VALUE-AT-RISK (VaR) AND DYNAMIC PORTFOLIO SELECTION

by

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CHAPTER I

Introduction

Value-at-Risk (VaR) has gained increasing popularity in risk management and regulation for a decade. However, the driving force for its use can be traced back much further than a decade. According to the brief history of VaR described in [12] [14], before the term "Value at Risk" was widely used in the mid 1990s, regulators developed capital requirements for banks to reduce risk. After the Great Depression and bank failures in the 1930s, the first regulatory capital requirement for banks were enacted. The Securities Exchange Commission (SEC), established by the Securities Exchange Act in 1934, required banks to keep their borrowings below 2000% of their net capital. In 1975, SEC's Uniform Net Capital Rule (UNCR) refined the capital requirement in which bank's financial assets were categorized into twelve classes according to the security types. Each class has different capital requirement represented by the haircut percentage. Depending on the risk, capital requirements ranged from 0% for short term treasuries to 30% for equities. In 1980, the SEC required financial firms to calculate the potential losses in different security classes with 95% confidence over a 30-day interval. The capital requirements were tied to this measure which was described as haircuts. Although the name "VaR" was not used, it was virtually the one-month 95% VaR and banks are required to hold enough capital to cover the potential loss. In the early 1990s, the Basel Committee updated its 1988 accord to add the capital requirements for market risk [4] [5]. The market risk capital requirement is calculated based on the 10-day VaR with 99% confidence level of the bank's risky assets portfolio. Now VaR is a widely used risk measure of the possible loss on a specific portfolio of financial assets. VaR is often used by commercial and investment banks to capture the potential loss in the value of their traded portfolio. In most of the applications, the VaR is used to determine the capital or cash reserves for ensuring that the future loss can be covered and the firm will remain solvent. Moreover, the VaR can be used for an individual asset, a portfolio of assets or an entire company. The risk can be specified more broadly or narrowly for special use. For example, the VaR in investment banks is specified in terms of market volatility, interest rate changes, and foreign exchange rate changes etc.

In most of the applications, the VaR estimations are always under the assumption that there is no trading or adjustment in the underlying portfolio during VaR horizon. As stated in Hull's book [9],

> *"VaR itself is invariably calculated on the assumption that the portfolio will remain unchanged during the time period."*

Apparently, this assumption is unrealistic in real life. For example, some insurance companies use one-year VaR as their risk measure. If we assume there is no trading in one year, it is unreasonable. Companies need to adjust their trading portfolios each day according to changes in the market. The distribution of the portfolio without trading is significantly different from the one with certain trading strategies. From a statistical view, VaR is a percentile of the portfolio loss distribution in the given investment horizon. The distribution of the portfolio value is the essential component in the VaR estimation. The distribution of the portfolio value depends on the portfolio selection strategy. Therefore, the selection strategy could have significant impact on the VaR estimation. To reflect the true risk of the portfolio, the portfolio adjustment during investment horizon must be incorporated in the VaR estimation, especially when the investment horizon is long.

The first goal of this study is to incorporate portfolio selection strategies and analyze the impact of those strategies in VaR estimation. For simplicity, we denote the VaR incorporating portfolio selection strategies by "*New VaR*" and the one with the assumption of no adjustment by "*Old VaR*". There are two types of portfolio selection strategies considered in this study. The first type represents the strategies derived based on the framework established by Merton (1971)[20]. In this case, the risk-averse investor is assumed to hold the portfolio over a fixed time interval $[0,T]$ and try to maximize the expected utility of the terminal wealth. The optimal portfolio weight can be expressed in terms of the solution of a nonlinear Partial Differential Equation (PDE), namely Hamilton-Jacobi-Bellman (HJB) equation. The second type represents the strategies in which the weights of each asset remain constant during the whole investment horizon. We call this *"simple portfolio selection strategy"*. Although this type of portfolio selection strategies is not as sophisticated as the first one, its simplicity has made it gain a lot of popularity among many institutional investors. We analyze the new VaR incorporating different portfolio selection strategies and compare the difference between the new VaR and old VaR.

The second goal of this study mainly concentrates on the application of VaR in dynamic portfolio selection. The theoretical applications of VaR in risk management and regulation can be divided roughly into two main categories [16]. The first category is to impose a limit on the VaR of the portfolio. Another is to set aside a VaR-based capital for the risky portfolio. For the first category, there are several papers analyzing the effects of the imposed VaR-limit. Vorst (2001) [26] analyzes the portfolios with options that maximize expected return under the VaR-limit constraint. Basak and Shapiro (2001)[7] comprehensively analyze the optimal portfolio policies of utility maximizing investors under the exogenously-imposed portfolio VaR-limit constraints. For the second category, the Basel Committee on Banking Supervision requires the banks to maintain a minimal level of eligible capital whose amount is a function of the portfolio VaR. Comparing with the VaR-limit, the VaR based capital risk management is conceptually different.

In the work of Basak and Shapiro, they consider the optimization problem with the VaR-limit constraint. The formulation of their framework is given by:

$$\begin{cases} \max_{P(T)\geqslant 0} E\left[U(P(T))\right], \\ E\left[\xi(T)P(T)\right] \leqslant P(0), \\ VaR_p(P,0,T) \leqslant \overline{VaR}. \end{cases} \tag{1.1}$$

In this setting, $P(0)$ and $P(T)$ are the initial and terminal value of the portfolio, $U(\cdot)$ is the investor's utility function, $\xi(T)$ is the state-price density at time T, $T > 0$ is the investment horizon which coincides with the VaR horizon, and $VaR_p(P,0,T)$ is the VaR of the terminal portfolio value $P(T)$ evaluated at time 0 with confidence level p and $\overline{VaR}$ is exogenously-imposed limit on the VaR. There are two constraints in this optimization framework. The first one is the constraint on the budget assuring the expectation of the discounted portfolio value is no larger than the initial investment in the unique martingale probability measure. The second constraint is the VaR-limit on the terminal portfolio value. The optimal terminal portfolio value can be described by a piecewise function of the state-price density $\xi(T)$. The possible range of terminal value of $\xi(T)$ is divided into three intervals: $(-\infty, \underline{\xi})$, $[\underline{\xi}, \overline{\xi})$, and $[\overline{\xi}, \infty)$

which correspond to “good states”, “intermediate states”, and “bad states” of the portfolio, respectively. Basak and Shapiro find that, whenever the constraint is binding, the VaR risk managers are forced to reduce losses in the “intermediate state” with the expense of increasing loss in the “bad states”. In other words, the VaR risk managers tend to choose a larger exposure to risky assets than they would have invested in the absence of the VaR-limit constraints. Consequently, this strategy leads the losses in the worst states when the large loss occurs. Similarly, Vorst (2001) [26] also shows that the optimal policies of maximizing expected portfolio return with VaR-limit lead to a larger exposure to extreme losses.

In risk management with VaR-based capital requirement, the risk measure VaR is applied in a completely different way. According to the financial agreement in Basel Accord issued by the Basel Committee on Banking Supervision, the banks must maintain a minimal level of eligible capital at all times as a function of the portfolio VaR. The purpose of Basel Accord is to strengthen the soundness and stability of the international banking system [3]. In 1996, an amendment on the market risk capital requirement was added to Basel Accord [4],[5],[6]. In this amendment, the bank's assets are separated into two categories: *Trading book* and *Banking book*. The trading book contains financial instruments that are intentionally held for short-term resale and marked-to-market [15]. The banking book consists of loans that are not marked-to-market and the major risk of this part is credit risk. By the amendment, the bank has to hold capital to cover the market risk of the portfolio of different traded instruments in the trading book. The market risk capital charge is equal to the maximum of the previous day's 10-day VaR and the average 10-day VaR over the last 60 business days times a multiplicative factor δ. The 10-day VaR is calculated at 99% confidence level. The multiplier δ is between 3 and 4 and it is determined by back testing results [6]. To summarize, the market risk capital charge on any day t is

$$(1.2) \qquad max\left(\delta\frac{1}{60}\sum_{i=1}^{60} VaR_{99\%}(P,t-i,10), VaR_{99\%}(P,t-1,10)\right),$$

where $VaR_{99\%}(P,t-i,10)$ is 10-day VaR of the portfolio on day $t-i$ with confidence level 99%.

Comparing with the practice of VaR-based capital risk management, the optimization framework (1.1) with VaR-limit has two major shortcomings. First, it assumes that the portfolio VaR is never reevaluated after the initial date. Many financial institutions with VaR-based risk management reevaluate VaR under certain frequency and adjust their investment portfolios according to the updated VaR. For example, banks complying with the Basel Accord are obligated to reevaluate the VaRs of the risky portfolios in the trading book daily and reserve the capital according to the updated VaRs. Therefore, the assumption of only one evaluation in VaR during the investment horizon is not realistic. Second, the formulation (1.1) does not incorporate the risk capital requirement. The required capital is part of the regulated portfolio and thus affects the portfolio VaR directly. Different trading strategies have different VaRs which require different amounts of risk capital. The decision of banks simultaneously influences the portfolio VaR and the required capital to cover the risk. Therefore, in order to reflect the realistic risk management practice, the optimization framework should incorporate the relationship between the risk-free asset (used as risk capital) and the VaR of the risky portfolio.

Several studies analyze Basel Accord's market risk requirement and develop optimization framework to incorporate some characteristics of it. Inspired by the work of Basak and Shapiro (2001), Kaplanski and Levy (2006) [16] analyze VaR-based capital requirement regulation under an optimization formulation which is very similar to (1.1). They transform the Basel's market risk capital requirement into an inequality constraint which puts a limit on the minimum of the portfolio terminal value. The solution of this optimization problem also has a similar form to the solution in formulation (1.1). Under their new framework, they analyze the efficiency of the VaR-based capital requirement regulation with different choices of multiplier δ. Their results show that there is an optimal level of required eligible capital from the regulation standpoint and the current Basel's range of δ is within the inefficient range. However, the VaR constraint in their framework is evaluated only at the end of the investment horizon. Cuoco, He, and Isaenko (2007) [11] derive the optimal portfolio selection subject to the VaR limit which is reevaluated dynamically. In their formulation, the trader must satisfy the specified risk limit during the investment horizon. They show that the concern expressed in the work of Basak and Shapiro do not apply. They also consider the formulation with tail conditional expectation limit as the constraint in the optimization which is suggested by Basak and Shapiro for correcting the shortcoming in VaR-limit formulation. Under the situation where the constraint is constantly reevaluated, the tail conditional expectation limit is equivalent to VaR limit. However, their analysis does not completely reflect Basel Accord's market risk requirement because their VaR constraint is not imposed on the amount of risk-free capital. Keppo, Kofman, and Xu (2010) [17] analyze the undesirable effect of Basel's credit and market risk requirements on the bank. They develop their banking model to account for the market risk capital requirement by restricting the holding of a risky portfolio within the certain range determined by the simplified formulation (1.2). In their formulation, the relationship between the holding of a risky portfolio and a risk-free asset is explicitly reflected in the constraints. That is, the buffer capital has to be larger than the product of δ and the VaR of the risky assets portfolio all the time. They show that if the expected return and volatility of the risky assets portfolio are high, the market risk requirement raises the default probability of the bank. That is, the market risk requirement is inefficient.

In this study, we extend those previous works in VaR application. There are two major improvements we expect to accomplish. First, we construct a sophisticated framework to develop the optimal portfolio selection strategy in which the Basel's VaR-based capital requirement is completely reflected. In other words, VaR-based capital requirement is formulated in terms of a lower bound on risk free asset and reevaluated all the time. Second, the framework can accommodate more complicated risk asset models such as Stochastic Volatility (SV) model as well as the simple Geometric Brownian Motion (GBM) model.

The rest of this paper is organized as follows. In Chapter II, we describe the general model setting from which two famous models (GBM and SV) can be derived. With this model setting, we apply Merton's framework to derive the optimal portfolio selection strategy when there is no constraint. In Chapter III, we analyze the VaRs incorporating portfolio selection strategies and compare the difference between the old VaR and new VaR. The strategies include optimal strategies derived from Chapter II and the simple ones with constant weight. In Chapter IV, we describe the framework for developing the optimal portfolio selection strategy in which the Basel's VaR-based capital requirement is completely reflected.

CHAPTER II

Dynamic Portfolio Selection

2.1 The Model Setting

We assume that the investor has two types of investment opportunities. The first one is a risk free asset $S_0(t)$ with constant interest rate r. The second one is a group of n risky assets whose prices is a vector process $S(t) = (S_1(t), ..., S_n(t))'$ ($'$ denotes transpose). Specifically, the asset prices satisfy the following stochastic differential equations:

$$dS_0(t) = rS_0(t)dt, \tag{2.1}$$

$$dS(t) = D(S(t))\mu^{(S)}(Y(t))dt + D(S(t))\sigma^{(S)}(Y(t))dW^{(S)}, \tag{2.2}$$

$$dY(t) = \mu^{(Y)}(Y(t))dt + \sigma^{(Y)}(Y(t))dW^{(Y)}. \tag{2.3}$$

In this setting, Y is a state variable for the presence of stochastic environment. In this study, Y is used to describe the market volatility. $W^{(S)}$ is an n-dimensional standard Brownian motion and $W^{(Y)}$ is a standard Brownian motion. The correlation between $dW^{(Y)}$ and $dW^{(S)}$ is ρdt where ρ is a $1 \times n$ row vector. $D(S(t))$ is a diagonal matrix with $(S_1(t), ..., S_n(t))$ on the diagonal. The instantaneous expected return $\mu^{(S)}(Y)$ is an $n \times 1$ vector and the instantaneous standard deviation of diffusive return $\sigma^{(S)}(Y)$ is an $n \times n$ matrix. Both of them are functions of a one-dimensional state variable Y. In the stochastic process of Y, the drift rate $\mu^{(Y)}(Y)$ and volatility $\sigma^{(Y)}(Y)$ are scalar functions of Y.

Moreover, we further assume that the instantaneous expected return $\mu^{(S)}(Y)$ and standard deviation of diffusive return $\sigma^{(S)}(Y)$ are formulated as follows

$$\begin{aligned} \mu^{(S)}(Y) &= r1_n + abY \\ \sigma^{(S)}(Y) &= a\sqrt{Y}, \end{aligned} \tag{2.4}$$

where 1_n is the n-dimensional column vector with 1 in all components i.e. $1_n = (1, ..., 1)'$, a is an invertible $n \times n$ matrix and b is an $n \times 1$ vector. As a result, the risk premium in (2.4) is abY. This form of risk premium is also used by Merton (1980)[21], Pan (2002)[24], and Liu (2007)[19]. For the state variable Y, the drift rate $\mu^{(Y)}(Y)$ and volatility $\sigma^{(Y)}(Y)$ in (2.3) are formulated as follows

$$\begin{aligned} \mu^{(Y)}(Y) &= d - cY \\ \sigma^{(Y)}(Y) &= g\sqrt{Y}, \end{aligned} \tag{2.5}$$

where the parameters c, d, and g are all assumed to be nonnegative. Moreover, we restrict the parameters d and g to satisfy $d > g^2/2$. If this inequality is violated, $Y(t)$ becomes 0 at some random time $\tau > 0$ with probability 1, and then Y(t)=0 for all $t > \tau$. $Y(t)$ is a mean reverting square root process. It is obvious that the state variable Y is always positive. The same process is used in the CIR (1985)[10] for the spot interest rate and Heston model(1993) [13] for the stochastic volatility.

The dynamics given in (2.1)-(2.3) is a generalized form which nests two models considered in this study. These two models are, in the order of complexity, Geometric Brownian Motion (GBM) model and Stochastic Volatility (SV) model.

Case I, Geometric Brownian Motion (GBM) model

This is the simplest form derived from the generalized model. In this case, the stochastic volatility is not incorporated in the model. That is, $Y = 1$. The instantaneous expected return $\mu^{(S)}(Y)$ and standard deviation of diffusive return $\sigma^{(S)}(Y)$ are then reduced to constants. The S process is formulated as follows

$$dS(t) = D(S(t))(r1_n + ab)dt + D(S(t))adW^{(S)}. \tag{2.6}$$

The risk premium R of the risky assets is a constant vector $R = ab$.

Case II, Stochastic Volatility (SV) model

In this case, the stochastic volatility Y is incorporated and the risk premium $R = abY$ is a time-varying random variable. Together with the setting (2.4)-(2.5), the asset price dynamics becomes

$$dS(t) = D(S(t))(r1_n + abY(t))dt + D(S(t))a\sqrt{Y(t)}dW^{(S)}, \tag{2.7}$$

$$dY(t) = (d - cY(t))dt + g\sqrt{Y(t)}dW^{(Y)}. \tag{2.8}$$

In this setting, the risky asset prices are driven by two sources of uncertainty: diffusion in S dynamics, $W^{(S)}$, and diffusion in volatility dynamics, $W^{(Y)}$. One should notice that the model becomes the Heston model (1993) [13] for single risky asset when a and b are reduced to scalars.

To construct the investor's portfolio, we follow the framework and assumptions in Merton (1971)[20]. The assumptions are:

1. there are no transaction costs;
2. short sales with full use of proceeds are allowed;
3. assets are traded continuously in time;
4. self-financing strategies are applied.

Given the initial wealth P_0, the dynamics of the investor's portfolio wealth $P(t)$ is given by

$$dP(t) = [\omega'\mu^{(S)}(Y(t)) + (1-\omega'1_n)r]P(t)dt + \omega'\sigma^{(S)}(Y(t))P(t)dW^{(S)}, \tag{2.9}$$

where ω is an $n \times 1$ vector which denotes the risky assets' relative weights. In this setting, ω represents the portfolio selection strategy used by the investor. It can be a constant vector, or a time-varying vector, or even a vector of functions of any other relevant variables such as Y. At time t, the investor's total wealth $P(t)$ is given by

$$\begin{aligned} P(t) = P_0 exp \Bigg\{ & \int_0^t \left(\omega'\mu^{(S)}(Y(u)) + (1-\omega'1_n)r - \frac{1}{2}\omega'\Sigma^{(S)}(Y(u))\omega \right) du \\ & + \int_0^t \omega'\sigma^{(S)}(Y(u))dW^{(S)}(u) \Bigg\}, \end{aligned} \tag{2.10}$$

where $\Sigma^{(S)}(Y(u)) = \sigma^{(S)}(Y(u))\sigma^{(S)}(Y(u))'$.

2.2 Optimal Dynamic Portfolio Selection

In this study, we assume that the risk-averse investor holds the portfolio over a fixed time interval $[0,T]$ and tries to maximize the expected utility $\mathbb{E}\left[U(P(T))\right]$ over terminal wealth. The objective function is only related to the portfolio value at time T. Arrow (1971)[1] argues that there are three desirable properties for the investor's utility function. Those three properties and their mathematical formulations are:

1. positive marginal utility for wealth, i.e. $\frac{dU}{dP} > 0$;
2. decreasing marginal utility for wealth, i.e. $\frac{d^2U}{dP^2} < 0$;
3. non-increasing absolute risk aversion, i.e. $d(-\frac{d^2U}{dP^2}/\frac{dU}{dP})/dP \leq 0$.

Logarithmic, power, and negative exponential utility functions have these three desired attributes. In this study, the power utility function is used and it has a constant relative risk aversion $((-\frac{d^2U}{dP^2}/\frac{dU}{dP})P = \gamma)$ over the terminal wealth. The utility function is defined as follows

$$U(P) = \begin{cases} \frac{P^{1-\gamma}}{1-\gamma}, & \text{if } P \geq 0, \\ -\infty, & \text{if } P < 0, \end{cases} \tag{2.11}$$

where γ is the risk aversion coefficient within [0,1] and is an indicator of investor's risk appetite. The second part of the utility function is a constraint that prevents the wealth from being negative. This utility function is a concave function and satisfies all three desirable properties of investor's utility function. The parameter γ would be different for different investors. The smaller the γ is, the less risk averse the investor is or the larger the investor's risk appetite is . Figure 2.1 shows different power utility functions with different choices of γ. This graph shows that as portfolio wealth P becomes larger, the utility of the investor with large γ grows slower than that of the investor with small γ. The utility function converges to P, $U(P) = P$, as γ converges to 0. This is the risk neutral case.

2.2.1 Optimization framework for portfolio selection

Following the framework established by Merton (1971)[20], we define a value function for the formulations (2.1)-(2.3)

$$V(P, Y, t) = \max_{\{\omega(s)\}_{s=t}^{T}} \mathbb{E}_t \left[U(P(T)) \right].$$

and the Hamilton-Jacobi-Bellman(HJB) equation for V is:

$$\begin{aligned}\max_{\{\omega(t)\}_{t=0}^{T}} \Big\{ & V_t + \frac{1}{2}\omega'\Sigma^{(S)}\omega P^2 V_{PP} + \omega'\Sigma^{(S,Y)} P V_{PY} + \frac{1}{2}\Sigma^{(Y)} V_{YY} \\ & + [\omega'\mu^{(S)} + (1-\omega' 1_n) r] P V_P + \mu^{(Y)} V_Y \Big\} = 0,\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}\Sigma^{(S)} &= \sigma^{(S)}\sigma^{(S)\prime}, \\ \Sigma^{(S,Y)} &= \sigma^{(S)}\rho'\sigma^{(Y)}, \\ \Sigma^{(Y)} &= \sigma^{(Y)}\sigma^{(Y)\prime}.\end{aligned} \tag{2.13}$$

The terminal condition is given by

$$V(P, Y, T) = \frac{P^{1-\gamma}}{1-\gamma}.$$

In order to solve for the optimal portfolio weight ω^*, we introduce the ansatz

$$V(P, Y, t) = \frac{1}{1-\gamma} P^{1-\gamma} f(Y, t), \tag{2.14}$$

where f is a function of Y and t satisfying the terminal condition $f(Y, T) = 1$. Then, the HJB equation (2.12) becomes

$$\begin{aligned}\max_{\{\omega(t)\}_{t=0}^{T}} \Big\{ & f_t + \left[(1-\gamma)\omega'\Sigma^{(S,Y)} + \mu^{(Y)}\right] f_Y + \frac{1}{2}\Sigma^{(Y)} f_{YY} \\ & + (1-\gamma)\left[-\frac{\gamma}{2}\omega'\Sigma^{(S)}\omega + \omega'(\mu^{(S)} - r 1_n) + r\right] f \Big\} = 0.\end{aligned} \tag{2.15}$$

The first order condition with respect to ω is

$$\Sigma^{(S,Y)} f_Y + \left(-\gamma \Sigma^{(S)} \omega + \mu^{(S)} - r1_n\right) f = 0.$$

Then, the optimal portfolio weight is given by

$$\omega^* = \frac{1}{\gamma} \Sigma^{(S)^{-1}} \left(\mu^{(S)} - r1_n\right) + \frac{1}{\gamma} \Sigma^{(S)^{-1}} \Sigma^{(S,Y)} \frac{f_Y}{f}. \tag{2.16}$$

The function f in (2.16) is an unknown function. To solve for f, we can substitute (2.16) back to (2.15) and obtain a PDE for f. The complete form of optimal portfolio weight can be derived by solving the PDE of f.

2.2.2 The solution of a general PDE

Before solving for the optimal portfolio weight, we first derive the solution of a general PDE. The special case of this PDE will be used in solving the portfolio selection problem for SV model. In this section, we consider a general PDE

$$f_t + C_1 f_{YY} + C_2 f_Y + C_3 \frac{f_Y^2}{f} + C_4 f = 0, \tag{2.17}$$

with the terminal condition $f(Y,T) = 1$. This PDE can not be solved analytically in general. We impose some conditions on the coefficients of this PDE in order to solve it. If all the coefficients C_1, C_2, C_3, C_4 of above PDE are linear in Y, say $C_i = h_i + l_i Y$ for $i = 1, ...4$, then the ansatz of f is $f(Y,t) = exp(A(t) + B(t)Y)$, where $A(t)$ and $B(t)$ are scalar functions. The corresponding partial derivatives of f are

$$\begin{aligned} f_t &= \left(A_t + B_t Y\right) f, \\ f_Y &= Bf, \\ f_{YY} &= B^2 f. \end{aligned} \tag{2.18}$$

When $f(Y,t) = exp(A(t) + B(t)Y)$ is substituted into the PDE (2.17), we have

$$(A_t + B_t Y)f + (h_1 + l_1 Y)B^2 f + (h_2 + l_2 Y)Bf + (h_3 + l_3 Y)B^2 f + (h_4 + l_4 Y)f = 0.$$

Combining all the like terms, we have:

$$(A_t + (h_1 + h_3)B^2 + h_2 B + h_4) + \left(B_t + (l_1 + l_3)B^2 + l_2 B + l_4\right) Y = 0$$

In order to hold the equation for all Y, the coefficients of Y and Y^0 (terms not related to Y) have to be zero, which leads to ordinary differential equations (ODEs) for $A(t)$ and $B(t)$. The PDE (2.17) can be solved by solving the following two Riccati equations:

$$A_t + (h_1 + h_3)B^2 + h_2 B + h_4 = 0 \tag{2.19}$$

$$B_t + (l_1 + l_3)B^2 + l_2 B + l_4 = 0 \tag{2.20}$$

with the terminal conditions $A(T) = 0$ and $B(T) = 0$. From the ODEs above, one can notice that the ODE (2.19) for $A(t)$ can be easily solved once $B(t)$ is given. The ODE for $B(t)$ is a Riccati equation. In order to solve (2.20), we define $q_0 = -l_4$, $q_1 = -l_2$, $q_2 = -(l_1 + l_3)$, and $\xi = \sqrt{(q_1)^2 - 4q_0 q_2}$. After the transformation the equation (2.20) becomes

$$M_{tt} - q_1 M_t + q_0 q_2 M = 0, \tag{2.21}$$

where $B(t) = -\frac{M_t}{M q_2}$. Depending on the value of ξ^2, there are two possible solutions for the equation (2.21).

Case I: $\xi^2 \geq 0$

The solution of (2.21) is given by $M(t) = u_1 e^{v_1 t} + u_2 e^{v_2 t}$ where $v_{1,2} = \frac{q_1 \pm \sqrt{{q_1}^2 - 4q_0 q_2}}{2}$ and u_1 and u_2 are certain constants to be determined by the terminal condition. From $B(t) = -\frac{M_t}{M q_2}$, we have

$$B(t) = -\frac{M_t}{M q_2} = -\frac{u_1 v_1 e^{v_1 t} + u_2 v_2 e^{v_2 t}}{(u_1 e^{v_1 t} + u_2 e^{v_2 t}) q_2}$$

By using the terminal condition, we have $u_1 = -u_2 \frac{v_2}{v_1} e^{(v_2-v_1)T}$ and the function $B(t)$ is given by

$$\begin{aligned}
B(t) &= -\frac{v_1 v_2(-e^{-v_1(T-t)} + e^{-v_2(T-t)})}{(-v_2 e^{-v_1(T-t)} + v_1 e^{-v_2(T-t)})q_2} \\
&= -\frac{q_0 q_2(-e^{(v_2-v_1)(T-t)} + 1)}{(-\frac{1}{2}(q_1+\xi)e^{(v_2-v_1)(T-t)} + \frac{1}{2}(q_1-\xi))q_2} \\
&= -\frac{2(e^{\xi(T-t)} - 1)q_0}{(q_1+\xi)(e^{\xi(T-t)} - 1) + 2\xi}.
\end{aligned}$$

Case II, $\xi^2 < 0$

By defining $\eta = \sqrt{4q_0q_2 - (q_1)^2}$, the solution of (2.21) is given by $M(t) = e^{\frac{1}{2}q_1 t}\left(u_1 cos(\frac{1}{2}\eta t) + u_2 sin(\frac{1}{2}\eta t)\right)$ where and u_1 and u_2 are certain constants to be determined by the terminal condition. Similarly, the function $B(t)$ is given by

$$B(t) = -\frac{M_t}{Mq_2} = -\frac{q_1}{2q_2} - \frac{\frac{\eta}{2}(-u_1 sin(\frac{1}{2}\eta t) + u_2 cos(\frac{1}{2}\eta t))}{q_2(u_1 cos(\frac{1}{2}\eta t) + u_2 sin(\frac{1}{2}\eta t))}$$

By using the terminal condition, we have $u_1 = u_2 \frac{q_1 sin(\frac{\eta}{2}T) + \eta cos(\frac{\eta}{2}T)}{\eta sin(\frac{\eta}{2}T) - q_1 cos(\frac{\eta}{2}T)}$ and

$$\begin{aligned}
B(t) &= -\frac{q_1}{2q_2} + \frac{\eta(\frac{q_1 sin(\frac{\eta}{2}T) + \eta cos(\frac{\eta}{2}T)}{\eta sin(\frac{\eta}{2}T) - q_1 cos(\frac{\eta}{2}T)} sin(\frac{\eta}{2}t) - cos(\frac{\eta}{2}t))}{2q_2(\frac{q_1 sin(\frac{\eta}{2}T) + \eta cos(\frac{\eta}{2}T)}{\eta sin(\frac{\eta}{2}T) - q_1 cos(\frac{\eta}{2}T)} cos(\frac{\eta}{2}t) + sin(\frac{\eta}{2}t))} \\
&= -\frac{q_1}{2q_2} + \frac{\eta q_1 cos[\frac{\eta}{2}(T-t)] - \eta^2 sin[\frac{\eta}{2}(T-t)]}{2q_2(q_1 sin[\frac{\eta}{2}(T-t)] + \eta cos[\frac{\eta}{2}(T-t)])} \\
&= -\frac{q_1^2 + \eta^2}{2q_2(q_1 + \eta cot[\frac{\eta}{2}(T-t)])} \\
&= -\frac{2q_0}{q_1 + \eta cot[\frac{\eta}{2}(T-t)]}.
\end{aligned}$$

Therefore, the complete form of $B(t)$ is

$$B(t) = \begin{cases} -\frac{2(e^{\xi\tau}-1)q_0}{(q_1+\xi)(e^{\xi\tau}-1)+2\xi}, & \text{if } \xi^2 \geq 0 \\ -\frac{2q_0}{q_1+\eta cot[\frac{\eta}{2}(T-t)]}, & \text{if } \xi^2 < 0 \end{cases} \tag{2.22}$$

where $\tau = T - t$, $\eta = \sqrt{4q_0q_2 - (q_1)^2}$, $q_0 = -l_4$, $q_1 = -l_2$, $q_2 = -(l_1 + l_3)$ and $\xi = \sqrt{(q_1)^2 - 4q_0q_2}$. Then we can substitute $B(t)$ into ODE (2.19), $A(t)$ can be easily solved simply by integrating both sides of the equation.

Liu [19] did more general work for a PDE similar to (2.17). He makes each coefficient in the PDE quadratic in Y and solves the PDE up to the solutions of ordinary differential equations. In order to get each coefficient quadratic in Y, a lot of complicated restrictions are imposed on the parameters which involve tensors calculation and require very tremendous computational effort for parameter calibration. Therefore, for practical purpose we use simpler constraints on the drift and diffusion terms by setting them as linear functions of the state variable Y .

2.2.3 Optimal portfolio weight solution

The optimal portfolio weight formula (2.16) is directly related to an unknown function f. In order to derive the complete form, we need to substitute this formula back to the HJB equation. Depending on the parameter settings of the two models under consideration, we have the following cases.

Case I, Geometric Brownian Motion (GBM) model

In this simple case, the parameter setting for (2.4) and (2.5) are:

$$Y = 1,\ c = d = g = 0 \tag{2.23}$$

The second term of (2.16) is gone, the optimal portfolio weight becomes

$$\omega^* = \frac{1}{\gamma}\Sigma^{(S)^{-1}}\left(\mu^{(S)} - r1_n\right) = \frac{1}{\gamma}(aa')^{-1}R. \tag{2.24}$$

where $R = \mu^{(S)} - r1_n$. Since $Y = 1$, there is no need to solve for the unknown function f in this case. When the investor has only one risky asset in the portfolio, the optimal weight ω^* is positively related to the risky asset risk premium R. It suggests that the investor should long the risky asset if its risk premium is positive or short the risky asset otherwise. The volatility parameter $\sigma^{(S)} = a$ affects the magnitude of the optimal weight . If the risky asset is very volatile the investor should reduce the amount of risky asset in the long or short position. Moreover, the optimal weight also depends on the risk-aversion of the given investor. If the investor is more risk averse (larger γ), the optimal weight decreases in its magnitude. If the investor has a great risk appetite (smaller γ), the optimal weight magnitude increases.

Under the optimal trading strategy, the risky asset weight is a constant vector when the asset price follows the GBM. Although the risky asset weight is kept constant throughout the whole time interval, it does not mean that there is no trading. On the contrary, the optimal strategy requires the investor to actively rebalance the investment portfolio in order to maintain the optimal risky asset weight. In other words, if the investor does not execute any trading within the given period, the quantity of the asset does not change but the risky asset weight will change with the asset price movement. In Figure 2.2, we show how the asset weight (dotted line) changes with the asset price (solid line). If there is no trading during the given time period, the relative asset weight increases (decreases) as the asset price increases (decreases). Therefore, in order to maintain the constant risky asset weight, the investor needs to buy or sell the risky assets according to the price movement as shown in the Figure 2.3.

Case II, Stochastic Volatility (SV) model

For SV model, in order to solve ω^*, one needs to substitute the optimal weight (2.16) into (2.15), then (2.15) becomes

$$
\begin{aligned}
&f_t + \frac{1}{2}\Sigma^{(Y)} f_{YY} + \left[\mu^{(Y)} + \frac{1-\gamma}{\gamma}\Sigma^{(S,Y)\prime}\Sigma^{(S)^{-1}}\left(\mu^{(S)} - r1_n\right)\right] f_Y \\
&+ \frac{1-\gamma}{2\gamma}\Sigma^{(Y,\rho)}\frac{f_Y^2}{f} \\
&+ (1-\gamma)\left[\frac{1}{2\gamma}\left(\mu^{(S)} - r1_n\right)'\Sigma^{(S)^{-1}}(\mu^{(S)} - r1_n) + r\right] f = 0,
\end{aligned}
\tag{2.25}
$$

where $\Sigma^{(Y,\rho)} = \sigma^{(Y)}\rho\rho'\sigma^{(Y)}$. This PDE is a special case of the general PDE (2.17) solved in the previous section. Together with (2.4), (2.5), and (2.13), the coefficients $C_1, ..., C_4$ of the PDE are given by

$$\begin{aligned}
C_1 &= h_1 + l_1 Y = \frac{1}{2}\Sigma^{(Y)} \\
&= \frac{1}{2}g^2 Y, \\
C_2 &= h_2 + l_2 Y = \mu^{(Y)} + \frac{1-\gamma}{\gamma}\Sigma^{(S,Y)'}\Sigma^{(S)^{-1}}\left(\mu^{(S)} - r1_n\right) \\
&= d + \left(-c + \frac{1-\gamma}{\gamma}\rho b g\right) Y, \\
C_3 &= h_3 + l_3 Y = \frac{1-\gamma}{2\gamma}\Sigma^{(Y,\rho)} \\
&= \frac{1}{2\gamma}(1-\gamma) g^2 \rho\rho' Y, \\
C_4 &= h_4 + l_4 Y = (1-\gamma)\left[\frac{1}{2\gamma}\left(\mu^{(S)} - r1_n\right)'\Sigma^{(S)^{-1}}(\mu^{(S)} - r1_n) + r\right] \\
&= (1-\gamma) r + \frac{1-\gamma}{2\gamma} b' b Y.
\end{aligned}$$

Since the solution of the PDE (2.17) has the form $f(Y,t) = exp(A(t) + B(t)Y)$, the optimal portfolio weight (2.16) becomes

$$\omega^* = \frac{1}{\gamma}a'^{-1}b + \frac{1}{\gamma}a'^{-1}\rho g B, \tag{2.26}$$

where B is given by (2.22) with $l_1 = \frac{1}{2}g^2$, $l_2 = -c + \frac{1-\gamma}{\gamma}\rho b g$, $l_3 = \frac{1-\gamma}{2\gamma}g^2\rho\rho'$, and $l_4 = \frac{1-\gamma}{2\gamma}b'b$.

The optimal risky asset weight in this case is time-varying due to the stochastic state variable Y. The risk premium $R = abY$ and the risky asset volatility $\sigma^{(S)} = a\sqrt{Y}$ are positively related to Y. To analyze the optimal weight with respect to risk premium, we define a constant vector $R_0 = ab$, namely risk premium coefficient. The optimal weight becomes

$$\omega^* = \frac{1}{\gamma}(aa')^{-1}R_0 + \frac{1}{\gamma}a'^{-1}\rho g B \tag{2.27}$$

The optimal weight of SV model is the sum of the myopic demand and the intertemporal hedging demand caused by the dynamics of the state variable [19]. The myopic demand is the risky asset weight that the investors would hold as if the state variable is constant. It is virtually the optimal weight in GBM model. The intertemporal hedging demand is the adjustment on myopic demand for the uncertainty of the state variable. When the correlation ρ between risky asset and state variable is zero, the intertemporal hedging demand is zero since there is no needs to hedge the uncertainty of Y. The intertemporal hedging demand converges to zero at the end of the investment horizon. In particular, we have several remarks regarding this time-varying function $B(t)$.

Remark II.1. The function $B(t)$ is non-negative and non-increasing on the interval $[0, T]$.

In the SV model, we assume that a is an invertible matrix and $g > 0$. Together with $b = a^{-1}R_0$, we have $l_1 = \frac{1}{2}g^2 > 0$, $l_2 = -c + \frac{1-\gamma}{\gamma}\rho g a^{-1}R_0$, $l_3 = \frac{1-\gamma}{2\gamma}g^2\rho\rho' \geq 0$, and $l_4 = \frac{1-\gamma}{2\gamma}R_0'(aa')^{-1}R_0 \geq 0$. Moreover, the ODE of B (2.20) can be revised as

$$B_t = Q(B) = -(l_1 + l_3)B^2 - l_2 B - l_4. \tag{2.28}$$

This ODE is an autonomous differential equation and can be analyzed on the phase line. On the phase line (Figure 2.4), the solution of the ODE moves along B axis. The number and positions of equilibrium points ($B_t = 0$) depend on the parameters: $l_1, ..., l_4$. The line can be segmented by the equilibrium points (grey circles) and the direction (solid arrows) of each segment is determine by the sign of $Q(B)$. Together with the terminal condition $B(T) = 0$, we can determined the possible direction (dotted arrows) for the solution $B(t)$ on the interval $[0, T]$. In the three panels of Figure 2.4, all possible positions of equilibrium points are demonstrated. Given the terminal condition $B(T) = 0$ (black star), the possible solution must move toward the black start on the phase line. Among all the segments shown on the diagram, only those with dotted arrows are possible solutions. When $R_0 = 0$, the terminal condition coincides with the equilibrium and B is zero on the interval $[0, T]$. The parameter l_2 determines the relative positions of equilibrium points on B axis. Another key parameter $\xi^2 = l_2^2 - 4(l_1 + l_3)l_4$ determines the number of the equilibrium points. It's obvious that all possible solutions always stay on the right-hand side of 0 on the phase line and point from right to left ($B \geq 0$ and $B_t \leq 0$). Therefore, $B(t)$ is non-negative and non-increasing on the interval $[0, T]$.

Remark II.2. In the case of the portfolio with only one single risky asset, if ρ and R_0 are non-negative, B is non-decreasing with respect to R_0 at any time t in $[0, T]$.

By taking the derivative with respect to R_0 on both sides of ODE (2.28), we have

(2.29)
$$B_{t,R_0} = -2(l_1 + l_3)BB_{R_0} - l_2 B_{R_0} - \frac{1-\gamma}{\gamma a} g\rho B - \frac{1-\gamma}{\gamma a^2} R_0.$$

Denote B_{R_0} by $H^{(R_0)}$. The ODE above becomes

(2.30)
$$H_t^{(R_0)} = [-2(l_1 + l_3)B - l_2]H^{(R_0)} - \frac{1-\gamma}{\gamma a} g\rho B - \frac{1-\gamma}{\gamma a^2} R_0.$$

with the terminal condition $H^{(R_0)}(T) = 0$. The solution of the ODE is given by

(2.31)
$$H^{(R_0)}(t) = \int_t^T \left(\frac{1-\gamma}{\gamma a} g\rho B(s) + \frac{1-\gamma}{\gamma a^2} R_0 \right) exp\left\{ \int_t^s [2(l_1 + l_3)B(u) + l_2] du \right\} ds.$$

Apparently, $H^{(R_0)}$ is non-negative when ρ and R_0 are non-negative at any time t in $[0, T]$. Therefore, B is non-decreasing with respect to R_0.

Remark II.3. In the case of the portfolio with one single risky asset, if ρ and R_0 are non-negative, B is non-increasing with respect to the volatility coefficient a at any time t in $[0, T]$.

Similarly, by taking the derivative with respect to a on both sides of ODE (2.28), we have the following ODE with $H^{(a)} = B_a$

$$H_t^{(a)} = [-2(l_1 + l_3)B - l_2]H^{(a)} + \frac{1-\gamma}{\gamma a^2} g\rho R_0 B + \frac{1-\gamma}{\gamma a^3} R_0^2, \tag{2.32}$$

with the terminal condition $H^{(a)}(T) = 0$. The solution of the ODE is given by

(2.33)
$$H^{(a)}(t) = \int_t^T \left(-\frac{1-\gamma}{\gamma a^2} g\rho R_0 B(s) - \frac{1-\gamma}{\gamma a^3} R_0^2 \right) exp\left\{ \int_t^s [2(l_1 + l_3)B(u) + l_2] du \right\} ds.$$

Apparently, $H^{(a)}$ is non-positive when ρ and R_0 are non-negative at any time t in $[0, T]$. Therefore, B is non-increasing with respect to a.

By taking the derivative of ω^* (2.27) with respect to R_0 and a respectively, we have

$$\omega^*_{R_0} = \frac{1}{\gamma a^2} + \frac{1}{\gamma a} \rho g H^{(R_0)},$$

and

$$\omega^*_a = -\frac{2}{\gamma a^3} R_0 - \frac{1}{\gamma a^2} \rho g B + \frac{1}{\gamma a} \rho g H^{(a)}.$$

Based on the previous three remarks, it is very straightforward that $\omega^*_{R_0} \geq 0$ and $\omega^*_a \leq 0$. Therefore, we have the following result for the optimal risky asset weight.

Remark II.4. In the case of the portfolio with one single risky asset, if ρ and R_0 are non-negative, ω^* is non-decreasing with respect to R_0 and is non-increasing with respect to the volatility coefficient a at any time t in $[0, T]$.

However, when the correlation coefficient ρ is negative, the analysis for the optimal weight is more complicated. As shown in Figure 2.5 - 2.6, the optimal weight could be decreasing with R_0 and increasing with a for certain parameter setting with negative ρ.

Through the analysis of the optimal weight on risky asset, one can notice that the GBM model and SV model share some common properties in optimal portfolio selection. First, the risky asset weight is positively related to risk premium R in GBM model. In SV model, the risky asset weight is positively related to risk premium coefficient R_0 when $\rho \geq 0$ and $R_0 \geq 0$. Second, when risk premium is non-negative, the risky asset weight decreases as the volatility (volatility coefficient in SV model) a increases ($\rho \geq 0$ is required in SV model). Some of these results can be used to analyze the expected utility with the optimal risky asset weight. The analysis on the expected utility is a fundamental element for optimal portfolio selection with the constraints of VaR-based capital requirement. However, those results might not be valid when ρ is negative in SV model. It induces lots of complexities in the analysis of the next step. Therefore, it becomes very difficult for us to analyze the relationship between the expected utility obtained by the optimal risky asset weight and the related parameters.

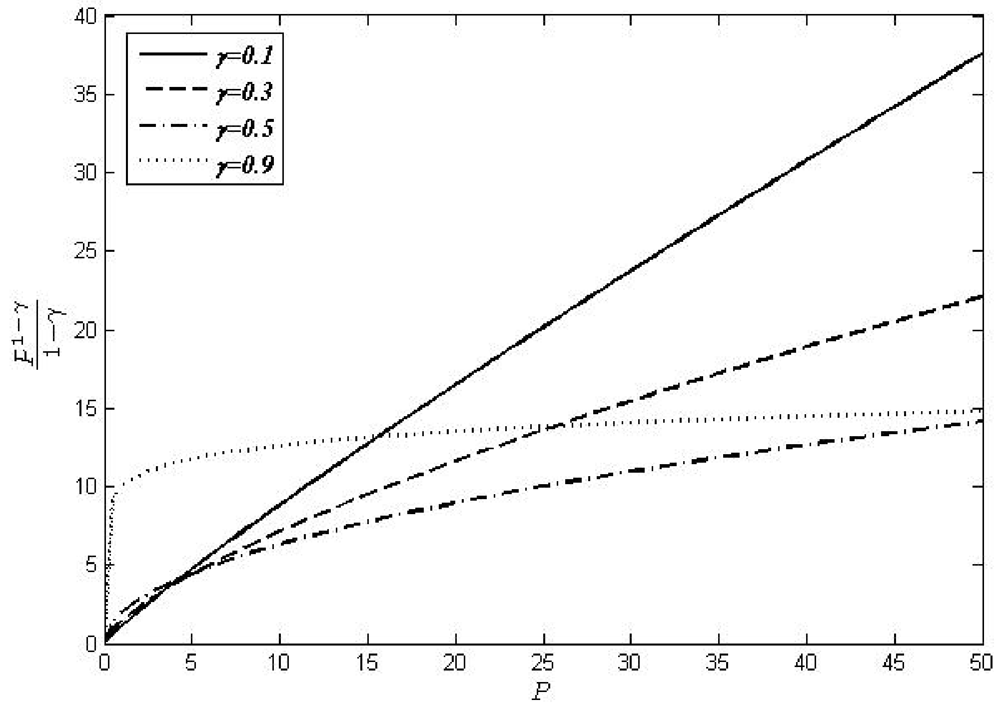


Figure 2.1: Utility function for non-negative portfolio value P with different choices of γ.

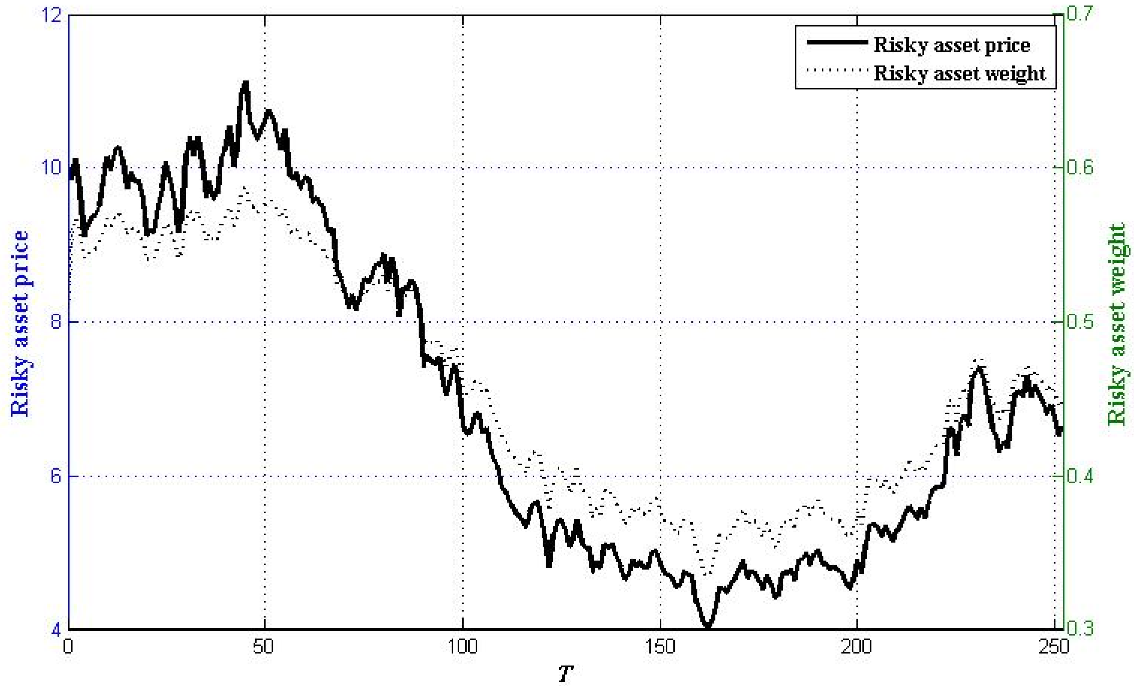


Figure 2.2: Risky asset weight (dotted line) changes as the asset price (solid line) changes when the share number of the asset is fixed. The risky asset price follows the GBM model with parameters: $\mu^{(S)} = 0.000278$, $\sigma^{(S)} = 0.0315$ and $T = 252$ days.

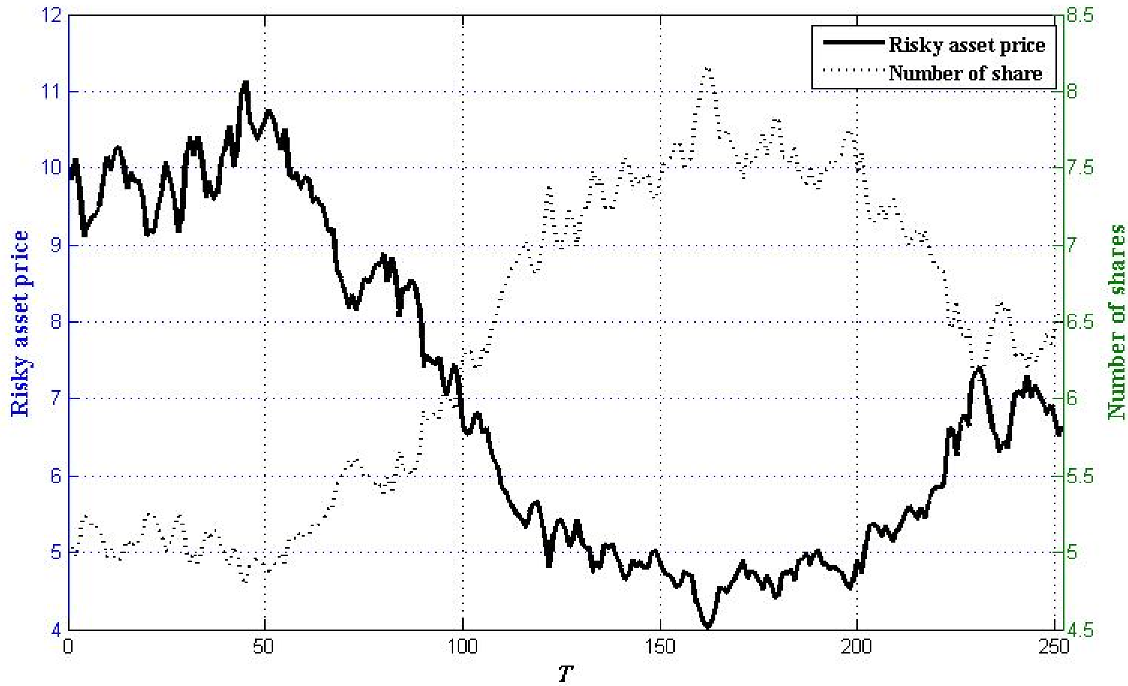


Figure 2.3: Risky asset share number (dotted line) changes as the asset price (solid line) changes when the weight of the asset is fixed. The risky asset price follows the GBM model with parameters: $\mu^{(S)} = 0.000278$, $\sigma^{(S)} = 0.0315$ and $T = 252$ days.

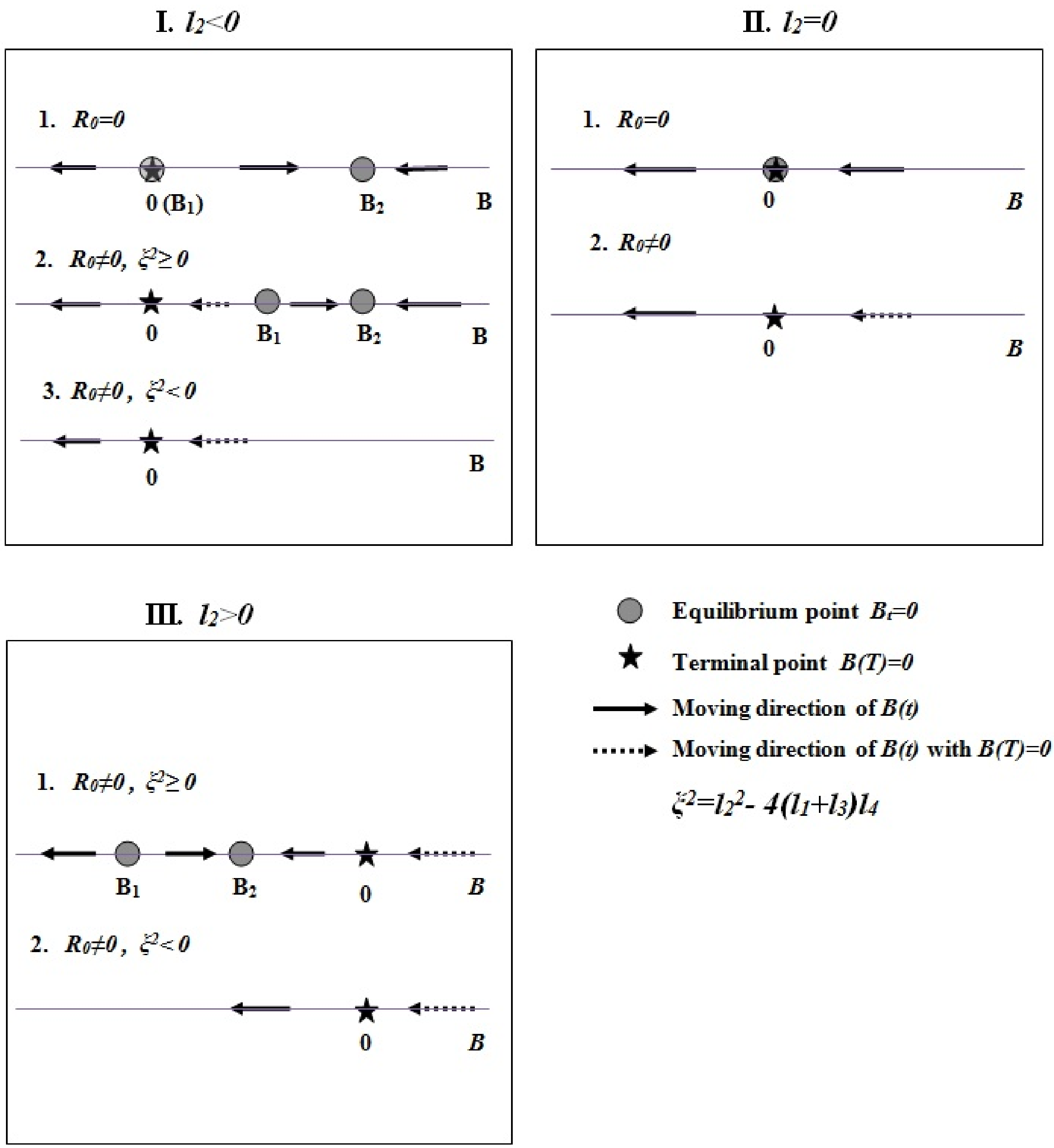


Figure 2.4: Phase line for the ODE of $B(t)$ with different parameter choices.

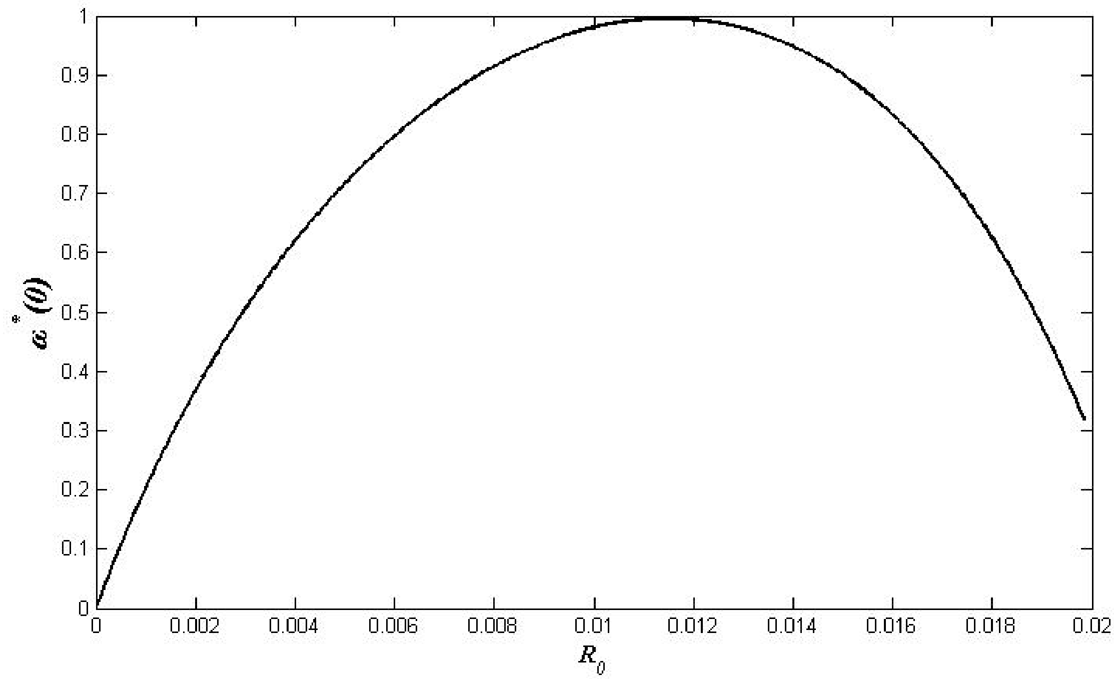


Figure 2.5: The optimal weight ω^* at time 0 changes with respect to R_0 under the SV model. When the correlation coefficient ρ is negative, the optimal weight could be decreasing with R_0. The parameters are set as: $\rho = -0.5$, $a = 0.21$, $c = 0.0015$, $d = 0.0015$, $g = 0.0525$ and $T = 252$ days.

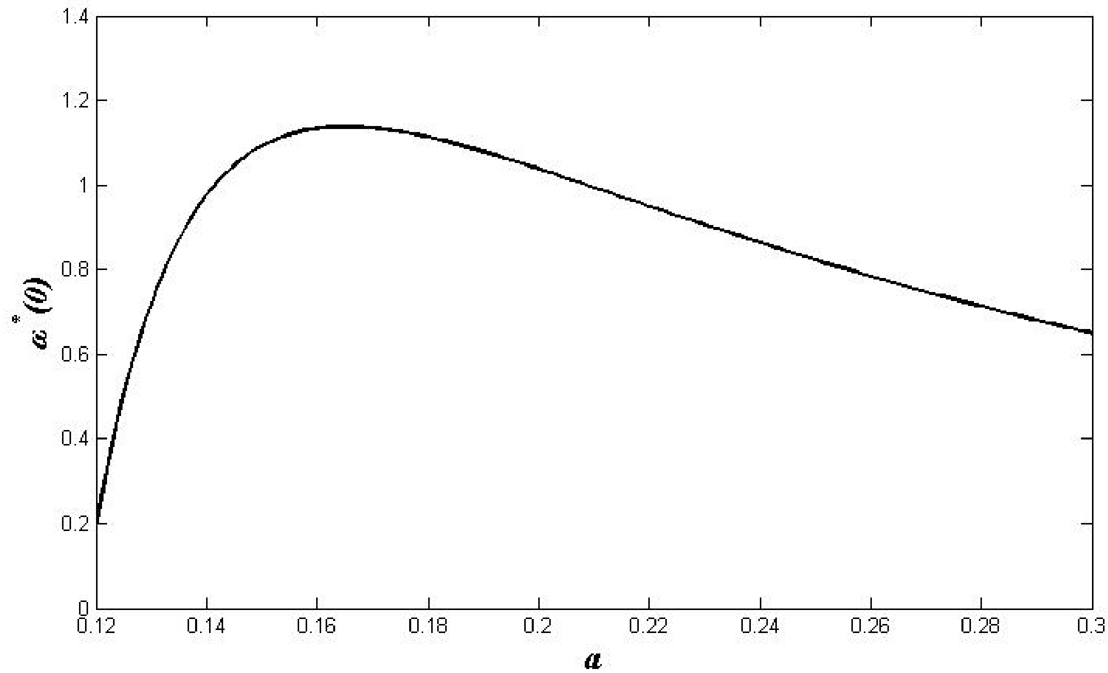


Figure 2.6: The optimal weight ω^* at time 0 changes with respect to a under the SV model. When the correlation coefficient ρ is negative, the optimal weight could be increasing with a. The parameters are set as: $\rho = -0.5$, $R_0 = 0.0119$, $c = 0.0015$, $d = 0.0015$, $g = 0.0525$ and $T = 252$ days.

CHAPTER III

Value-at-Risk (VaR) Incorporating Portfolio Selection Strategies

3.1 Value-at-Risk (VaR) Overview

To clarify the definition of VaR, we consider a portfolio whose value $P(t)$ is a time-dependent stochastic process. Given the portfolio value $P(t)$ at any time t and the investment horizon $\tau > 0$, VaR with confidence level p is the loss in value corresponding to p-quantile of the distribution of the portfolio loss $P(t) - P(t+\tau)$ over the investment horizon. The confidence level p is usually a number slightly less than 1 such as 99% in practice. In other words, if $F_{P(t)-P(t+\tau)}$ denotes the cumulative distribution function (cdf) of the loss in value over $[t, t+\tau]$, the VaR of the portfolio at time t is

$$VaR_p\,(P,t,\tau) = F^{-1}_{P(t)-P(t+\tau)}(p).$$

Another equivalent formulation of VaR can be derived as

$$VaR_p\,(P,t,\tau) = P(t) - F^{-1}_{P(t+\tau)}(1-p), \tag{3.1}$$

where $F_{P(t+\tau)}$ is the cdf of the portfolio value $P(t+\tau)$ at time $t+\tau$. Both formulations provide the same information to the investors. That is, the loss of the portfolio over the next investment horizon is no more than $VaR_p\,(P,t,\tau)$ with probability p. For example, if a portfolio's two-week VaR with confidence level 95% is \$1 million, it means there is a 95% chance that the value of the portfolio will drop no more than \$1 million over any given two-week period. From a statistical view, this value at risk measures the $1-p$ critical value of the probability distribution of the changes in market value. Apparently, there are three key elements in VaR definition, a confidence level p, the distribution of $P(t+\tau)$ and a fixed time interval over which the risk is assessed.

In the VaR estimation, the distribution of portfolio value $P(t+\tau)$ at the end of the investment horizon is the key component. The $P(t+\tau)$ distribution is affected by the initial value $P(t)$ and the portfolio selection strategy applied during the investment horizon $[t, t+\tau]$. In most applications of VaR, the VaR estimation is always under the assumption that there is no trading during the VaR horizon. Apparently, this assumption is unrealistic in real life. The distribution of a portfolio without trading is significantly different from the one with certain trading strategy. In Figure 3.1, the comparison of portfolio distributions is demonstrated. Two portfolios are both constructed with one risky asset and one risk-free asset. The risky asset price follows the GBM. Both portfolios start with the same risky asset weight which is the optimal risky asset weight and the same initial value which is \$100. However, one portfolio is not subject to any change during the investment horizon ($T = 252$ days). Another portfolio is always adjusted by the investor in order to maintain the optimal risky asset weight. For all levels of risk aversion parameter γ, these two portfolio value distributions are very different. Since the distributions are different, VaR estimations are also different. To reflect the true risk of the portfolio, the adjustment during the investment horizon must be incorporated in the VaR estimation, especially when the investment horizon is long. In the rest of study, we refer to the old VaR as the VaR with the assumption of no trading or rebalancing during the VaR horizon, and the new VaR as the VaR incorporating certain trading strategy. The trading strategy could be an optimal selection strategy or just as simple as the one with constant weight on each asset.

The rest of this chapter is organized as follows. In the second section, we demonstrate the new VaR estimation incorporating the optimal portfolio selection strategy and analyze the difference between the old VaR and new VaR. In the third section, we develop a theoretical framework to analyze the VaR incorporating a simple portfolio selection strategy.

3.2 VaR With Optimal Dynamic Portfolio Selection

The main objective of this section is to demonstrate the VaR estimation incorporating the portfolio selection strategy and the difference between the new VaR and the old VaR. The VaR analysis is based on the portfolio consisting of a risk-free asset and one risky asset. One of the advantages of the analysis with only one risky asset lies in the parameter dimension and the identification of essential factors. With one single risky asset, the number of parameters is greatly reduced comparing to the problem with multiple risky assets. Moreover, those important factors such as drift rate and volatility can be easily identified as scalars. On the contrary, the drift rate is a vector and the diffusion term is a matrix in the case of multiple risky assets. Another advantage with one single risky asset is that analytical forms of VaRs can be derived under certain model such as GBM model. Even with a simple portfolio consisting of a risk-free asset and one single risky asset, the difference between the old VaR and the new VaR is very significant. The difference between these two VaRs depends on many factors such as drift rate, volatility, the length of VaR horizon, and the investor's risk-averseness. In the case of multiple risky assets, more variations will be added on the VaR difference due to the increasing number of parameters. Therefore, the simple case with only one risky asset is used to describe the impact. In this study, two different models are used to describe the dynamics of the risky asset: Geometric Brownian Motion (GBM) model and Stochastic Volatility (SV) model. Under these two models, the analytical formulation for optimal portfolio selection can be derived given that the investor's utility over terminal wealth is a concave function (2.11) with constant relative risk aversion.

Both old VaR and new VaR are calculated in this section. For the purpose of comparison, we construct two portfolios. In the first portfolio, there is no trading activity during the investment horizon. On the contrary, the second portfolio is actively adjusted according to certain selection strategy. $P^{(1)}$ and $P^{(2)}$ denote the portfolio value for the first portfolio and second portfolio, respectively. Both portfolios start with the same initial wealth and initial risky asset weight. The VaR estimation is based on the distribution of the wealth at the end of the investment horizon. It is obvious that the VaR estimation based on $P^{(1)}$ gives the old VaR ($VaR^{(1)}$) and the VaR estimation based on $P^{(2)}$ gives the new VaR ($VaR^{(2)}$). The analysis of the impact of the portfolio selection on VaR is based on the difference between these two VaRs. The VaR difference is represented by

$$G = VaR^{(1)} - VaR^{(2)}.$$

Then, if G is positive (negative), it means the old VaR overestimates (underestimates) the true risk of the portfolio (the new VaR). For simplicity, the initial wealth is set to be \$100. Therefore, the estimations from either VaRs or VaR difference can be viewed as the the percentage of the initial wealth. From all the following numerical experiments, we notice that the VaR difference changes sign with various parameter settings. The sign (positive or negative) of VaR difference shows whether the old VaR overestimates or underestimates the risk of the portfolio (the new VaR). Even with the GBM model in which analytical forms of VaR difference can be derived, the relationship between VaR difference and relevant parameters is very complicated. The numerical results suggest that the old VaR with no-trading assumption is not suitable in a volatile market (large volatility)or for a long investment horizon.

3.2.1 The case with Geometric Brownian Motion (GBM) model

In this case, parameters and optimal risky asset weight are constants. In particular, the optimal risky weight ω^* (2.24) is positively proportional to the risk premium $R = \mu^{(S)} - r$ with the multiplier $1/(\gamma{\sigma^{(S)}}^2)$. Without loss of generality, we can first start the analysis of VaR difference on the $R-\omega$ plane and concentrate on the effects of general ω and other variables. After that, we can easily extend our analysis to incorporate the optimal risky asset weight ω^* which is represented by a straight line on the $R-\omega$ plane.

The portfolio value $P^{(2)}(t)$ with constant risky asset weight ω also follows the GBM

$$dP^{(2)}(t) = [\omega(\mu^{(S)} - r) + r]P^{(2)}(t)dt + \omega\sigma^{(S)}P^{(2)}(t)dW^{(S)}.$$

Over the investment horizon [0,T], the analytical solution of $P^{(2)}(T)$ is given by:

$$P^{(2)}(T) = P^{(2)}(0)exp\left(\left[\omega(\mu^{(S)} - r) + r - \frac{1}{2}\omega^2{\sigma^{(S)}}^2\right]T + \omega\sigma^{(S)}\sqrt{T}Z\right),$$

where Z is a random variable following a normal distribution with mean zero and standard deviation one. The weight ω in this portfolio is fixed. In order to keep ω fixed, the investor needs to actively trade based on the market change. The VaR calculation based on this strategy is essentially the new VaR, $VaR^{(2)}$.

For the purpose of comparison, we construct another portfolio with the same initial risky asset weight ω and assume the investor will not execute any trading to adjust the portfolio. The VaR calculation based on this assumption is essentially the old VaR, $VaR^{(1)}$. Since the investor will not make any adjustment to the portfolio, the number of shares in the risky asset does not change during the given time interval and remains at $\omega P^{(1)}(0)/S(0)$. The portfolio value $P^{(1)}(t)$ at time $t = T$ is given by:

$$P^{(1)}(T) = P^{(1)}(0)\left\{\omega\ exp\left[(\mu^{(S)} - \frac{1}{2}{\sigma^{(S)}}^2)T + \sigma^{(S)}\sqrt{T}Z\right] + (1-w)exp(rT)\right\}.$$

VaR with confidence level p is the p-quantile of the loss distribution in portfolio value over the time interval [0,T]. The confidence level p is usually set no less than 0.95. In Basel's regulation [4], p is equal to 0.99 for the VaR estimation. By definition, we can calculate the VaRs for the above two portfolios as follows:

$$\begin{aligned} VaR_p^{(1)} &= P^{(1)}(0)\left\{1-\omega\ exp\left[(\mu^{(S)}-\frac{1}{2}\sigma^{(S)^2})T+\sigma^{(S)}\sqrt{T}Z_{1-p}\right]-(1-\omega)exp(rT)\right\} \\ VaR_p^{(2)} &= P^{(2)}(0)\left\{1-exp\left(\left[\omega(\mu^{(S)}-r)+r-\frac{1}{2}\omega^2\sigma^{(S)^2}\right]T+\omega\sigma^{(S)}\sqrt{T}Z_{1-p}\right)\right\}, \end{aligned}$$

where Z_{1-p} is the $(1-p)$-quantile of the normal distribution with mean zero and standard deviation one. In general, Z_{1-p} is a negative number. In practice, we usually use the 99th percentile of the loss and therefore Z_{1-p} is the 1st percentile of the normal distribution which is -2.3263.

For simplicity, we assume $P(0) = P^{(1)}(0) = P^{(2)}(0) = 100$. The difference F between these two VaRs is:

$$\begin{aligned} G &= VaR_p^{(1)} - VaR_p^{(2)} \\ &= P(0)e^{rT}\left[e^{g_1}-\omega e^{g_2}-(1-w)\right], \end{aligned}$$

where

$$\begin{aligned} g_1 &= \left(\omega R-\frac{1}{2}w^2\sigma^{(S)^2}\right)T+\omega\sigma^{(S)}\sqrt{T}Z_{1-p} \\ g_2 &= \left(R-\frac{1}{2}\sigma^{(S)^2}\right)T+\sigma^{(S)}\sqrt{T}Z_{1-p} \\ R &= \mu^{(S)}-r. \end{aligned}$$

The difference between these two VaRs is a function with multiple variables such as r, R, ω, $\sigma^{(S)}$ and T. If we consider the optimal trading strategy ω^* (2.24), the difference is also related to the risk aversion coefficient γ. To analyze the effects of all these parameters, we plot the contour map of VaR difference on the $R-\omega$ plane (Figures 3.2- 3.3) with different choices of $\sigma^{(S)}$ and T. In all the numerical experiments, parameters r, R and $\sigma^{(S)}$ are set in terms of one trading day and we assume that there are 252 trading days in one year. The Table 3.1 shows the range of parameters and the corresponding measurement per year.

Parameter	Value per trading day	Value per year
r	0.00019841	5%
R	$[-0.00019841,\ 0.00039683]$	$[-5\%,\ 10\%]$
$\sigma^{(S)}$	[0.0094, 0.0441]	$[15\%,\ 70\%]$

Table 3.1: Parameter setting for numerical experiments with the GBM model.

There are some common patterns in Figures 3.2-3.3. First, it is obvious that $G = 0$ when $\omega = 0$ or $\omega = 1$. When $\omega = 0$ or $\omega = 1$, it means that the investor does not hold any risky asset or the investor spends all the wealth in the risky asset. In either case, the strategy with the fixed weight is the same as the passive strategy (no trading strategy) and therefore the difference in VaR between these two strategies is zero. Second, when the weight is between 0 and 1, G is less than zero. $G < 0$ implies that the old VaR is less than the new VaR. Third, when $\omega > 1$ or $\omega < 0$, VaR difference G is greater than zero. That is, the old VaR is greater than the new VaR.

As we discussed before, even the simple trading strategy with fixed constant ω requires the investor constantly to adjust the share number of the risky asset to maintain the preset weight. To understand the common patterns shown in those VaR difference contour maps, we need to derive the formula of risky asset share number movement under the circumstance where the investor maintains constant risky asset weight. At any given time t, the risky asset weight can be represented as

$$\omega = \frac{k(t)S(t)}{P^{(2)}(t)}, \tag{3.2}$$

where $k(t)$ is the share number of the risky asset. Since the risky asset weight remains fixed, the same formula holds at time $t+\tau$

$$\omega = \frac{k(t+\tau)S(t+\tau)}{P^{(2)}(t+\tau)}. \tag{3.3}$$

The ratio of share number between t and $t+\tau$ is given by

$$\begin{aligned} \frac{k(t+\tau)}{k(t)} &= \frac{S(t)}{S(t+\tau)}\frac{P^{(2)}(t+\tau)}{P^{(2)}(t)} \\ &= \frac{exp\left\{\left(\omega R + r - \frac{1}{2}\omega^2\sigma^{(S)^2}\right)\tau + \omega\sigma^{(S)}\sqrt{\tau}Z\right\}}{exp\left\{\left(R + r - \frac{1}{2}\sigma^{(S)^2}\right)\tau + \sigma^{(S)}\sqrt{\tau}Z\right\}} \\ &= exp\left\{-(1-\omega)\left[R\tau - \frac{1}{2}(1+\omega)\sigma^{(S)^2}\tau + \sigma^{(S)}\sqrt{\tau}Z\right]\right\} \end{aligned} \tag{3.4}$$

where Z is a random variable following a normal distribution with mean zero and standard deviation one. Z is also the key variable that determines whether the risky asset price S moves up or down between t and $t+\tau$. If Z satisfies

$$Z > -\frac{\left(R + r - \frac{1}{2}\sigma^{(S)^2}\right)\sqrt{\tau}}{\sigma^{(S)}}, \tag{3.5}$$

the risky asset price increases from t to $t+\tau$. Otherwise, it decreases. Regarding the share number ratio, we have the following results.

Lemma III.1. *On the interval $[t, t+\tau]$, the share number ratio $k(t+\tau)/k(t)$ in (3.4) has the following properties:*

1. *When $\omega > 1$, the ratio $k(t+\tau)/k(t) < 1$ if risky asset price S decreases between t and $t+\tau$;*

2. *When $0 < \omega < 1$, the ratio $k(t+\tau)/k(t) > 1$ if risky asset price S decreases between t and $t+\tau$;*

3. *When $\omega < 0$, the ratio $k(t+\tau)/k(t) < 1$ if risky asset price satisfies*

$$S(t+\tau) > S(t)exp\left(r\tau + \frac{1}{2}\omega\sigma^{(S)^2}\tau\right). \tag{3.6}$$

Proof: When risky asset price S decreases between t and $t+\tau$, Z satisfies

$$Z < -\frac{\left(R + r - \frac{1}{2}\sigma^{(S)^2}\right)\sqrt{\tau}}{\sigma^{(S)}}. \tag{3.7}$$

Substituting the above inequality into (3.4), we have the first two properties.

When (3.6) holds, Z satisfies the following inequality

$$Z > -\frac{\left(R - \frac{1}{2}(1+\omega)\sigma^{(S)^2}\right)\sqrt{\tau}}{\sigma^{(S)}}. \tag{3.8}$$

Substituting the above inequality into (3.4), we have the last property. □

These three properties give a clear picture of the movement of the risky asset share number under the undesirable price change. Three plots in Figure 3.4 show the numerical results paralleling the properties listed in Lemma III.1. When ω is positive, the investor is holding the long position of the risky asset and hoping that the price of the risky asset will go up. Under the undesirable price change (price goes down) of the risky asset, the investor's behaviors are completely different for $\omega > 1$ and $0 < \omega < 1$. If $\omega > 1$, the investor will reduce the holding of the risky asset. On the contrary, the investor will increase the holding of the risky asset if $0 < \omega < 1$. According to the definition of VaR, VaR is a measurement of the loss in the extremely undesirable scenario given that the confidence level p is large enough. In this case, the extremely undesirable scenario is that the risky asset price continuously keeps falling. The investor with $\omega > 1$ will reduce the loss in the worst scenario. The investor with $0 < \omega < 1$ will have much larger loss in the worst scenario. Therefore, the old VaR overestimates the risk (VaR difference G is positive) when $\omega > 1$ and underestimates the risk (VaR difference G is negative) when $0 < \omega < 1$. With $\omega < 0$, the investor shorts the risky asset ($k < 0$) and anticipates the asset price will decrease. Loss occurs when the risky asset price increases. According to the third property, investor will reduce the number of the risky asset in short position when the risky asset price is above certain level relative to the previous price. In other words, the investor's behavior reduces risk in the extremely undesirable scenario in which the risky asset price continuously increases. Therefore, the old VaR overestimates the risk and the VaR difference G is positive. These results imply that the strategies used by investors could greatly change the risk characteristics of the portfolio. Since the old VaR does not account for the investor's strategy, it cannot reflect the true risk of portfolio.

Another important result observed from Figures 3.2-3.3 is that the absolute magnitude of the VaR difference increases as the volatility $\sigma^{(S)}$ or VaR horizon T increases for all levels of ω and R. Under a highly volatile market or a long investment horizon, the old VaR may largely underestimate or overestimate the risk. Based on the results shown on the contour map, we found that the types of trading strategies (long or short the risky asset) corresponding to different levels of ω determine whether the old VaR overestimates or underestimates the true risk. However, the volatility and the VaR horizon determine how big is the difference between the old VaR and the true risk.

With all the analysis based on general ω above, the VaR difference with optimal portfolio selection is straightforward. Based on the formula (2.24), the optimal weight is linear to the risk premium with the slope equal to $1/\gamma\sigma^2$. On the contour map (Figures 3.2- 3.3), the optimal weight can be represented by a straight line passing through the origin. With small γ (less risk averse investor), the optimal weight line is steeper. VaR differences caused by the optimal portfolio selection strategy for different investors can be observed on the corresponding straight line on the map. For those aggressive investors with small γ, a small change in risk premium would cause large change in VaR difference due to the large change in risky asset weight. Moreover, given a fixed risk premium, VaR difference may be positive or negative for different γ. This reflects the reality that different investors have various strategies to achieve their investment goals and those strategies significantly impact the VaR estimation. The old VaR without accounting for the investor's strategy cannot distinguish the risks among the investors with different levels of risk aversion.

3.2.2 The case with Stochastic Volatility (SV) model

Comparing with the GBM model, there are several obstacles for analyzing the VaR difference in the SV model ((2.7)- (2.8)). First, there is no analytical form for either the asset price or VaR. Therefore, the analysis cannot be done through deriving formulas for some important variables such as VaR difference G and risky asset share number k. Second, those important variables such as risk premium, volatility, and optimal risky asset weight are time-varying. The previous analysis based on the $R - \omega$ plane is not applicable in this case. Third, there are many parameters in the SV model. Actually, there are 9 parameters included in the analysis. The relationship among these parameters could be complicated. In order to overcome these difficulties, we analyze the VaR difference based on a large random sample from the parameter space. We draw a large sample (size of 50000) which is randomly sampled in the 9-dimensional cube with the uniform distribution. The 9-dimensional cube is constructed by specifying the range for each parameter. For each sample, the optimal risky asset weight and VaR difference are calculated. The analysis is then based on the observation of VaR difference distribution variation along different parameters.

The parameters of the SV model can be categorized into two groups. The first group is the group of parameters for the stochastic process of the risky asset and investor's decision: a, b, ρ, γ and T. Since $R_0 = ab$ is the risky premium coefficient, the parameter R_0 is used instead of using b alone. The second group is the parameters for the stochastic process of the state variable Y: c, d, g and the initial value Y_0. Since the Y process is a mean-reverting process, the parameter d/c represents the equilibrium of the process and c is the rate by which the variable reverts towards the equilibrium. In the numerical experiments, we use the ratio d/c as a parameter instead of using d alone. All the parameters are set in terms of one trading day and we assume that there are 252 trading days in one year. The ranges of parameters in both groups are given by the Tables 3.2-3.3. The ranges of all those 9 parameters form a multi-dimensional cube in the parameter space. A large sample (size of 50000) is drawn randomly from the uniform distribution. All the numerical experiments for the VaR difference in the SV model are conducted based on this sample.

Parameter	Range
a	$[0.001,\ 0.15]$
R_0	$[-0.000794,\ 0.002]$
ρ	$[-1,\ 1]$
γ	$[0.1,\ 0.9]$
T	$[22,\ 252]$

Table 3.2: Parameter setting for the stochastic process of the risky asset price and investor decision in numerical experiments (first group).

Parameter	Range
c	$[0.005,\ 0.1]$
d/c	$[0.05,\ 2]$
g	$[0.01,\ 1]$
Y_0	$[0.05,\ 1.5]$

Table 3.3: Parameter setting for the stochastic process of the state variable Y in numerical experiments (second group).

The process for identifying the relationship between the VaR difference and any specific parameter can be divided into three steps. First, the optimal weight and VaR difference are calculated for each element in the sample. The sample of VaR difference is then formed. Second, the range of the specific parameter is evenly divided into 20 subintervals. Since the sampling is based on uniform distribution, the number of samples falling into each subinterval is roughly 2500. Third, the VaR difference sample is aligned with the subintervals of the specific parameter. The statistics such as mean and quantiles of the 20 sub-samples of VaR difference are estimated. Unlike the GBM model, the optimal risky asset weight is a function of time in the SV model. In order to reveal the relationship between the VaR difference and the optimal risky asset weight, the average value of ω^* over the entire VaR horizon is used. Figures 3.5- 3.6 show the numerical results of the process for all the parameters. In each plot, three statistics: mean, 5^{th}-percentile and 95^{th}-percentile of the VaR difference are plotted against the given parameters. These three curves reveal the changes in value and the range of the VaR difference changing with the parameters. The 6 plots in Figure 3.5 demonstrate the relationship between the average optimal risky weight $\omega^*_{average}$ and parameters of risky asset price process and investor's decision in the first group. Graphs in Figure 3.6 demonstrate the results for the 4 parameters of the state variable Y process in the second group. Compared with the results in the first group, the curves of VaR difference related to second group are much flatter. Therefore, the VaR difference is less sensitive to the parameters in the second group.

Apparently, the 6 parameters in the first group differ a lot in the relationship with VaR difference. The results shown in the panels for parameters: $\omega^*_{average}$, T, and a share some similarities with the results in the GBM case. Some common properties lie in the panel of parameter $\omega^*_{average}$. First,the VaR difference tends to be zero when the average weight, $\omega^*_{average}$, is close to 0 or 1. The mean of VaR difference is also very close to zero when $\omega^*_{average}$ is 0 or 1. Moreover, the range of VaR difference is relatively narrow when $\omega^*_{average}$ is 0 or 1. Second, when $\omega^*_{average}$ is between 0 and 1, most of the VaR difference samples are negative. Third, the value of VaR difference tends to be positive when $\omega^*_{average}$ is less than 0 or greater than 1. Although the range of VaR difference increases dramatically when $\omega^*_{average}$ is less than 0, most samples of the VaR difference are positive. Other significant similarities with the GBM model lie in the results with the VaR horizon T and volatility coefficient a. It is evident that the range of VaR difference increases as T or a increases. In the other three plots, three curves imply that both the value and range of the VaR difference change along with the corresponding parameters. The range of VaR difference shrinks as γ increases. That is because the holding of risky asset is less (in absolute magnitude) for the investor with higher γ. In such case, the VaR differences tend to zero. Similar pattern can be observed when the risk premium coefficient R_0 is close to zero. For the correlation coefficient ρ, the curves of VaR difference are relatively flat compared to the other five plots. One can still observe that the range of VaR difference narrows when ρ tends to 1.

No matter how different are the VaR difference patterns shown in all the plots, they all convey the same information. That is, the old VaR with assumption of no trading during the VaR horizon could not reflect the true risk when the investor applies some trading strategy (sometimes the strategy is optimal) in his/her investment. Moreover, the difference between two VaRs is sensitive to some of the parameters. When the model for risky asset price is more complicated (more parameters), the number of key parameters increases and the range of VaR difference varies dramatically.

3.3 VaR With Simple Portfolio Selection Strategy

In this section, the main objective is to analyze VaR incorporating simple portfolio selection strategy for the portfolio consisting of risky assets only . Simple portfolio selection strategy is a strategy in which the weight of each asset remains constant during the whole investment horizon. Although this portfolio selection may not be optimal, many investors always maintain their portfolio according to certain preset structure in practice. Institutional investors have their own purposes and investment guidelines for the portfolio. For example, in some insurance companies, an investment portfolio could be used as capital for the loss reserve and the main goal of the portfolio is not for the aggressive return. For all kinds of investment purposes, many financial institutions have investment guideline documents in which the structure of the portfolio is specified and the boundaries of all types of investment are given. In some cases, the portfolio is maintained to match the preset target weight of each type of asset. Therefore, the simple portfolio selection strategy is one of the popular strategies in the real world. In 1996 amendment of Basel Accord [4],[5],[6], the banks are required to calculate the VaR of the risky assets for market risk capital requirement. The VaR estimation is essentially based on the portfolio consisting of risky assets only.

To match reality, we construct a risky portfolio consisting of n risky assets: $S(t) = (S_1(t), ..., S_n(t))'$ with self-financing. The portfolio value is denoted by $P_R(t)$. With the general setting in (2.2) and (2.4), the stochastic process for $P_R(t)$ is given by

$$dP_R(t) = (r + \omega' abY(t))P_R(t)dt + \omega' a\sqrt{Y(t)}P_R(t)dW^{(S)}, \tag{3.9}$$

where ω satisfies the constraint $\omega' 1_n = 1$. Since the term $\omega' adW^{(S)}$ is essentially a linear combination of a random vector with multivariate normal distribution, $\omega' adW^{(S)}$

is equivalent to a normal random variable with zero mean and standard deviation $\sqrt{\omega' a a' \omega} dt$. Therefore, we can define a 1-dimensional Brownian motion $W^{(X)}$ by

$$dW^{(X)} = \frac{\omega' a}{\sqrt{\omega' a a' \omega}} dW^{(S)}. \tag{3.10}$$

The stochastic process $P_R(t)$ can be equivalently represented with a 1-dimensional Brownian motion $W^{(X)}$ instead of the original n-dimensional Brownian motion $W^{(S)}$

$$dP_R(t) = (r + R_\omega Y(t))P_R(t)dt + \sigma_\omega \sqrt{Y(t)} P_R(t) dW^{(X)}, \tag{3.11}$$

where $R_\omega = \omega' ab$ and $\sigma_\omega = \sqrt{\omega' a a' \omega}$. Since the correlation between $dW^{(Y)}$ and $dW^{(S)}$ is ρ, the correlation between $dW^{(Y)}$ and $dW^{(X)}$ is $\rho_\omega = \rho a' \omega / \sqrt{\omega' a a' \omega}$.

3.3.1 The case with Geometric Brownian Motion (GBM) model

Under the model of GBM, the risky portfolio $P_R(t)$ is given by

$$dP_R(t) = (r + R_\omega)P_R(t)dt + \sigma_\omega P_R(t) dW^{(X)}. \tag{3.12}$$

Over the investment horizon $[t, t+\tau]$, the analytical solution of $P_R(t+\tau)$ is given by

$$P_R(t+\tau) = P_R(t) exp\left(\left[r + R_\omega - \frac{1}{2}\sigma_\omega^2\right]\tau + \sigma_\omega \sqrt{\tau} Z\right),$$

where Z is a random variable following a normal distribution with mean zero and standard deviation one. By definition, the VaR of the risky portfolio $P_R(t)$ at time t with confidence level p and VaR horizon $\tau > 0$ is given by

$$VaR_p(P_R, t, \tau) = P_R(t)\left\{1 - exp\left(\left[r + R_\omega - \frac{1}{2}\sigma_\omega^2\right]\tau + \sigma_\omega \sqrt{\tau} Z_{1-p}\right)\right\}, \tag{3.13}$$

where Z_{1-p} is the $(1-p)$-quantile of the normal distribution with mean zero and standard deviation one. Based on the above VaR formulation, we can define a VaR percentage (VaR%) function ϕ

$$\phi(p, \tau, \omega) = 1 - exp\left(\left[r + R_\omega - \frac{1}{2}\sigma_\omega^2\right]\tau + \sigma_\omega \sqrt{\tau} Z_{1-p}\right). \tag{3.14}$$

The VaR% (ϕ) is not related to the size of the portfolio and only gives the relative loss for the risky portfolio with simple portfolio strategy ω. This function is related to the portfolio selection strategy ω, VaR confidence level p, and VaR horizon τ. The VaR% (ϕ) is a good measurement for evaluating the risk of a certain portfolio selection strategy.

Theorem III.2. *For the simple portfolio selection policy ω, a risky portfolio P_R consists of n risky assets whose value processes follow GBM in (2.6) and the portfolio value follows the stochastic process in (3.12). The VaR% (ϕ) (3.14) of the risky portfolio P_R with confidence level p and VaR horizon $\tau > 0$ is a decreasing function of R_ω and an increasing function of σ_ω if $p > 50\%$.*

Proof: The derivatives of ϕ (3.14) with respect to R_ω and σ_ω are

$$\begin{aligned} \phi_{R_\omega} &= -\tau exp\left(\left[r + R_\omega - \frac{1}{2}\sigma_\omega^2\right]\tau + \sigma_\omega\sqrt{\tau}Z_{1-p}\right), \\ \phi_{\sigma_\omega} &= (\sigma_\omega\tau - \sqrt{\tau}Z_{1-p})exp\left(\left[r + R_\omega - \frac{1}{2}\sigma_\omega^2\right]\tau + \sigma_\omega\sqrt{\tau}Z_{1-p}\right). \end{aligned} \tag{3.15}$$

It is evident that ϕ_{R_ω} is negative. If $p > 50\%$, Z_{1-p} is negative and then ϕ_{σ_ω} is positive. Therefore, the VaR% is a decreasing function of R_ω and is an increasing function of σ_ω if $p > 50\%$. □

In practice, the VaR confidence level p is always above 90%. This theorem will be applicable in all the VaR applications.

3.3.2 The case with Stochastic Volatility (SV) model

With the presence of state variable $Y(t)$, the analytical solution of $P_R(t+\tau)$ over the investment horizon $[t, t+\tau]$ is more complicated. From (3.11), $P_R(t+\tau)$ is given by

(3.16)

$$P_R(t+\tau) = P_R(t)exp\left(\int_t^{t+\tau}\left[r + R_\omega Y(s) - \frac{1}{2}\sigma_\omega^2 Y(s)\right]ds + \int_t^{t+\tau}\sigma_\omega\sqrt{Y(s)}dW^{(X)}(s)\right).$$

Since $Y(t)$ is also a stochastic process, these two integrations

$$k_1(t,\tau) = \int_t^{t+\tau} Y(s)ds \quad and \quad k_2(t,\tau) = \int_t^{t+\tau} \sqrt{Y(s)}dW^{(X)}(s), \tag{3.17}$$

are random variables with unknown distributions. The analytical form of VaR is not available in this case. However, we still can analyze the VaR based on two key parameters: R_ω and σ_ω. First, by definition, the VaR and VaR% (ϕ) of the risky portfolio P_R have the following forms

$$\begin{aligned} VaR_p(P_R,t,\tau) &= P_R(t)\left\{1 - exp\left[r\tau + F^{-1}_{K(t,\tau)}(1-p)\right]\right\}, \\ \phi(p,t,\tau,\omega) &= 1 - exp\left[r\tau + F^{-1}_{K(t,\tau)}(1-p)\right], \end{aligned} \tag{3.18}$$

where

$$K(t,\tau) = \left(R_\omega - \frac{1}{2}\sigma_\omega^2\right)k_1(t,\tau) + \sigma_\omega k_2(t,\tau). \tag{3.19}$$

and $F_{K(t,\tau)}$ is the cdf of the random variable $K(t,\tau)$. The key component in VaR estimation is the $(1-p)$-quantile of $K(t,\tau)$, i.e. $F^{-1}_{K(t,\tau)}(1-p)$. The relationship between $F^{-1}_{K(t,\tau)}(1-p)$ and parameters R_ω can be summarized in the following lemma.

Lemma III.3. *$F^{-1}_{K(t,\tau)}$ is a non-decreasing function of R_ω.*

Proof: For simplicity, we denote $K(t,\tau)$, $k_1(t,\tau)$, and $k_2(t,\tau)$ by simpler symbols K, k_1, and k_2 respectively. Based on (3.19), the random variable K is a linear combination of two random variables: k_1 and k_2. The distribution function $F_K(x)$ can be formulated

$$F_K(x) = Prob\left\{(k_1,k_2) \mid \left(R_\omega - \frac{1}{2}\sigma_\omega^2\right)k_1 + \sigma_\omega k_2 \leq x\right\}.$$

For the given pair of $R_{\omega,1}$ and $R_{\omega,2}$ satisfying $R_{\omega,1} < R_{\omega,2}$, we have

$$\left(R_{\omega,1} - \frac{1}{2}\sigma_\omega^2\right)k_1 + \sigma_\omega k_2 < \left(R_{\omega,2} - \frac{1}{2}\sigma_\omega^2\right)k_1 + \sigma_\omega k_2,$$

because k_1 is positive. For a real number x, the corresponding two sets of (k_1, k_2) pairs satisfy

$$\left\{(k_1,k_2) \mid \left(R_{\omega,2} - \frac{1}{2}\sigma_\omega^2\right) k_1 + \sigma_\omega k_2 \leq x\right\} \subseteq \left\{(k_1,k_2) \mid \left(R_{\omega,1} - \frac{1}{2}\sigma_\omega^2\right) k_1 + \sigma_\omega k_2 \leq x\right\}.$$

Therefore, the corresponding two cdfs F_{K_1} and F_{K_2} satisfy

$$F_{K_2}(x) \leq F_{K_1}(x). \tag{3.20}$$

By the definition of the inverse distribution function, we have

$$F_{K_1}^{-1}(q) = \inf_{x\in\mathbb{R}} \{F_{K_1}(x) \geq q\},$$
$$F_{K_2}^{-1}(q) = \inf_{x\in\mathbb{R}} \{F_{K_2}(x) \geq q\},$$

for any $q \in [0,1]$. Based on (3.20), we have

$$\{F_{K_2}(x) \geq q\} \subseteq \{F_{K_1}(x) \geq q\}.$$

Therefore, the corresponding two inverse cdfs, $F_{K_1}^{-1}$ and $F_{K_2}^{-1}$, satisfy

$$F_{K_2}^{-1}(q) \geq F_{K_1}^{-1}(q).$$

and the statement of this lemma is then proven. □

Based on this lemma and the formulation of VaR% in (3.18), the following theorem can be directly derived.

Theorem III.4. *For the simple portfolio selection policy ω, a risky portfolio P_R consists of n risky assets whose value processes follow the SV mdoel in (2.7-2.8) and the portfolio value follows the stochastic process in (3.11). The VaR% (ϕ) (3.18) of the risky portfolio P_R with confidence level p and VaR horizon $\tau > 0$ is a non-increasing function of R_ω.*

Proof: By (3.18), VaR% is a decreasing function of $F^{-1}_{K(t,\tau)}(1-p)$. Together with the Lemma III.3, The VaR% is a non-increasing function of R_ω. $\square$

Unfortunately, we can not determine the relationship between the VaR% and σ_ω in this case. First, the parameter σ_ω appears in both coefficients of random variables k_1 and k_2. The range of random variable k_2 covers all the real numbers. Second, the correlation between k_1 and k_2 is unknown because the distributions of k_1 and k_2 are unknown. In this study, we rely on numerical experiments to unfold the relationship between the VaR% and σ_ω. Figure 3.7 shows the VaR% (ϕ) with different choices of R_ω, σ_ω and ρ_ω. The range of those three parameters are summarized in the following table. All the plots in Figure 3.7 indicate that VaR% (ϕ) decreases with R_ω with all

Parameter	Range
R_ω	[0, 0.004]
σ_ω	[0.01, 0.25]
ρ_ω	$\{-1, -0.5, 0, 0.5, 1\}$
c	0.05
d/c	1
g	0.1
Y_0	1
τ	10 days

Table 3.4: Parameter setting for the numerical experiments of observing the relationships between VaR% ϕ and parameters: R_ω, σ_ω and ρ_ω.

the choices of σ_ω and ρ_ω which matches the statement in Theorem III.4 and increases with σ_ω with all the choices of R_ω and ρ_ω. Moreover, the effect of σ_ω on VaR% is much larger than the effect of R_ω.

By the theorems and lemma (III.2-III.4) and the numerical results shown in Figure 3.7, we establish the relationship between the VaR% (ϕ) and two key parameters: R_ω and σ_ω for both GBM and SV models. By the risky portfolio formulation (3.11), R_ω and σ_ω are essentially the risk premium (risk premium coefficient in the SV model) and volatility (volatility coefficient in the SV model) of the risky portfolio.

The market risk capital requirement in Basel Accord puts an upper bound on the amount of capital that the banks are allowed to invest in the risky portfolio. This upper bound is directly related to the VaR of the risky portfolio. The portfolio selection for the risky portfolio directly affects these two key parameters by which the VaR and the maximal amount of capital on the risky assets are determined. If the investors apply a very aggressive strategy (pursuing high risk premium) to build the risky portfolio, their portfolio will be inevitably with high volatility and large VaR. It leads to higher market risk capital requirement and smaller amount of capital allocated to the risky assets. It may hurt the profitability of the portfolio overall. The relationship revealed in the theorems above give banks the guidelines to build the risky portfolio in the proper way such that enough amount of capital is allocated to a risky portfolio with balanced risk premium and volatility. The theorems are the important building blocks in developing optimal portfolio selection under the Basel's market risk capital requirement.

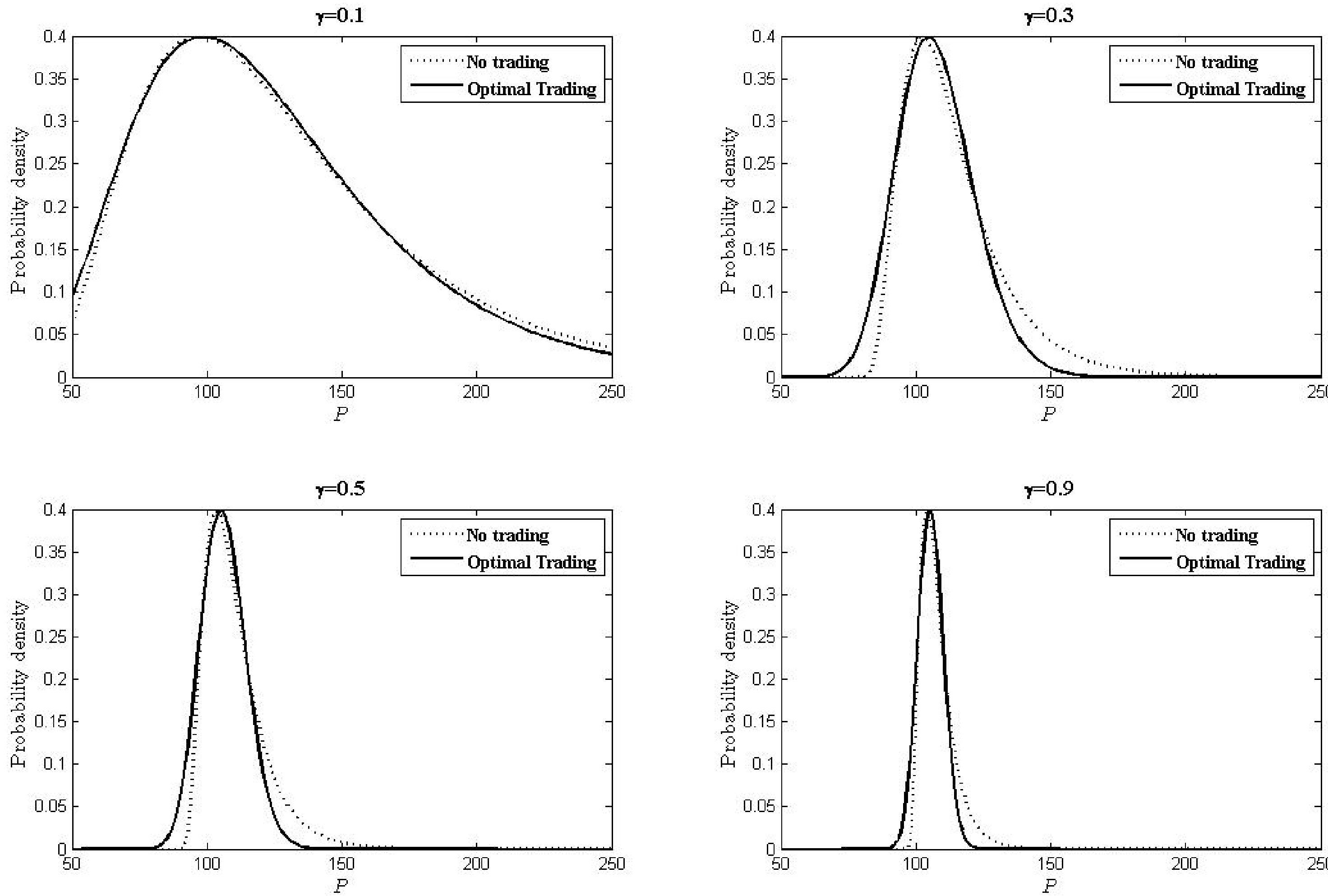


Figure 3.1: Portfolio distributions comparison between the one without any trading (dotted line) and the one with optimal trading strategy (solid line). Each panel corresponds to different risk aversion parameter γ. The risky asset price follows GBM with parameters: $\mu^{(S)} = 0.000278$, $\sigma^{(S)} = 0.0315$, and $T = 252$ days.

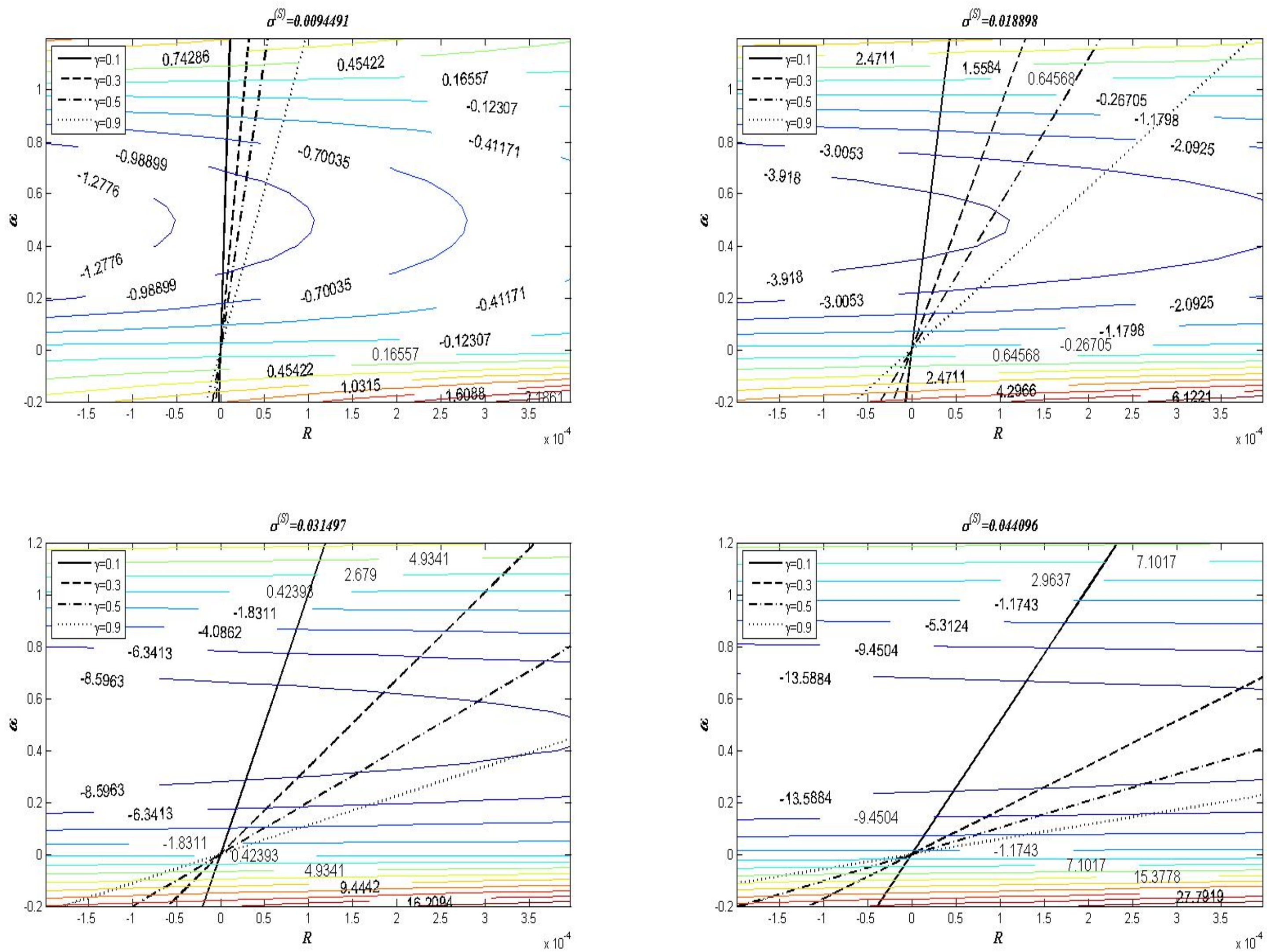


Figure 3.2: VaR difference contour map with different volatility levels $\sigma^{(S)}$. The four straight lines in each panel represent the optimal risky asset weight with different level of risk aversion parameter γ. The risky asset price follows GBM with VaR horizon $T = 252$ days.

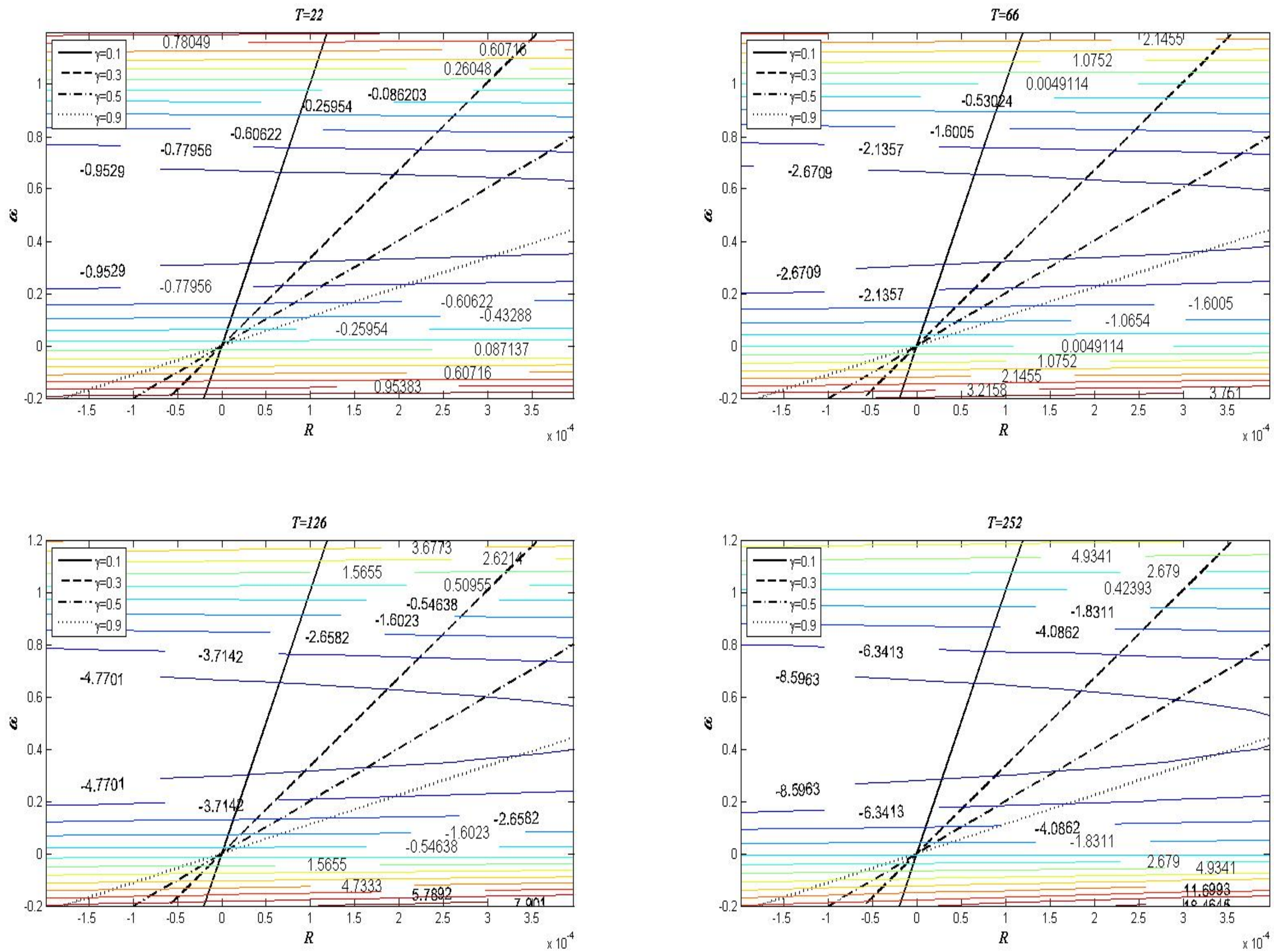


Figure 3.3: VaR difference contour map with different VaR horizons T. The four straight lines in each panel represent the optimal risky asset weight with different level of risk aversion parameter γ. The risky asset price follows GBM with volatility $\sigma^{(S)} = 0.031497$.

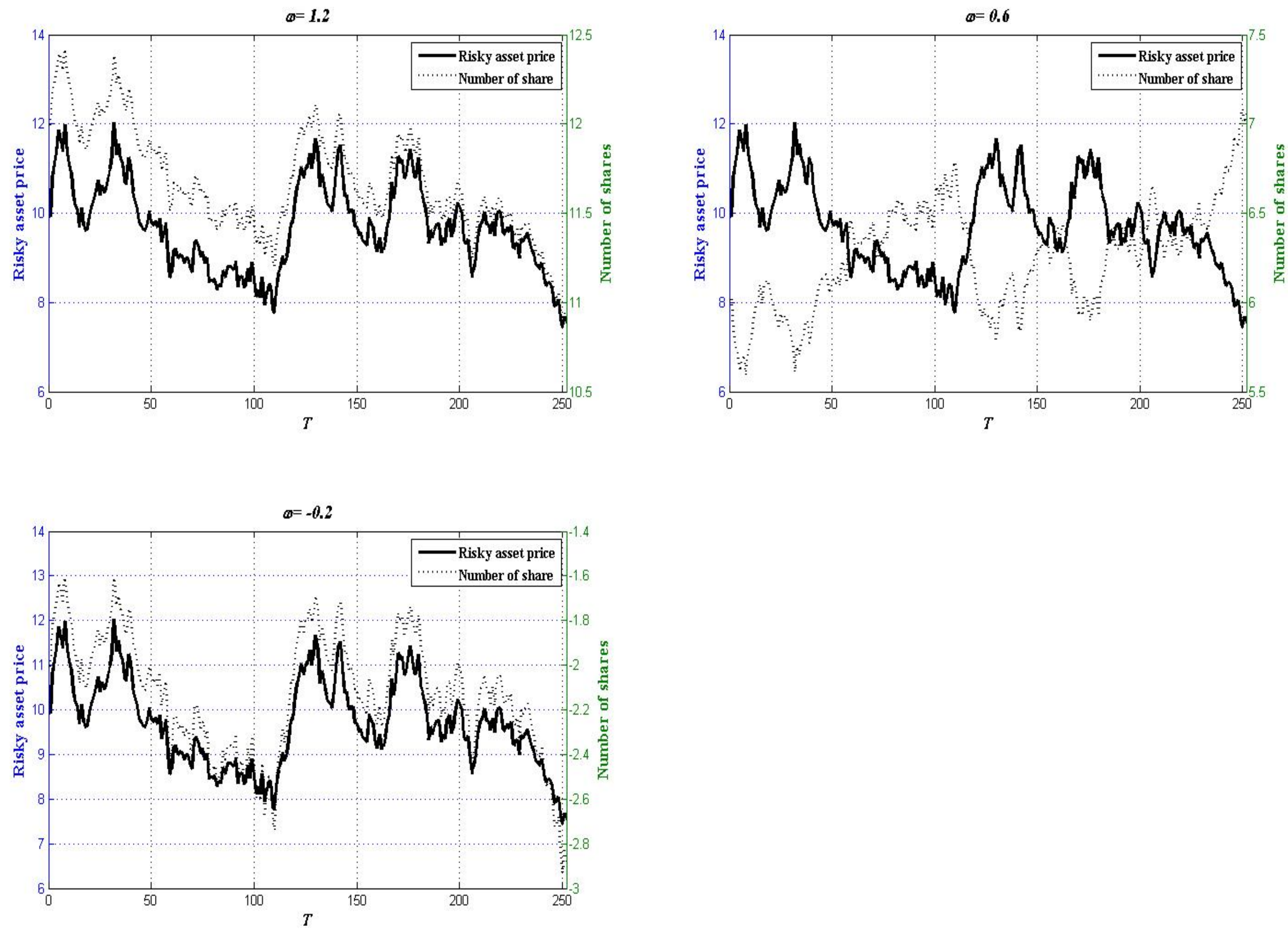


Figure 3.4: Risky asset share number (dotted line) changes as the asset price (solid line) changes with different risky asset weight. The risky asset weight is maintained during the investment horizon. The risky asset price follows GBM with parameters: $\mu^{(S)} = 0.000278$, $\sigma^{(S)} = 0.031497$, and $T = 252$ days.

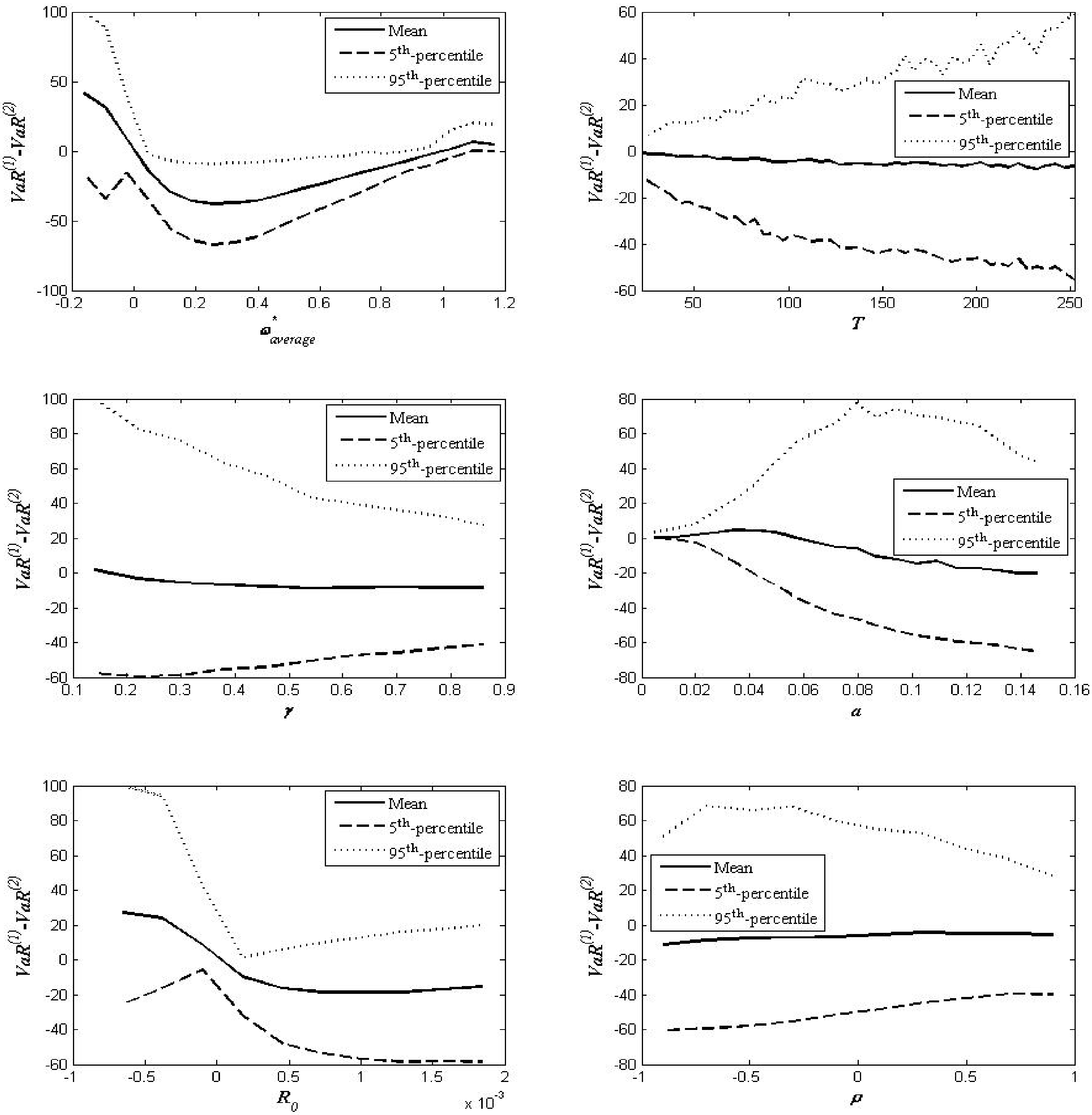


Figure 3.5: VaR difference with respect to the parameters for the stochastic process of risky asset and investor decision.

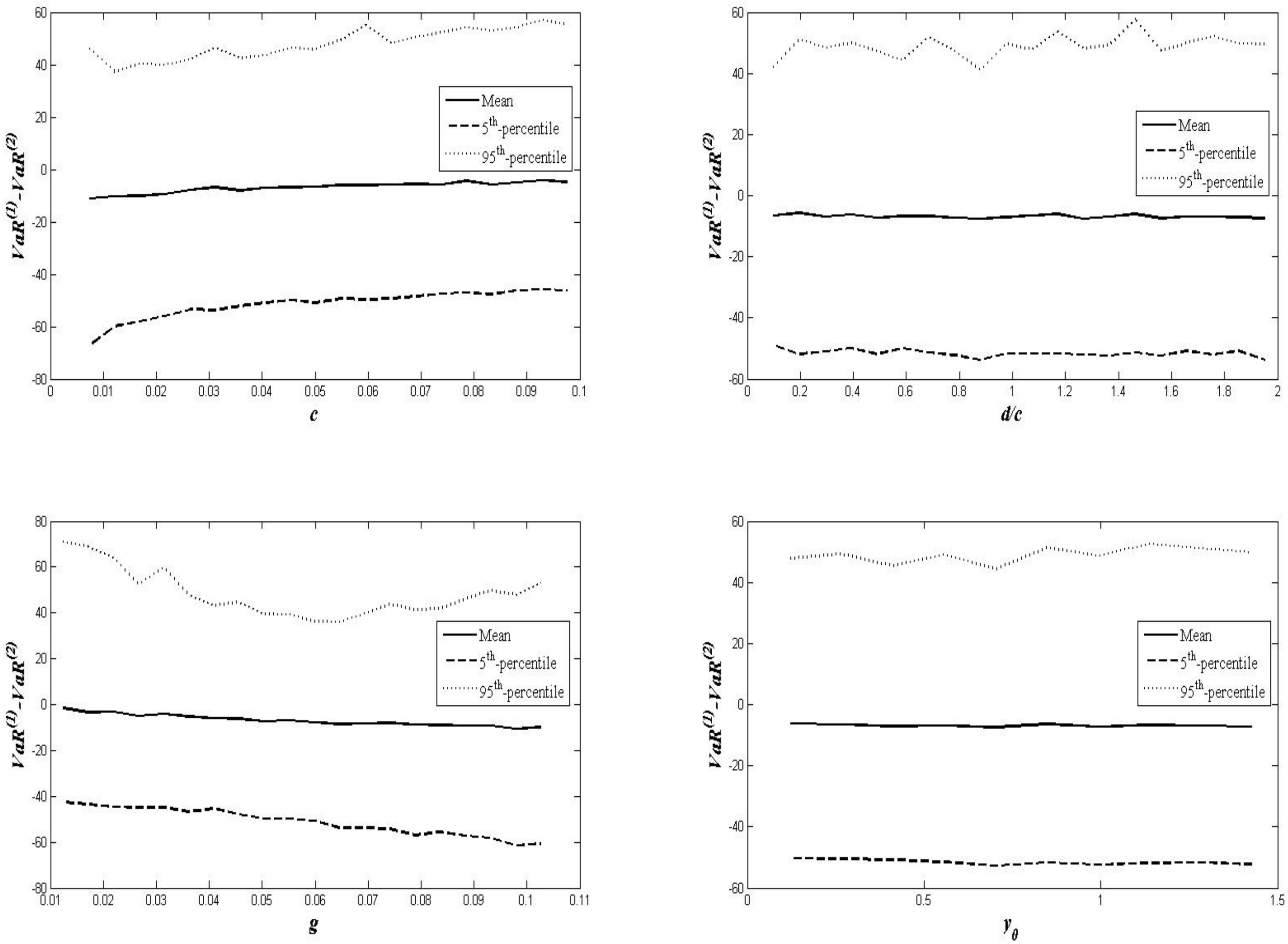


Figure 3.6: VaR difference with respect to the parameters for the stochastic process of the state variable Y.

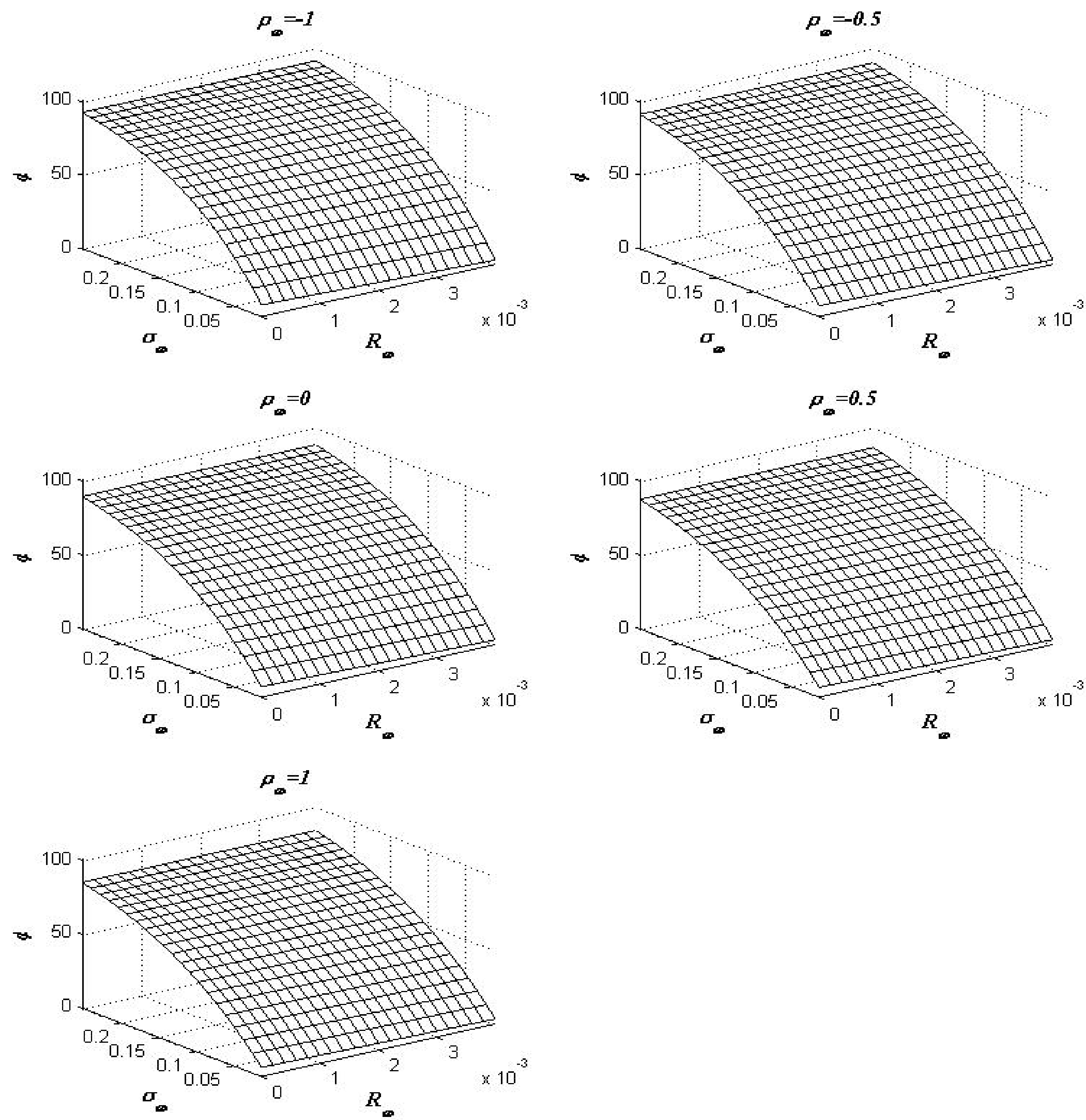


Figure 3.7: VaR% ϕ (in percentages) for difference choices of R_ω, σ_ω, and ρ_ω.

CHAPTER IV

Dynamic Portfolio Selection With VaR Capital requirement

4.1 VaR-based Risk Management Overview

In this study, we develop numerical schemes to find optimal portfolio selection strategies of the trading book under Basel's VaR-based capital requirement. The trading book contains financial instruments that are intentionally held for short-term resale and marked-to-market[15]. Ignoring the capital requirement for credit risk from the banking book, we consider the portfolio consisting of risk-free asset (used as capital) and marked-to-market risky assets. The group of risky assets forms the risky portfolio. According to the Basel's market risk capital requirement, the amount of risk-free asset (1.2) is equal to the maximum of the previous day's 10-day VaR and the average 10-day VaR over the last 60 business days times a multiplicative factor δ which is between 3 and 4. The VaR is estimated based on the risky portfolio with 99% confidence level. For simplicity, we assume the market risk capital charge at any time t is given by the positive part of the current 10-day VaR at 99% confidence level multiplied by δ, i.e.

$$\delta VaR_{99\%}(P, t, 10)^{+}, \tag{4.1}$$

where $x^{+} = max(x, 0)$. The positive operator $^{+}$ mathematically eliminates the possibilities of negative VaRs which lead negative risk capital requirement. Some of the assets in the trading book might also have credit risk (e.g. corporate bond) and counterparty risk (e.g. OTC derivative). In this study, since we mainly analyze the effect of market risk in the trading book, we assume that the capital charges for the risk other than market risk are zero.

By the Basel's VaR-based capital requirement, the portfolio of the trading book can be separated into two components: risk-free asset and risky portfolio consisting of risky assets. The investment strategy can be decomposed into two allocation problems. The first one is the capital allocation between the risk-free asset and risky portfolio. The second one is the portfolio selection among the risky portfolio. These two problems are actually coupled with each other. The decision on the second problem determines the upper bound of the capital allocated in the risky portfolio. On the other hand, the constraint in the first problem affects the bank's decision on the selection strategy of the risky portfolio. Aggressive strategy may give a high return on the risky portfolio. However, the consequence of high volatility leads to large VaR and reduces the amount of capital allocated to the risky portfolio and the return of the total wealth.

The rest of this chapter is organized as follows. Section 2 describes the formulation of the Basel's VaR-based capital requirement. Section 3 derives the optimal allocation between risk-free asset and risky portfolio under Basel's market risk capital requirement. Section 4 analyzes the expected utility obtained by the optimal allocation derived in section 3. Section 5 develops the process to find the optimal allocation within the risky portfolio.

4.2 Basel's VaR-based Capital Requirement Formulation

We assume that the trading book of the bank has one risk-free asset and n risky assets. The price of risk-free asset is denoted by $S_0(t)$. The risk-free asset is used as risk capital for the Basel's VaR-based capital requirement. The prices of those n risky assets are denoted by a vector process $S(t) = (S_1(t), ..., S_n(t))'$ ($'$ denotes transpose). The asset prices follow the general form in (2.1)-(2.3) which nests two models (GBM and SV) under consideration in this study. The total wealth $P(t)$ of the whole trading book can be divided into two parts: risk-free capital $P_C(t)$ and risky portfolio $P_R(t)$. The risk-free capital $P_C(t)$ is a portfolio consisting of only one asset. That is the risk-free asset. The amount of capital allocated to the risk-free capital or the share number of risk-free asset depends on the VaR of risky portfolio P_R. For the second part, the risky portfolio is a portfolio consisting of n risky assets whose prices follow the process $S(t)$. At any time t, the total wealth $P(t)$ is the sum of the risk-free capital $P_C(t)$ and the risky portfolio $P_R(t)$.

According to the Basel's market risk capital requirement, at any time t, the amount of risk-free capital should satisfy the following constraint

$$P_C(t) \geq \delta VaR_p \left(P_R, t, \tau\right)^+ , \tag{4.2}$$

where δ is a positive number between 3 and 4, τ=10, and $p = 99\%$. Because of this constraint, the bank needs to adjust the capital allocation between P_C and P_R from time to time to achieve two goals. The first goal is to satisfy the Basel's market risk capital requirement and the second one is to maximize the return or utility of the whole trading book. The optimization involves two allocation problems. The first allocation problem is the allocation between P_C and P_R. It can be denoted by ψ

which is the relative weight of P_R and satisfies

$$\psi = \frac{P_R}{P}.$$

The variable ψ is affected by the constraint in (4.2) and the bank's decision which depends on many factors such as risk aversion and investment purpose. The second one is the allocation within the risky portfolio P_R which can be denoted by an n-dimensional vector ω. ω is the relative weight vector of n risky assets in P_R and satisfies $\omega' 1_n = 1$. In this study, we assume that the simple portfolio selection strategy is applied when the bank constructs the risky portfolio in the trading book. That is, the relative weight vector ω of n risky assets is a constant vector. Under this assumption, the bank always maintains the preset structure in the risky portfolio P_R.

In this setting, the risky portfolio P_R is equivalent to a portfolio with only one hypothetical asset or index. This hypothetical index is constructed based on those n risky assets by using the simple trading strategy ω and this constant relative weight vector ω is always maintained. Therefore, with the general setting in (2.2) and (2.4), the index value can be described by the following stochastic process

$$dX(t) = (r + \omega' a b Y(t)) X(t) dt + \omega' a \sqrt{Y(t)} X(t) dW^{(S)}, \tag{4.3}$$

with initial value $X(0) = 1$. Since the term $\omega' a dW^{(S)}$ is essentially a linear combination of a random vector with multivariate normal distribution, $\omega' a dW^{(S)}$ is equivalent to $\sqrt{\omega' a a' \omega} dW^{(X)}$ where the 1-dimensional standard Brownian motion $W^{(X)}$ is defined as in (3.10)

$$dW^{(X)} = \frac{\omega' a}{\sqrt{\omega' a a' \omega}} dW^{(S)}. \tag{4.4}$$

The stochastic process $X(t)$ can be equivalently represented with the 1-dimensional standard Brownian motion $W^{(X)}$ instead of the original n-dimensional Brownian motion $W^{(S)}$

$$dX(t) = (r + R_\omega Y(t))X(t)dt + \sigma_\omega \sqrt{Y(t)}X(t)dW^{(X)}, \tag{4.5}$$

where $R_\omega = \omega' ab$ and $\sigma_\omega = \sqrt{\omega' a a' \omega}$. Since the correlation between $dW^{(Y)}$ and $dW^{(S)}$ is ρ, the correlation between $dW^{(Y)}$ and $dW^{(X)}$ is $\rho_\omega = \rho a'\omega/\sqrt{\omega' a a' \omega}$. One can notice that R_ω and σ_ω are essentially the risk premium coefficient and volatility coefficient of the index X.

The index dynamics in (4.5) reflects the return value on each unit of capital invested in those n risky assets with the trading strategy ω. In other words, at any time t, one unit of capital invested in the index will have

$$exp\left\{\int_t^{t+\tau}\left(r + R_\omega Y(s) - \frac{1}{2}\sigma_\omega^2 Y(s)\right)ds + \int_t^{t+\tau}\sigma_\omega\sqrt{Y(s)}dW^{(X)}(s)\right\}$$

in return after the investment horizon τ. Therefore, the index value $X(t)$ can be viewed as the "price" of the portfolio with strategy ω. The risky portfolio with strategy ω is equivalent to buying the index $X(t)$. If the bank invests $P_R(t)$ in the index $X(t)$ at time t, at the end of the investment interval $[t, t+\tau]$, the value of $P(t+\tau)$ is given by

$$P_R(t+\tau) = P_R(t)exp\left(\int_t^{t+\tau}\left[r + R_\omega Y(s) - \frac{1}{2}\sigma_\omega^2 Y(s)\right]ds + \int_t^{t+\tau}\sigma_\omega\sqrt{Y(s)}dW^{(X)}(s)\right). \tag{4.6}$$

Based on this formulation, the τ-day VaR of $P_R(t)$ with confidence level p has the following form

$$VaR_p\left(P_R, t, \tau\right) = P_R(t)\phi(p, Y, t, \tau, \omega), \tag{4.7}$$

where $\phi(p, Y, t, \tau, \omega)$ is the VaR percentage (VaR%) function defined in Chapter 3 and represents the risk of each capital unit invested on the index X. Based on the theorems (III.2-III.4) and numerical experiments in the previous chapter, the VaR% function ϕ is a decreasing function of R_ω and an increasing function of σ_ω if $p > 50\%$. These results give a very important guideline for the bank to construct its risky portfolio with balanced return and volatility for achieving maximal utility of the whole trading book.

With the hypothetical risky index X, we can view the total wealth $P(t)$ from a different perspective. The portfolio $P(t)$ in the trading book is essentially a portfolio consisting of two assets: S_0 and X. The relative weight ψ of risky portfolio P_R is essentially the relative weight on the risky index X. The portfolio selection strategy ω in risky portfolio directly affects the risk premium coefficient R_ω, volatility coefficient σ_ω and the correlation ρ_ω of the index X. The VaR of risky portfolio is the product of P_R and the VaR% function ϕ of X. Under the assumption of self-financing, the total wealth $P(t)$ can be described by the following stochastic process

$$dP(t) = (\psi R_\omega Y(t) + r)P(t)dt + \psi\sigma_\omega\sqrt{Y(t)})P(t)dW^{(X)}. \tag{4.8}$$

In this formulation, the relative weight ψ also represents the allocation strategy between risk-free capital and risky portoflio. It could be a function of time or any other relevant factors. Given the capital allocation between the risk-free capital and the risky portfolio, we have $P_R(t) = \psi P(t)$. Together with (4.7), the τ-day VaR with probability p of the risky portfolio is

$$VaR_p\,(P_R, t, \tau) = \psi P(t)\phi(p, Y, t, \tau, \omega).$$

Since $P_C(t) = (1 - \psi)P(t)$, the inequality (4.2) derived from Basel's market risky capital requirement becomes

$$\psi \leq \frac{1}{1 + \delta\phi(p, Y, t, \tau, \omega)^+}. \tag{4.9}$$

Therefore, to construct the portfolio with risk-free capital and risky portfolio, the bank has to cope with two control variables ψ and ω. ω represents the asset allocation within the risky portfolio consisting of n risky assets and $\omega' 1_n = 1$. ψ is the relative weight on the risky portfolio with the strategy ω. The inequality (4.9) gives the constraint on the risky portfolio construction strategy ω and the capital allocation ψ on the risky portfolio. If the bank chooses a very aggressive strategy ω, it leads a high VaR of P_R which lowers the upper bound of the capital allocation ψ on the risky portfolio and the return of the total wealth will be limited. On the other hand, a conservative strategy ω can increase the upper bound for the capital allocated to the risky portfolio. However, the low return on P_R could reduce the return of the total wealth. Therefore, to find a ω with balanced return and volatility in P_R is the essential part in the optimization process.

The process of finding optimal allocation to maximize the bank's utility in trading book can be divided into two steps. The first step is to develop the optimal $\psi^*(\omega)$ for any given simple portfolio selection ω in the risky portfolio. In this step, we only concentrate on deriving the optimal allocation between risk-free capital and risky portfolio under the constraint (4.9). The optimal solution ψ^* is a function of ω. Moreover, the expected utility $\mathbb{E}\left[U(P(T))\right]$ obtained by ψ^* is also a function of ω. In further analysis, we notice that ω affects ψ^* and expected utility only through three parameters: R_ω, σ_ω and ρ_ω. Instead of performing analysis based on ω directly, we develop the relationship between the expected utility $\mathbb{E}\left[U(P(T))\right]$ with strategy ω and the parameters R_ω, σ_ω and ρ_ω of the risky index. The relationship is the foundation for the second step. The second step is to find the optimal ω^* based on the results in first step. Since ω is an n-dimensional vector, searching for an optimal solution in a high dimensional space could be very difficult. However, the results from the first step can help us convert the searching space into a parameter space formed by R_ω, σ_ω and ρ_ω. That is, the result derived in the first step is used to reduce the dimension of the problem.

4.3 Optimal Allocation Between Risky Portfolio And Risk-free Capital

In this section, the goal is to derive the optimal allocation strategy between the risky portfolio and risk-free capital with the given simple portfolio selection strategy ω in the risky portfolio. The allocation between the risky portfolio and risk-free capital is represented by the relative weight ψ on the risky portfolio. As discussed in the previous section, when ω is fixed, the risky portfolio with n risky assets is equivalent to the portfolio with only one hypothetical risky index X. The stochastic process of X is given by (4.5). The relative weight ψ of the risky portfolio is essentially the relative weight on the risky index X. The fixed ω may not be the optimal choice for the allocation within risky portfolio. It just represents any one of the admissible portfolio selection policies used by the bank in the risky portfolio. Since ω affects the risk premium coefficient, volatility coefficient and correlation of the risky index, the optimum ψ^* is a function of the vector ω. With the optimum $\psi^*(\omega)$, the optimal choice of ω can be derived in the next step.

The capital allocated in the risky portfolio is limited by the constraint (4.9) based on Basel's market risk capital requirement. According to the constraint (4.9), the upper bound of the risky portfolio weight ψ is

$$\frac{1}{1+\delta\phi_(p,Y,t,\tau,\omega)^+},$$

where $\phi(p,Y,t,\tau,\omega)$ is the VaR% function of X. For simplicity, we define

$$G(\delta,p,Y,t,\tau,\omega)=\frac{1}{1+\delta\phi_(p,Y,t,\tau,\omega)^+}. \tag{4.10}$$

Under the objective of maximizing utility over the investment horizon $[0,T]$, the optimization problem for identifying the optimal policy ψ can be formulated as the following framework

$$
\begin{cases}
\max_{\{\psi(t)\}_{t=0}^{T}} \mathbb{E}\left[U(P(T))\right] \\
\psi \leq G(\delta, p, Y, t, \tau, \omega),
\end{cases}
\tag{4.11}
$$

where the utility function is given by (2.11). The corresponding value function is defined as

$$
V\left(P,t\right) = \max_{\{\psi(t)\}_{t=0}^{T}} \mathbb{E}\left[\frac{P(T)^{1-\gamma}}{1-\gamma}\right].
$$

Apparently, this is a constrained optimization problem. We first derive the formulation of optimum ψ^* under the general forms (4.5) and (4.8) with the state variable Y dynamics (2.3). After that, the generalized solution is applied in the further discussion in those two models (GBM and SV) under consideration. In GBM model, the analytical form of ψ^* can be obtained. However, in SV model, the complete form of ψ^* comes from a system of two nonlinear PDEs with free boundary. Therefore, we have to rely on numerical methods to obtain ψ^*.

Combining the framework established by Merton (1971)[20] with the constraint, the HJB equation for the problem in (4.11) is:

$$
\begin{cases}
\max\limits_{\{\psi(t)\}_{t=0}^{T}} \Big\{ V_t + \frac{1}{2}\psi^2 \Sigma^{(X)} P^2 V_{PP} + \psi \Sigma^{(X,Y)} P V_{PY} + \frac{1}{2}\Sigma^{(Y)} V_{YY} \\
\qquad\qquad + (\psi R_\omega Y + r) P V_P + \mu^{(Y)} V_Y \Big\} = 0, \\
V\left(P,T\right) = \frac{P^{1-\gamma}}{1-\gamma}, \\
\psi \leq G(\delta, p, Y, t, \tau, \omega).
\end{cases}
\tag{4.12}
$$

In this setting, the terms $\Sigma^{(X)}$, $\Sigma^{(X,Y)}$, and $\Sigma^{(Y)}$ are given by

$$
\begin{aligned}
\Sigma^{(X)} &= \sigma_\omega^2 Y, \\
\Sigma^{(X,Y)} &= \sigma_\omega \rho_\omega g Y, \\
\Sigma^{(Y)} &= g^2 Y.
\end{aligned}
\tag{4.13}
$$

The terminal condition is given by

$$V(P,Y,T) = \frac{P^{1-\gamma}}{1-\gamma}.$$

To solve the above optimization problem, we define a function

$$\begin{aligned} L = V_t + \frac{1}{2}\psi^2\Sigma^{(X)}P^2V_{PP} + \psi\Sigma^{(X,Y)}PV_{PY} + \frac{1}{2}\Sigma^{(Y)}V_{YY} \\ + (\psi R_\omega Y + r)PV_P + \mu^{(Y)}V_Y + \alpha(G-\psi), \end{aligned}$$

where α is a non-negative Karush-Kuhn-Tucker (KKT) multiplier (dual feasibility). In the KKT conditions, the first-order condition $L_\psi = 0$ (stationarity) leads

$$R_\omega YPV_P + P^2V_{PP}\Sigma^{(X)}\psi + \Sigma^{(X,Y)}PV_{PY} - \alpha = 0. \tag{4.14}$$

Moreover, in the KKT conditions, the primal feasibility and complementary slackness conditions are

$$G(\delta,p,Y,t,\tau,\omega) - \psi \geq 0, \tag{4.15}$$

$$\alpha\left(G(\delta,p,Y,t,\tau,\omega) - \psi\right) = 0. \tag{4.16}$$

From equation (4.14), we have

$$\begin{aligned} \psi =& -\frac{1}{\Sigma^{(X)}PV_{PP}}\left[R_\omega YV_P + \Sigma^{(X,Y)}V_{PY} - \frac{\alpha}{P}\right] \\ =&\psi_0 + \frac{\alpha}{\Sigma^{(X)}P^2V_{PP}}, \end{aligned} \tag{4.17}$$

with

$$\psi_0 = -\frac{1}{\Sigma^{(X)}PV_{PP}}\left[R_\omega YV_P + \Sigma^{(X,Y)}V_{PY}\right]. \tag{4.18}$$

One can notice that ψ_0 is the optimal solution for the problem without constraint. Therefore, the effect of the constraint is represented by the second term $\alpha/\Sigma^{(X)}P^2V_{PP}$ in (4.17). From the complementary slackness condition (4.16), we have two possible cases: $\alpha = 0$ or $\psi = G(\delta, p, Y, t, \tau, \omega)$. The first case implies that $\psi = \psi_0$ and ψ_0 must satisfy the primal feasibility condition $G(\delta, p, Y, t, \tau, \omega) - \psi_0 \geq 0$. In this case, the constraint is not binding since the optimal solution of the unconstrained problem satisfies the constraint. In the second case, the constraint is binding and the solution is at the boundary of the constrain $\psi = G(\delta, p, Y, t, \tau, \omega)$. It implies

$$\alpha = \left(G(\delta, p, Y, t, \tau, \omega) - \psi_0\right) \Sigma^{(X)} P^2 V_{PP}$$

Therefore, we have the optimal solution ψ^* as following

$$\psi^* = \begin{cases} \psi_0 & \text{if } \psi_0 < G(\delta, p, Y, t, \tau, \omega) \\ G(\delta, p, Y, t, \tau, \omega) & \text{otherwise} \end{cases}$$

or

$$\psi^* = min\left(\psi_0, G(\delta, p, Y, t, \tau, \omega)\right) \tag{4.19}$$

Substituting (4.19) into (4.12), the PDE of the value function V is given by

$$0 = \begin{cases} V_t + (\psi_0 R_\omega Y + r) P V_P + \frac{1}{2}\psi_0^2 \Sigma^{(X)} P^2 V_{PP} & \\ \quad + \psi_0 \Sigma^{(X,Y)} P V_{PY} + \frac{1}{2}\Sigma^{(Y)} V_{YY} + \mu^{(Y)} V_Y, & \text{if } \psi_0 < G(\delta, p, Y, t, \tau, \omega) \\ V_t + (G R_\omega Y + r) P V_P + \frac{1}{2} G^2 \Sigma^{(X)} P^2 V_{PP} & \\ \quad + G \Sigma^{(X,Y)} P V_{PY} + \frac{1}{2}\Sigma^{(Y)} V_{YY} + \mu^{(Y)} V_Y, & \text{otherwise} \end{cases} \tag{4.20}$$

with the terminal condition $V(P, T) = \frac{P^{1-\gamma}}{1-\gamma}$. In order to solve for V, we introduce the ansatz

$$V(P, Y, t) = \frac{P^{1-\gamma}}{1-\gamma} f(Y, t). \tag{4.21}$$

Then, ψ_0 in (4.18) is given by

$$\psi_0 = \frac{1}{\gamma} \Sigma^{(X)^{-1}} \left(R_\omega Y + \Sigma^{(X,Y)} \frac{f_Y}{f} \right). \tag{4.22}$$

Substituting the ansatz of V (4.21) and ψ_0 (4.22) into the PDE (4.20), we obtain

(4.23)

$$0 = \begin{cases} f_t + \frac{1}{2}\Sigma^{(Y)} f_{YY} + \left[\mu^{(Y)} + \frac{1-\gamma}{\gamma}\Sigma^{(X,Y)\prime}\Sigma^{(X)^{-1}} R_\omega Y\right] f_Y & \\ \quad + \frac{1-\gamma}{2\gamma}\Sigma^{(Y,\rho)}\frac{f_Y^2}{f} + (1-\gamma)\left[\frac{1}{2\gamma}\Sigma^{(X)^{-1}} R_\omega^2 Y^2 + r\right] f, & \text{if } \psi_0 < G(\delta, p, Y, t, \tau, \omega) \\ f_t + \frac{1}{2}\Sigma^{(Y)} f_{YY} + \left[(1-\gamma)G\Sigma^{(X,Y)} + \mu^{(Y)}\right] f_Y & \\ \quad + \left[(1-\gamma)(GR_\omega Y + r) - \frac{\gamma(1-\gamma)}{2}G^2\Sigma^{(X)}\right] f, & \text{otherwise} \end{cases}$$

where $\Sigma^{(Y,\rho)} = \sigma^{(Y)}\rho_\omega^2\sigma^{(Y)}$ and the terminal condition is $f(Y,T) = 1$.

From formulas (4.19)-(4.23), the structure of the generalized solution of ψ^* is revealed. There are two components in the solution: ψ_0 and G. The function G is relatively straightforward once the VaR% function ϕ is given. The main obstacle for us to derive ψ^* is the component ψ_0. The function ψ_0 is related to a function f which can be solved from PDE system (4.23). With different risky asset models, the PDE system (4.23) could be completely different. In the rest of this section, we further develop the procedure for obtaining ψ^* with GBM and SV models.

In the GBM model the state variable Y is not present. f is a function of time t and the PDEs in (4.23) are reduced to be ODEs. Moreover, ψ_0 and G are just constants. The formulation of the optimal solution ψ^* is straightforward. That is

$$\psi^* = min\left(\frac{R_\omega}{\gamma\sigma_\omega^2}, G\right). \tag{4.24}$$

In the SV model, the solution becomes much more complicated and the analytical form can be obtained only when $\rho_\omega = 0$. In the case of $\rho_\omega = 0$, the optimal solution ψ^* is also straightforward because there is no need to solve the PDE for f. That is,

$$\psi^* = min\left(\frac{R_\omega}{\gamma\sigma_\omega^2}, G(\delta, p, Y, t, \tau, \omega)\right). \tag{4.25}$$

However, when $\rho_\omega \neq 0$, solving for ψ_0 is very difficult because the key component f is a solution of two PDEs (4.23) based on the switch condition $\psi_0 < G$. Unfortunately, the switch condition itself depends on the unknown function f. In other words, the boundary between those two PDEs is to be determined and varies with Y and t. This is a free boundary problem. Moreover, the first PDE is also a nonlinear PDE. Solving such PDEs is very intractable even with numerical methods. In this study, instead of solving the PDEs in (4.23), we develop a numerical scheme based on dynamic programming to obtain the optimal solution ψ^* for the SV model.

First, we discretize the continuous time model (4.8) and (2.3). For the stochastic process of the total wealth, we discretize the stochastic process of the portfolio growth rate $R_P(t)$ instead of the portfolio $P(t)$. The relationship between $R_P(t)$ and $P(t)$ are formulated as follows

$$P(t) = P(0)exp(R_P(t)) \quad and \quad R_P(0) = 0.$$

By Îto lemma, the SDE of $R_P(t)$ is given by

$$dR_P(t) = \left(r + \psi R_\omega Y(t) - \frac{1}{2}\psi^2\sigma_\omega^2 Y(t)\right)dt + \psi\sigma_\omega\sqrt{Y(t)}dW^{(X)}. \tag{4.26}$$

To match setting of the numerical experiment, we represent the investment horizon T in trading days and one time step corresponds to one trading day, i.e. $\Delta t = 1$. All time-varying processes or functions are estimated at the sequence of discrete time points $t = 0, 1, ..., T$. To distinguish the discretized variables from the original ones, we add " $^-$ " on top of the corresponding notation of each variable. With explicit Euler method, the recurrence relations of $R_P(t)$ and $Y(t)$ are given by

$$\begin{aligned} \bar{R}_{P,t+1} &= \bar{R}_{P,t} + \left(r + \bar{\psi}_t R_\omega \bar{Y}_t - \frac{1}{2}\bar{\psi}_t^2\sigma_\omega^2\bar{Y}_t\right) + \bar{\psi}_t\sigma_\omega\sqrt{\bar{Y}_t}z_t^{(X)} \\ \bar{Y}_{t+1} &= \bar{Y}_t + d - c\bar{Y}_t + g\sqrt{\bar{Y}_t}z_t^{(Y)}. \end{aligned} \tag{4.27}$$

where z_0^X, z_1^X,..., are i.i.d. (identical independent distributed) standard normal and z_0^Y, z_1^Y,..., are also i.i.d. standard normal. In the setting above, $z_t^{(X)}$ and $z_t^{(Y)}$ are two correlated standard normal random variables with correlation coefficient ρ_ω. Moreover, $(z_t^{(X)}, z_t^{(Y)})$ and $(z_s^{(X)}, z_s^{(Y)})$ are independent with each other when $t \neq s$. For simplicity, we can always represent $z_t^{(X)}$ in terms of $z_t^{(Y)}$ and an auxiliary standard normal random variable $z_t^{(A)}$

$$z_t^{(X)} = \rho_\omega z_t^{(Y)} + \sqrt{1-\rho_\omega^2} z_t^{(A)}. \tag{4.28}$$

where $z_t^{(A)}$ is independent with $z_t^{(Y)}$. At any time point t, the total wealth $\bar{P}_t$ is given by

$$\bar{P}_t = \bar{P}_{t-1} exp(\bar{R}_{P,t} - \bar{R}_{P,t-1}). \tag{4.29}$$

Second, we apply the dynamic programming technique to find the optimal solutions $\left\{\bar{\psi}_0^*, ..., \bar{\psi}_{T-1}^*\right\}$ maximizing the expected utility of $\bar{P}_T$. The optimization can be expressed as

$$\max_{\{\bar{\psi}_t\}_{t=0}^{T-1}} \mathbb{E}\left[\frac{\bar{P}_T^{1-\gamma}}{1-\gamma}\right],$$

with the constraint

$$\bar{\psi}_t \leq G(\delta, p, \bar{Y}_t, t, \tau, \omega).$$

For simplicity, we denote $G(\delta, p, \bar{Y}_t, t, \tau, \omega)$ by $G(\bar{Y}_t, t)$. The main principle of dynamic programming is that the optimal control sequence $\left\{\bar{\psi}_0^*, ..., \bar{\psi}_{T-1}^*\right\}$ must also be optimal for the remaining sequence at any intermediate time point t [8]. That is, $\left\{\bar{\psi}_t^*, ..., \bar{\psi}_{T-1}^*\right\}$ must be optimal for conditional expectation $\mathbb{E}_t\left[\frac{\bar{P}_T^{1-\gamma}}{1-\gamma}\right]$ given all the information at any time point $0 < t < T-1$. This property is very well known, namely *Principle of Optimality.* Define the value function by

$$\bar{V}(t, \bar{P}_t, \bar{Y}_t) = \mathbb{E}_t\left[\frac{\bar{P}_T^{1-\gamma}}{1-\gamma} | \left\{\bar{\psi}_t^*, ..., \bar{\psi}_{T-1}^*\right\}\right].$$

This value function gives the conditional expectation of the utility of the terminal portfolio value given all the information up to time t and the optimal control sequence starting from t. With the value function, the principle of optimality can be summarized by the Bellman equation

$$\bar{V}(t, \bar{P}_t, \bar{Y}_t) = \max_{\bar{\psi}_t < G(\bar{Y}_t, t)} \mathbb{E}_t \left[\bar{V}(t+1, \bar{P}_{t+1}, \bar{Y}_{t+1}) | \bar{\psi}_t \right],$$

with the terminal condition $\bar{V}(T, \bar{P}_T, \bar{Y}_T) = \frac{\bar{P}_T^{1-\gamma}}{1-\gamma}$. With the Bellman equation, we can develop a numerical scheme to derive the optimal solution sequence $\{\bar{\psi}_t^*\}_{t=0}^{T-1}$ backwards over time.

Apparently, the optimal solution ψ^* is also a function of Y. The approximate solution $\bar{\psi}^*$ generated by a numerical scheme cannot cover all possible Y at each time step t. In this study, a trinomial tree model introduced by Nawalkha and Beliaeva [22] is used to approximate the stochastic process of Y. Binomial and trinomial tree approaches are widely accepted techniques for derivative pricing. The stochastic evolution of the underlying variable is represented by a tree structure in which the tree nodes and links are associated with the discretized states and transition probabilities. When the volatility of the stochastic process is constant, the corresponding tree is recombining such that the number of tree nodes is significantly reduced and computation based on the tree is tractable. However, the tree is non-recombing when the volatility is time-varying. In the Nawalkha and Beliaeva's work, an efficient trinomial tree for the Cox, Ingersoll, and Ross (CIR) model is developed based on the solution presented by Nelson and Ramaswamy (NR)[23]. In our model setting, the stochastic process of $Y(t)$ is essentially a CIR model. In order to achieve the recombining property of the tree for the square root process, Nelson and Ramaswamy suggest using a transformation of the underlying variable such that the volatility of the new variable is constant. For the CIR model, the transformation is given by

$$y(t) = \frac{2\sqrt{Y(t)}}{g}.$$

By taking the inverse of the above equation, the state variable $Y(t)$ is given by

$$Y(t) = \frac{y(t)^2 g^2}{4}. \tag{4.30}$$

By Îto lemma, the SDE of $y(t)$ is given by

$$dy(t) = \mu^{(y)} dt + dW^{(Y)},$$

where

$$\mu^{(y)} = \frac{1}{y}\left[\frac{1}{2}c\left(\frac{4d}{cg^2} - y^2\right) - \frac{1}{2}\right].$$

The trinomial tree is actually constructed based on $y(t)$ instead of $Y(t)$. The resulted tree can be easily transformed into the one for $Y(t)$ via the formula (4.30). To distinguish from the original notations, the value of y on the tree node is denoted by $\hat{y}$. In the trinomial tree, the branches of each tree node $\hat{y}$ (except those at the ending time point) go up to $\hat{y} + v\sqrt{\Delta t}$, stay at $\hat{y}$ and go down to $\hat{y} - v\sqrt{\Delta t}$ in the next time step. According to Nawalkha and Beliaeva's work, the parameter v is determined by

$$v = \begin{cases} v_c & \text{if } |v_c - \sqrt{1.5}| < |v_e - \sqrt{1.5}| \\ v_e & \text{otherwise} \end{cases} \tag{4.31}$$

where v_c and v_e are given by

$$\begin{aligned} v_e &= \frac{y_0/\sqrt{\Delta t}}{Floor(y_0/\sqrt{1.5\Delta t})} \\ v_c &= \frac{y_0/\sqrt{\Delta t}}{Floor(y_0/\sqrt{1.5\Delta t} + 1)}. \end{aligned} \tag{4.32}$$

y_0 is the initial value of $y(t)$ at time $t = 0$. Between v_c and v_e, the final value of v is the one closest to the starting value of $\sqrt{1.5}$ in absolute distance. Moreover, there is an additional constraint on v which is

$$1 \leqslant v \leqslant \sqrt{2}. \tag{4.33}$$

This particular setting of v ensures that the probabilities are non-negative for three different moves. For each tree node (except those at the end), the up, middle and down moves are given by

$$\begin{aligned} \hat{y}_u &= \hat{y} + v(J+1)\sqrt{\Delta t} \\ \hat{y}_m &= \hat{y} + vJ\sqrt{\Delta t} \\ \hat{y}_d &= \hat{y} + v(J-1)\sqrt{\Delta t} \end{aligned} \tag{4.34}$$

where J is the multiple node jump indicator and is defined later. The corresponding probabilities are given by

$$\begin{aligned} p_u(\hat{y}) &= \frac{1}{2v^2} - \frac{J}{2} + \frac{1}{2v}\mu^{(y)}(\hat{y})\sqrt{\Delta t} \\ p_d(\hat{y}) &= \frac{1}{2v^2} + \frac{J}{2} - \frac{1}{2v}\mu^{(y)}(\hat{y})\sqrt{\Delta t} \\ p_m(\hat{y}) &= 1 - \frac{1}{v^2}. \end{aligned} \tag{4.35}$$

The jump indicator J is set as follows

$$J = Floor\left(\frac{\mu^{(y)}(\hat{y})\sqrt{\Delta t}}{v} + \frac{1}{v^2}\right). \tag{4.36}$$

To check whether p_u, p_m and p_d are valid probabilities, we make several observations based on the formulas (4.33)-(4.36). First, it is obvious that the sum of p_u, p_m and p_d is 1. Second, based on the inequality (4.33), p_m is a number between 0 and 1. Third, from the formula of J (4.36), we have

$$\frac{\mu^{(y)}(\hat{y})\sqrt{\Delta t}}{v} - \frac{1}{v^2} \leqslant J \leqslant \frac{\mu^{(y)}(\hat{y})\sqrt{\Delta t}}{v} + \frac{1}{v^2}. \tag{4.37}$$

Substituting the inequality above into the formulas of p_u and p_d, we have

$$0 \leqslant p_u, p_d \leqslant \frac{1}{v^2}.$$

Therefore, the formulas of p_u, p_m and p_d are valid probabilities for three different movements in the trinomial tree.

Moreover, the parameter J essentially controls the evolution of the tree and prevents the branches move into the infeasible range of the process. The formula of J above ensures that J can take some integer value. When $J = 0$, there is no multiple node jumps and the branches extend in three directions: up, middle and down. When $J \geq 1$, the tree branches reach the bottom level of the tree and move upward. When $J \leq -1$, the tree branches reach the upper bound of the tree and move downward.

Depending on the parameters, it is possible that the tree nodes could reach zero if the number of time steps becomes large. When a tree node $\hat{y}$ becomes exactly zero, the above formulations of branch movements and probabilities are not valid. Nawalkha and Beliaeva provide a special treatment for it. In this case, the middle and down moves are the same and they stay at zero, i.e.

$$\hat{y}_m = \hat{y}_d = 0. \tag{4.38}$$

For the upward move, the up node is selected as the node closest to the zero line where the following inequality is satisfied

$$\hat{y}_u = v(J+1)\sqrt{\Delta t} \geq \frac{2}{g}\sqrt{d\Delta t}. \tag{4.39}$$

The corresponding probabilities are given by

$$\begin{aligned} p_u &= \frac{4d\Delta t}{\hat{y}_u^2 g^2} \\ p_m &= 0 \\ p_d &= 1 - p_u. \end{aligned} \tag{4.40}$$

After the trinomial tree of $\hat{y}$ is constructed, it can be directly converted into the tree for $Y(t)$ by the formula (4.30). Figure 4.1 demonstrates the construction of a trinomial tree for Y process with the first 21 time steps. The method recognizes the upper and lower bounds of stochastic process and restricts the branches evolving within the range. Denote the Y value on the tree nodes by $\hat{Y}$. At any time t ($t = 0, ..., T$), there are n_t tree nodes and the corresponding Y values are denoted by $\hat{Y}_{t,1}, ..., \hat{Y}_{t,n_t}$.

On top of the trinomial tree, we develop the numerical scheme to derive the optimal solution sequence $\{\bar{\psi}_t^*\}_{t=0}^{T-1}$ backwards over time. Starting with time $T-1$, by the Bellman equation, the value function is given by

(4.41)

$$\begin{aligned}
\bar{V}(T-1, \bar{P}_{T-1}, \bar{Y}_{T-1}) &= \max_{\bar{\psi}_{T-1} < G(\bar{Y}_{T-1}, T-1)} \mathbb{E}_{T-1}\left[\bar{V}(T, \bar{P}_T, \bar{Y}_T) | \bar{\psi}_{T-1}\right] \\
&= \max_{\bar{\psi}_{T-1} < G(\bar{Y}_{T-1}, T-1)} \mathbb{E}_{T-1}\left[\frac{\bar{P}_T^{1-\gamma}}{1-\gamma} | \bar{\psi}_{T-1}\right] \\
&= \max_{\bar{\psi}_{T-1} < G(\bar{Y}_{T-1}, T-1)} \mathbb{E}_{T-1}\left[\frac{\bar{P}_{T-1}^{1-\gamma}}{1-\gamma} exp\left\{(1-\gamma)\left(r + \bar{\psi}_{T-1} R_\omega \bar{Y}_{T-1}\right.\right.\right. \\
&\qquad \left.\left.\left. - \frac{1}{2}\bar{\psi}_{T-1}^2 \sigma_\omega^2 \bar{Y}_{T-1} + \bar{\psi}_{T-1}\sigma_\omega \sqrt{\bar{Y}_{T-1}} z_{T-1}^{(X)}\right)\right\} | \bar{\psi}_{T-1}\right] \\
&= \max_{\bar{\psi}_{T-1} < G(\bar{Y}_{T-1}, T-1)} \frac{\bar{P}_{T-1}^{1-\gamma}}{1-\gamma} exp\left\{(1-\gamma)\left(r + \bar{\psi}_{T-1} R_\omega \bar{Y}_{T-1} - \frac{\gamma}{2}\bar{\psi}_{T-1}^2 \sigma_\omega^2 \bar{Y}_{T-1}\right)\right\}.
\end{aligned}$$

Therefore, the optimal solution at time $T-1$ is given by

(4.42) $$\bar{\psi}_{T-1}^* = min\left(\frac{R_\omega}{\gamma\sigma_\omega^2}, G(\bar{Y}_{T-1}, T-1)\right).$$

Since $\bar{\psi}_{T-1}^*$ is a function of $\bar{Y}_{T-1}$, on the trinomial tree, the corresponding optimal weight $\bar{\psi}_{T-1}^*$ for $\{\hat{Y}_{T-1,1}, ..., \hat{Y}_{T-1,n_{T-1}}\}$ are denoted by $\hat{\psi}_{T-1,i}^*$

(4.43) $$\hat{\psi}_{T-1,i}^* = min\left(\frac{R_\omega}{\gamma\sigma_\omega^2}, \hat{G}_{T-1,i}\right), \quad i = 1, ..., n_{T-1},$$

where $\hat{G}_{T-1,i} = G(\hat{Y}_{T-1,i}, T-1)$. Similarly, the value function for each $\hat{Y}_{T-1,i}$ can be formulated by

(4.44) $$\hat{V}_{T-1,i}(\bar{P}_{T-1}) = \frac{\bar{P}_{T-1}^{1-\gamma}}{1-\gamma} A_{T-1,i},$$

where

(4.45) $$A_{T-1,i} = exp\left\{(1-\gamma)\left(r + \hat{\psi}_{T-1,i}^* R_\omega \hat{Y}_{T-1,i} - \frac{\gamma}{2}\hat{\psi}_{T-1,i}^{*^2} \sigma_\omega^2 \hat{Y}_{T-1,i}\right)\right\}.$$

At time $T-2$, the computation is much more complicated because the correlation between $z^{(X)}$ and $z^{(Y)}$ needs to be considered in this step. In the previous step, the expectation is estimated only based on $z^{(X)}_{T-1}$ because the terminal condition does not depends on Y. However, the expectation at time $T-2$ must be estimated based on $z^{(X)}_{T-2}$ and $z^{(Y)}_{T-2}$ together. On the trinomial tree, the tree node $\hat{Y}_{T-2,i}$ can reach three possible tree nodes at time $T-1$: $\hat{Y}_{T-1,i_1}$, $\hat{Y}_{T-1,i_2}$ and $\hat{Y}_{T-1,i_3}$ where i_1, i_2 and i_3 are integers between 1 and n_{T-1}. The corresponding probabilities can be denoted by $q^{(T-2)}_{i,i_1}$, $q^{(T-2)}_{i,i_2}$ and $q^{(T-2)}_{i,i_3}$ which are determined by the formulas in (4.35) and (4.40). Given the transition from $\hat{Y}_{T-2,i}$ to $\hat{Y}_{T-1,i_j}$ $(j = 1, 2, 3)$, $z^{(Y)}$ is approximated by

$$\hat{z}^{(Y)}_{T-2,i,i_j} = \frac{\hat{Y}_{T-1,i_j} - \hat{Y}_{T-2,i} - d + c\hat{Y}_{T-2,i}}{g\sqrt{\hat{Y}_{T-2,i}}}. \tag{4.46}$$

Then, by formula (4.28) $z^{(X)}$ can be approximated by

$$\hat{z}^{(X)}_{T-2,i,i_j} = \rho_\omega \hat{z}^{(Y)}_{T-2,i,i_j} + \sqrt{1-\rho_\omega^2}\, z^{(A)}. \tag{4.47}$$

With this setting and the given condition $\bar{Y}_{T-2} = \hat{Y}_{T-2,i}$, the expectation of $\bar{V}$ at time $T-2$ can be calculated as

$$\begin{aligned}
&\mathbb{E}_{T-2}\left[\bar{V}(T-1,\bar{P}_{T-1},\bar{Y}_{T-1})|\bar{\psi}_{T-2}\right] \\
=&\mathbb{E}_{T-2}\left[\sum_{j=1}^{3} q^{(T-2)}_{i,i_j}\frac{\bar{P}^{1-\gamma}_{T-1}}{1-\gamma}A_{T-1,i_j}|\bar{\psi}_{T-2}\right] \\
=&\mathbb{E}_{T-2}\left[\frac{\bar{P}^{1-\gamma}_{T-2}}{1-\gamma}\sum_{j=1}^{3} q^{(T-2)}_{i,i_j}A_{T-1,i_j}exp\left\{(1-\gamma)\left(r+\bar{\psi}_{T-2}R_\omega\hat{Y}_{T-2,i}\right.\right.\right. \\
&\left.\left.\left.-\frac{1}{2}\bar{\psi}^2_{T-2}\sigma^2_\omega\hat{Y}_{T-2,i}+\bar{\psi}_{T-2}\sigma_\omega\sqrt{\hat{Y}_{T-2,i}}\bar{z}^{(X)}_{T-2,i,i_j}\right)\right\}|\bar{\psi}_{T-2}\right] \\
=&\frac{\bar{P}^{1-\gamma}_{T-2}}{1-\gamma}\sum_{j=1}^{3} q^{(T-2)}_{i,i_j}A_{T-1,i_j}exp\left\{(1-\gamma)\left(r+\bar{\psi}_{T-2}R_\omega\hat{Y}_{T-2,i}\right.\right. \\
&\left.\left.+\frac{(1-\gamma)(1-\rho^2_\omega)-1}{2}\bar{\psi}^2_{T-2}\sigma^2_\omega\hat{Y}_{T-2,i}+\bar{\psi}_{T-2}\sigma_\omega\rho_\omega\sqrt{\hat{Y}_{T-2,i}}\bar{z}^{(Y)}_{T-2,i,j}\right)\right\}
\end{aligned} \tag{4.48}$$

For simplicity, we define a function

$$B_{i,i_j}^{(T-2)}(x) = exp\left\{(1-\gamma)\left(r + xR_\omega \hat{Y}_{T-2,i} + \frac{(1-\gamma)(1-\rho_\omega^2)-1}{2}x^2\sigma_\omega^2\hat{Y}_{T-2,i} + x\sigma_\omega\rho_\omega\sqrt{\hat{Y}_{T-2,i}}\bar{z}_{T-2,i,i_j}^{(Y)}\right)\right\} \tag{4.49}$$

Then, the expectation can be revised as

$$\mathbb{E}_{T-2}\left[\bar{V}(T-1,\bar{P}_{T-1},\hat{Y}_{T-2,i})|\bar{\psi}_{T-2}\right] = \frac{\bar{P}_{T-2}^{1-\gamma}}{1-\gamma}\sum_{j=1}^{3} q_{i,i_j}^{(T-2)} A_{T-1,i_j} B_{i,i_j}^{(T-2)}(\bar{\psi}_{T-2}). \tag{4.50}$$

According to the Bellman equation, we have

$$\bar{V}(T-2,\bar{P}_{T-2},\hat{Y}_{T-2,i}) = \max_{\bar{\psi}_{T-2}<G_{T-2,i}} \mathbb{E}_{T-2}\left[\bar{V}(T-1,\bar{P}_{T-1},\bar{Y}_{T-1})|\bar{\psi}_{T-2}\right]. \tag{4.51}$$

The optimal solution $\hat{\psi}_{T-2,i}^*$ can be obtained by the following optimization expression

$$\hat{\psi}_{T-2,i}^* = arg\max_{\bar{\psi}_{T-2}<G_{T-2,i}} \frac{\bar{P}_{T-2}^{1-\gamma}}{1-\gamma}\sum_{j=1}^{3} q_{i,i_j}^{(T-2)} A_{T-1,i_j} B_{i,i_j}^{(T-2)}(\bar{\psi}_{T-2}). \tag{4.52}$$

The objective function is highly nonlinear. The traditional method of finding extreme points by differentiation gives rise to a very complicated transcendental equation. Therefore, this optimization problem has to be solved numerically. In this study, we directly use optimization tools in *MATLAB* to find the maximizer in (4.52). In *MATLAB*, the function *fminbnd* finds the minimum of a function in the given interval. In order to utilize *fminbnd*, the negative of the original objective function is used instead and the searching interval is set as $[0, G_{T-2,i}]$. With the optimal solution $\hat{\psi}_{T-2,i}^*$, we define

$$\begin{aligned} B_{i,i_j}^{(T-2)*} &= B_{i,i_j}^{(T-2)}(\hat{\psi}_{T-2}^*) \\ A_{T-2,i} &= \sum_{j=1}^{3} q_{i,i_j}^{(T-2)} A_{T-1,i_j} B_{i,i_j}^{(T-2)*}. \end{aligned} \tag{4.53}$$

Then, the value function at time $T-2$ with $\bar{Y}_{T-2} = \hat{Y}_{T-2,i}$ can be formulated as

$$\hat{V}_{T-2,i}(\bar{P}_{T-2}) = \frac{\bar{P}_{T-2}^{1-\gamma}}{1-\gamma}A_{T-2,i}. \tag{4.54}$$

In particular, when $\rho_\omega = 0$, the function $B^{(T-2)}_{i,i_j}$ is simplified as

$$B^{(T-2)}_{i,i_j}(x) = exp\left\{(1-\gamma)\left(r + xR_\omega \hat{Y}_{T-2,i} - \frac{\gamma}{2}x^2\sigma^2_\omega \hat{Y}_{T-2,i}\right)\right\}. \tag{4.55}$$

The optimal solution $\hat{\psi}^*_{T-2,i}$ can be derived analytically, i.e.

$$\hat{\psi}^*_{T-2,i} = min\left(\frac{R_\omega}{\gamma\sigma^2_\omega}, G_{T-2,i}\right). \tag{4.56}$$

This result matches the formulation in (4.25).

As we continue to move backwards (shown in Figure 4.2), the numerical scheme can be formulated similarly at each step. That is, at time point $T-k$ ($k = 2, ..., T$) and the tree node $\hat{Y}_{T-k,i}$ ($i = 1, ..., n_{T-k}$), the optimal solution $\hat{\psi}^*_{T-k,i}$ and value function $\hat{V}_{T-k,i}(\bar{P}_{T-k})$ can be obtained by

$$\begin{aligned}
\hat{\psi}^*_{T-k,i} &= arg \max_{\bar{\psi}_{T-k} < G_{T-k,i}} \frac{\bar{P}^{1-\gamma}_{T-k}}{1-\gamma} \sum_{j=1}^{3} q^{(T-k)}_{i,i_j} A_{T-k+1,i_j} B^{(T-k)}_{i,i_j}(\bar{\psi}_{T-k}) \\
B^{(T-k)*}_{i,i_j} &= B^{(T-k)}_{i,i_j}(\hat{\psi}^*_{T-k}) \\
A_{T-k,i} &= \sum_{j=1}^{3} q^{(T-k)}_{i,i_j} A_{T-k+1,i_j} B^{(T-k)*}_{i,i_j} \\
\hat{V}_{T-k,i}(\bar{P}_{T-k}) &= \frac{\bar{P}^{1-\gamma}_{T-k}}{1-\gamma} A_{T-k,i}.
\end{aligned} \tag{4.57}$$

Eventually, the complete $\hat{\psi}^*_{t,i}$ for each t and possible i can be calculated. Moreover, the expected utility with $\hat{\psi}^*$ is given by $\hat{V}_{0,1}(\bar{P}_0)$.

4.4 Expected Utility With Optimal Allocation

Now we have the optimal allocation strategy ψ^* between risky portfolio and risk-free capital. ψ^* is closely related the simple portfolio selection strategy ω. However, the effect of ω only takes place through three parameters: R_ω, σ_ω and ρ_ω. R_ω and σ_ω are the risk premium coefficient and volatility coefficient of the risky index X which is constructed by simple portfolio selection strategy ω on n risky assets. Those three parameters affect not only the optimum ψ^* but also the maximal expected utility obtained through ψ^*. To analyze the effect of ω on the expected utility, we don't need to perform the analysis directly on ω. Instead, we just need to analyze the effects from those three parameters.

In order to further analyze those parameters, we define a function Φ which is the expected utility of terminal total wealth in which the simple portfolio selection strategy ω is used to construct the risky portfolio and the optimal weight ψ^* in (4.19) is used for allocation between risk-free capital and risky portfolio. That is,

$$\Phi(R_\omega, \sigma_\omega, \rho_\omega) = \mathbb{E}\left[\frac{P(T)^{1-\gamma}}{1-\gamma}\right] \tag{4.58}$$

and the terminal total wealth $P(T)$ has the following form derived from (4.8)

$$\begin{aligned} P(T) = P(0)exp\Bigg\{ & \int_0^T \left[r + \psi^*(s,Y)R_\omega Y(s) - \frac{1}{2}\sigma_\omega^2 {\psi^*}^2(s,Y)Y(s)\right] ds \\ & + \int_0^T \sigma_\omega \psi^*(s,Y)\sqrt{Y(s)}dW^{(X)}(s)\Bigg\}. \end{aligned} \tag{4.59}$$

Apparently, Φ is the maximal expected utility for the given simple portfolio selection strategy ω. The term "maximal" is only for the fixed strategy ω. To avoid confusion, we call Φ the ω-utility. Moreover, Φ is a function of three parameters: R_ω, σ_ω, and ρ_ω. In the next step, we analyze the effect of these parameters on the ω-utility for two different models: GBM and SV.

4.4.1 The case with Geometric Brownian Motion (GBM) model

In this case, all the parameters are constants and state variable Y is not considered in the model. Therefore, the parameter ρ_ω is irrelevant. The other two parameters R_ω and σ_ω are actually the risky premium and volatility of the index X. Based on (3.14), VaR% (ϕ) is given by

$$\phi(p, \tau, \omega) = 1 - exp\left(\left[r + R_\omega - \frac{1}{2}\sigma_\omega^2\right]\tau + \sigma_\omega\sqrt{\tau}Z_{1-p}\right). \tag{4.60}$$

The Theorem (III.2) states that the VaR% (ϕ) of the risky portfolio P_R with confidence level p and VaR horizon $\tau > 0$ decreases with R_ω and increases with σ_ω if $p > 50\%$. Based on the characteristics of ϕ, the upper bound G of the risky portfolio weight (4.10) increases with R_ω and decreases with σ_ω. Since ρ_ω is not in the model, the component ψ_0 (4.18) becomes

$$\psi_0 = \frac{1}{\gamma}\frac{R_\omega}{\sigma_\omega^2}. \tag{4.61}$$

It is evident that ψ_0 also increases with R_ω and decreases with σ_ω if $R_\omega > 0$. Since the optimal policy ψ^* is the minimum of ψ_0 and G, ψ^* has the same relationship to R_ω and σ_ω as ψ_0 and G. Namely, ψ^* increases with R_ω and decreases with σ_ω. Since the purpose of holding the risky portfolio is pursuing the higher return over the risk-free rate for the normal investor, the risky index is constructed in such a way that its risk premium is positive. It is possible that there are some risky assets with negative risk premium in the index. However, the bank can short those assets to make the overall risk premium positive. Therefore, we can always assume that the risk premium R_ω of the risky index is positive. Based on this assumption, ψ_0 is always positive and so is ψ^*. With all these properties of ψ^*, we can develop the relationship between the ω-utility and the parameters: R_ω and σ_ω.

Theorem IV.1. *Considering all the risky assets in the trading book follow the GBM model (2.6), if the capital allocation of the trading book satisfies*

1. *the simple portfolio selection strategy (constant vector ω) is applied with $R_\omega > 0$;*

2. *the confidence level p of the risky portfolio VaR is greater than 50%;*

3. *the optimal policy ψ^* from (4.19) is used to allocate capital between risky portfolio and risk-free asset,*

the expected utility of total wealth

$$\Phi(R_\omega, \sigma_\omega) = \mathbb{E}\left[\frac{P(T)^{1-\gamma}}{1-\gamma}\right] \tag{4.62}$$

is an increasing function of the index risk premium R_ω and a decreasing function of the index volatility σ_ω.

Proof: First, we derive the formula for the expected utility of total wealth under the optimal policy ψ^*. Based on (4.8), the stochastic process of total wealth $P(t)$ with ψ^* under GBM model is given by

$$dP(t) = (\psi^* R_\omega + r)P(t)dt + \psi^* \sigma_\omega dW^{(X)}. \tag{4.63}$$

The analytic form of $P(T)$ is given by

$$P(T) = P(0)exp\left\{\left(\psi^* R_\omega + r - \frac{1}{2}{\psi^*}^2 \sigma_\omega^2\right) T + \psi^* \sigma_\omega \sqrt{T} Z\right\}, \tag{4.64}$$

where Z is a standard normal random variable. Then, the expected utility of $P(T)$ is

$$\begin{aligned} \Phi(R_\omega, \sigma_\omega) &= \mathbb{E}\left[\frac{P(T)^{1-\gamma}}{1-\gamma}\right] \\ &= \frac{P(0)^{1-\gamma}}{1-\gamma} exp\left\{(1-\gamma)\left(\psi^* R_\omega + r\right) T - \frac{1}{2}(1-\gamma)\gamma {\psi^*}^2 \sigma_\omega^2 T\right\} \\ &= \frac{P(0)^{1-\gamma}}{1-\gamma} exp\left\{(1-\gamma) T\left[\psi^*\left(R_\omega - \frac{1}{2}\gamma\psi^*\sigma_\omega^2\right) + r\right]\right\}. \end{aligned} \tag{4.65}$$

Since $\psi^* = min(\psi_0, G)$, we have

$$\psi^* \leq \psi_0 \quad or \quad \gamma\psi^*\sigma_\omega^2 \leq R_\omega \tag{4.66}$$

Given any pairs of $(R_{\omega,1}, R_{\omega,2})$ satisfying $R_{\omega,1} < R_{\omega,2}$, the corresponding ψ_1^* and ψ_2^* satisfy $\psi_1^* < \psi_2^*$. Together with (4.66), we have

$$\begin{aligned}
&\psi_2^*\left(R_{\omega,2} - \frac{1}{2}\gamma\psi_2^*\sigma_\omega^2\right) - \psi_1^*\left(R_{\omega,1} - \frac{1}{2}\gamma\psi_1^*\sigma_\omega^2\right)\\
=&(\psi_2^* - \psi_1^*)\left(R_{\omega,2} - \frac{1}{2}\gamma\psi_2^*\sigma_\omega^2 - \frac{1}{2}\gamma\psi_1^*\sigma_\omega^2\right) + \psi_1^*(R_{\omega,2} - R_{\omega,1})\\
\geq&(\psi_2^* - \psi_1^*)\left(\frac{R_{\omega,2}}{2} - \frac{R_{\omega,1}}{2}\right) + \psi_1^*(R_{\omega,2} - R_{\omega,1})\\
=&\frac{1}{2}(\psi_2^* + \psi_1^*)(R_{\omega,2} - R_{\omega,1}) > 0.
\end{aligned} \tag{4.67}$$

Similarly, for any pair of $(\sigma_{\omega,1}, \sigma_{\omega,2})$ satisfying $\sigma_{\omega,1} < \sigma_{\omega,2}$, the corresponding ψ_1^* and ψ_2^* satisfy $\psi_1^* > \psi_2^*$. Together with (4.66), we have

$$\begin{aligned}
&\psi_2^*\left(R_\omega - \frac{1}{2}\gamma\psi_2^*\sigma_{\omega,2}^2\right) - \psi_1^*\left(R_\omega - \frac{1}{2}\gamma\psi_1^*\sigma_{\omega,1}^2\right)\\
=&(\psi_2^* - \psi_1^*)\left(R_\omega - \frac{1}{2}\gamma\psi_2^*\sigma_{\omega,2}^2 - \frac{1}{2}\gamma\psi_1^*\sigma_{\omega,1}^2\right) + \frac{1}{2}\gamma\psi_1^*\psi_2^*(\sigma_{\omega,1}^2 - \sigma_{\omega,2}^2)\\
<&(\psi_2^* - \psi_1^*)\left(R_\omega - \frac{R_\omega}{2} - \frac{R_\omega}{2}\right) + 0\\
=&0.
\end{aligned} \tag{4.68}$$

Based on the inequalities (4.67) and (4.68), the term $\psi^*\left(R_\omega - \frac{1}{2}\gamma\psi^*\sigma_\omega^2\right)$ is an increasing function of the index risk premium R_ω and a decreasing function of the index volatility σ_ω. Therefore, the same property holds for the expected utility $\Phi(R_\omega, \sigma_\omega)$.

□

4.4.2 The case with Stochastic Volatility (SV) model

In this case, the analysis is much more difficult due to several reasons. First, the optimal weight ψ^* does not have an analytical formulation except when $\rho_\omega = 0$. Due to the additional state variable Y, we have to rely on a numerical scheme to derive ψ^*

by using dynamic programming approach. Moreover, the numerical scheme based on a trinomial tree is used to approximate the Y process. Therefore, there is no analytical closed form for the expected utility of the terminal total wealth. Second, the risky portfolio selection strategy ω affects three parameters: R_ω, σ_ω, and ρ_ω. All of them affect the expected utility of the terminal total wealth. In the study, we derive the analytical result on the relationship between Φ and parameter R_ω with the condition $\rho_\omega = 0$. For the case of $\rho_\omega \neq 0$, we have to rely on numerical experiments.

First, we define a set of admissible simple portfolio selection strategies ω satisfying $\rho_\omega = x_\rho$ and $\sigma_\omega = x_\sigma$ for any given real numbers $x_\rho \in [-1, 1]$ and $x_\sigma \geq 0$

$$\Gamma^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma} = \{\omega | \omega' 1_n = 1 \;\; and \;\; \rho_\omega = x_\rho \;\; and \;\; \sigma_\omega = x_\sigma\}. \tag{4.69}$$

The set of R_ω on $\Gamma^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma}$ can be defined as

$$\Omega^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma} = \{R_\omega | \omega \in \Gamma^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma}\}. \tag{4.70}$$

If the set $\Omega^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma}$ is not empty, we define the restriction of function Φ to $\Omega^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma}$ as

$$\Phi|_{\Omega^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma}}(R_\omega) = \Phi(R_\omega, x_\sigma, x_\rho) \; where \; R_\omega \in \Omega^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma}. \tag{4.71}$$

The following theorem for the relationship between Φ and parameters R_ω is then restricted on the sets $\Omega^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma}$ where $x_\rho = 0$.

Theorem IV.2. *Considering all the risky assets in the trading book follow the SV model (2.7), if the capital allocation of the trading book satisfies*

1. *the simple portfolio selection strategy (constant vector ω)is applied with $R_\omega > 0$ and $\Omega^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma}$ is not empty with $x_\rho = 0$;*

2. *the optimal policy ψ^* from (4.19) is used to allocate capital between risky portfolio and risk-free asset;*

the ω-utility Φ of total wealth restricted on the set $\Omega_{x_\rho,x_\sigma}^{(\rho_\omega,\sigma_\omega)}$ defined in (4.71), $\Phi|_{\Omega_{x_\rho,x_\sigma}^{(\rho_\omega,\sigma_\omega)}}$, is a non-decreasing function of the index risk premium coefficient R_ω.

Proof: Based on (4.59), Φ can be formulated as

(4.72)

$$\Phi(R_\omega,\sigma_\omega,\rho_\omega)=\frac{P(0)^{1-\gamma}}{1-\gamma}\mathbb{E}\Big[exp\Big\{(1-\gamma)\int_0^T\Big(r+\psi^*(s,Y(s))R_\omega Y(s)-\frac{1}{2}\sigma_\omega^2{\psi^*}^2(s,Y(s))Y(s)\Big)\,ds$$
$$+(1-\gamma)\int_0^T\sigma_\omega\psi^*(s,Y(s))\sqrt{Y(s)}dW^{(X)}(s)\Big\}\Big].$$

For simplicity, we define

(4.73)

$$\begin{aligned}K(R_\omega,\sigma_\omega,\rho_\omega)=&(1-\gamma)\int_0^T\Big(r+\psi^*(s,Y(s))R_\omega Y(s)-\frac{1}{2}\sigma_\omega^2{\psi^*}^2(s,Y(s))Y(s)\Big)\,ds\\&+(1-\gamma)\int_0^T\sigma_\omega\psi^*(s,Y(s))\sqrt{Y(s)}dW^{(X)}(s).\end{aligned}$$

Since the state variable Y follows a stochastic process, K is a random variable with unknown distribution. However, $K|Y$ is a random variable with well known distribution. When $Y(t)$ is given over $[0,T]$, $\int_0^T\sigma_\omega\psi^*(s,Y(s))\sqrt{Y(s)}dW^{(X)}(s)$ is a normal random variable with zero mean and variance $\int_0^T\sigma_\omega^2{\psi^*}^2(s,Y(s))Y(s)ds$ because the correlation coefficient ρ_ω between $dW^{(X)}$ and $dW^{(Y)}$ is zero. Therefore, $K|Y$ is a normal random variable with mean

$$(1-\gamma)\int_0^T\Big(r+\psi^*(s,Y(s))R_\omega Y(s)-\frac{1}{2}\sigma_\omega^2{\psi^*}^2(s,Y(s))Y(s)\Big)\,ds$$

and variance

$$(1-\gamma)^2\int_0^T\sigma_\omega^2{\psi^*}^2(s,Y(s))Y(s)ds.$$

Then, $\mathbb{E}[exp(K|Y)]$ has closed analytical form

$$\mathbb{E}[exp(K|Y)]=exp\Big\{(1-\gamma)\int_0^T\Big[r+Y(s)\Big(\psi^*(s,Y(s))R_\omega-\frac{\gamma}{2}\sigma_\omega^2{\psi^*}^2(s,Y(s))\Big)\Big]\,ds\Big\}.$$

The ω-utility Φ can be formulated as

$$\Phi = \frac{P(0)^{1-\gamma}}{1-\gamma}\mathbb{E}_Y\left[\mathbb{E}[exp(K|Y)]\right].$$

Based on this formula, Φ and $\mathbb{E}[exp(K|Y)]$ have the same relationship with parameters R_ω and σ_ω when $\rho_\omega = 0$. Within the formulation of $\mathbb{E}[exp(K|Y)]$, the term $\psi^* R_\omega - \frac{\gamma}{2}\sigma_\omega^2 \psi^{*^2}$ is the key component for our analysis.

When the simple portfolio selection strategy is restricted on the set $\Gamma_{x_\rho,x_\sigma}^{(\rho_\omega,\sigma_\omega)}$ with $x_\rho = 0$ ($\rho_\omega = 0$), from (4.25), the optimal policy ψ^* satisfies the following inequality

$$\psi^* \le \psi_0 = \frac{1}{\gamma}\frac{R_\omega}{\sigma_\omega^2}. \tag{4.74}$$

Moreover, ψ_0 increases with R_ω and decreases with σ_ω. Since the risky portfolio upper bound G is reciprocal with VaR% function (ϕ), G is a non-decreasing function of R_ω according to the Theorem III.4. Therefore, the optimal policy ψ^* is a non-decreasing function of R_ω. Together with (4.74), we can prove the terms $\psi^* R_\omega - \frac{\gamma}{2}\sigma_\omega^2 \psi^{*^2}$ is a non-decreasing function of the index risk premium coefficient R_ω by the same method used in the proof of Theorem IV.1. Therefore, the same property holds for the expected utility of total wealth restricted on the set $\Gamma_{x_\rho,x_\sigma}^{(\rho_\omega,\sigma_\omega)}$ with $x_\rho = 0$ i.e. $\Phi|_{\Omega_{x_\rho,x_\sigma}^{(\rho_\omega,\sigma_\omega)}}$. $\square$

For the case of $\rho_\omega \neq 0$, we have to rely on the numerical experiments to unfold the relationship between Φ and those two parameters: R_ω and σ_ω. Figure 4.3 shows the ω-utility Φ with different choices of R_ω, σ_ω and ρ_ω. The range of those three parameters are summarized in Table 4.1. All the plots of Figure 4.3 indicate the same relationship between Φ and R_ω, σ_ω. First, Φ increases with R_ω with all the choices of σ_ω and ρ_ω. Theorem IV.2 only reveals the relation between Φ and R_ω when $\rho_\omega = 0$. Actually, the numerical results shown in Figure 4.3 suggest that the same property holds for non-zero ρ_ω. Second, Φ decreases with σ_ω with all the choices of

R_ω and ρ_ω. These results give the banks the same guidelines as derived in the GBM model. That is, the optimal allocation ω^* lies in the balanced combination of risk premium coefficient R_ω and volatility coefficient σ_ω.

Parameter	Range or Value
R_ω	$[0,\ 0.004]$
σ_ω	$[0.01,\ 0.25]$
ρ_ω	$\{-1, -0.5, 0, 0.5, 1\}$
c	0.05
d/c	1
g	0.1
Y_0	1
γ	0.5
δ	3.5
T	252 days

Table 4.1: Parameter setting for the numerical experiments of observing the relationship between Φ and parameters: R_ω, σ_ω and ρ_ω.

In brief, the results from Theorems IV.1-IV.2 and the numerical results shown in Figure 4.3 give a very important guideline in the next step for finding the optimal allocation ω^* within the risky portfolio. First, for any simple portfolio selection strategy ω in the risky portfolio, the bank can achieve the maximal expected utility (ω-utility Φ) of the total wealth through the allocation strategy ψ^* formulated in (4.19). Second, the strategy ω affects Φ of the total wealth through three parameters: R_ω, σ_ω and ρ_ω. Especially, when certain condition is satisfied, Φ is an increasing function of R_ω and a decreasing function of σ_ω. In the next step, since the optimal allocation ω^* within the risky portfolio is an n-dimensional vector, in order to avoid searching in the n-dimensional space, the characteristics of ω-utility Φ must be utilized in the optimization process.

4.5 Optimal Allocation Within Risky Portfolio

The goal of this section is to develop an algorithm to search for the optimal allocation ω^* within the risky portfolio. The results from Theorems IV.1-IV.2 and numerical experiments in previous section lay down the groundwork for this step. Although two different algorithms are developed based on different risky asset models, both of them utilize the relationship between ω-utility and parameters: R_ω and σ_ω.

4.5.1 The case with Geometric Brownian Motion (GBM) model

In this case, the simple portfolio selection strategy ω affects the ω-utility Φ of the total wealth through only two parameters: R_ω and σ_ω. According to the Theorem IV.1, Φ is an increasing function of the index risk premium R_ω and a decreasing function of the index volatility σ_ω. Therefore, when searching for the optimal ω^*, the banks have two optimization processes simultaneously. The first one is to maximize R_ω and the second one is to minimize σ_ω. R_ω is linear with ω and σ_ω^2 is quadratic with ω i.e.

$$R_\omega = \omega' ab \ \ and \ \ \sigma_\omega^2 = \omega' a a' \omega.$$

Unfortunately, these two parameters move together with ω. Usually, higher R_ω comes with higher σ_ω. Therefore, the optimal ω^* lies in the balanced combination of R_ω and σ_ω. In order to avoid searching in an n-dimensional space, we first develop a method to find the ω for a given R_ω with the minimal σ_ω.

Lemma IV.3. *Considering the constrained minimization problem*

$$\min_{\Theta\omega=\theta} \omega' A \omega$$

where A is an $n \times n$ positive definite matrix, Θ is an $m \times n$ $(m < n)$ matrix and

θ is an m-dimensional vector, if there exist at least one solution ω satisfying the constraint

$$\Theta\omega = \theta, \tag{4.75}$$

the solution of the problem is given by

$$\hat{\omega} = -\frac{1}{2}A^{-1}\Theta'\alpha, \tag{4.76}$$

where α is an m-dimensional vector, i.e. $\alpha = (\alpha_1, ..., \alpha_m)'$. α is the solution of the following linear equation

$$-\frac{1}{2}\Theta A^{-1}\Theta'\alpha = \theta. \tag{4.77}$$

In particular, if the rank of matrix Θ is m, α is given by

$$\alpha = -2(\Theta A^{-1}\Theta')^{-1}\theta. \tag{4.78}$$

Proof: With the Lagrangian multiplier $\alpha = (\alpha_1, ..., \alpha_m)'$, we define the Lagrange function

$$L = \omega' A\omega + \alpha'(\Theta\omega - \theta).$$

The formula (4.76) can be directly derived from the first order condition of L

$$L_\omega = 2A\omega + \Theta'\alpha = 0.$$

Substituting (4.76) into the constraint (4.75), we have the linear equation (4.77). If there is at least one solution for the constrain (4.75), the vector θ is in the column space of the matrix Θ. Since matrix A is positive definite, so is its inverse A^{-1}. By Cholesky decomposition, we have

$$A^{-1} = UU',$$

where U is a lower triangular and invertible matrix. Therefore, the matrix $\Theta A^{-1}\Theta'$ can be formulated as

$$\Theta U U' \Theta'.$$

By Theorem A.1 and A.5, the vector θ is also in the column space of the matrix $\Theta U U' \Theta'$ which guarantees the existence of solution in the linear equation (4.77). If the rank of the matrix Θ is m, the matrix $\Theta U U' \Theta'$ is invertible by Theorem A.1 and A.6 and the solution of the linear equation can be represented in (4.78). $\square$

This lemma enables us to choose ω from the set

$$\Gamma_{x_R}^{(R_\omega)} = \{\omega | 1_n' \omega = 1 \ and \ R_\omega = x_R\}$$

with the lowest possible σ_ω. Since the ω-utility Φ decreases with σ_ω by Theorem IV.1, the lemma essentially enables us to find the ω for the highest Φ within the pool of strategies giving the same risk premium x_R. We denote this lowest possible σ_ω for the given x_R by $\hat{\sigma}(x_R)$ and the corresponding ω by $\hat{\omega}(x_R)$. According to Lemma IV.3, $\hat{\sigma}^2(x_R)$ and $\hat{\omega}(x_R)$ can be formulated as

$$\begin{aligned} \hat{\omega}(x_R) &= A^{-1}\Theta'(\Theta A^{-1}\Theta')^{-1}\theta, \\ \hat{\sigma}^2(x_R) &= \theta'(\Theta A^{-1}\Theta')^{-1}\theta, \end{aligned} \tag{4.79}$$

where $A = aa'$, $\Theta = (1_n, ab)'$ and $\theta = (1, x_R)'$. The optimal allocation ω^* within the risky portfolio lies in the set

$$\Gamma = \{\hat{\omega}(x) | x \in \hat{\mathbb{R}}\},$$

where $\hat{\mathbb{R}}$ is the set of all possible R_ω. Therefore, only searching in the interval of possible R_ω is needed for the optimal ω^*. In other words, the optimization process is essentially to search the optimal x_R^* in the interval of R_ω instead of searching in the n-dimension space of ω. In order to develop the optimization framework, we also need the following notations

$$(4.80)\qquad \begin{aligned}
\hat{\psi}_0(x_R) &= \frac{1}{\gamma}\frac{x_R}{\hat{\sigma}(x_R)^2},\\
\hat{G}(x_R) &= G(x_R, \hat{\sigma}(x_R)),\\
\hat{\phi}(x_R) &= \phi(x_R, \hat{\sigma}(x_R)),\\
\hat{\psi}(x_R) &= min(\hat{\psi}_0(x_R), \hat{G}(x_R)),\\
\hat{\Phi}(x_R) &= \Phi(x_R, \hat{\sigma}(x_R))
\end{aligned}$$

Equipped with the results in Lemma IV.3 and formulation of Φ in (4.65), we have

$$(4.81)\qquad \hat{\Phi}(x_R) = \frac{P(0)^{1-\gamma}}{1-\gamma} exp\left\{(1-\gamma)T\left[\hat{\psi}(x_R)\left(x_R - \frac{1}{2}\gamma\hat{\psi}(x_R)\hat{\sigma}^2(x_R)\right) + r\right]\right\}.$$

The optimization process is essentially finding the suitable risk premium x_R^* to maximize $\hat{\Phi}(x_R)$ and the maximum $\hat{\Phi}^*$ is the global maximum of Φ.

If the function $\hat{\Phi}(x_R)$ is differentiable, it is very straightforward to find the optimum x_R^*. Unfortunately, $\hat{\Phi}$ is not differentiable for all x_R because $\hat{\psi}$ is not differentiable. $\hat{\psi}$ is the minimum of $\hat{\psi}_0$ and $\hat{G}$. Based on these two components of $\hat{\psi}$, we define

(4.82)

$$\begin{aligned}
\hat{\Phi}^{(\hat{\psi}_0)}(x_R) &= \frac{P(0)^{1-\gamma}}{1-\gamma} exp\left\{(1-\gamma)T\left[\hat{\psi}_0(x_R)\left(x_R - \frac{1}{2}\gamma\hat{\psi}_0(x_R)\hat{\sigma}^2(x_R)\right) + r\right]\right\}\\
\hat{\Phi}^{(\hat{G})}(x_R) &= \frac{P(0)^{1-\gamma}}{1-\gamma} exp\left\{(1-\gamma)T\left[\hat{G}(x_R)\left(x_R - \frac{1}{2}\gamma\hat{G}(x_R)\hat{\sigma}^2(x_R)\right) + r\right]\right\}
\end{aligned}$$

Then, $\hat{\Phi}$ can be formulated in a different form

$$(4.83)\qquad \hat{\Phi}(x_R) = \begin{cases} \hat{\Phi}^{(\hat{\psi}_0)}(x_R) & \text{if } \hat{\psi}_0(x_R) < \hat{G}(x_R) \\ \hat{\Phi}^{(\hat{G})}(x_R) & \text{otherwise} \end{cases}$$

Therefore, the maximizer x_R^* of $\hat{\Phi}$ can be one of the following three cases:

1. the maximizer $x_R^{\hat{\psi}_0 *}$ of $\hat{\Phi}^{(\hat{\psi}_0)}$;

2. the maximizer $x_R^{\hat{G}*}$ of $\hat{\Phi}^{(\hat{G})}$;

3. elements in $\{x_R^I | \hat{\psi}_0(x_R^I) = \hat{G}(x_R^I)\}$.

In brief, the optimization in this case can be reduced to the problem of finding solutions for three simpler problems which can be solved either analytically or numerically.

For the first case, $x_R^{\hat{\psi}_0 *}$ can be derived by solving the equation $\frac{d\hat{\Phi}^{(\hat{\psi}_0)}}{dx_R} = 0$. By (4.82), we have

$$\hat{\Phi}^{(\hat{\psi}_0)}(x_R) = \frac{P(0)^{1-\gamma}}{1-\gamma} exp\left\{(1-\gamma)T\left(\frac{x_R^2}{2\gamma\hat{\sigma}^2(x_R)} + r\right)\right\}.$$

The equation $\frac{d\hat{\Phi}^{(\hat{\psi}_0)}}{dx_R} = 0$ is equivalent to

$$d\left(\frac{x_R^2}{\hat{\sigma}^2(x_R)}\right)/dx_R = 0,$$

where $\hat{\sigma}^2$ is defined in (4.79). Then, we have

$$\frac{2x_R\hat{\sigma}^2(x_R) - x_R^2\frac{d\hat{\sigma}^2(x_R)}{dx_R}}{\hat{\sigma}^4(x_R)} = 0.$$

It is equivalent to the following equation

$$2\hat{\sigma}^2(x_R) - x_R\frac{d\hat{\sigma}^2(x_R)}{dx_R} = 0.$$

We set $(\Theta A^{-1}\Theta')^{-1} = \Lambda$, where

$$\Lambda = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}.$$

Then, the equation becomes

$$\begin{aligned} 2\hat{\sigma}^2(x_R) - x_R\frac{d\hat{\sigma}^2(x_R)}{x_R} &= 2\alpha_{11} + 2(\alpha_{12}+\alpha_{21})x_R + 2\alpha_{22}x_R^2 - 2\alpha_{21}x_R - 2\alpha_{22}x_R^2 \\ &= 2\alpha_{1,1} + 2\alpha_{1,2}x_R \\ &= 0. \end{aligned}$$

To get $x_R^{\hat{\psi}_0 *}$, we just need to solve

$$\alpha_{1,1} + \alpha_{1,2} x_R = 0,$$

where $\alpha_{i,j}$ is the element on i^{th} row and j^{th} column of the matrix Λ. Therefore, the maximizer $x_R^{\hat{\psi}_0 *}$ of $\hat{\Phi}^{(\hat{\psi}_0)}$ is given by

$$x_R^{\hat{\psi}_0 *} = -\frac{\alpha_{1,1}}{\alpha_{1,2}}. \tag{4.84}$$

For the second case, it is more complicated than the previous one. By (4.10), the key component $\hat{G}$ is given by

$$\hat{G}(x_R) = \frac{1}{1 + \delta \hat{\phi}(x_R)^+}$$

and is piecewise differentiable with the exceptions on the set $\{x_R | \hat{\phi}(x_R) = 0\}$. Moreover, $\hat{G} = 1$ for all x_R with $\hat{\phi}(x_R) \leq 0$. We define

$$\bar{G}(x_R) = \frac{1}{1 + \delta \hat{\phi}(x_R)}.$$

Then, the function $\hat{\Phi}^{(\hat{G})}$ can be formulated as

(4.85)

$$\hat{\Phi}^{(\hat{G})}(x_R) = \begin{cases} \frac{P(0)^{1-\gamma}}{1-\gamma} exp\left\{(1-\gamma)T\left(x_R - \frac{1}{2}\gamma\hat{\sigma}^2(x_R) + r\right)\right\} & \text{if } \hat{\phi}(x_R) \leq 0 \\ \frac{P(0)^{1-\gamma}}{1-\gamma} exp\left\{(1-\gamma)T\left[\bar{G}(x_R)\left(x_R - \frac{1}{2}\gamma\bar{G}(x_R)\hat{\sigma}^2(x_R)\right) + r\right]\right\} & \text{otherwise} \end{cases}$$

From the second component of formulation (4.85), the traditional method of finding extreme points by differentiation gives rise to a very complicated transcendental equation. Therefore, there is no analytical solution for this case. In this study, we directly use optimization tools in *MATLAB* to find the maximizer of $\hat{\Phi}^{(\hat{G})}$. In *MATLAB*, the function *fminbnd* finds the minimum of a function in the given interval. In order to utilize *fminbnd*, the objective function is set as $-\hat{\Phi}^{(\hat{G})}$ instead. The searching interval is the possible range of x_R.

For the third case, it is essentially finding the solution of a nonlinear equation $\hat{\psi}_0(x_R) = \hat{G}(x_R)$ which can also be solved numerically. Denote this solution by x_R^I. Again, we use the numerical analysis tools in *MATLAB* to find the solution x_R^I. In *MATLAB*, the function *fsolve* finds the roots of nonlinear equation with a given initial starting point. The solution obtained by the function *fsolve* varies with the starting point. When there are multiple solutions for the equation, in order to identify all the solutions in the given interval, we evenly sample multiple points in the interval of x_R and apply each of them as the starting point in the function *fsolve*. Then, all possible solutions can be obtained.

In summary, the maximizer of $\hat{\Phi}$ comes from one of $x_R^{\hat{\psi}_0 *}$, $x_R^{\hat{G}*}$ and x_R^I. The first candidate can be derived analytically. The other two can be obtained by the numerical tools in *MATLAB*. When $x_R^{\hat{\psi}_0 *}$, $x_R^{\hat{G}*}$ and x_R^I are available, we just need to choose the one with the highest value of $\hat{\Phi}$ as shown in Figure 4.4. In the numerical experiments, the risky portfolio is constructed based on 10 risky assets. The risk premium $R = ab$ and diffusion coefficient a are a vector and a matrix, respectively. The setting of the parameters are shown in the Table 4.2

Figure 4.4 shows the process of x_R selection for maximizing $\hat{\Phi}$. In the upper panel, the dashed and dotted curves represent functions $\hat{\psi}_0(x_R)$ and $\hat{G}(x_R)$, respectively. The red continuous curve represents the function $\hat{\psi}(x_R)$ which is the minimum of $\hat{\psi}_0(x_R)$ and $\hat{G}(x_R)$. The vertical lines passing the intersections of $\hat{\psi}_0(x_R)$ and $\hat{G}(x_R)$ identify the locations of x_R^I. In the lower panel, the three curves (dashed, dotted and red continuous) represent the expected utilities when $\hat{\psi}_0(x_R)$, $\hat{G}(x_R)$ and $\hat{\psi}(x_R)$ are applied in the calculation. The vertical lines identify the locations of the possible maximizers: $x_R^{\hat{\psi}_0 *}$, $x_R^{\hat{G}*}$ and x_R^I. The markers are the corresponding expected utilities $\hat{\psi}(x_R)$ of those candidates. It is obvious that the optimal x_R^* is one of those four candidates shown in the plot.

R=ab	*Matrix a*									
0.0003173	0.0626496	-0.070878	-0.1098374	0.1631608	0.139467	-0.2015021	0.1479187	-0.1697228	0.2024644	-0.2088622
0.0001197	-0.0782711	-0.2362922	0.000722	-0.0932083	0.1542436	0.2312093	-0.2216554	-0.0894305	0.1020508	0.0110455
0.0003287	0.0239289	-0.0977015	0.2020814	0.0232815	-0.0514767	-0.0048977	-0.0086564	0.086885	0.1504532	0.0850411
7.781E-05	-0.1549096	-0.0504227	0.1884213	0.2388747	-0.2373807	0.2454812	0.1799054	0.1806524	0.1581545	-0.1980857
0.0003379	-0.175279	-0.0034205	0.0998584	0.0193355	0.0827974	-0.2124373	0.1562256	-0.2483105	0.1171904	0.0904432
0.0001335	0.1289043	-0.217522	0.148953	-0.10808	-0.0945365	-0.1432644	-0.0970945	0.0631051	-0.143456	0.1764293
0.0001786	0.1115987	-0.1081273	0.1583162	0.180756	0.0743993	0.129495	-0.0141598	0.2203424	0.1370031	-0.2381252
0.0001409	-0.1955598	0.0294412	0.0878679	-0.2212206	0.0487879	0.1931541	-0.1009883	0.0498431	0.2407553	0.0865858
9.299E-05	0.0695712	-0.2237422	0.1568434	-0.117598	0.2497888	0.1269126	-0.0118719	-0.0616532	0.1483013	-0.1267268
3.839E-05	0.1534574	-0.1496479	0.1983876	-0.2334585	-0.0326023	-0.1874744	0.1654843	0.0079164	-0.1063363	-0.0094377

Parameter	Range or Value
n	10
γ	0.3
δ	3.5
T	252 days

Table 4.2: The parameter setting of the numerical experiment for finding the optimal x_R^* with GBM model.

4.5.2 The case with Stochastic Volatility (SV) model

In this case, the optimization is much more complicated than the case with GBM model. First, the simple portfolio selection strategy ω affects the ω-utility Φ of the total wealth through three parameters: R_ω, σ_ω, and ρ_ω. Their formulas are given as follows

$$R_\omega = \omega' ab \;\; and \;\; \sigma_\omega^2 = \omega' aa' \omega \;\; and \;\; \rho_\omega = \rho a' \omega / \sqrt{\omega' aa' \omega}.$$

According to the numerical results shown in Figure 4.3, the relation between Φ and those three parameters can be well determined. For any given ρ_ω, Φ is an increasing function of R_ω and a decreasing function of σ_ω. Second, the time complexity of expected utility estimation for any given simple portfolio selection strategy ω is very high. Due to the additional state variable Y, there is no closed-form expression for the ω-utility Φ. The estimation of the ω-utility Φ relies on the numerical scheme based on the trinomial CIR-tree and dynamic programming. In the numerical procedure, we calculate the ψ^* as well as the expected utility at each tree node iteratively. Based on the formulas in (4.57), the calculation at each tree node involves an optimization. At each time step, the number of tree nodes varies from 1 to n_{max} where n_{max} is the maximal number of tree nodes at any time step. If the investment horizon is long enough, most of the time steps are associated with n_{max} tree nodes. For example, in order to determine the optimal ω^* for one year investment horizon, the simulation is performed based on the discretized model with 252 time steps. With the numerical example shown in Figure 4.1, there are 25 tree nodes at each time step after $t = 17$. The total number of tree nodes could be very large. Therefore, the estimation of ω-utility Φ for one trial of ω is very time consuming. In practice, the total number of trials is limited for the searching algorithm. When the number n of risky assets is large, searching in an n-dimensional space is impossible because it needs large number of trials to achieve acceptable solution. The main purpose of this section is to reduce the scale of the optimization problem by reducing the dimension of the searching space.

As mentioned before, the simple portfolio selection strategy ω only affects three parameters: R_ω, σ_ω, and ρ_ω. Moreover, the numerical results shown in Figure 4.3 imply that Φ is an increasing function of R_ω and a decreasing function of σ_ω for any given ρ_ω. Based on this fact, we first develop a method to find the ω with the maximal R_ω for the given parameters ρ_ω and σ_ω. Then, the searching space can be reduced to the 2-dimensional parameter space. In such way, the number of trials can be significantly reduced, especially when $n \gg 3$.

Lemma IV.4. *Considering the constrained maximization problem*

$$\max_{\substack{\Theta\omega=\theta \\ \omega' A\omega = x^2}} \beta'\omega$$

where A is an $n \times n$ positive definite matrix, Θ is an $m \times n$ $(m < n)$ matrix with rank m, θ is an m-dimensional vector, β is an n-dimensional non-zero vector and x is a scalar, if the following conditions are satisfied

1. *β is not in the row space of Θ (or column space of Θ');*
2. *$x^2 > \theta'(\Theta A^{-1}\Theta')^{-1}\theta$,*

the solution of the problem is given by

$$\hat{\omega} = -\frac{1}{2\lambda}A^{-1}(\beta + \Theta'\alpha), \tag{4.86}$$

where λ is a scalar and α is an m-dimensional vector, i.e. $\alpha = (\alpha_1, ..., \alpha_m)'$. α is given by

$$\alpha = -\kappa_0 - \kappa_1\lambda. \tag{4.87}$$

and λ is one of the two solutions from the quadratic equation

$$\lambda^2(4x^2 - 2\theta'\kappa_1) + \beta'A^{-1}\Theta'\kappa_0 - \beta'A^{-1}\beta = 0, \tag{4.88}$$

where κ_0 and κ_1 are given by

$$\kappa_0 = (\Theta A^{-1}\Theta')^{-1}\Theta A^{-1}\beta \quad \textit{and} \quad \kappa_1 = 2(\Theta A^{-1}\Theta')^{-1}\theta. \tag{4.89}$$

Proof: With the Lagrangian multipliers $\alpha = (\alpha_1, ..., \alpha_m)'$ and λ, we define the Lagrange function

$$L = \beta'\omega + \alpha'(\Theta\omega - \theta) + \lambda(\omega'A\omega - x^2).$$

The formula (4.86) can be directly derived from the first order condition of L

$$L_\omega = \beta + \Theta'\alpha + 2\lambda A\omega = 0.$$

Substituting (4.86) into the constraint $\Theta\omega = \theta$, we have

$$\Theta A^{-1}\Theta'\alpha = -2\lambda\theta - \Theta A^{-1}\beta.$$

Since matrix A is positive definite, so is its inverse A^{-1}. By Cholesky decomposition, we have

$$A^{-1} = UU',$$

where U is a lower triangular and invertible matrix. Therefore, the matrix $\Theta A^{-1}\Theta'$ can be formulated as

$$\Theta UU'\Theta'.$$

Since the rank of matrix Θ is m, so is the rank of ΘU. Therefore, the matrix $\Theta UU'\Theta'$ is invertible by Theorem A.6 and α can be represented in (4.87) with (4.89).

Substituting (4.86) and (4.87) into the constraint $\omega' A\omega = x^2$, we have the quadratic equation of λ in (4.88). Together with (4.89), the coefficient of λ^2 is

$$4x^2 - 2\theta'\kappa_1 = 4\left[x^2 - \theta'(\Theta A^{-1}\Theta')^{-1}\theta\right].$$

Based on the second condition, we have $4x^2 - 2\theta'\kappa_1 > 0$. Moreover, by Theorem A.9, we have

$$\beta' A^{-1}\Theta'\kappa_0 - \beta' A^{-1}\beta \leq 0,$$

and the equality holds if and only if $\beta \in Row(\Theta)$. Since β is not in the row space of Θ, the term $\beta' A^{-1}\Theta'\kappa_0 - \beta' A^{-1}\beta$ is always negative. That implies that the equation in (4.88) always has two distinct solutions in real numbers. Moreover, between these two solutions, one is the minimizer and another is the maximizer of $\beta'\omega$ with the constraints $\Theta\omega = \theta$ and $\omega' A\omega = x^2$. $\square$

When the ρ_ω and σ_ω are given, the set of ω is defined as

$$\Gamma^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma} = \left\{\omega | 1_n'\omega = 1 \ and \ \rho_\omega = x_\rho \ and \ \sigma_\omega = x_\sigma\right\}.$$

If $\Gamma^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma}$ is not empty for the given x_ρ and x_σ, the results of Lemma IV.4 can be applied with the following adjustment

$$\beta = ab, \quad \Theta = (1_n, a\rho')', \quad \theta = (1, x_\rho x_\sigma)', \quad A = aa', \quad x^2 = x_\sigma^2.$$

The lemma enable us to choose the ω from the set $\Gamma^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma}$ with highest possible R_ω. By the numerical results in previous section, the ω-utility Φ increases with the R_ω. Therefore, the lemma essentially enables us to find the ω for the highest Φ within the pool $\Gamma^{(\rho_\omega,\sigma_\omega)}_{x_\rho,x_\sigma}$ of strategies with correlation and volatility equal to x_ρ and x_σ, respectively. We denote this highest possible R_ω for the given x_ρ and x_σ by $\hat{R}(x_\rho, x_\sigma)$ and the corresponding ω and Φ by $\hat{\omega}(x_\rho, x_\sigma)$ and $\hat{\Phi}(x_\rho, x_\sigma)$, respectively. The optimal allocation ω^* within the risky portfolio lies in the set

$$\Gamma = \left\{ \hat{\omega}(x_\rho, x_\sigma) | x_\rho \in [-1, 1] \ \ and \ \ x_\sigma \in \hat{\Sigma} \right\},$$

where $\hat{\Sigma}$ is the set of all possible σ_ω. Therefore, the optimization process is essentially finding the suitable pair of (x_ρ, x_σ) to maximize $\hat{\Phi}(x_\rho, x_\sigma)$ and the maximum $\hat{\Phi}^*$ is the global maximum of Φ. The searching space is indeed 2-dimensional.

In this study, Simulated Annealing (SA) algorithm is used to search for the optimal (x_ρ^*, x_σ^*). SA is a generic probabilistic metaheuristic method of locating a good approximation to the global optimum for the global optimization problem. The method was independently described by Kirkpatrick in 1983 [18] and by Černý in 1985 [25]. SA starts with a randomly selected point in the searching space. At each step, the algorithm randomly selects a neighbor point of the current point and probabilistically decides whether moves to the new position or stays in the current location. The probability leads the selection to move to the area with high objective value ($\hat{\Phi}$ in this case). In the pure SA method, only the current position is stored and the algorithm stops when the preset conditions are met (Maximal number of steps is used in this study). In this study, the SA algorithm always keeps track of the best solution found so far because the estimation of $\hat{\Phi}$ for one trial (x_ρ, x_σ) is time-consuming and should not be thrown away easily. When a new trial solution is selected, its utility is compared to the utility of current solution. If the new solution has higher utility, it replaces the current one and the algorithm starts from it in the next step. However, if the utility of the new solution is lower, the algorithm accepts the new solution with probability $exp(\hat{\Phi}_i - \hat{\Phi}_{i-1})$ where $\hat{\Phi}_{i-1}$ and $\hat{\Phi}_i$ are utility of current solution and new solution, respectively. Apparently, with this setting, the new solution is definitely accepted if its utility is higher than the current one. On the contrary, $exp(\hat{\Phi}_i - \hat{\Phi}_{i-1})$ is less than 1 and the new solution is accepted with this probability. The procedure is outlined in the Algorithm 1. In Figure 4.5, an example of SA searching is demonstrated. In this numerical experiment, a risky portfolio of 10 risky assets is used. The risk premium coefficients of all the risky assets can be condensed in a 10-dimensional vector: $R_0 = ab$. The diffusion term a is a 10×10 matrix. The correlation coefficient between $dW^{(Y)}$ and $dW^{(S)}$ is also represented in a 10-dimensional vector ρ. The setting of parameters are shown in the Table 4.3.

The upper panel of the Figure 4.5 shows the surface of $\hat{\Phi}$ with respect to ρ_ω and σ_ω. The lower panel shows the corresponding contour map. In the contour map, we use different shapes to demonstrate how the SA procedure evolves and pushes the sampling moving to the area around the global optimum. First of all, the square represents the global optimal solution. Since our modified SA method keeps track on the best solution found so far at each iteration, the best solution changes in each step and moves toward to the optimal solution during the whole procedure. The small circles represent the best solutions found so far at each iteration. The triangle is the starting point and the diamond is the best solution at the end of the procedure. In the numerical experiment shown in the figure, the diamond is very close to the square at the end of 500 iterations. The accuracy of this algorithm mainly depends on the number of iterations which is m in the Algorithm 1. Although SA algorithm may not be the best way for searching the optimal solution, the main purpose of the section is to reduce the dimension of the optimization problem. Moreover, any possible better searching algorithm can be applied in this case. However, before using any searching method, the dimension reduction technique we present in this study is crucial since searching in a 2-dimensional space is obviously much more efficient than searching in an n-dimensional space with $n \gg 2$.

$R_0=ab$	Matrix a									
0.0001326	0.00387	-0.0459433	-0.0185517	-0.0105684	0.0260352	0.0172111	-0.0412106	-0.0271253	0.005814	0.0215506
0.0001891	0.0287057	0.0381341	0.0248439	0.0133595	-0.0264963	0.018642	-0.0173175	-0.0122223	-0.0431979	0.0335689
2.353E-05	-0.0053111	-0.0374862	-0.0311738	0.0214768	0.0147802	0.0444421	-0.0372234	-0.0243769	-0.0109793	0.0118933
0.0001213	-0.027875	0.0075362	-0.0195224	-0.0084563	0.0114393	0.0322593	0.0448447	0.0291538	0.0035963	0.0174601
0.0001005	0.0491091	-0.0426392	0.0141249	0.0322939	0.0037324	-0.0171147	0.0489983	0.028198	0.0307798	-0.0121687
9.503E-05	-0.0202735	0.0369394	0.0008932	-0.0335226	0.0267225	-0.0113819	-0.0411571	-0.016266	-0.0355871	-0.0304688
0.0001959	0.0199535	0.0492473	-0.0301183	-0.002405	-0.0469163	-0.0336005	0.0179626	-0.0231018	-0.0040397	0.0403909
2.203E-05	-0.0347713	-0.0321552	0.0389211	-0.0208479	0.0216986	0.0428666	-0.0440794	0.025352	0.0414342	-0.017997
1.93E-05	0.0234692	-0.0153077	-0.0388692	-0.0247979	0.0246954	0.0080615	-0.0410258	-0.0353977	0.0204052	-0.040609
1.995E-05	-0.0293993	-0.0008679	-0.0089329	0.0400546	-0.0117041	0.0168251	0.0326376	-0.0233654	-0.0326629	-0.0289366

ρ									
0.2406111	-0.1887555	0.5197634	0.0851476	-0.0542671	-0.0443096	0.3669109	0.2559898	0.2017156	-0.5421026

Parameter	Range or Value
n	10
c	0.05
d/c	1
g	0.1
Y_0	1
γ	0.5
δ	3.5
T	252 days

Table 4.3: Parameter setting of the numerical experiments of SA method with SV model.

Algorithm 1 Find the optimal allocation ω_1^* for SV model

Initialization:

Set $\beta = ab$; $A = aa'$; $\Theta = (1_n, a\rho')'$;
Set the iteration number m;
Randomly select $(x_{\rho,0}, x_{\sigma,0})$ from the feasible space $[0,1] \times \hat{\Sigma}$;

Procedure:

Set $\theta = (1, x_{\rho,0} x_{\sigma,0})'$;
Calculate $\hat{R}_0 = \hat{R}(x_{\rho,0}, x_{\sigma,0})$ and $\hat{\omega}_0 = \hat{\omega}(x_{\rho,0}, x_{\sigma,0})$ and $\hat{\Phi}_0 = \Phi(\hat{R}_0, x_{\sigma,0}, x_{\rho,0})$;
Set $x_\rho^* = x_{\rho,0}$ and $x_\sigma^* = x_{\sigma,0}$ and $\hat{\Phi}^* = \hat{\Phi}_0$ and $\omega^* = \omega_0$;
for all $i = 1$ to m **do**
  Randomly select $(x_{\rho,i}, x_{\sigma,i})$ from the neighborhood of $(x_{\rho,i-1}, x_{\sigma,i-1})$;
  Set $\theta = (1, x_{\rho,i} x_{\sigma,i})'$;
  Calculate $\hat{R}_i = \hat{R}(x_{\rho,i}, x_{\sigma,i})$ and $\hat{\omega}_i = \hat{\omega}(x_{\rho,i}, x_{\sigma,i})$ and $\hat{\Phi}_i = \Phi(\hat{R}_i, x_{\sigma,i}, x_{\rho,i})$;
  Randomly select ε from $[0,1]$;
  if $exp(\hat{\Phi}_i - \hat{\Phi}_{i-1}) < \varepsilon$ **then**
    set $x_{\rho,i} = x_{\rho,i-1}$ and $x_{\sigma,i} = x_{\sigma,i-1}$;
  else
    if $\hat{\Phi}_i > \hat{\Phi}^*$ **then**
      Set $x_\rho^* = x_{\rho,i}$ and $x_\sigma^* = x_{\sigma,i}$ and $\hat{\Phi}^* = \hat{\Phi}_i$ and $\omega^* = \omega_i$;
    end if
  end if
end for
Output the solution ω^*;

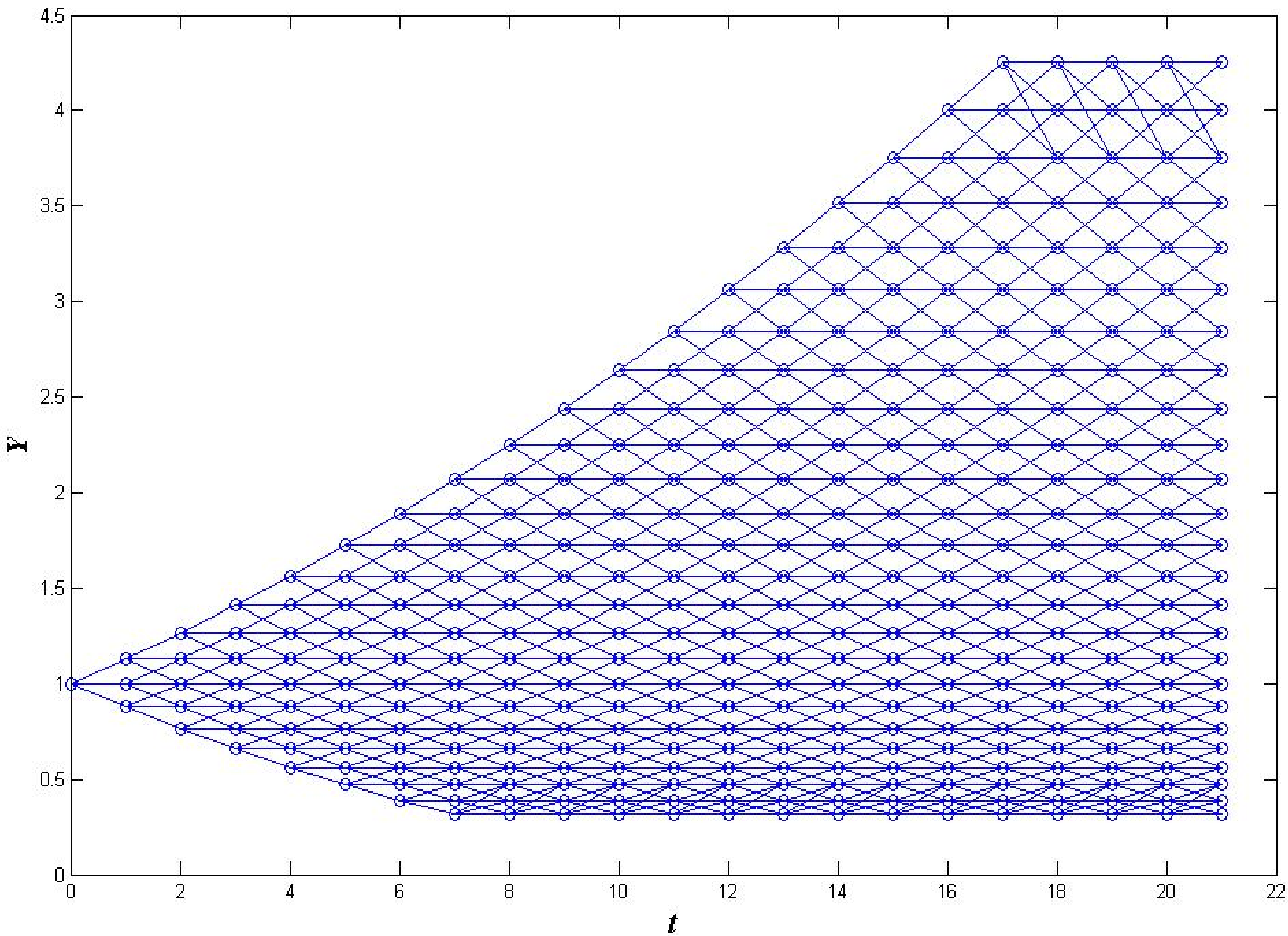


Figure 4.1: The trinomial tree for the stochastic process of the state variable Y. The parameters of Y process are: $c = 0.05$, $d = 0, 05$, $g = 0.1$ and $Y(0) = 1$. The graph demonstrates the first 21 time steps of the tree construction.

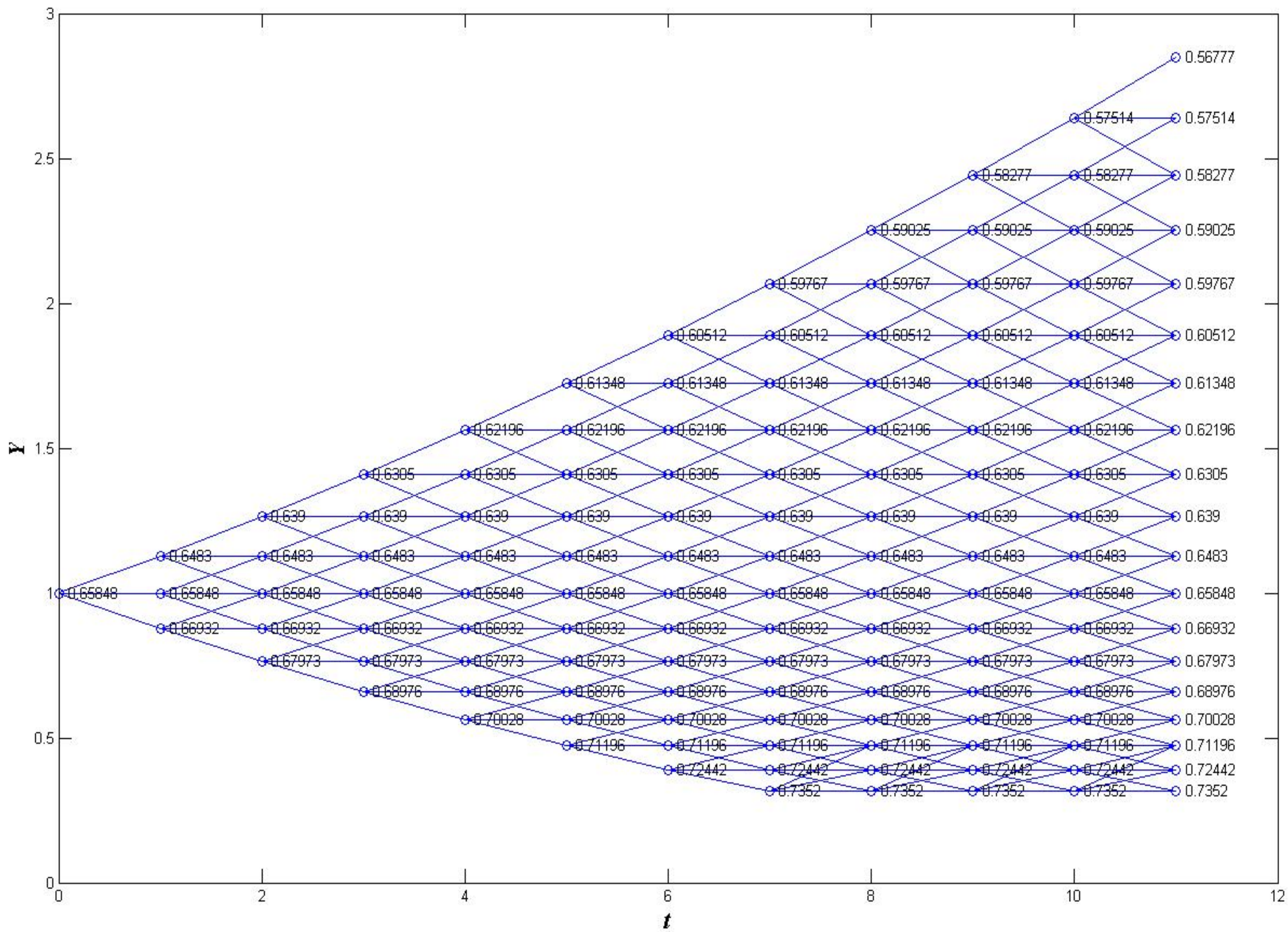


Figure 4.2: The dynamic programming procedure of finding optimal allocation ψ^* on top of the trinomial tree for the stochastic process of the state variable Y. The parameters of Y process are: $c = 0.05$, $d = 0.05$, $g = 0.1$ and $Y(0) = 1$. The parameters for the risky portfolio are: $R_\omega = 0.00051587$, $\sigma_\omega = 0.021$ and $\rho_\omega = -0.2$. The Basel multiplier is $\delta = 3.5$ and investor risk-aversion parameter is $\gamma = 0.5$. The graph demonstrates the first 11 time steps of the procedure. The numbers shown on each tree node are the optimal weight ψ^*.

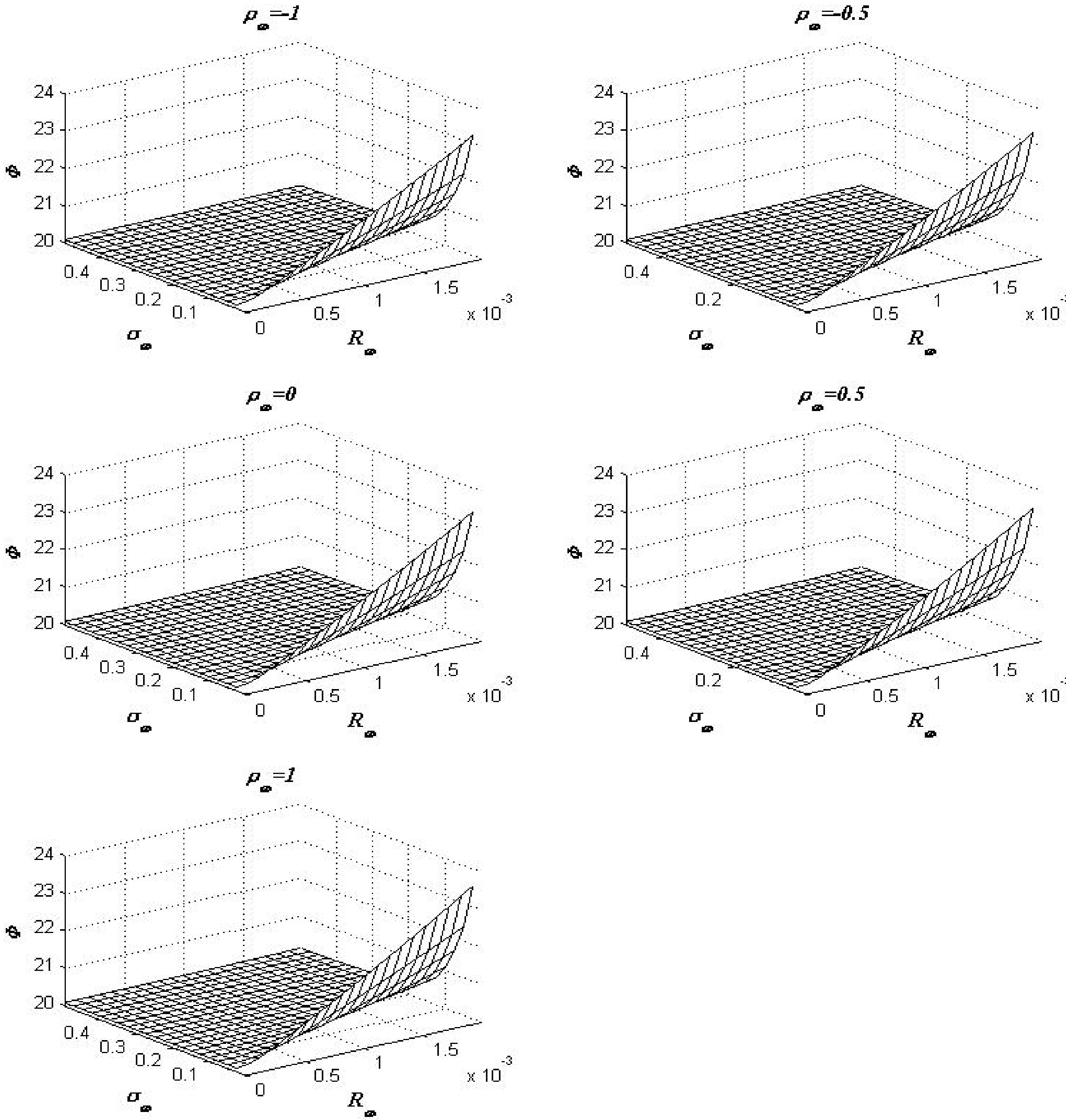
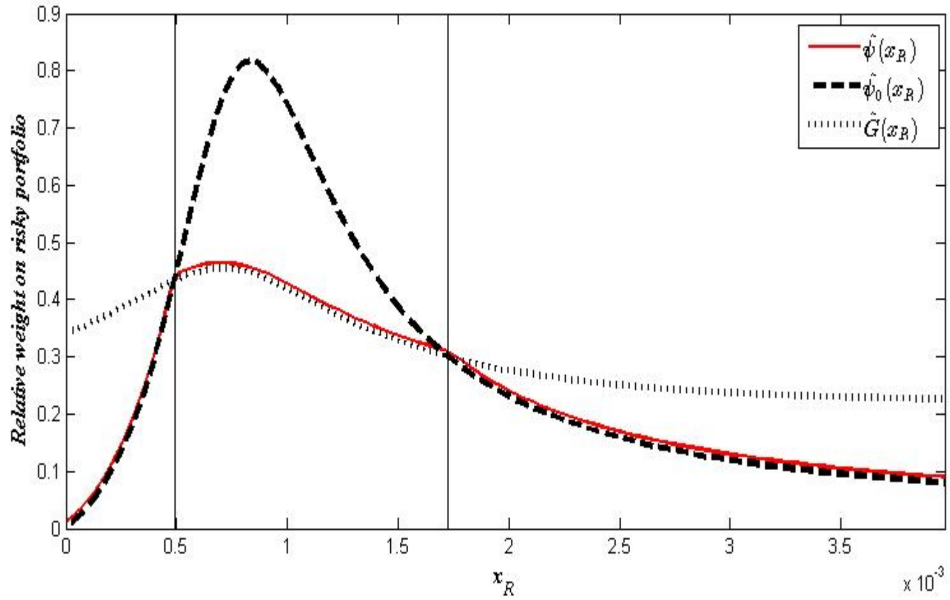


Figure 4.3: Expected utility Φ for difference choices of R_ω, σ_ω, and ρ_ω.



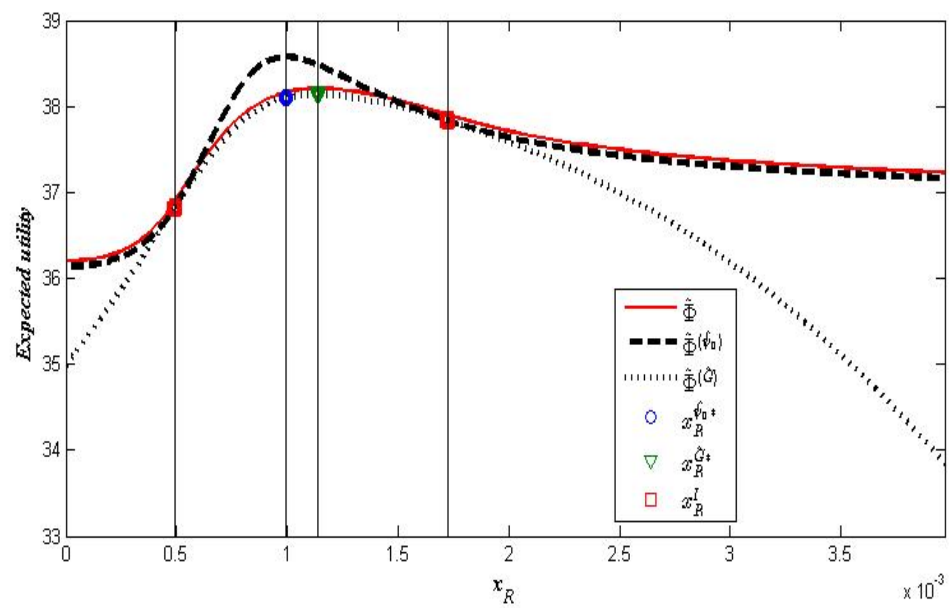


Figure 4.4:
Optimal x_R selection for $\hat{\Phi}$ in the GBM model. In the upper panel, the dashed and dotted curves represent functions $\hat{\psi}_0(x_R)$ and $\hat{G}(x_R)$, respectively. The red continuous curve represent the function $\hat{\psi}(x_R)$ which is the minimum of $\hat{\psi}_0(x_R)$ and $\hat{G}(x_R)$. The vertical lines passing the intersections of $\hat{\psi}_0(x_R)$ and $\hat{G}(x_R)$ identify the locations of x_R^I. In the lower panel, the three curves (dashed, dotted and red continuous) represent the expected utilities when $\hat{\psi}_0(x_R)$, $\hat{G}(x_R)$ and $\hat{\psi}(x_R)$ are applied in calculation. The vertical lines identify the locations of the possible maximizers: $x_R^{\hat{\psi}_0*}$, $x_R^{\hat{G}*}$ and x_R^I. The markers are the corresponding expected utility $\hat{\psi}(x_R)$ of those candidates.

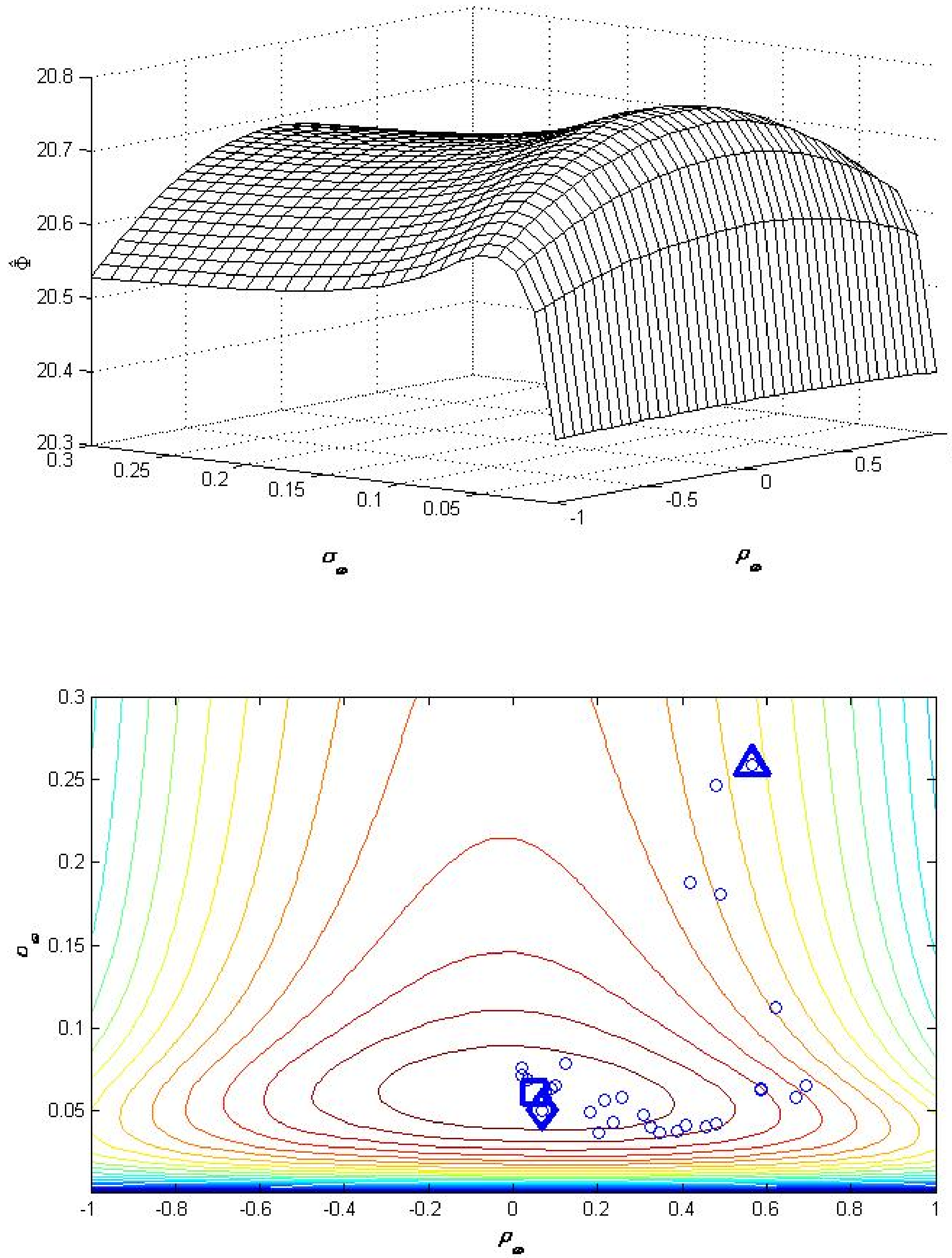


Figure 4.5:
The surface and contour map of $\hat{\Phi}$ with respect to ρ_ω and σ_ω in the SV model. In the lower panel, the square represents the global optimal solution. The circles represent the best trial solutions so far at each iteration during the SA procedure. The triangle is the starting point of the procedure. The diamond is the best solution at the end of the procedure.

CHAPTER V

Conclusion

In this study, we analyze the combination of VaR and dynamic portfolio selection strategy. First, we notice that the VaR estimation in most of the applications is based on the assumption of no adjustment in the portfolio during the VaR horizon. Under this assumption, the VaR does not reflect the risk of the underlying portfolio. In order to reflect the risk induced by the portfolio adjustment during the VaR horizon, the VaR incorporating portfolio selection strategy is developed. The analysis of the new VaR reveals that any adjustment during the VaR horizon could have significant impact on the risk of the portfolio. Second, when VaR is used as a constraint in dynamic portfolio selection, it is not just applied on the terminal portfolio value. According to the Basel' market risk capital requirement, the 10-day VaR needs to be estimated daily and the banks need to adjust the capital reserve according to the updated VaR. The process of finding the optimal selection strategy can be divided into two parts. The first one is the allocation problem between risk-free capital and risky portfolio. The second one is the allocation within the risky portfolio. Each part itself is also an optimization problem and involves sophisticated theoretical analysis and numerical algorithm to achieve the optimal solution.

In Chapter II, we apply the framework established by Merton (1971)[20] and the extended work by Liu [19] to derive the optimal portfolio selection strategies for the GBM and SV models. In this case, the investor is assumed to be risk-averse and try to maximize the expected utility over the terminal wealth. The optimal solution is expressed in terms of the solution of HJB equation. Depending on the choice of the model, the optimal weight vector of risky assets could be as simple as a constant for the GBM model, or could be a very complicated and time-varying function for the SV model. Moreover, for the portfolio with only one risky asset, the optimal solution has a very simple relationship with risk premium and volatility (increases with risk premium and decreases with volatility) in the GBM model. However, in the SV model, the same relationship only holds if the correlation between risky asset price and state variable Y is non-negative. The analysis in this chapter reveals the complexity of the optimal solution when stochastic volatility is present.

In Chapter III, we analyze the VaR estimation incorporating the portfolio selection strategy and the difference between the new VaR and the old VaR. Starting with one single risky asset in the portfolio, we calculate the new VaR with the assumption that the investors apply the optimal portfolio selection strategy in their portfolios. In order to apply the preset strategy, the investors have to actively adjust their portfolios according to the price movement of risky asset. The share number of risky asset changes with the adjustment. Ultimately, the selection strategy could lead to a significant change in the distribution of the portfolio value. Since VaR is essentially a percentile of the portfolio loss, the VaR will be different when the portfolio adjustment is incorporated in the estimation. Moreover, we also calculate the old VaR and difference between those two different VaRs. Based on all the numerical experiments, we notice that the VaR difference changes sign with various parameter setting. The sign (positive or negative) of VaR difference shows whether the old VaR overestimates or underestimates the risk of the portfolio. Moreover, the relationships between VaR difference and relevant parameters are very complicated. The numerical results suggest that the old VaR with no-trading assumption is not suitable in the volatile market (large volatility) or for a long investment horizon.

Besides the optimal selection strategies, we also analyze the VaR with the simple portfolio selection strategy in which the weight of each risky asset remains constant. By applying the simple strategy in the risky portfolio, we can identify several important parameters: R_ω, σ_ω and ρ_ω. These three parameters are functions of the risky asset weight vector and other parameters of the risky asset value process. If the risky portfolio is viewed as one single asset, these parameters are the risky premium (risk premium coefficient in the SV model), volatility (volatility coefficient in the SV model) and correlation coefficient (with state variable Y) of the risky portfolio. Basically, the risky asset weight vector affects the VaR estimation through those three parameters. By theoretical analysis and numerical experiments, we can observe the relationship between VaR and these parameters. That is, VaR decreases with R_ω and increases with σ_ω. This finding is an important building block for finding the optimal portfolio selection strategy with Basel's market risk capital requirement.

In Chapter IV, we develop numerical schemes to find the optimal portfolio selection strategy under Basel's VaR-based capital requirement. Essentially, the Basel's market risk capital requirement gives an upper bound on the weight of the risky portfolio. In this study, we assume that the simple portfolio selection strategy is applied when the bank constructs the risky portfolio in the trading book. The investment strategy can be decomposed into two allocation problems. The first one is the capital allocation between risk-free capital and risky portfolio. The second one is the portfolio selection among the risky assets in the risky portfolio.

By viewing the risky portfolio as a portfolio of one hypothetical asset or index constructed by the simple portfolio selection strategy, we first solve the problem of finding optimal allocation between risk-free capital and risky portfolio. Incorporating the Basel's market risk requirement, it becomes a constrained optimization problem. For the GBM model, the analytical solution can be derived. For the SV model, we have to rely on the dynamic programming technique to develop an iterative numerical scheme to find the optimal solution based on the CIR-tree of state variable Y. With the optimal allocation between risk-free capital and risky portfolio, we define the ω-utility which is the maximal expected utility with the given simple portfolio strategy ω. We also find that the ω-utility increases with R_ω and decreases with σ_ω. With this finding, we can reduce the dimension of the searching space in the optimization for ω.

To identify the optimal ω^*, the main difficulty is the dimension of the search space. For a risky portfolio with n risky assets, searching in an n-dimensional space makes the problem intractable when n is large. With the results in the previous step, the searching space can be reduced to 1 or 2 dimensional space. Instead of finding the ω to maximize the complicated ω-utility function, we first identify a set Γ of special ω. In the GBM model, Γ includes all the ω giving minimal σ_ω for each possible R_ω. In the SV model, Γ includes all the ω giving maximal R_ω with each possible pair of $(\rho_\omega,\sigma_\omega)$. Since the parameter R_ω is a linear function of ω and σ_ω is a quadratic function of ω, the element in Γ can be analytically derived. The optimal ω^* lies in the set Γ. Moreover, the searching space is only the domain of R_ω or $(\rho_\omega,\sigma_\omega)$. The dimension of the search space is significantly reduced. In conclusion, we provide a tractable solution to the dynamic portfolio selection strategy under the Basel's VaR-capital requirement.

APPENDICES

APPENDIX A

This appendix gives proofs and related definitions of several theorems used in this study.

Theorem A.1. *If A is an $n \times m$ matrix and B is an $m \times m$ invertible matrix, the column space of A is the same as the column space of AB, i.e.*

$$Col(A) = Col(AB).$$

Proof: For any $x \in Col(A)$, there exists a vector $u \in \mathbb{R}^{m\times 1}$ such that

$$Au = x.$$

Since B is invertible, we have

$$x = Au = ABB^{-1}u.$$

Therefore, we have $x \in Col(AB)$.

On the other hand, for any $x \in Col(AB)$, there exists a vector $u \in \mathbb{R}^{m\times 1}$ such that

$$ABu = x.$$

It is evident that $x \in Col(A)$. Therefore, $Col(A) = Col(AB)$. $\square$

Definition A.2. Let W be the subspace of the vector space $\mathbb{R}^m$, the orthogonal complement of W (denoted by $W^\perp$) is the set of vectors which are orthogonal to all elements of W, i.e.

$$W^\perp = \{v | v'x = 0 \ for \ any \ x \in W\}.$$

Theorem A.3. *Let W be a subspace of the vector space $\mathbb{R}^m$ and $W^\perp$ be the orthogonal complement of W, any vector $x \in \mathbb{R}^m$ can be represented as*

$$x = u + v,$$

where $u \in W$ and $v \in W^\perp$.

Proof: The detail of the proof is provided in [2],Page 111. □

Theorem A.4. *If A is an $n \times m$ matrix, the null space $Null(A)$ of A is orthogonal complement of the row space $Row(A)$ of A.*

Proof: For any vector $x \in Row(A)$, there exist a vector $u \in \mathbb{R}^n$ s.t. $u'A = x$. Given any vector $v \in Null(A)$, we have

$$x'v = u'Av = 0.$$

Therefore, $v \in Row(A)^\perp$ and it leads to $Null(A) \subseteq Row(A)^\perp$.

On the other hand, for any vector $v \in Row(A)^\perp$, we have $Av = 0$. It leads to $v \in Null(A)$ and $Row(A)^\perp \subseteq Null(A)$.

Therefore, the null space $Null(A)$ of A is orthogonal complement of the row space $Row(A)$ of A. □

Theorem A.5. *If A is an $n \times m$ matrix, the column space of A is same as the column space of AA', i.e.*

$$Col(A) = Col(AA').$$

Proof: For any vector $y \in Col(A)$, there exists a vector $x \in \mathbb{R}^m$ such that

$$Ax = y.$$

By Theorem A.3 and A.4, we have

$$x = u + v,$$

where $u \in Row(A)$ and $v \in Null(A)$. Since $u \in Row(A)$, there exists a vector $z \in \mathbb{R}^n$ such that

$$z'A = u' \text{ or } A'z = u.$$

Therefore, we have

$$y = Ax = A(u + v) = Au = AA'z,$$

which implies $y \in Col(AA')$.

For any vector $y \in Col(AA')$, there exists a vector $x \in \mathbb{R}^n$ such that

$$AA'x = y,$$

which implies $y \in Col(A)$. $\square$

Theorem A.6. *If A is an $n \times m$ $(n \leq m)$ matrix satisfying $Rank(A) = n$, the matrix AA' is invertible.*

Proof: From the theorem A.5, we have $Col(A) = Col(AA')$. Therefore, we have $Rank(AA') = Rank(A) = n$. Since AA' is $n \times n$ matrix, AA' is invertible. $\square$

Definition A.7. If a matrix A satisfies the following conditions

1. $Av_0 = v_0$ for any vector v_0 in the subspace W;

2. $Av_1 = 0$ for any vector v_1 in the subspace $W^{\perp}$;

then A is a perpendicular projection operator onto the subspace W.

Theorem A.8. *If A is an $m \times n$ $(m \leq n)$ matrix satisfying $Rank(A) = m$, the matrix $A'(AA')^{-1}A$ is a perpendicular projection operator onto the row space $Row(A)$ of A.*

Proof: According to the definition of perpendicular projection operator, the proof is divided into two steps. First, for any vector $v \in Row(A)^{\perp}$, we have $Av = 0$ by the Definition A.2. Therefore, we have $A'(AA')^{-1}Av = 0$.

Second, for any vector $v \in Row(A)$, there exist a vector $x \in \mathbb{R}^m$ satisfying $A'x = v$. For an arbitrary vector $y \in \mathbb{R}^n$, by Theorem A.3,we can decompose y as following

$$y = y_0 + y_1,$$

where $y_0 \in Row(A)$ and $y_1 \in Row(A)^{\perp}$. Since $y_0 \in Row(A)$, we can find a vector $z \in \mathbb{R}^n$ satisfying $A'z = y_0$. Therefore, we have

$$\begin{aligned} y'A'(AA')^{-1}Av =&(z'A + y_1')A'(AA')^{-1}Av \\ =&z'AA'(AA')^{-1}Av \\ =&z'Av \\ =&z'Av + y_1'A'x \\ =&z'AA'x + y_1'A'x \\ =&(z'A + y_1')Ax \\ =&y'v. \end{aligned}$$

Since the vector y is chosen arbitrarily, the result above implies that $A'(AA')^{-1}Av = v$. Therefore, $A'(AA')^{-1}A$ is a perpendicular projection operator onto the row space of A. $\square$

Theorem A.9. *If A is an $n \times n$ positive definite matrix and X is a $m \times n (m \leq n)$ matrix satisfying $Rank(X) = m$, then the following inequality holds for any vector $\beta \in \mathbb{R}^n$*

$$\beta' AX'(XAX')^{-1}XA\beta - \beta' A\beta \leq 0.$$

Moreover, the equality holds if and only if $\beta \in Row(X)$.

Proof: Since matrix A is positive definite, by Cholesky decomposition, we have

$$A = UU',$$

where U is an lower triangular and invertible matrix. Therefore, the matrix XAX' can be formulated as

$$XUU'X'.$$

Since the rank of $m \times n$ matrix X is m, so is the rank of XU. Therefore, by Theorem A.8, the matrix

$$U'X'(XUU'X')^{-1}XU$$

is a perpendicular projection operator onto row space $Row(XU)$ of XU.

For any vector $\beta \in \mathbb{R}^n$, the vector $U'\beta$ can be decomposed as

$$U'\beta = y_0 + y_1$$

where $y_0 \in Row(XU)$ and $y_1 \in Row(XU)^{\perp}$. Moreover, y_0 and y_1 are orthogonal to each other, i.e. $y_0'y_1 = 0$. Then, we have

$$\begin{aligned}\beta'AX'(XAX')^{-1}XA\beta =& \beta'UU'X'(XUU'X')^{-1}XUU'\beta \\ =& (y_0 + y_1)'y_0 \\ =& y_0'y_0.\end{aligned}$$

Since U' is invertible and $U'\beta = y_0 + y_1$, we have

$$\begin{aligned}\beta'AX'(XAX')^{-1}XA\beta - \beta'A\beta =& y_0'y_0 - (y_0 + y_1)'U^{-1}A(U')^{-1}(y_0 + y_1) \\ =& y_0'y_0 - y_0'y_0 - y_1'y_1 \\ =& - y_1'y_1 \leq 0.\end{aligned}$$

Based on previous inequality, the equality holds if and only if $y_1 = 0$ which is equivalent to $U'\beta \in Row(XU)$. The condition $U'\beta \in Row(XU)$ holds if and only if there exists a vector $x \in \mathbb{R}^m$ such that $(XU)'x = U'\beta$. Since U' is invertible, $(XU)'x = U'\beta$ is equivalent to $X'x = \beta$ which implies $\beta \in Row(X)$. $\square$

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