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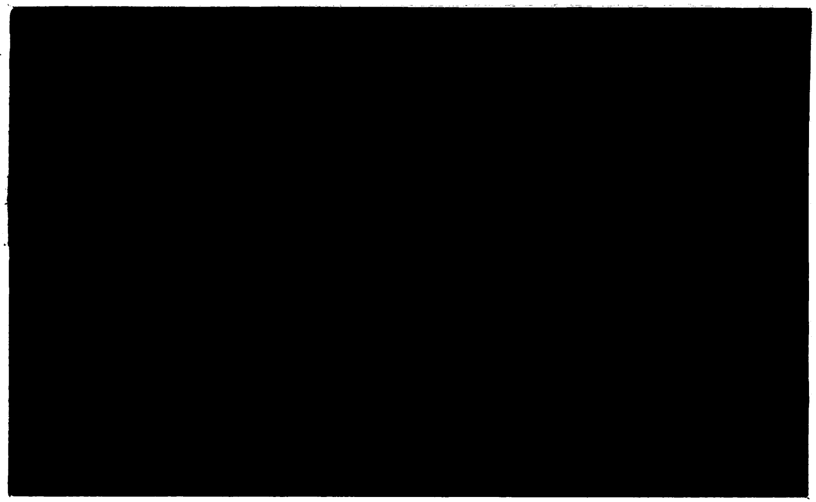

INCOME AND INTERGROUP SUBSTITUTION EFFECTS WHEN PREFERENCES ARE STRONGLY SEPARABLE

by

N. Sören Blomquist*

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*University of Stockholm and University of Michigan. I would like to thank Ted Bergstrom, John Laitner and Hal Varian for helpful comments on an earlier version of this paper.

I. Introduction

In empirical demand and supply studies it is common to assume that preferences are strongly separable. (See Brown and Deaton (1972) for references.) The assumption of strongly separable preferences is also often used in theoretical studies, such as optimal taxation models and intertemporal models. (See, for example, Atkinson and Stiglitz (1980) and Deaton (1981) for discussions of the role of separability assumptions in the optimal taxation literature.) It is well known that this restricts intergroup substitution effects to a very special form. If i and j are two goods in different groups, then the intergroup substitution effects can be expressed as a scalar function times the income effects of the two goods. This is described in, for example, Goldman and Uzawa (1964), Brown and Deaton (1972), Deaton and Muellbauer (1980), and Barten and Böhm (1982). However, none of these studies discuss the sign of the scalar function in any detail, nor do they give a characterization of it in terms of properties of the direct utility function. Obviously, such a characterization could be of value in comparative statics exercises.

In the present paper I provide an alternative to earlier proofs that intergroup substitution effects can be characterized in terms of income effects. An advantage of the new proof is that it gives an explicit form for the scalar function. I also show that there exist exceptions to the rule that intergroup substitution effects can be expressed in terms of the income effects.

As a preliminary to the study of strongly separable preferences, in section 2 of this paper I derive "A Minor Theorem". This theorem is a basic tool used in the rest of the paper. The theorem might also be useful in other comparative statics problems. Section 3 contains the derivation of the relationship between intergroup substitution effects and income effects.

2. A Minor Theorem

Let A be a $K \times K$ symmetric matrix with the following structure

$$A = \begin{bmatrix} B_1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & V_1 \\ 0 & B_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & V_2 \\ \vdots & \vdots & \ddots & & & & & & \vdots \\ \vdots & \vdots & & B_h & & & & & V_h \\ \vdots & \vdots & & & \ddots & & & & \vdots \\ \vdots & \vdots & & & & B_g & & & V_g \\ \vdots & \vdots & & & & & \ddots & & \vdots \\ \vdots & \vdots & & & & & & B_N & V_N \\ V_1' & V_2' & \cdots & V_h' & \cdots & V_g' & \cdots & V_N' & 0 \end{bmatrix} \tag{1}$$

where B_i are $m_i \times m_i$ matrices, V_i are $m_i \times 1$ column matrices, and V_i' is the transpose of V_i. We can then state the following theorem.

Theorem 1. (Minor Theorem) Let H_{ij} $(H_{ij} \neq H_{KK})$ be the minor associated with one of the zero elements in the matrix A. Then

$$H_{ij} H_{KK} = H_{Ki} H_{Kj} \tag{2}$$

If moreover all the matrices B_i are nonsingular, then

$$H_{ij} = \frac{H_{Ki} H_{Kj}}{H_{KK}} \tag{2'}$$

Proof: We first calculate each of the minors in (2) and then we show the equality. Without loss of generality we assume $i < j$, A_{ii} is an element of B_h, and A_{jj} an element of B_g.

i) Calculation of H_{KK}

Using basic rules for calculating a minor we find that[1]

[1]In the proof we repeatedly make use of the fact that the determinant of a blocktriangular matrix is equal to the product of the determinants of the submatrices on the diagonal.

$$H_{KK} = |B_1||B_2|\cdots|B_N|$$

ii) <u>Calculation of H_{Ki}</u>

Since A is a symmetric matrix $H_{Ki} = H_{iK}$. Below we calculate the minor H_{iK}. Deleting the K:th column and the i:th row of A we are left with a determinant of the form

$$H_{iK} = \begin{vmatrix} B_1 & 0 \cdots & 0 \cdots & 0 & 0 \\ \vdots & B_2 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \bar{B}_h & \vdots & \vdots \\ \vdots & \vdots & \vdots & B_g & \vdots \\ 0 & 0 \ldots & 0 \ldots & 0 \ldots & B_N \\ V_1' & V_2' \ldots & V_h' \ldots & V_g' \ldots & V_N' \end{vmatrix}$$

where $\bar{B}_h$ equals B_h, but with the i:th row deleted. Exchange $V' = \{V_1', V_2', \ldots, V_N'\}$ successively with the other rows so the following determinant is obtained.

$$\bar{H}_{iK} = \begin{vmatrix} B_1 & 0 \quad \ldots & 0 \quad \ldots & 0 \quad \ldots & 0 \\ 0 & B_2 & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 \quad \ldots & 0 \quad \ldots & \bar{B}_h & \vdots & \vdots \\ V_1' \quad \ldots & V_2' \quad \ldots & V_h' & V_g' & V_h' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & B_g & \vdots \\ 0 & 0 & 0 & 0 & B_N \end{vmatrix}$$

$$= |B_1||B_2| \cdots |\hat{B}_h| \cdots |B_g| \cdots |B_N| \qquad (3)$$

where

$$|\hat{B}_h| = \begin{vmatrix} \bar{B}_h \\ V_h' \end{vmatrix}$$

If we have had to do p row changes, then

$$H_{iK} = (-1)^p \overline{H}_{iK} \tag{3'}$$

iii) Calculation of H_{Kj}—

When we delete the K:th row and the j:th column of A we are left with a determinant of the following form

$$H_{Kj} = \begin{vmatrix} B_1 & 0 \;\ldots & 0 \;\ldots & 0 \;\ldots & 0 & V_1 \\ 0 & B_2 & & & & V_2 \\ & & \ddots\; B_h & & & \vdots \; V_h \\ & & & \ddots\; \overline{B}_g & & \vdots \; V_g \\ & & & & \ddots & \vdots \\ 0 & 0 \;\ldots & 0 \;\ldots & 0 \;\ldots & B_N & V_N \end{vmatrix}$$

where $\overline{B}_g$ is B_g with the j:th column deleted. Exchange V successively to obtain

$$\overline{H}_{Kj} = \begin{vmatrix} B_1 & 0 \;\ldots & 0 \;\ldots & 0 & V_1 & \ldots & 0 \\ 0 & B_2 & & & V_2 & \ldots & 0 \\ & & \ddots\; B_h & & V_h & \ldots & 0 \\ & & & \ddots\; \overline{B}_g & V_g & \ldots & 0 \\ & & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 \;\ldots & 0 \;\ldots & 0 & V_N & \ldots & B_N \end{vmatrix}$$

$$= |B_1||B_2| \cdots |B_h| \cdots |\hat{B}_g| \cdots |B_N| \tag{4}$$

where $|\hat{B}_g| = |\overline{B}_g \; V_g|$

Suppose we have had to interchange rows r times, then

$$H_{Kj} = (-1)^r \; \overline{H}_{Kj} \tag{4'}$$

iv) Calculation of H_{ij}

When the i:th row and the j:th column of A are deleted we obtain the determinant

$$H_{ij} = \begin{vmatrix} B_1 & 0 & & & & V_1 \\ 0 & B_2 & & & & V_2 \\ & & \bar{B}_h & & & \bar{V}_h \\ & & & \bar{B}_g & & V_g \\ & & & & B_N & V_N \\ V_1' & V_2' \cdots & V_h' \cdots & \bar{V}_g' \cdots & V_N' & 0 \end{vmatrix}$$

where $\bar{B}_h$ and $\bar{B}_g$ are as defined above. $\bar{V}_h$ is V_h with the i:th element deleted and $\bar{V}_g'$ is V_g' with the j:th element deleted. Exchange $[V_1' \ldots \; \bar{V}_g' \ldots V_n' \;\; 0]$ successively with the other rows, and the column $[V_1 \;\; V_2 \;\; \bar{V}_h \;\; V_n \;\; 0]$ with the other columns to obtain

$$\bar{H}_{ij} = \left|\begin{array}{cccc|ccccc} B_1 & 0 & \cdots & 0 & \cdots & 0 & V_1 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 & \cdots & 0 & V_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots & & & & & \\ & & & \bar{B}_h & \cdots & 0 & \bar{V}_h & & 0 \\ V_1' & V_2' & \cdots & V_h' & \cdots & V_g' & 0 & & V_N' \\ \hline & & & & & \bar{B}_g & V_g & \cdots & 0 \\ & & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & & 0 & V_N & \cdots & B_N \end{array}\right|$$

$$= |B_1||B_2| \cdots |\hat{B}_h| \cdots |\hat{B}_g| \cdots |B_N| \qquad (5)$$

where $\hat{B}_h$ and $\hat{B}_g$ are as defined above.[1] We have had to change rows and columns $p+r$ times so

$$H_{ij} = (-1)^{p+r}\,\overline{H}_{ij} \qquad (5')$$

v) Combining the minors:

We can now put the above results together and obtain

$$H_{Ki}H_{Kj} = (-1)^{p}|B_1|\cdot|B_2|\cdots|\hat{B}_h|\cdots|B_g|\cdots|B_N|\cdot(-1)^{r}|B_1|\cdot|B_2|\cdots|B_h|\cdots|\hat{B}_g|\cdots|B_N| =$$

$$= |B_1||B_2|\cdots|B_h|\cdots|B_g|\cdots|B_N|\cdot(-1)^{p+r}|B_1||B_2|\cdots|\hat{B}_g|\cdots|\hat{B}_h|\cdots|B_N| =$$

$$= H_{KK}H_{ij}.$$

If all B_i are nonsingular, then $H_{KK} \neq 0$, and (2') is obtained.

Q.E.D.

3. Intergroup Substitution Effects

Assume that we have a strongly separable preference ordering that can be represented by the utility function $V(X) = V(X^1, X^2, \ldots, X^N)$, where X^k is the commodity vector belonging to the k:th group, and k_r denotes the number of commodities in the k:th group. Goldman and Uzawa (1964) have proved that V(X) must be of the form

$$V(X) = F(U^1(X^1) + U^2(X^2) + \ldots + U^N(X^N)) \qquad (6')$$

where $F(\cdot)$ is a monotone increasing function. In order to simplify the notation, in the following we will use the explicitly additive form:

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[1] In order to derive (5), we use the property that the determinant of a block triangular matrix is equal to the product of the determinants of the submatrices on the diagonal three times. The overall matrix whose determinant we are to calculate is an upper block triangular matrix, as indicated by the lines. To evaluate the upper left and lower right blocks we note that both blocks are lower block triangular matrices.

$$U(X) = U^1(X^1) + U^2(X^2) + \ldots + U^N(X^N) \tag{6}$$

Suppose further that the preferences are convex. This implies that $U(X)$ and all the subutility functions are quasiconcave. Debreu and Koopmans (1982) have shown that it also implies that all the subutility functions $U^k(X^k)$ are continuous, and that at most one of the subutility functions $U^k(X^k)$ can be nonconcave.[1] If indeed one of the subutility functions is nonconcave, then all the other subutility functions must be strictly concave. In the analysis below we will use the following assumption about preferences.

A1: The preferences can be represented by (6), where X is a (K-1) vector of goods, and U(X) is an increasing, quasiconcave, and twice differentiable function.

Let an individual's preferences be as described in A1 and suppose he faces the budget constraint

$$P'X = y \tag{7}$$

where P is a (K-1) vector of strictly positive prices, and y is income. We assume there exists an interior unique global optimum to the implied utility maximization problem, satisfying the first order conditions

$$\frac{\partial U(X)}{\partial X_i} - \lambda P_i = 0 \qquad i = 1,\ldots,(K-1) \tag{8}$$
$$P'X = y,$$

as well as the second order sufficient conditions. Deriving the comparative static expressions, we obtain the substitution and income effects

$$\left.\frac{\partial X_i}{\partial P_j}\right|_{\bar{u}} = (-1)^{i+j} \lambda \frac{H_{ij}}{D} \tag{9a}$$

$$\frac{\partial X_i}{\partial y} = (-1)^{K+i+1} \frac{H_{Ki}}{D} \tag{9b}$$

[1] Koopmans and Debreu (1982) contains references to earlier studies having shown similar theorems, but using stronger assumptions than they use.

where H denotes the associated bordered Hessian, H_{ij} the ij:th minor, and D the determinant of H. The bordered Hessian has a structure like (1), with

$$B_k = \begin{bmatrix} U^k_{11} & U^k_{12} & \cdots & U^k_{1k_r} \\ U^k_{21} & U^k_{22} & \cdots & U^k_{2k_r} \\ \vdots & \vdots & & \vdots \\ U^k_{k_r1} & U^k_{k_r2} & \cdots & U^k_{k_rk_r} \end{bmatrix} \tag{10}$$

and $V = -P$. Thus, the bordered Hessian has a structure such that the Minor Theorem can be applied. A straightforward application of the theorem on eqs. (9) yields:

<u>Theorem 2</u>. Given A1 and an assumption that the utility maximization problem has a unique interior optimum, if goods i and j belong to two separate groups, then intergroup substitution effects can be expressed as

$$\left.\frac{\partial X_i}{\partial P_j}\right|_{\bar{u}} H_{KK} = \lambda D \frac{\partial X_i}{\partial y} \frac{\partial X_j}{\partial y} \tag{11}$$

If moreover all the matrices B_k are nonsingular, then

$$\left.\frac{\partial X_i}{\partial P_j}\right|_{\bar{u}} = \gamma(X) \frac{\partial X_i}{\partial y} \frac{\partial X_j}{\partial y} \tag{11'}$$

and

$$\gamma(X) = \frac{\lambda D}{H_{KK}} \tag{12}$$

The form of the scalar function given by (12) can be quite useful in comparative statics problems. We are often willing to make assumptions about the signs of the income effects. If the sign of $\gamma(X)$ also could be established, this would imply that the sign of the substitution effect could be determined.

Below we study in greater detail under what conditions (11) can be rewritten as (11'). We also study the sign of $\gamma(X)$, and the possible patterns of income effects. In order to do this it is convenient to segment the utility maximization problem into two stages. There are N submaximization problems of type (S) shown below, and a master maximization problem (M). It is easy to establish that the first order conditions (8) and those generated by (S) and (M) are identical. Hence, if there exists a unique optimum, given by (8), the solution to (S) and (M) yield the same optimum.

Let e_k denote expenditures on the k:th group. We can then formulate the submaximization problems

$$(S) \qquad \begin{array}{l} \text{Max } U^k(X^k) \\ \text{s.t. } \; P^k X^k \leq e_k \end{array} \qquad k = 1,\ldots,N$$

The solution to these problems yield the conditional demand functions $X^k(P^k, e_k)$. These conditional demand functions have all the normal properties of demand functions. Since we have made no assumptions about the subutility functions, except that they are increasing and quasiconcave, the income effects $\partial X_i^k/\partial e_k$ can be either negative or positive for a good i belonging to group k. The solutions also imply the existence of indirect utility functions $V_k(e_k)$, where I have suppressed P^k in order to ease the notation. It is easy to show that $V_k'(e_k)$ is positive.

The second stage optimization problem is formulated as

$$(M) \qquad \begin{array}{l} \text{Max } \Sigma V_k(e_k) \\ \text{s.t. } \; \Sigma e_k \leq y \end{array}$$

The F.O.C.'s for an interior optimum to this problem are

$$\begin{array}{ll} V_k'(e_k) = \lambda & k = 1,\ldots,N \\ \Sigma e_k = y & \end{array} \tag{13}$$

and the SOSC are that all border preserving minors of order m of the associated bordered Hessian have sign $(-1)^m$, $m = 2,\ldots,N$. We assume the SOSC to be satisfied at the optimum. This implies (among other things) that

$$\begin{vmatrix} V_i'' & 0 & -1 \\ 0 & V_j'' & -1 \\ -1 & -1 & 0 \end{vmatrix} = -V_i'' - V_j'' > 0,$$

or

$$V_i'' + V_j'' < 0 \tag{14}$$

for any i,j. The implication of (14) is that $V_k''(e_k)$ can be non-negative for <u>at most</u> one k. We state this as Lemma 1:

<u>Lemma 1</u>: $V_k''(e_k)$ can be non-negative for at most one k.

Let us next consider the relationship between the sign of $V_k''(e_k)$ and $|B_k|$. Straightforward calculations for one of the submaximization problems show that $V_k'(e_k) = \lambda_k$, where λ_k is the lagrange multiplier of the submaximization problem. Letting D_k denote the determinant of the associated bordered Hessian one can further show that $\partial\lambda_k/\partial e_k = -|B_k|/D_k$. Hence, $V_k''(e_k) = -|B_k|/D_k$. Now, if the number of commodities in group k is even (odd), then D_k is positive (negative). This follows from the SOSC. Hence we can state the following Lemma:

<u>Lemma 2</u>: If the number of commodities in a group is even, then

$$V_k''(e_k) \gtrless 0 \text{ as } |B_k| \lessgtr 0$$

If the number of commodities in a group k is odd, then

$$V_k''(e_k) \gtrless 0 \quad \text{as} \quad |B_k| \gtrless 0.$$

$$V_k''(e_k) = 0 \quad \text{iff} \quad |B_k| = 0.$$

There are three interesting combinations of subutility functions to study: i) at the optimum one subutility function has the property that $|B_k| = 0$, ii) at the optimum all $|B_k| \neq 0$ and all subutility functions are concave, iii) at the optimum all $|B_k| \neq 0$, one subutility function is nonconcave and all the others are strictly concave. We start with a study of case i, which is the most complicated one.

Case i can occur either when all subutility functions are concave, or when one of the subutility functions is nonconcave. Let us index the group with $|B_k| = 0$ with s. It follows from Lemma 1 and 2 that $V''_s = 0$ and $V''_k < 0$ for all other groups. Since B_s is singular we cannot rewrite the substitution effect as (11'), but only as (11). However, since $H_{KK} = \prod_{i=1}^{N} |B_k| = 0$, there is really no relationship between intergroup substitution effects and income effects. Since $H_{KK} = 0$, it also follows from (11) that at least one of the income effects $\partial X_i/\partial y$ and $\partial X_j/\partial y$ are zero. In fact, doing some comparative statics for the master maximization problem one can show that $\partial e_k/\partial y$ is zero for all groups with $V''_k(e_k) < 0$, and $\partial e_s/\partial y > 0$ for the group with $V''_s(e_s) = 0$. The intuition behind this result is illustrated in figure 1, where I for simplicity just have two groups, and let $V''_s(e_s) = 0$ for all values of e_s.

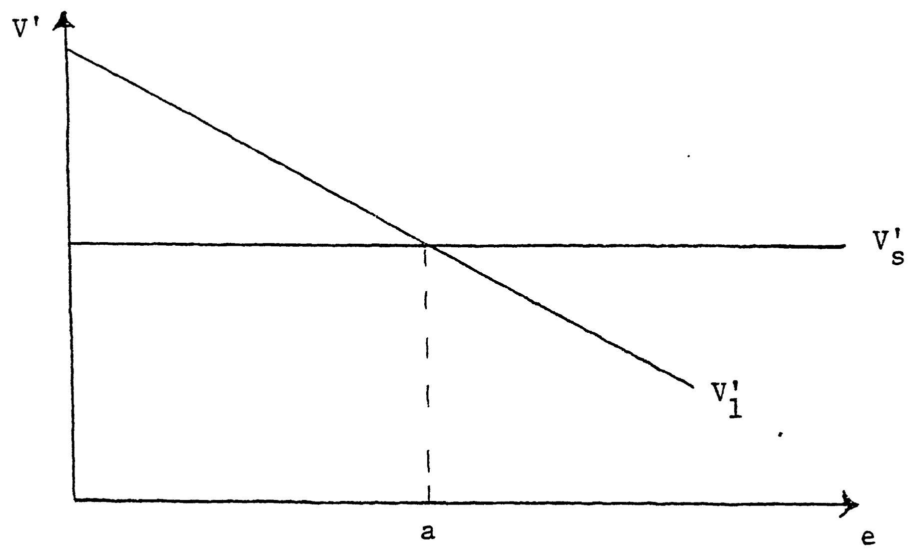
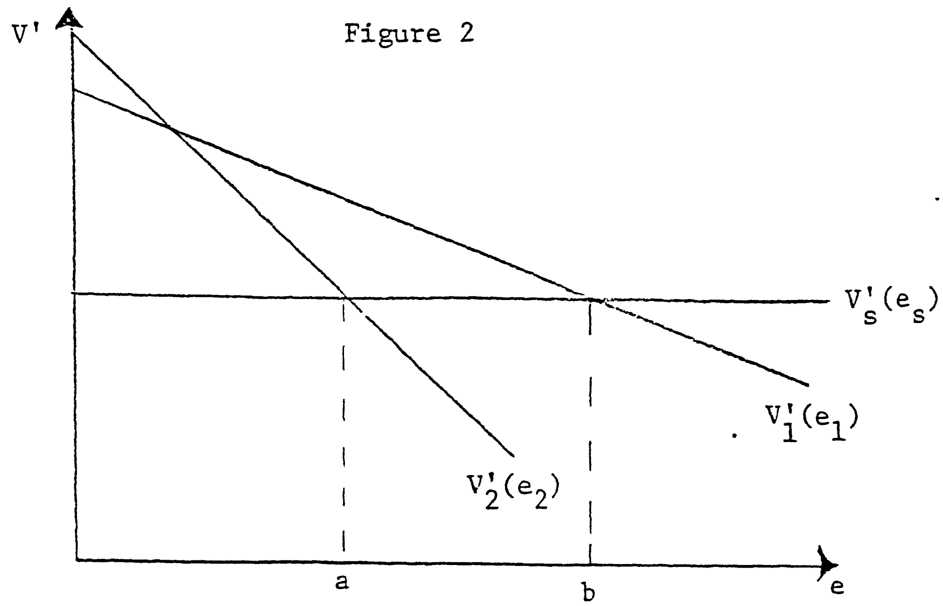


Figure 1

For $y < a$ a corner solution obtains, with all income being spent on group 1. Thus, we see that for this particular example, in order for an interior solution to exist, V_1' and V_s' must be such that $V_1'(0) > V_s'(0)$, and $y > a$. For values of y larger than a, expenditures on group 1 are always a. Increases in income are then exclusively spent on group s. Hence, at an interior solution, $\partial X_i/\partial y = (\partial X_i/\partial e_1)(\partial e_1/\partial y) = 0$ for a good i belonging to group 1. For goods in group s $\partial X_i/\partial y$ can be either positive or negative. However, on average goods in group s are normal goods in the sense that $\partial e_s/\partial y > 0$.

What about intergroup substitution effects when $|B_s| = 0$? Inspection of (5) and (5') tells us that $H_{ij} = 0$ if $s \neq g,h$, because then $|B_s|$ is one of the terms of which H_{ij} is a product. If $s = g$ or $s = h$, then $|B_s|$ will be replaced by $|\hat{B}_s|$ in the product of terms, where $\hat{B}_s$ is B_s with one row (or column) deleted and one row (or column) of prices added. The determinant $|\hat{B}_s|$ will in general be nonsingular. Thus, we conclude that an intergroup substitution effect $(\partial X_i/\partial P_j)_{\bar{u}}$ is zero if i and j both belong to groups other than s. If either i or j belong to group s, then the substitution effect will be different from zero. The intuition behind this result is illustrated in figure 2, where we have three groups.

Figure 2



For incomes larger than $y = a + b$ there will be an interior solution such that $e_2 = a$, $e_1 = b$, and $e_s = y - a - b$. Suppose the price of a good in group 1 changes. This will shift the $V_1'(e_1)$ schedule. Let us assume the shift is such that point b shifts to the left. That is, expenditures on group 1 will decrease. Expenditures on group 2 will be unchanged, and expenditures on group s will increase. These are the uncompensated changes. Let us assume that we after these changes still are at an interior solution. To restore utility to the original level an increase in lump-sum income is needed. However, since we are at an interior solution all this income will be spent on group s. Thus, we see that both the total and intergroup substitution effects are zero if none of the goods belong to group s. If one of the goods belong to group s the total effects and the substitution effects will both be nonzero.

An increase in the price of one of the goods in group s will shift $V_s'(e_s)$ downwards. The direct effect of this is to increase both e_1 and e_2, and to decrease e_s. The lumpsum income to restore utility to the original level will be spent exclusively on group s, given that the price increase did not cause a corner solution. Thus, the intergroup substitution effects are nonzero and equal to the total effects. We summarize the results above in Theorem 3.

Theorem 3: If there exists an interior unique optimum such that $|B_s| = 0$ ($V_s''(e_s) = 0$) for one group s, then the income effects for goods belonging to groups other than s are all zero. Goods in group s can be either inferior or normal. However on average they are normal in the sense that $\partial e_s/\partial y > 0$. Intergroup substitution effects $(\partial X_i/\partial P_j)_{\bar{u}}$ can not be expressed in terms of income effects and are nonzero only if one of the goods i or j belong to group s.

We next study the case where all the subutility functions are concave and $|B_k| \neq 0$ for all k. Since $V_k''(e_k) \leq 0$ if $U^k(\cdot)$ is concave, it follows from Lemma 2 that all V_k'' are strictly negative. The fact that all B_k are nonsingular implies that the intergroup substitution effects can be written as (11'). In expression (12) for the scalar function $\gamma(X)$ the sign of D is determined from the SOSC. D is positive if $(K-1)$ is an even number, and negative if $(K-1)$ is an odd number. The lagrange multiplier λ, which we can interpret as the marginal utility of income is positive. From Lemma 2 we know that $|B_k|$ is positive if the number of goods in the group is even, and negative if the number of goods in the group is odd. This implies that $H_{KK} = \prod_{k=1}^{N} |B_k|$ and D are of the same sign, and $\gamma(X)$ positive.

It can also be of interest to study what combinations of income effects there can be. Suppose there exists an original interior solution. At this solution all V_k' are equal. Suppose income y is increased. Since all V_k'' are negative it is obvious that the increase in y will be used to increase expenditures for <u>all</u> groups. That is $\partial e_k/\partial y$ is positive for all k. Since, $\partial X_i/\partial y = (\partial X_i/\partial e_k)(\partial e_k/\partial y)$, it is equally obvious that $\partial X_i/\partial y$ can be either positive or negative. We summarize these results in Theorem 4.

<u>Theorem 4</u>: If all subutility functions are concave and there exists an interior optimum such that $|B_k| \neq 0$ ($V_k'' < 0$) for all k, then intergroup substitution effects can be written as (11'). The scalar function $\gamma(X)$ will be positive. The income effects $\partial X_i/\partial y$ can be either negative or positive. For each group k it is true that the goods in that group on average are normal goods in the sense that $\partial e_k/\partial y$ is positive.

We finally consider the case where $|B_k| \neq 0$ for all k, and one of the subutility functions is nonconcave. Let us denote the group with a nonconcave subutility function as group 1. Since $|B_k| \neq 0$ we know that all V_k'' are different from zero. They might all be negative. If this is true we obtain the results stated in theorem 3. However, since $U^1(\cdot)$ is nonconcave it is possible that $V_1''(e_1) > 0$. If this is the case we obtain the following results.

Using Lemma 2 to evaluate the different possible combinations of odd and even numbers of commodities in the various groups we establish that $\gamma(X)$ is negative if $V_1''(e_1) > 0$. It can also be of interest to study the possible combinations of income effects. Suppose we originally have an interior solution with $e_k > 0$ for all k. At this optimum all V_k' are equal. Suppose the income y is increased. At the new optimum, which we also assume to be interior, we once again have equality of V_k' for all k. But this implies that e_1 must have increased and all other e_k decreased. That is, $\partial e_1/\partial y > 1$ and $\partial e_k/\partial y < 0$ for $k \neq 1$. We summarize these results in theorem 5.

Theorem 5: Suppose $|B_k| \neq 0$ for all k and that the subutility function for group 1 is nonconcave. If V_1'' is negative, the results of theorem 4 applies. If on the other hand V_1'' is positive, then the scalar function $\gamma(X)$ is negative. The income effects $\partial X_i/\partial y$ can be either negative or positive. However, for each group with a concave subutility function it is true that goods in that group on the average are inferior goods in the sense that $\partial e_k/\partial y$ is negative. Goods in the group with a nonconcave subutility function are on average luxuries in the sense that $\partial e_k/\partial y > 1$.

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