Lu and u in L^2 and similarly for $\Omega\partial_x|D|^{-1}u$. We conclude

$$|u(t,x)| \le \frac{C}{t^{\frac{1}{2}}}\|Lu\|_{L^2(\mathbb{R}_x)} + \frac{C}{t^{\frac{1}{2}}}\|\Omega u\|_{L^2(\mathbb{R}_x)}.$$

Case V.2.2 ($|x| \le t$)**.** In order to show the bounds, we will need to use a different bound on u. First observe that

$$\frac{t^2}{4}u = xLv - \frac{t}{2}\Omega v - x^2\partial_x v.$$

Then, in absolute value we can control the first two terms using standard Sobolev norms and the last term using (5.3):

$$
\begin{aligned}
\frac{t^2}{4}|u(x)| &\leq x|Lv| + \frac{t}{2}|\Omega v| + x^2|\partial_x v| \\
&\leq x\left(\|Lv\|_{L^2} + \|\partial_x Lv\|_{L^2}\right) + \frac{t}{2}\left(\|\Omega v\|_{L^2} + \|\partial_x \Omega v\|_{L^2}\right) \\
&\quad + x^2\left(\frac{1}{|x|^{\frac{1}{2}}}\|L\partial_x(u+iv)\|_{L^2} + \frac{t}{|x|^{\frac{3}{2}}}\|\Omega\partial_x(u+iv)\|_{L^2}\right) \\
|u(t,x)| &\leq \frac{4x}{t^2}\left(\|Lv\|_{L^2} + \|\partial_x Lv\|_{L^2}\right) + \frac{2}{t}\left(\|\Omega v\|_{L^2} + \|\partial_x \Omega v\|_{L^2}\right) \\
&\quad + \frac{4x^2}{t^2}\left(\frac{1}{|x|^{\frac{1}{2}}}\|L\partial_x(u+iv)\|_{L^2} + \frac{t}{|x|^{\frac{3}{2}}}\|\Omega\partial_x(u+iv)\|_{L^2}\right) \\
&\leq \frac{4}{t}\left(\|Lv\|_{L^2} + \|\partial_x Lv\|_{L^2}\right) + \frac{2}{t}\left(\|\Omega v\|_{L^2} + \|\partial_x \Omega v\|_{L^2}\right) \\
&\quad + \left(\frac{4|x|^{\frac{3}{2}}}{t^2}\|L\partial_x(u+iv)\|_{L^2} + \frac{4|x|^{\frac{1}{2}}}{t}\|\Omega\partial_x(u+iv)\|_{L^2}\right)
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
|u(t,x)| &\leq \frac{C}{t}\left(\|Lv\|_{L^2} + \|\partial_x Lv\|_{L^2} + \|\Omega v\|_{L^2} + \|\partial_x \Omega v\|_{L^2}\right) \\
&\quad + \frac{C}{t^{\frac{1}{2}}}\left(\|L\partial_x(u+iv)\|_{L^2} + \|\Omega\partial_x(u+iv)\|_{L^2}\right)
\end{aligned} \tag{5.4}
$$

□

5.1.2 Energy bounds

In order to turn the Klainerman-type estimates into L^∞ bounds on the solution in terms of the initial data, we use the energy estimates for (5.1).

Lemma V.3. *Let $u(t,x)$ be a solution of (5.1) with $(u(0,x), u_t(0,x)) = (u_0(x), u_1(x))$ and $u_i \in \mathcal{S}(\mathbb{R})$ for $i = 0, 1$. In addition, let $\Gamma = \{\partial_t, \partial_x, L = \frac{t}{2}\partial_t + x\partial_x\}$ and $\Gamma_1 = \Gamma \cup \{\Omega = x\partial_t + \frac{t}{2}\partial_x|D|^{-1}\}$ and let k be a multiindex. Define the energy functional as*

$$E[v](t) = \int |\partial_t v(t,x)|^2 + ||D|^{\frac{1}{2}} v(t,x)|^2 dx.$$

Then, $E[u](t) = E[u](0)$ *and*

$$\|\Gamma_1^k u(t)\|_{L^2} \leq \|\partial_t |D|^{-\frac{1}{2}} \Gamma_1^k u(0)\|_{L^2} + \|\Gamma_1^k u(0)\|_{L^2}.$$

Remark V.4. This equality holds for a variety of classes of initial data. However, considering the data in Schwartz class allows us to use density arguments when the natural space for the data appears in our analysis.

Proof. Since ∂_t, ∂_x, and Ω are invariant under the operator $\partial_t^2 + |D|$ and $[\partial_t^2 + |D|, L] = \partial_t^2 + |D|$, we can bound $\|\Gamma^\alpha u\|_{L^2}$ using the energy:

$$\begin{aligned}\|\Gamma^\alpha u(t)\|_{L^2} &\leq E[|D|^{-\frac{1}{2}}\Gamma^\alpha u](t)^{\frac{1}{2}} \\ &= E[|D|^{-\frac{1}{2}}\Gamma^k u](0)^{\frac{1}{2}} \leq \left(\int |\partial_t |D|^{-\frac{1}{2}} \Gamma^\alpha u(0,x)|^2 + |\Gamma^\alpha u(0,x)|^2 dx\right)^{\frac{1}{2}}.\end{aligned}$$

A similar calculation holds for $\Omega\Gamma^\alpha$. □

Remark V.5. It is worth noting that the bound on $\Omega u(t,x)$ is not ideal:

$$\begin{aligned}\|\Omega u(t)\|_{L^2} &\leq \|\partial_t |D|^{-1/2} \Omega u(0)\|_{L^2} + \|\Omega u(0)\|_{L^2} \\ &\leq \||D|^{-\frac{1}{2}} x |D| u_0\|_{L^2} + \|\partial_x |D|^{-1} u_0\|_{L^2} + \|x u_1\|_{L^2}.\end{aligned}$$

We expect that $|D|^{-1/2} u_1$ has roughly the same regularity as u_0. However, the term involving $|D|^{-\frac{1}{2}} u_0$ requires regularity on the antiderivative of u_0. This issue is precisely what caused the dependence of the data in [20] on initial height and energy as well as initial slope. However, if $\widehat{u_0}$ was supported outside a ball centered at zero, we could control the bad term by the L^2 norm of the data.

5.1.3 L^∞ decay for the Linearized Water Wave problem

The combination of Lemma V.3 and Lemma V.1 yields the following L^∞ bound.

Proposition V.6. *Let $u(t,x)$ be a solution of* (5.1) *with*

$$(u(0,x), u_t(0,x)) = (u_0(x), u_1(x)) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}^n).$$

Then,

$$|u(t,x)| \leq \frac{C}{t^{\frac{1}{2}}} \left(\sum_{1\leq|k|\leq 2} (\|\partial_t |D|^{-\frac{1}{2}} \Gamma^k u(0)\|_{L^2} + \|\Gamma^k u(0)\|_{L^2}) \right) + \frac{1}{t^{\frac{1}{2}}} \left(\sum_{|k|\leq 1} (\|\partial_t |D|^{-\frac{1}{2}} \Gamma^k \Omega u(0)\|_{L^2} + \|\Gamma^k \Omega u(0)\|_{L^2}) \right). \tag{5.5}$$

Remark V.7. The inequality (5.5) has concise notation but it obscures the precise bounds on the right hand side. Using commutators, we can write each of these sums explicitly. The first two terms contain L^2 bounds of derivatives up to first order and homogeneous operators (such as $x\partial_x$) of the initial data. More interesting are the bounds on the second two terms:

$$\sum \|\Gamma^k \Omega u(0)\|_{L^2} \lesssim \|xu_1\|_{L^2} + \|(x\partial_x)(xu_1)\|_{L^2} + \|\partial_x |D|^{-1} u_0\|_{L^2} + \|x|D|u_0\|_{L^2} + \|x\partial_x u_1\|_{L^2} + \|u_1\|_{L^2} \tag{5.6}$$

$$\begin{aligned} \sum \|\partial_t |D|^{-\frac{1}{2}} \Gamma^k \Omega u(0)\|_{L^2} \lesssim{} & \|x|D||D|^{-\frac{1}{2}} u_0\|_{L^2} + \|\partial_x |D|^{-1} |D|^{-\frac{1}{2}} u_0\|_{L^2} \\ & + \|(x\partial_x)(\partial_x |D|^{-1})|D|^{-\frac{1}{2}} u_0\|_{L^2} + \|(x\partial_x)(x|D|)|D|^{-\frac{1}{2}} u_0\|_{L^2} \\ & + \|\partial_x |D|^{-1} u_1\|_{L^2} + \||D|^{\frac{1}{2}} u_0\|_{L^2} + \||D| u_0\|_{L^2} \\ & + \|\partial_x |D|^{-1} u_0\|_{L^2} + \|x\partial_x \partial_x |D|^{-1} u_0\|_{L^2} \\ & + \|x|D||D|^{-\frac{1}{2}} u_1\|_{L^2} + \|\partial_x |D|^{-1} |D|^{-\frac{1}{2}} u_1\|_{L^2}. \end{aligned} \tag{5.7}$$

These terms contain the troublesome terms involving $|D|^{-\frac{1}{2}} u_0$, as mentioned in Remark V.5.

These results give a decay of $t^{-\frac{1}{2}}$ for certain classes of data. The inequality in Proposition V.6 along with the observation about $\Omega u(t)$ in Remark V.5 suggests that for data bounded away from the origin in frequency, Proposition V.6 gives the desired $t^{-\frac{1}{2}}$ decay. On the other hand, Theorem IV.3 for $a = \frac{1}{2}$ gives the desired decay in the low frequency regime. Theorem IV.3 implies $t^{-\frac{1}{2}}$ decay whenever $|y| \geq 1$ or other constant, and thus only when $|y| < 1$ do we have an undesirable decay rate. Previous sharpness results for that theorem also suggest that singularity comes from a singularity in norm around $1/y$ in frequency. Combining these two observations suggests that Theorem IV.3 is the right choice for initial data concentrated in low frequency.

5.2 Analysis of Initial Data

The argument above suggests that we should examine data supported away from the origin in frequency and data supported near the origin in frequency independently. We begin with the first of these cases.

5.2.1 Data Supported away from a Ball of fixed radius in Frequency

If instead we consider data supported in $|\xi| > R$, we can conclude the following corollary to Proposition V.6 :

Corollary V.8. *For $w(t,x)$ a solution of (5.1) with $(\widehat{w_0}(\xi), \widehat{w_1}(\xi))$ each supported in $|\xi| \geq R$, we have*

$$\sup_y |w(t, yt^2)| \leq \frac{1 + R^{-1/2}}{t^{\frac{1}{2}}} \left(\sum_{|k| \leq 2} (\| |D|^{-\frac{1}{2}} \Gamma^k w_1 \|_{L^2} + \|\Gamma^k w_0\|_{L^2}) \right). \tag{5.8}$$

Proof. The inequality above follows directly from the pointwise Klainerman bound, Proposition V.6:

$$\sup|w(t,yt^2)| \leq \frac{1}{t^{\frac{1}{2}}}\left(\sum_{1\leq|k|\leq 2}(\||D|^{-\frac{1}{2}}\Gamma^k w_1\|_{L^2} + \|\Gamma^k w_0\|_{L^2}) + \sum_{|k|\leq 1}(\||D|^{-\frac{1}{2}}\Gamma^k \Omega w_1\|_{L^2} + \|\Gamma^k \Omega w_0\|_{L^2})\right).$$

Since $\widehat{w_0}$ is supported in $|\xi| > R$, we can bound $\|\Omega w_0\|$ by

$$\|\Omega w_0\|_{L^2} \leq \sum_{|k|=1} \|\Gamma^k w_0\|_{L^2} + R^{-1/2}\|w_0\|_{L^2}.$$

Then, the full bound on the first term is the equation in the statement of the proposition. □

5.2.2 Data Supported in a Ball centered at the origin in Frequency

On the other hand, we can use results from Chapter IV to control a solution with data concentrated at low frequency, as in the following proposition:

Proposition V.9. *Let $u(t,x)$ be a solution to the differential equation (5.1) with initial data $u_i(x) \in \mathcal{S}(\mathbb{R})$ such that supp $\widehat{u_i}(\xi) \subset B_R(0)$. Then,when $|y| \leq (8R^{1/2}T)^{-1}$*

$$\sup_{|y|\leq(8R^{1/2}T)^{-1}} |u(t,yt^2)| \leq \frac{1}{T^{\frac{1}{2}}}\sum_{|k|\leq 1}\left(\|L^k u_0\|_{\dot{H}^{1/4}} + \|L^k u_1\|_{\dot{H}^{-1/4}}\right).$$

This proposition follows from Lemma IV.2 and this proposition on the L^2 norm:

Proposition V.10. *Let $\widehat{v} \in C_0^\infty(\mathbb{R})$ with supp $\widehat{v} \subseteq B_R(0)$. Let $1/4 \leq \sigma \leq 1/2$. Then, for $y < Y = (8R^{1/2}T)^{-1}$,*

$$\|\mathcal{S}v\|_{L^2(T,2T)} \leq C\|v\|_{\dot{H}^\sigma}.$$

Proof. Let $\varphi(t) \in C_0^\infty(\mathbb{R})$ with $\varphi = 1$ for $t \in (1,2)$ and $\varphi = 0$ for $t \in (1/2, 5/2)^C$ and

$\varphi_T(t) = \varphi(t/T)$, and let $\chi(\xi)$ be identically 1 on $B_R(0)$ and 0 on $B_{2R}(0)^C$. Now,

$$\begin{aligned}\|\mathcal{S}v\|_{L^2(T,2T)} &= \sup_{g\in L^2, \|g\|=1} |\langle \mathcal{S}v(t)\varphi_T(t), g(t)\rangle| \\ &= \sup_{g\in L^2, \|g\|=1} |\langle \widehat{v}(\xi), \mathcal{T}(g\varphi_T)(\xi)\rangle| \\ &\leq \|v\|_{\dot{H}^\sigma}\left(\int (1+|\xi|^2)^{-\sigma}|\chi(\xi)\mathcal{T}(g\varphi_T)(\xi)|^2 d\xi\right)^{\frac{1}{2}}.\end{aligned}$$

As before, we rewrite the operator squared as a product and reorder the integral:

$$\int |\chi(\xi)\mathcal{T}(g\varphi_T)(\xi)|^2 d\xi = \int g\varphi_T(t)\overline{g\varphi_T}(s)\int e^{-i(y(t^2-s^2)\xi+(t-s)|\xi|^{1/2})}\frac{|\chi(\xi)|^2}{(1+|\xi|^2)^\sigma}d\xi$$

The stationary point of the oscillatory integral is at $-1/(4y^2(t+s)^2)$, so if $2R < 1/(4y^2(t+s)^2)$, we can simply integrate by parts. Then, $y < Y$ implies that $\xi_0 \in B_{2R}(0)^C$, and we can control the integral in ξ by integration by parts:

$$\int e^{-i(y(t^2-s^2)\xi+(t-s)|\xi|^{1/2})}|\chi(\xi)|^2|\xi|^{-\frac{1}{2}}d\xi = \int_{-2R}^{2R}\frac{-|\xi|^{-\frac{1}{2}}\partial_\xi\left(e^{-i(y(t^2-s^2)\xi+(t-s)|\xi|^{\frac{1}{2}})}\right)d\xi}{i(y(t^2-s^2)+(1/2)(t-s)|\xi|^{-\frac{1}{2}}\mathrm{sgn}\xi)}$$

$$\begin{aligned}&= \left.\frac{e^{-i(y(t^2-s^2)\xi+(t-s)|\xi|^{\frac{1}{2}})}|\chi(\xi)|^2}{-i(y(t^2-s^2)|\xi|^{1/2}+(1/2)(t-s)\mathrm{sgn}\xi)}\right|_{-\infty}^{\infty} \\ &\quad - \int e^{-i(y(t^2-s^2)\xi+(t-s)|\xi|^{\frac{1}{2}})}\partial_\xi\left(\frac{|\chi(\xi)|^2}{-i(y(t^2-s^2)|\xi|^{\frac{1}{2}}+(1/2)(t-s)\mathrm{sgn}\xi)}\right)d\xi \\ &= \frac{4i}{t-s} - \int e^{-i(y(t^2-s^2)\xi+(t-s)|\xi|^{\frac{1}{2}})}\partial_\xi\left(\frac{\chi(|\xi)|^2}{-i(y(t^2-s^2)|\xi|^{\frac{1}{2}}+(1/2)(t-s)\mathrm{sgn}\xi)}\right)d\xi\end{aligned}$$

The remaining term is also bounded by $|t-s|^{-1}$. In fact, following the same arguments as an earlier proof we can show the second term is a standard kernel with constant independent of y and we can use the $T1$ theorem to show it is the kernel of a bounded operator. In fact, all the necessary bounds on the kernels are independent of the size of the support of $\widehat{v}$. Therefore, we have:

$$\left(\int |\chi(\xi)\mathcal{T}(g\varphi_T)(\xi)|^2 d\xi\right)^{1/2} \leq C\|g\varphi_T\|_{L^2}$$

with C independent of y, T, and R. □

5.2.3 A First Attempt at Optimality

Since the results from the invariant vector field-type bounds and those from the Klainerman type bounds have difficulty controlling the solution in different areas, we combine the two results in order to improve the decay rate. We combine Theorem IV.3 in the case $a = 1/2$ with Proposition V.6 to get the following theorem:

Theorem V.11. *Let $u(t,x)$ be a solution of $\partial_t^2 u + |D|u = 0$ with $(u(0,x), u_t(0,x)) = (u_0, u_1) \in \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$. Then,*

$$\sup_{y\in\mathbb{R}} |u(t, yt^2)| \leq \frac{1}{t^{5/14}} \sum_{|k|\leq 2} \left(\|\Gamma^k u_0\|_{L^2} + \|\Gamma^k |D|^{-\frac{1}{2}} u_1\|_{L^2} \right) \tag{5.9}$$

Proof. Fix $t > 1$. Let $\chi(\xi)$ denote the indicator function for the ball of radius t^p centered at 0. Then, let $v(t,x)$ be the solution to

$$(\partial_t^2 + |D|)v = 0 \text{ with } (\widehat{v}(0,\xi), \partial_t \widehat{v}(0,\xi)) = (\widehat{u_0}\chi, \widehat{u_1}\chi)$$

and $w(t,x)$ be the solution to

$$(\partial_t^2 + |D|)w = 0 \text{ with } (\widehat{w}(0,\xi), \partial_t \widehat{w}(0,\xi)) = (\widehat{u_0}(1-\chi), \widehat{u_1}(1-\chi)).$$

Notice that since all of these differential equations are linear, $u(t,x) = w(t,x) + v(t,x)$. Therefore,

$$|u(t, yt^2)| \leq |w(t, yt^2)| + |v(t, yt^2)| \tag{5.10}$$

Since the initial data for the first term is bounded away from 0 in frequency, we will use the pointwise bound:

$$\begin{aligned} |w(t, yt^2)| \leq & \frac{1}{t^{\frac{1}{2}}} \left(\sum_{1\leq|k|\leq 2} (\||D|^{-\frac{1}{2}} \Gamma^k w_1\|_{L^2} + \|\Gamma^k w_0\|_{L^2}) \right) \\ & + \frac{1}{t^{\frac{1}{2}}} \left(\sum_{|k|\leq 1} (\||D|^{-\frac{1}{2}} \Gamma^k \Omega w_1\|_{L^2} + \|\Gamma^k \Omega w_0\|_{L^2}) \right) \end{aligned}$$

Since $\widehat{w_0}$ is supported in $|\xi| > t^p$, we can bound $\|\Omega w_0\|$ by

$$\|\Omega w_0\|_{L^2} \leq \sum_{|k|=1} \|\Gamma^k w_0\|_{L^2} + t^{-p/2}\|w_0\|_{L^2}.$$

Then, the full bound on the first term is

$$|w(t, yt^2)| \leq \frac{1 + t^{-p/2}}{t^{\frac{1}{2}}} \left(\sum_{|k|\leq 2} (\||D|^{-\frac{1}{2}}\Gamma^k w_1\|_{L^2} + \|\Gamma^k w_0\|_{L^2}) \right). \tag{5.11}$$

For the function $v(t, x)$, first observe that if $|y| > 1$, the decay is $t^{-\frac{1}{2}}$. The choice of 1 here is slightly arbitary; what will matter more is a lower bound on $|y|$ from the analysis of the critical point. Since $v(t, x)$ has initial compactly supported on the Fourier transform side, for sufficiently small values of $|y|$, we also have $t^{-\frac{1}{2}}$ decay. Precisely, if $|y| \leq (8T^{p/2+1})^{-1}$

$$|v(t, yt^2)| \leq \frac{1}{t^{\frac{1}{2}}} \sum_{|k|\leq 1} \left(\|L^k v_0\|_{H^{\frac{1}{4}}} + \|L^k v_1\|_{H^{\frac{-1}{4}}} \right).$$

Notice that because we have compact support in the Fourier transform, we can rewrite the right hand side here as

$$|v(t, yt^2)| \leq \frac{t^{p/4}}{t^{\frac{1}{2}}} \sum_{|k|\leq 1} \left(\|L^k v_0\|_{L^2} + \||D|^{-\frac{1}{2}} L^k v_1\|_{L^2} \right).$$

If $(8t^{p/2+1})^{-1} < |y| < 1$, we use Theorem IV.3 and have

$$|v(t, yt^2)| \leq \frac{(1 + t^{\frac{p+2}{8}})t^{p/4}}{t^{\frac{1}{2}}} \sum_{|k|\leq 1} \left(\|L^k v_0\|_{L^2} + \||D|^{-\frac{1}{2}} L^k v_1\|_{L^2} \right). \tag{5.12}$$

Different values of p will cause different terms to dominate. When $p > 0$, the contribution from (5.11) will be $t^{-\frac{1}{2}}$, but (5.12) decays like $t^{\frac{3p}{8}} t^{\frac{1}{4}} t^{-\frac{1}{2}}$. These cannot be equal for any positive value of p. On the other hand, if $-2 < p < 0$, we have $t^{-\frac{p+1}{2}}$ from (5.11) and $t^{\frac{3p}{8}} t^{\frac{1}{4}} t^{-\frac{1}{2}}$ from (5.12), which are equal for $p = -2/7$.

Notice that choosing $p = -2/7$ improves the decay in the case $|y| > 1$ to $T^{-4/7}$ times L^2 norms. By taking $p = -2/7$, we can conclude that

$$\sup_{y\in\mathbb{R}} |u(t, yt^2)| \leq \frac{1}{t^{5/14}} \sum_{|k|\leq 2} \left(\|\Gamma^k u_0\|_{L^2} + \|\Gamma^k |D|^{-\frac{1}{2}} u_1\|_{L^2} \right).$$

□

Remark V.12. Observe that in almost every term we can get the desired decay. In the case of $|\xi| > T^p$, whenever $p > 0$, we get better than $T^{-\frac{1}{2}}$ decay, but at the cost of severely worse decay in the $|\xi| < T^p$ part. We might as well decompose around $|\xi| \sim 1$, which gives the desired decay from the Klainerman type bounds with the smallest penalty on the remainder. In that remainder, only certain values of $|y|$ contribute to the growth, namely $(8T)^{-1} < |y| < 1$. It is worth noting that this range is barely larger than the region described by the optimal choice $p = -2/7$, where we have $(8T)^{-6/7} < |y| < 1$, but the miniscule reduction in the range of y introduces $T^{1/7}$ of growth on the Klainerman term. Clearly there is more to understand with data compactly supported in frequency.

5.3 Data Compactly Supported in Frequency, a Second Attempt

What truly matters in this regime is whether or not the initial data has a singularity at the origin and how rapidly that singularity grows as the frequency approaches 0.

Theorem V.13. *Let $u(t,x)$ be a solution of*

$$\begin{cases} \partial_t u - i|D|^{\frac{1}{2}} u &= 0 \\ u(0,x) &= u_0(x) \end{cases} \tag{5.13}$$

with $u_0(x) \in \dot{H}^{\frac{1}{4}}$ and supp $\widehat{u_0}(\xi) \subseteq (-1,1)$. In addition, let $\frac{C}{t} \leq |y| \leq 1$. Then

$$u(t,x) \in L^q(\mathbb{R}_x) \text{ where } \begin{cases} q \in \left(\frac{2}{1-2|\gamma|}, \infty\right) & \text{when } \gamma < 0 \\ q \in (2,\infty) & \text{when } \gamma > 0. \end{cases} \tag{5.14}$$

Remark V.14. Heuristically, we expect $u(t,x) \in L^q(\mathbb{R})$ to decay like $|x|^{-\frac{1}{q}}$. In the case of $-1/4 < \gamma < 0$, the reduced range of q gives $|u(t,x)|$ would decay no faster than $|x|^{-\frac{1}{2}+|\gamma|}$, which would prevent $u(t,x)$ from being in L^2. In fact, if we consider $|u(t,yt^2)|$ and $\gamma = -1/4$, we would get that $|u(t,yt^2)| \leq y^{-\frac{1}{4}}t^{\frac{1}{2}}$, precisely the growth factor from our previous results. These heuristics suggest that the size of the singularity at the origin in frequency is what generates the troublesome growth factor. On the other hand, it appears that the solution to this issue is to consider data in L^2, which is an /improvement over previous results.

The $L^q(\mathbb{R})$ bounds on $u(t,x)$ are uniform in compact sets of t. We relate the function $u(t,x)$ to $u(t,yt^2)$ to take advantage of scaling in t but return to $u(t,x)$ after manipulations have recast the problem in a nicer form.

Theorem V.13 follows from this proposition relating $u(t,yt^2)$ to a singular integral of ψ.

Proposition V.15. *Let u be a solution to* (4.2) *with $u_0 \in \dot{H}^{\frac{1}{4}}(\mathbb{R})$ and supp $\widehat{u_0}(\xi) \subseteq (-1,1)$. Then,*

$$|u(t,yt^2)| \leq C\int \frac{1}{|y-z|^{\frac{1}{2}}}\left|t|D|^{\frac{1}{2}}\mathcal{H}u_0(t^2z)\right|dz. \tag{5.15}$$

Proof. Since $u_0 \in \dot{H}^{\frac{1}{4}}(\mathbb{R})$ and is compactly supported in frequency, $\widehat{u_0}$ must take the following form for $-1/4 < \gamma$ and $\psi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\psi \subseteq (-1,1)$:

$$\widehat{u_0} = |\xi|^{-\frac{1}{2}}\text{sgn}\xi|\xi|^{\gamma}\widehat{\psi}(\xi). \tag{5.16}$$

The range of γ is easily deduced by considering where the $\dot{H}^{\frac{1}{4}}$ norm of u_0 is finite. We make no additional assumptions on ψ.

To prove the theorem, we begin by rewriting the solution in a different form.

Observe that

$$
\begin{aligned}
u(t, yt^2) &= \int e^{i(yt^2\xi + t|\xi|^{\frac{1}{2}})}\widehat{u_0}(\xi)d\xi \\
&= \int e^{i(y\xi+|\xi|^{\frac{1}{2}})}|\xi|^{-\frac{1}{2}}\mathrm{sgn}\xi|\xi|^{\frac{1}{2}}\mathrm{sgn}\xi\frac{1}{t^2}\widehat{u_0}\left(\frac{\xi}{t^2}\right)d\xi \\
&= \int e^{i(y\xi+|\xi|^{\frac{1}{2}})}|\xi|^{-\frac{1}{2}}\mathrm{sgn}\xi\int \delta(\xi-\eta)|\eta|^{\frac{1}{2}}\mathrm{sgn}\eta\frac{1}{t^2}\widehat{u_0}\left(\frac{\eta}{t^2}\right)d\eta d\xi \\
&= \int e^{i(y\xi+|\xi|^{\frac{1}{2}})}|\xi|^{-\frac{1}{2}}\mathrm{sgn}\xi\iint Ce^{-iz(\xi-\eta)}dz|\eta|^{\frac{1}{2}}\mathrm{sgn}\eta\frac{1}{t^2}\widehat{u_0}\left(\frac{\eta}{t^2}\right)d\eta d\xi \\
&= \iint e^{i((y-z)\xi+|\xi|^{\frac{1}{2}})}|\xi|^{-\frac{1}{2}}\mathrm{sgn}\xi d\xi\int Ce^{iz\eta}|\eta|^{\frac{1}{2}}\mathrm{sgn}\eta\frac{1}{t^2}\widehat{u_0}\left(\frac{\eta}{t^2}\right)d\eta dz \\
&= \int k(y-z)t|D|^{\frac{1}{2}}\mathcal{H}u_0(t^2z)dz.
\end{aligned}
$$

The second line comes from rescaling in t and $k(y-z) = \int e^{i((y-z)\xi+|\xi|^{\frac{1}{2}})}|\xi|^{-\frac{1}{2}}\mathrm{sgn}\xi d\xi$.

To complete the proof of this theorem, we simply need to show that $|k(y-z)| \leq C|y-z|^{-\frac{1}{2}}$. We will conduct our analysis on $k(x)$ for simplicity in notation. Now, if we change variables $\zeta = |\xi|^{\frac{1}{2}}$,

$$
\begin{aligned}
k(x) &= \int e^{i(x\xi+|\xi|^{\frac{1}{2}})}|\xi|^{-\frac{1}{2}}\mathrm{sgn}\xi d\xi \\
&= 2\int_0^\infty e^{i(x\zeta^2+\zeta)}d\zeta - 2\int_0^\infty e^{-i(x\zeta^2-\zeta)}d\zeta \\
&= 2\int_0^\infty e^{i\zeta}\left(e^{ix\zeta^2} - e^{-ix\zeta^2}\right)d\zeta. \qquad (5.17)
\end{aligned}
$$

From (5.17), it is clear that $k(-x) = -k(x)$. Since the phase functions in the calculation above are quadratic, it is possible to solve exactly for pieces of the kernel. Now,

$$
\begin{aligned}
\mathrm{sgn}x k(x) &= 2e^{\frac{-i}{4|x|}}\int_0^\infty e^{i|x|\left(\zeta+\frac{1}{2|x|}\right)^2}d\zeta - 2e^{\frac{i}{4|x|}}\int_0^\infty e^{-i|x|\left(\zeta-\frac{1}{2|x|}\right)^2}d\zeta \\
&= 2e^{\frac{-i}{4|x|}}\int_{\frac{1}{2|x|}}^\infty e^{i|x|\zeta^2}d\zeta - 2e^{\frac{i}{4|x|}}\int_{-\frac{1}{2|x|}}^\infty e^{-i|x|\zeta^2}d\zeta \\
&= -2e^{\frac{i}{4|x|}}\int_{-\infty}^\infty e^{-i|x|\zeta^2}d\zeta + 2e^{\frac{i}{4|x|}}\int_{-\infty}^{-\frac{1}{2|x|}} e^{-i|x|\zeta^2}d\zeta + 2e^{\frac{-i}{4|x|}}\int_{\frac{1}{2|x|}}^\infty e^{i|x|\zeta^2}d\zeta \\
&= -k_1(|x|) + k_2(|x|).
\end{aligned}
$$

The term $k_1(|x|)$ is straightforward to solve exactly using contours:

$$\begin{aligned}
-2e^{\frac{i}{4|x|}}\int_{-\infty}^{\infty} e^{-i|x|\zeta^2}d\zeta &= 2e^{\frac{i}{4|x|}}\int_{e^{\frac{-i\pi}{4}}\mathbb{R}} e^{-i|x|\zeta^2}d\zeta \\
&= 2e^{\frac{i}{4|x|}}e^{-\frac{i\pi}{4}}\int_{-\infty}^{\infty} e^{-|x|\zeta^2}d\zeta \\
&= \frac{2\sqrt{\pi}e^{\frac{i}{4|x|}}e^{-\frac{i\pi}{4}}}{|x|^{\frac{1}{2}}}.
\end{aligned}$$

Therefore, $|k_1(|x|)| \leq 2\sqrt{\pi}|x|^{-\frac{1}{2}}$. It only remains to control the second term, $k_2(|x|)$. Observe that $k_2(|x|)$ is the sum of integral and its complex conjugate, so it is sufficient to consider just one of the terms and show it is bounded by $C|x|^{-\frac{1}{2}}$. By construction, we can use integration by parts on the terms of k_2 as they avoid the critical point of the phase function. Thus,

$$\begin{aligned}
\left|2e^{\frac{-i}{4|x|}}\int_{\frac{1}{2|x|}}^{\infty} e^{i|x|\zeta^2}d\zeta\right| &= 2\left|\int_{\frac{1}{2|x|}}^{\infty} e^{i|x|\zeta^2}d\zeta\right| \\
&= 2\left|\int_{\frac{1}{2|x|}}^{\infty} \frac{\partial_\zeta\left(e^{i|x|\zeta^2}\right)}{2i|x|\zeta}d\zeta\right| \\
&= 2\left|\left.\frac{e^{i|x|\zeta^2}}{2i|x|\zeta}\right|_{\frac{1}{2|x|}}^{\infty} + \int_{\frac{1}{2|x|}}^{\infty}\frac{e^{i|x|\zeta^2}}{2i|x|\zeta^2}d\zeta\right| \leq 4
\end{aligned}$$

If $|x| \leq \pi/16$, this calculation implies that $k_2(|x|)$ is bounded by $C|x|^{-\frac{1}{2}}$. In order to see the behavior of this term for large x, we will calculate it using contours as we did for k_1. Then,

$$2e^{\frac{-i}{4|x|}}\int_{\frac{1}{2|x|}}^{\infty} e^{i|x|\zeta^2}d\zeta = 2e^{\frac{-i}{4|x|}}\left(e^{\frac{i\pi}{4}}\int_{\frac{1}{2|x|}}^{\infty} e^{-|x|\zeta^2}d\zeta + \int_{\Gamma} e^{i|x|\zeta^2}d\zeta\right) \tag{5.18}$$

where $\Gamma = \{|\zeta| = \frac{1}{2|x|}, \theta \in (0, \pi/4)\}$. By a similar argument similar to the one used for k_1, the first term is bounded by $e^{\frac{-1}{8|x|}}|x|^{-\frac{1}{2}}$. The second term can be rewritten as an integral in θ and bounded like so:

$$\left|\int_{\Gamma} e^{i|x|\zeta^2} d\zeta\right| = \left|\int_0^{\frac{\pi}{4}} e^{\frac{ie^{2i\theta}}{4|x|}} \frac{ie^{i\theta}}{2|x|} d\theta\right|$$
$$\leq \frac{\pi}{8|x|}$$

If $|x| > \frac{\pi}{16}$, we get precisely that the sum of these two terms is less than $2\sqrt{\pi}|x|^{-\frac{1}{2}}$. Therefore,

$$|k(x)| \leq C|x|^{-\frac{1}{2}}. \tag{5.19}$$

and we can conclude that

$$|u(t, yt^2)| \leq C \int \frac{t||D|^{\frac{1}{2}}\mathcal{H}u_0(t^2 z)|}{|y-z|^{\frac{1}{2}}} dz. \tag{5.20}$$

□

Given the relationship between $u(t, yt^2)$ and the fractional integral of $|D|^{\frac{1}{2}}\mathcal{H}u_0$, the proof of Theorem V.13 reduces to careful application of the Hardy-Littlewood-Sobolev lemma.

Proof of Theorem V.13. By a straightforward change of variables,

$$C \int \frac{t||D|^{\frac{1}{2}}\mathcal{H}u_0(t^2 z)|}{|y-z|^{\frac{1}{2}}} dz = C \int \frac{||D|^{\frac{1}{2}}\mathcal{H}u_0(z)|}{|yt^2 - z|^{\frac{1}{2}}} dz.$$

Let $F(x) = \int \frac{||D|^{\frac{1}{2}}\mathcal{H}u_0(z)|}{|x-z|^{\frac{1}{2}}} dz$. Observe that $|u(t,x)| \leq |F(x)|$, independent of t.

By the Hardy-Littlewood-Sobolev lemma, we know that for $n \geq 1$, $1 < p < q < \infty$, and the operator I_β defined by

$$I_\beta g(x) = \int_{\mathbb{R}^n} \frac{1}{|x-z|^\beta} g(z) dz, \tag{5.21}$$

we have the bounds

$$\|I_\beta g\|_{L^q(\mathbb{R}^n)} \leq \|g\|_{L^p(\mathbb{R}^n)} \text{ when } \frac{1}{q} = \frac{1}{p} - \frac{n-\beta}{n}.$$

In this case, we have $\|F\|_{L^q} \leq \||D|^{\frac{1}{2}}\mathcal{H}u_0\|_{L^p}$ for $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$. To complete the proof, it suffices to identify to which L^p spaces $|D|^{\frac{1}{2}}\mathcal{H}u_0$ belongs.

Recall $\widehat{u_0}$ is of the form (5.16). When $\gamma > 0$, $|D|^{\frac{1}{2}}\mathcal{H}u_0$ is some order derivative of ψ. Since $\widehat{\psi} \in C_0^\infty$, we know that ψ is in Schwartz class, and thus $|D|^{\frac{1}{2}}\mathcal{H}u_0$ is in L^p for all p. Then, we can conclude that $F \in L^q$ for all $q \in (2, \infty)$.

When $-\frac{1}{4} < \gamma < 0$, we will need to apply the Hardy-Littlewood-Sobolev lemma for a second time. By definition, $|D|^{\frac{1}{2}}\mathcal{H}u_0(z) = I_{1-|\gamma|}\psi(z)$, so we know that for $1 < r < p < \infty$

$$\||D|^{\frac{1}{2}}\mathcal{H}u_0\|_{L^p} \leq \|\psi\|_{L^r} \text{ when } \frac{1}{p} = \frac{1}{r} - \frac{|\gamma|}{1}.$$

By combining the bound on F with the bound on $|D|^{\frac{1}{2}}\mathcal{H}u_0$, we have

$$\|F\|_{L^q} \leq \||D|^{\frac{1}{2}}\mathcal{H}u_0\|_{L^r} \text{ for } \frac{1}{q} = \frac{1}{r} - |\gamma| - \frac{1}{2}.$$

Since ψ is Schwartz, we know that $\psi \in L^r$ for $1 \leq r \leq \infty$. In order to satisfy the lemma, we limit r to the range $(1, \frac{1}{\frac{1}{2}+|\gamma|})$, which implies that $q \in (\frac{2}{1-2|\gamma|}, \infty)$ □

In addition to the upper bounds found above, we also have the following lower bounds for more specialized data.

5.3.1 Sharp lower bounds compactly supported data

Theorem V.16. *Let u be a solution to the initial value problem*

$$\begin{cases} \partial_t u - i|D|^{\frac{1}{2}}u &= 0 \\ u(0,x) &= u_0(x) \end{cases}$$

where $\widehat{u_0}(\xi) = |\xi|^{-\frac{1}{2}} sgn\xi \widehat{\varphi}(\xi)$ and $\varphi(x)$ compactly supported in the interval $(-\frac{1}{M}, \frac{1}{M})$, $M \in \mathbb{N}$ and $\varphi(x)$ does not change sign. Let $\frac{C}{t} \leq |y| \leq \delta$ where $\delta > 0$ and independent of t. In addition, assume that $\delta + \frac{1}{M} \leq \frac{\pi}{64}$. Then,

$$|u(t, yt^2)| \geq \frac{\sqrt{\pi}}{2} \int \frac{t|\varphi(t^2 z)|}{|y-z|^{\frac{1}{2}}} dz. \tag{5.22}$$

Remark V.17. The sign assumption on φ is a technical condition which allows us to move absolute values inside the integral without changing the value. It may be possible to avoid this condition through other techniques.

Proof. We will show that given sufficient large M, this part of the kernel convolved with the scaled initial data bounds the solution below. To begin, we verify that $\frac{1}{2}|k_1| > |k_2|$. Observe that $k_2(|x|)$ is the sum of integral and its complex conjugate, so it is sufficient to consider just one of the terms and show it is bounded by $\frac{1}{4}|k_1|$. Then,

$$\begin{aligned}\left|2e^{\frac{-i}{4|x|}}\int_{\frac{1}{2|x|}}^{\infty} e^{i|x|\zeta^2}d\zeta\right| &= 2\left|\int_{\frac{1}{2|x|}}^{\infty} e^{i|x|\zeta^2}d\zeta\right| \\ &= 2\left|\int_{\frac{1}{2|x|}}^{\infty} \frac{\partial_\zeta\left(e^{i|x|\zeta^2}\right)}{2i|x|\zeta}d\zeta\right| \\ &= 2\left|\left.\frac{e^{i|x|\zeta^2}}{2i|x|\zeta}\right|_{\frac{1}{2|x|}}^{\infty} + \int_{\frac{1}{2|x|}}^{\infty}\frac{e^{i|x|\zeta^2}}{2i|x|\zeta^2}d\zeta\right| \le 4.\end{aligned}$$

As long as $|x| \le \pi/4$, k_1 is the dominant part of the kernel. In our convolution operator, $x = y - z$ with $|z| \le \frac{1}{Mt^2}$, so $|y - z| \le |y| + |z| \le \delta + \frac{1}{Mt^2}$. By our assumption on M and δ , k_1 is the dominant part of the kernel.

Now, since

$$\begin{aligned}|u(t, yt^2)| &= \left|\int k(y-x)t\varphi(t^2z)dz\right| \\ &\ge \left|\int k_1(y-x)t\varphi(t^2z)dz\right| - \left|\int k_2(y-x)t\varphi(t^2z)dz\right|,\end{aligned}$$

in order to show we have a lower bound we also need that

$$\left|\int k_2(y-x)t\varphi(t^2z)dz\right| \le \frac{1}{2}\left|\int k_1(y-x)t\varphi(t^2z)dz\right|.$$

In fact we will show first that $\left|\int k_1(y-x)t\varphi(t^2z)dz\right|$ is bounded below and then show that $\left|\int k_2(y-x)t\varphi(t^2z)dz\right|$ is bounded by half of this lower bound for the k_1 term.

Claim V.18. *Assume that φ does not change sign (that is, it is strictly positive or negative inside its support). For $k_1(x)$ defined as above, we have*

$$\left|\int k_1(y-z)t\varphi(t^2z)dz\right| \geq \sqrt{\pi}\int \frac{t|\varphi(t^2z)|}{|y-z|^{\frac{1}{2}}}dz. \tag{5.23}$$

Proof. Recall that $k_1(y-z) = \frac{2\sqrt{\pi}e^{\frac{i}{4|y-z|}}e^{-\frac{i\pi}{4}}\operatorname{sgn}(y-z)}{|y-z|^{\frac{1}{2}}}$. Then,

$$\begin{aligned}
\int k_1(y-z)t\varphi(t^2z)dz &= \int \frac{2\sqrt{\pi}e^{\frac{i}{4|y-z|}}e^{-\frac{i\pi}{4}}\operatorname{sgn}(y-z)}{|y-z|^{\frac{1}{2}}}t\varphi(t^2z)dz \\
&= \int \frac{2\sqrt{\pi}e^{\frac{i}{4|y|}}e^{-\frac{i\pi}{4}}\operatorname{sgn}y}{|y-z|^{\frac{1}{2}}}t\varphi(t^2z)dz \\
&\quad + \int \frac{2\sqrt{\pi}\left(e^{\frac{i}{4|y-z|}}\operatorname{sgn}(y-z) - e^{\frac{i}{4|y|}}\operatorname{sgn}y\right)e^{-\frac{i\pi}{4}}}{|y-z|^{\frac{1}{2}}}t\varphi(t^2z)dz \\
\left|\int k_1(y-z)t\varphi(t^2z)dz\right| &\geq \left|\int \frac{2\sqrt{\pi}e^{\frac{i}{4|y|}}e^{-\frac{i\pi}{4}}\operatorname{sgn}y}{|y-z|^{\frac{1}{2}}}t\varphi(t^2z)dz\right| \\
&\quad - \left|\int \frac{2\sqrt{\pi}\left(e^{\frac{i}{4|y-z|}}\operatorname{sgn}(y-z) - e^{\frac{i}{4|y|}}\operatorname{sgn}y\right)e^{-\frac{i\pi}{4}}}{|y-z|^{\frac{1}{2}}}t\varphi(t^2z)dz\right| \\
&\geq \left|\int \frac{2\sqrt{\pi}t\varphi(t^2z)}{|y-z|^{\frac{1}{2}}}dz\right| \\
&\quad - \left|\int \frac{2\sqrt{\pi}\left(e^{\frac{i}{4|y-z|}}\operatorname{sgn}(y-z) - e^{\frac{i}{4|y|}}\operatorname{sgn}y\right)e^{-\frac{i\pi}{4}}}{|y-z|^{\frac{1}{2}}}t\varphi(t^2z)dz\right|
\end{aligned}$$

Since φ doesn't change sign, we can move the absolute value inside the first term, and it is sufficient to show

$$\left|\int \frac{2\sqrt{\pi}\left(e^{\frac{i}{4|y-z|}}\operatorname{sgn}(y-z) - e^{\frac{i}{4|y|}}\operatorname{sgn}y\right)e^{-\frac{i\pi}{4}}}{|y-z|^{\frac{1}{2}}}t\varphi(t^2z)dz\right| \leq \int \frac{\sqrt{\pi}t|\varphi(t^2z)|}{|y-z|^{\frac{1}{2}}}dz. \tag{5.24}$$

To show (5.24), we will use the assumption of compact support for φ. Assume that $2\delta > \frac{1}{M}$. Then, $\operatorname{sgn}(y-z) = \operatorname{sgn}y$. By the mean value theorem, we have

$$e^{\frac{i}{4|y-z|}}\operatorname{sgn}(y-z) - e^{\frac{i}{4|y|}}\operatorname{sgn}y = \frac{-ize^{\frac{i}{4p(z)}}}{p(z)^2}$$

for $p(z) \in (y-z, y+z)$ gives the correct point in this interval for the Mean Value Theorem. Then,

$$\begin{aligned}
&\left| \int \frac{2\sqrt{\pi}\left(e^{\frac{i}{4|y-z|}}\mathrm{sgn}(y-z) - e^{\frac{i}{4|y|}}\mathrm{sgn} y\right) e^{-\frac{i\pi}{4}} t\varphi(t^2 z)dz}{|y-z|^{\frac{1}{2}}} \right| \\
&\leq \left| \int \frac{-2i\sqrt{\pi} z e^{\frac{i}{4p(z)}} e^{-\frac{i\pi}{4}}}{p(z)^2 |y-z|^{\frac{1}{2}}} t\varphi(t^2 z)dz \right| \\
&\leq \int \frac{2\sqrt{\pi}|z|}{|p(z)|^2 |y-z|^{\frac{1}{2}}} t|\varphi(t^2 z)|dz
\end{aligned}$$

Since $|z| \leq \frac{1}{Mt^2}$, $p(z) \in (y-z, y+z)$, and $y > \frac{1}{t}$, we have

$$\frac{|z|}{p(z)^2} \leq \frac{\frac{1}{Mt^2}}{|y - \frac{1}{Mt^2}|^2} \leq \frac{Mt^2}{|Mt^2 y - 1|^2} \leq \frac{M}{(M-1)^2}.$$

For $M > 4$, we have $\frac{|z|}{p(z)^2} < \frac{1}{2}$, and so (5.23) holds. ☐

Finally, we need only to show that $\left|\int k_2(y-x)t\varphi(t^2 z)dz\right|$ is less than or equal to half of this lower bound on the k_1 term.

Claim V.19. *Let φ and k_2 be defined as above. Then*

$$\left| \int k_2(y-x)t\varphi(t^2 z)dz \right| \leq \frac{\sqrt{\pi}}{2} \int \frac{t|\varphi(t^2 z)|}{|y-z|^{\frac{1}{2}}} dz$$

Proof. Clearly,

$$\left| \int k_2(y-x)t\varphi(t^2 z)dz \right| \leq 4 \int t|\varphi(t^2 z)|dz.$$

Now, all we need to show is that

$$4 \leq \frac{\sqrt{\pi}}{2|y-z|^{\frac{1}{2}}}.$$

By the triangle inequality and our assumptions on y and M in the statement of the theorem, we have $|y-z| < |y| + |z| < \frac{\pi}{64}$. Then we have precisely that

$$\frac{\sqrt{\pi}}{2|y-z|^{\frac{1}{2}}} \geq \frac{\sqrt{\pi}}{2} \frac{8}{\sqrt{\pi}} \geq 4,$$

proving the claim. ☐

By combining the two claims, we see that

$$|u(t, yt^2)| \geq \frac{\sqrt{\pi}}{2} \int \frac{t|\varphi(t^2 z)|}{|y-z|^{\frac{1}{2}}} dz,$$

which completes the proof of the theorem. □

The key ingredient in the proof above is the smallness of the support of φ. That assumption allows us to treat the kernel without worrying about the oscillatory factor $e^{\frac{i}{|y-z|}}$, which may contribute some cancellation in the region where $t^2 z \sim y$.

CHAPTER VI

Conclusions and Future Work

6.1 Conclusions

We began by identifying an area in need of improvement in the proof of almost global existence for the water wave in two dimensions. Adapting the techniques of Keel, Smith, and Sogge to a general class of one dimensional equations lead to Theorem IV.3 which identified the possibility for growth along certain trajectories. The presence of this growth impedes the ideal decay of $t^{-\frac{1}{2}}$ for the full water wave problem, and its appearance was unexpected. The sharpness result of Theorem IV.5 emphasizes that there is something even in the linear problem which keeps the solution from decaying. However, the specificity of the assumptions in Theorem IV.5 leaves open the possibility that a more restrictive class of data, but still larger than that allowed by Wu's results, could overcome the obstacles.

While a promising direction, the obvious combination of Theorem IV.3 and the Klainerman-type bounds used by Wu does not achieve the ideal decay rate. The implication here is something non-trivial is keeping solution to the linearized problem from decaying as we would like. In particular, the part of the solution intitially at small frequency and propagating in the wedge $t < |x| < t^2$ is not decaying quickly enough. A promising first step towards a complete analysis of the effect of initial data on long time decay is Theorem V.13. This theorem relates, in some sense, the spatial decay of a solution to the linearized water wave problem to the size of the singularity at the origin in frequency.

6.2 Future Work

The immediate goal is to sharpen the bounds of Chapter V and use them to produce a long time existence result for the full water wave problem with a more general class of initial data than [20]. Once the bounds in Theorem V.13 are sharp, we will be able to completely characterize the relationship between singularities at the origin in frequency to growth and decay in the linearized problem. A full understanding of the linearized problem will clarify the class of initial data neccessary for long time existence in the nonlinear problem.

In addition, all of the theorems in Chapter IV can be generalized in some form to higher spatial dimensions. We plan to generalize those theorems and continue the analysis of initial data to the three dimensional water wave problem. As a consequence, we will have global solutions in the three dimensional case for a more general class of allowable initial conditions than in existing work.

APPENDICES

APPENDIX A

Results for the 2D Wave Equation

During our study of the techniques of Keel, Smith, and Sogge, we extended their results to the two dimensional case. Since this result is not easily found in the literature, we include it here.

Consider the following initial value problem for the inhomogeneous wave equation in two space dimensions:

$$\text{(A.1)} \qquad \begin{cases} \Box v = \partial_t^2 v - \Delta v = (\partial_1 v)^3 \\ v(0,x) = f(x) \\ v_t(0,x) = g(x). \end{cases}$$

The full complement of the vector fields used by Klainerman is $\Gamma = \{\partial_i, L, \Omega_{jk} : 0 \leq i \leq 2, 1 \leq j < k \leq 2\}$ where $L = t\partial_t + x_1\partial_1 + x_2\partial_2$ and $\Omega_{jk} = x_k\partial_{x_j} - x_j\partial_k$. Instead of using all the invariant vector fields of the d'Alembertian, the techniques of Keel, Smith, and Sogge restrict to the collection $Z = \{\partial_t, \partial_1, \partial_2, \Omega_{12}\}$. Let $\|F\| = \|F\|_{L^2(\mathbb{R}^2)}$.

Theorem A.1. *Let* $(f,g) \in C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2)$ *with*

$$\sum_{|\alpha|\leq 6} \|Z^\alpha g\| + \sum_{|\alpha|\leq 7} \|Z^\alpha f\| \leq \epsilon.$$

Then there is a unique solution $u(t,x)$ *to* (A.1) *with* $u(t,x) \in C^\infty([0,T*] \times \mathbb{R}^2)$ *where* $T^* = e^{\frac{c}{\epsilon}}$.

Proof. First, recall the energy estimate for inhomogeneous wave equation with forcing $G(s,x)$:

$$\|v'(t)\| \le C\left(\|v'(0)\| + \int_0^T \|G(s)\|ds\right).$$

We have an additional weighted energy estimate:

$$\||x|^{-\frac{1}{2}}v'\|_{L^2([0,T]\times[R,2R])} \le C\left(\|v'(0)\| + \int_0^t \|G(s)\|ds\right).$$

This particular bound was first found by Metcalfe in [13]. A proof of this bound can also be found in the Appendix B of [2].

Notice that the right hand side is the same in both of these cases. We will use the second of these to derive a weighted energy estimate for $\||x|^{-\frac{1}{2}}Z^\alpha v'\|_{L^2([0,T]\times\{|x|>1\})}$. For $|x|>T$, we have the following:

$$\begin{aligned}
\||x|^{-\frac{1}{2}}v'\|_{L^2([0,T]\times\{|x|>T\})} &\le T^{-1/2}\left(\int_0^T\int_{|x|>T}|v'(t,x)|^2dxdt\right)^{1/2} \\
&\le T^{-1/2}\left(T\sup_{[0,T]}\int_{|x|>T}|v'(t,x)|^2dxdt\right)^{1/2} \\
&\le \sup_{[0,T]}\|v'(t)\| \\
&\le C\left(\|v'(0)\| + \int_0^t \|G(s)\|ds\right)
\end{aligned}$$

For $T>|x|>1$, we decompose $\||x|^{-\frac{1}{2}}v'\|_{L^2([0,T]\times\{T>|x|>1\})}$ into annuli $R_j = \{x : 2^j < |x| < 2^{j+1}\}$ The necessary range of j is 0 to $\beta = \lfloor \frac{\ln T}{\ln 2}\rfloor$. Then, we have:

$$\begin{aligned}
\||x|^{-\frac{1}{2}}v'\|_{L^2([0,T]\times\{T>|x|>1\})} &= \left(\sum_{j=0}^{\beta}\||x|^{-\frac{1}{2}}v'\|^2_{L^2([0,T]\times R_j)}\right)^{1/2} \\
&\le (\beta+1)^{1/2}C\left(\|v'(0)\| + \int_0^t \|G(s)\|ds\right) \\
&\le \ln(T+2)^{1/2}C\left(\|v'(0)\| + \int_0^t \|G(s)\|ds\right)
\end{aligned}$$

By combining these two and noticing that $\ln(T+2)\ge c>0$, we get

$$\sum_{|\alpha|\le 6}\||x|^{-\frac{1}{2}}Z^\alpha v'\|_{L^2([0,T]\times\{|x|>1\})} \lesssim \ln(T+2)^{\frac{1}{2}}\left(\sum_{|\alpha|\le 6}\|Z^\alpha v'(0)\| + \sum_{|\alpha|\le 6}\int_0^t \|Z^\alpha G(s)\|ds\right).$$

For $|x| < 1$, we recall the work of Hidano-Yokoyama [2]. Let $\langle x\rangle = (1+|x|)^{1/2}$. We state without proof the following proposition found in [2, Appendix B]:

Proposition A.2. *[2, Proposition B.1] Let $n \geq 1$, $(f,g) \in \mathcal{S}(\mathbb{R}^n)\times\mathcal{S}(\mathbb{R}^n)$ and $\delta > 0$. Let u be a solution of* (A.1) *with data $(u(0), \partial_t u(0)) = (f,g)$. Then the solution satisfies*

$$\||x|^{-1/2+\delta}\langle x\rangle^{-\delta}u'\|_{L^2(0<t<T,|x|<1)} \leq C(\delta)\left(\|v'(0)\| + \int_0^t \|G(s)\|ds\right).$$

The constant $C(\delta) = \frac{2^{2\delta}}{2^{2\delta}-1}$ blows up as δ goes to zero.

For simplicitity, we will use functions $A(T)$, $B_\delta(T)$ and $m(T)$ defined

$$\begin{aligned}
A(T) &= \ln(T+2)^{-1/2}\textstyle\sum_{|\alpha|\leq 6}\||x|^{-\frac{1}{2}}Z^\alpha v'\|_{L^2([0,T]\times\{|x|>1\})}\\
m(T) &= \sup_{[0,T]}\textstyle\sum_{|\alpha|\leq 6}\|Z^\alpha v'(t)\|\\
B_\delta(T) &= \textstyle\sum_{|\alpha|\leq 6}\||x|^{-1/2+\delta}\langle x\rangle^{-\delta}Z^\alpha u'\|_{L^2(0<t<T,|x|<1)}.
\end{aligned}$$

We want to show that we can bound $A(T)+m(T)+B_\delta(T)$ by various powers of $A(T)$, $m(T)$, and $B_\delta(T)$. First, notice by the above facts that we have

$$A(T)+m(T)+B_\delta(T) \leq C(\delta)\left(\sum_{|\alpha|\leq 6}\|Z^\alpha v'(0)\| + \sum_{|\alpha|\leq 6}\int_0^T \|Z^\alpha(\partial_1 v)^3(s)\|ds\right).$$

We need to work on the integral term. We first decompose the L^2 norm into $|x| < 1/2$ and R_j for $j \geq 0$. Then, using the product rule to expand $Z^\alpha(\partial_1 u)^3$, we get

$$\begin{aligned}
\sum_{|\alpha|\leq 6}\|Z^\alpha(\partial_1 v)^3(t)\| &= \left(\sum_{|\alpha|\leq 6}\|Z^\alpha(\partial_1 u)^3(t)\|^2_{L^2(|x|<1/2)} + \sum_{j=-1}^{\infty}\|Z^\alpha(\partial_1 u)^3(t)\|^2_{L^2(R_j)}\right)^{1/2}\\
&\leq \left(\left(\sum_{|\alpha|\leq 3}|Z^\alpha u'(t)|_{L^\infty(|x|<1/2)}\right)^4\sum_{|\alpha|\leq 6}\|Z^\alpha u'(t)\|^2_{L^2(|x|<1/2)}\right.\\
&\quad\left.+\sum_{j=-1}^{\infty}\left(\sum_{|\alpha|\leq 3}|Z^\alpha u'(t)|_{L^\infty(R_j)}\right)^4\sum_{|\alpha|\leq 6}\|Z^\alpha u'(t)\|^2_{L^2(R_j)}\right)^{1/2}\\
&= (I+II)^{1/2}
\end{aligned}$$

where I is the term involving the norms on $|x| < 1/2$ and II is the term with the sum over j. We will consider the first of these terms to start. Notice that by standard Sobolev lemmas, we can bound the L^∞ norms by L^2 norms with a slight increase in derivatives and in the size of the domain. Therefore,

$$\begin{aligned} I &\leq \left(\sum_{|\alpha|\leq 5} \|Z^\alpha u'(t)\|_{L^2(|x|<1)} \right)^4 \sum_{|\alpha|\leq 6} \|Z^\alpha u'(t)\|^2_{L^2(|x|<1/2)} \\ &\lesssim \left(\sum_{|\alpha|\leq 5} \||x|^{-1/2+\delta}\langle x\rangle^{-\delta} Z^\alpha u'(t)\|_{L^2(|x|<1)} \right)^4 \sum_{|\alpha|\leq 6} \|Z^\alpha u'(t)\|^2_{L^2(|x|<1/2)} \end{aligned}$$

as long as $\delta < 1/2$.

For the second term, we recall the weighted Sobolev estimate Lemma II.6:

$$|f|_{L^\infty(R<|x|<2R)} \lesssim R^{-1/2} \sum_{|\beta|\leq 2} \|Z^\beta f\|_{L^2(R/2<|x|<4R)}.$$

Using this inequality, we derive the following:

$$\begin{aligned} II &\lesssim \sum_{j=-1}^{\infty} \left(\sum_{|\alpha|\leq 5} \||x|^{-\frac{1}{2}} Z^\alpha u'(t)|_{L^2(\widetilde{R}_j)} \right)^4 \sum_{|\alpha|\leq 6} \|Z^\alpha u'(t)\|^2_{L^2(R_j)} \\ &\lesssim \sup_j \sum_{|\alpha|\leq 6} \|Z^\alpha u'(t)\|^2_{L^2(R_j)} \sum_{j=-1}^{\infty} \left(\sum_{|\alpha|\leq 5} \||x|^{-\frac{1}{2}} Z^\alpha u'(t)|_{L^2(\widetilde{R}_j)} \right)^4 \end{aligned}$$

Now we plug these back into the time integral, and use Hölder's inequality:

$$\int_0^T (I+II)^{\frac{1}{2}} dt \le \int_0^T \sum_{|\alpha|\le 6} \|Z^\alpha u'(t)\|_{L^2(|x|<\frac{1}{2})} \left(\sum_{|\alpha|\le 5} \||x|^{\frac{-1}{2}+\delta} \langle x\rangle^{-\delta} Z^\alpha u'(t)\|_{L^2(|x|<1)} \right)^2 dt$$

$$+ \int_0^T \sup_j \sum_{|\alpha|\le 6} \|Z^\alpha u'(t)\|_{L^2(R_j)} \left(\sum_{j=-1}^{\infty} \left(\sum_{|\alpha|\le 5} \||x|^{-\frac{1}{2}} Z^\alpha u'(t)|_{L^2(\widetilde{R}_j)} \right)^4 \right)^{\frac{1}{2}} dt$$

$$\le m(T) \int_0^T \left(\sum_{|\alpha|\le 6} \||x|^{-1/2+\delta} \langle x\rangle^{-\delta} Z^\alpha u'(t)\|_{L^2(|x|<1)} \right)^2 dt$$

$$+ m(T) \int_0^T \sum_{j=-1}^{\infty} \left(\sum_{|\alpha|\le 5} \||x|^{-\frac{1}{2}} Z^\alpha u'(t)|_{L^2(\widetilde{R}_j)} \right)^2 dt$$

$$\le C m(T) B_\delta(T)^2 + C m(T) \int_0^T \sum_{|\alpha|\le 6} \||x|^{-\frac{1}{2}} Z^\alpha u'(t)|^2_{L^2(|x|>1)} dt$$

$$\le C(m(T) B_\delta(T)^2 + \ln(T+2) m(T) A(T)^2)$$

If we plug this bound into the original inequality, we have

$$A(T) + m(T) + B_\delta(T) \lesssim \sum_{|\alpha|\le 6} \|Z^\alpha u'(0)\| + C(m(T)B_\delta(T)^2 + \ln(T+2) m(T) A(T)^2).$$

An application of a continuity argument implies the solution exists for times on the order of $e^{c/\epsilon}$. ☐

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] Remi Carles. Linear vs. Nonlinear Effects for Nonlinear Schrodinger Equations with Potential. *Communications in Contemporary Mathematics*, 7(4):483–508, 2005.

[2] Kunio Hidano and Kazuyoshi Yokoyama. A Remark on the Almost Global Existence Theorems of Keel, Smith and Sogge. *Funkcialaj Ekvacioj*, 48(1):1–34, 2005.

[3] Geri Izbicki Jennings. *Efficient Numerical Methods for Water Wave Propagation in Unbounded Domains*. PhD thesis, University of Michigan, 2012.

[4] Jean-lin Journe and Guy David. A boundedness criterion for generalized Calderon Zygmund operators. *Annals of Mathematics*, 120(2):371–397, 1984.

[5] Markus Keel, Hart F Smith, and Christopher D. Sogge. Almost Global Existence for Some Semilinear Wave Equations. *Journal d'Analyse Mathematique*, 87:265–279, 2002.

[6] Markus Keel, Hart F Smith, and Christopher D. Sogge. Global Existence for a Quasilinear Wave Equation Outside of Star-shaped Domains. *Journal of Functional Analysis*, 189(1):155–226, 2002.

[7] Markus Keel, Hart F Smith, and Christopher D. Sogge. Almost Global Existence for Quasilinear Wave Equations in Three Space Dimensions. *Journal of the American Mathematical Society*, 17(1):109–153, 2004.

[8] Carlos E Kenig, Gustavo Ponce, and Luis Vega. Oscillatory Integrals and Regularity of Dispersive Equations. *Indiana University Mathematics Journal*, 40(1):33–69, 1991.

[9] Sergiu Klainerman. Weighted L^∞ and L^1 Estimates for Solutions to the Classical Wave Equation in Three Space Dimensions. *Communications on Pure and Applied Mathematics*, 37:269–288, January 1984.

[10] Sergiu Klainerman. Global Existence of Small Amplitude Solutions to Nonlinear Klein-Gordon Equations in Four Space-Time Dimensions. *Communications on Pure and Applied Mathematics*, 38:631–641, 1985.

[11] Sergiu Klainerman. Uniform Decay Estimates and the Lorentz Invariance of the Classical wave equation. *Communications on Pure and Applied Mathematics*, 38:321–332, January 1985.

[12] Jason L Metcalfe. Global existence for semilinear wave equations exterior to nontrapping obstacles. *Houston Journal of Mathematics*, 30:259–281, 2004.

[13] Jason L Metcalfe. Global Strichartz Estimates for Solutions to the Wave Equation Exterior to a Convex Obstacle. *Transcripts of the American Mathematical Society*, 356:4839–4855, 2004.

[14] Jason L Metcalfe and Christopher D. Sogge. Hyperbolic trapped rays and global existence of quasilinear wave equations. *Inventiones Mathematicae*, 159:75–117, 2005.

[15] Jason L Metcalfe and Christopher D. Sogge. Long-Time Existence of Quasilinear Wave Equations Exterior to Star-Shaped Obstacles via Energy Methods. *SIAM Journal on Mathematical Analysis*, 38(1):188, 2006.

[16] Jason L Metcalfe and Daniel Tataru. Decay estimates for Variable coefficient wave equation in Exterior domains. In Antonio Bove, Daniele Del Santo, and M.K. Venkatesha Murthy, editors, *Advances in Phase Space Analysis of Partial Differential Equations*, pages 201–216. Birkhäuser Boston, Boston, 2009.

[17] Ronald Smith. Giant waves. *Journal of Fluid Mechanics*, 77(03):417–431, April 1976.

[18] Christopher D. Sogge. *Lectures on Non-Linear Wave Equations.* International Press of Boston, 2nd edition, 2008.

[19] Elias Stein. *Harmonic Analysis: Real-variable methods, Orthogonality, and Oscillatory integrals.* Princeton University Press, 1993.

[20] Sijue Wu. Almost global wellposedness of the 2-D full water wave problem. *Inventiones mathematicae*, 177(1):45–135, February 2009.