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NONLINEAR TAXES AND LABOR SUPPLY

A GEOMETRIC ANALYSIS

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by

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I. Introduction

In a one-period model economic theory yields no predictions about the total effect on labor supply of variations of a proportional income tax. To obtain specific results one must turn to empirically estimated labor supply functions. However, the empirical literature on labor supply functions contain a wide range of estimated wage rate elasticities, both positive and negative. The only really robust results seem to be that leisure is a normal good, and that the own wage rate substitution effect is positive.[1] Does this imply that economics has nothing to contribute to the evaluation of various proposed changes in tax schedules? Fortunately, this is not the case. In the present paper I show that for many types of variations in a nonlinear tax, smooth or piecewise linear, one can derive quite detailed results about the effect on labor supply, especially if one is willing to assume that leisure is a normal good.

The theory of labor supply with nonlinear taxes has not been developed much so far. Hausman (1983) contains a few results for a one-period model, and Blomquist (1984b) contains some results for a two-period model. In the present paper I study the theory of nonlinear taxes and labor supply in a one-period model using simple geometric arguments. In Section 2 of the paper I derive results where only information of the form of the tax schedules and the assumptions that preferences are strictly convex and leisure is a normal good are used. In Section 3 I give results where information of the optimum position on the original budget constraint is used. Section 4 presents results regarding the effect of variations of parameters of a progressive piecewise linear tax. Finally, in section 5 I discuss what the results of

[1] Blomquist (1984a) surveys studies that have tried to model the influence of taxes on individuals' budget constraints in a careful way. Almost all of these studies show a positive own wage rate effect and a negative income effect.

the paper imply about the interpretation of estimated labor supply functions.

2. Results Using the General Shape of the Budget Constraint

In the following we always assume preferences can be represented by a strictly quasiconcave utility function $U(C,h)$, increasing in C (consumption), and decreasing in h (hours of work). Maximization of $U(\cdot)$ s.t. the linear budget constraint $C \leq y + wh$, where y is nonlabor income and w the wage rate, yields the labor supply function $h(y,w)$.

Definition 1: Leisure (labor) is a normal good if $y_1 > y_2$ implies $h(y_1,w) < h(y_2,w)$.

In definition 2 below we introduce nonlinear budget constraints of the form $C \leq g(h)$ and $C \leq G(h)$. However, in order to simplify the notation we simply write the budget constraints as $g(h)$ and $G(h)$. Throughout the paper these functions are assumed to be continuous.

Definition 2: Suppose we have a budget constraint $g(h)$, $0 \leq h \leq \bar{H}$, which might be either smooth or piecewise linear, and replace this with another budget constraint $G(h)$, $0 \leq h \leq \bar{H}$, such that $G(h) > g(h)$ for all h, and $G'(h) \leq g'(h)$ at all points where the derivative exists, then $G(h)$ is said to be an ALS (Above Less Slope) budget constraint relative to $g(h)$.

Proposition 1: If leisure is a normal good, and a convex and smooth budget constraint $g(h)$ is replaced by an ALS budget constraint, which does not have to be convex and/or smooth, then hours of work will decrease.

Note that since $G(h)$ does not have to be convex, there might exist multiple optima for this budget constraint. However, all optima will lie to the left of the old optimum point. We use figure 1 to give a proof of the proposition.

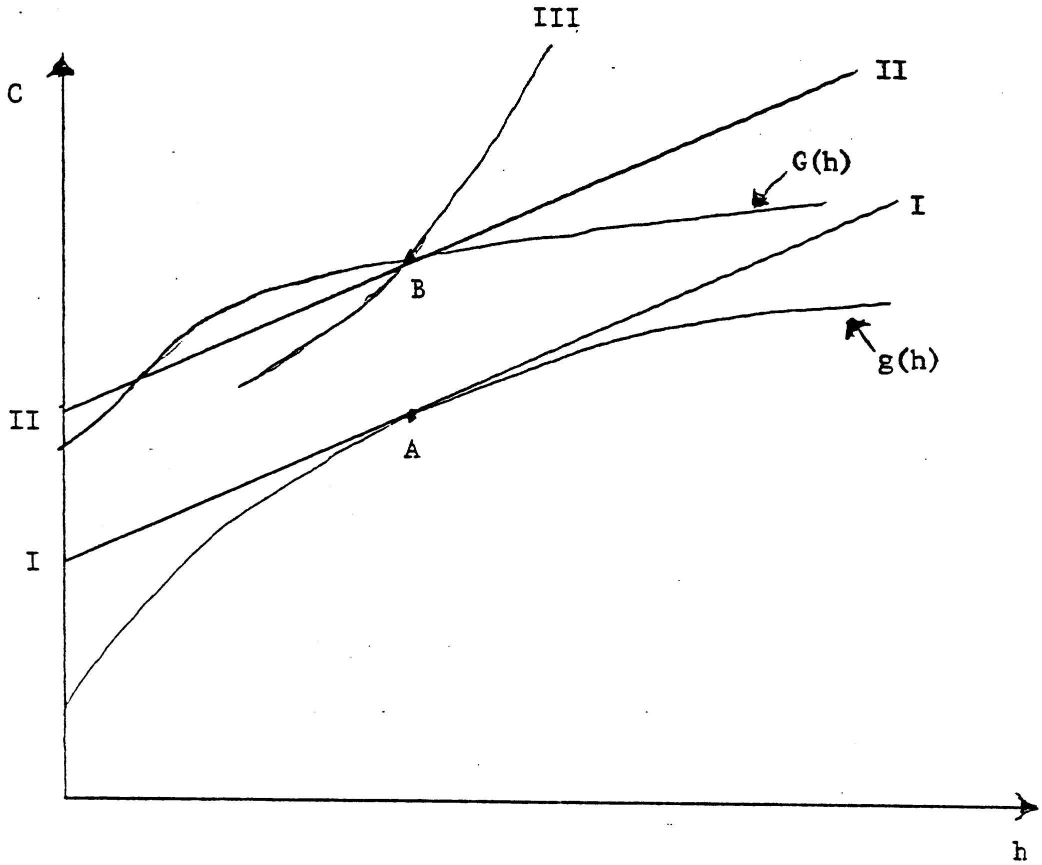


Figure 1

Proof: In figure 1, g(h) is the original budget constraint. G(h) is the ALS budget constraint. A is the original optimum point. B is the point on G(h) right above A. I-I is a linearized budget constraint passing through A and II-II is a parallel budget line passing through B. Since leisure is a normal good, there exists a tangency point on II-II to the left of B. This implies there is an indifference curve passing through B, cutting the budget line II-II from below. We denote this indifference curve III. Since, to the right of B, III lies above II-II, which lies above G(h), it is true that B is preferred to all points on G(h), which are to the right of B. In a small neighborhood to the left of B we are sure that III lies below G(h). This implies that there exists feasible points to the left of B, which are preferred to B. Hence, the new optimum point must be to the left of B and A. //

<u>Proposition 2</u>: If leisure is a normal good, and a convex piecewise linear budget constraint is replaced by an ALS budget constraint, then

i) if the optimum h^* on $g(h)$ is at a kink, and the ALS budget constraint also has a kink at h^*, then hours of work will not increase.

ii) if the conditions under i do not apply, then hours of work will decrease.

Proposition 2 can be proved in a way very similar to the way proposition 1 was proved. The proof is therefore omitted.

The reversals of Propositions 1 and 2 are obviously also true. That is, if there initially is a budget constraint $G(h)$, and this is replaced by a budget constraint $g(h)$, where $G(h)$ is an ALS budget constraint relative to $g(h)$, then if leisure is a normal good, labor supply will increase.

In propositions 1 and 2 we assumed $g(h)$ to be convex, but allowed $G(h)$ to be nonconvex. We could equally well have assumed that $G(h)$ is convex, and $g(h)$ nonconvex. The important thing is that at least one of the two budget sets is convex. The joint assumptions that $G'(h) \leq g'(h)$ and one of the two sets is convex ensures that the budget line II-II lies everywhere above $G(h)$, to the right of B.

3. Results Using Information About the Optimum Position

When evaluating the effect on labor supply of a change in a tax schedule, we often are in the situation that we know or easily can obtain information about individuals' optimum positions, given the present tax schedule. Below I study how such information can be used to assess the effect of changes in the tax schedule.

Proposition 3: Let $g(h)$ be a continuous budget constraint of any shape, and let C^*, h^* be the unique optimizing point for this budget constraint. Suppose that $g(h)$ is differentiable at h^*. Let $G(h)$ be another continuous budget constraint. If leisure is a normal good, $G(h^*) > g(h^*)$, the straight line with slope $g'(h^*)$ passing through the point $(h^*, G(h^*))$ lies nowhere below $G(h)$ for $h > h^*$, and not above $G(h)$ for (at least) some small neighborhood of h^* to the left of h^*, then hours of work will decrease as $g(h)$ is replaced by $G(h)$.

Proof: In figure 2, $g(h)$ is the original budget constraint. A is the optimum point given $g(h)$. I-I is a straight line tangent to $g(h)$ at h^*. II-II is a parallel straight line passing through h^*, $G(h^*)$. The fact that leisure is a normal good implies there is a tangency solution on II-II to the left of B. This implies there is an indifference curve passing through B, cutting the line II-II from below. We denote this indifference curve III. To the right of B this indifference curve lies above II-II, which never lies below $G(h)$. Hence, it follows that B is preferred to all points on $G(h)$ to the right of B. In a small neighborhood to the left of B we are sure the indifference curve III lies below $G(h)$. This implies there exist feasible points to the left of B, which are preferred to B. Hence, the new optimum point must be to the left of h^*. //

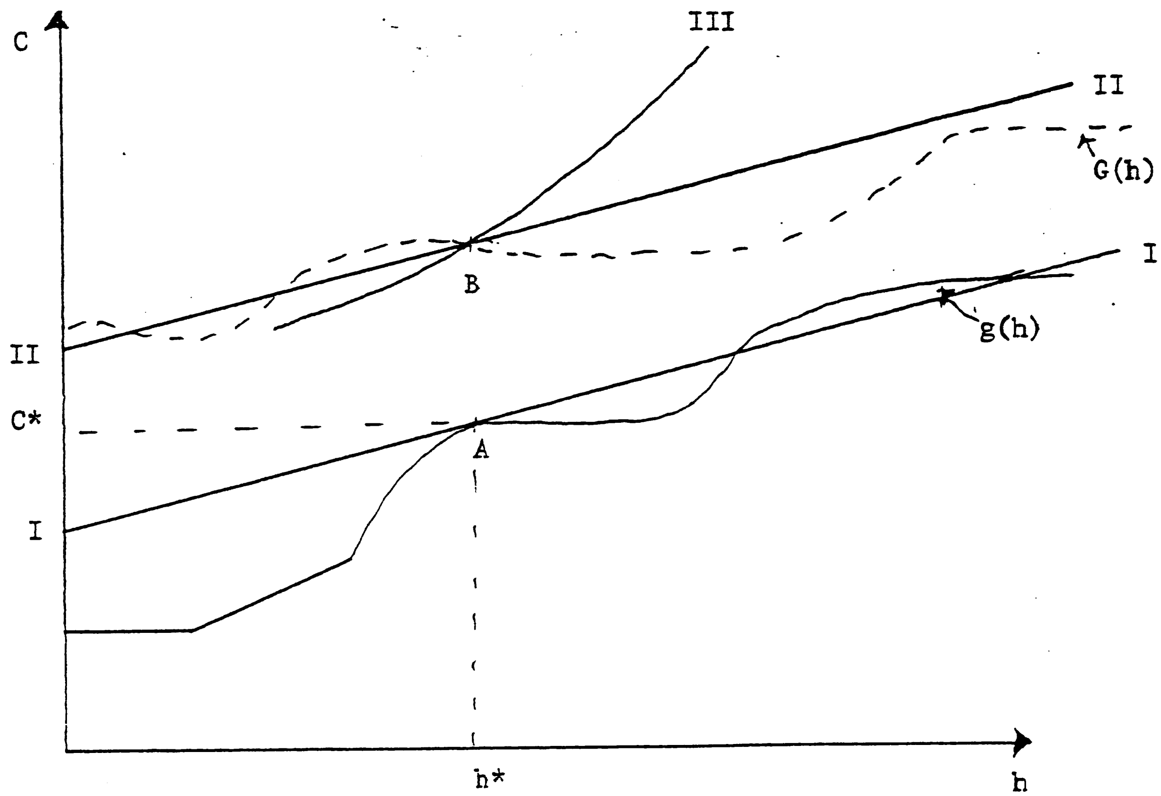


Figure 2

One can make modifications of proposition 3 to take account of kinks in either g(h) or G(h). Likewise one could state a similar proposition for the case where G(h) lies below g(h). In order not to burden the reader with repetitious details, these modifications are not spelled out here.

The results of proposition 3 make it possible to draw (at least partial) conclusions about the effect of changes in tax schedules on the labor supply of a population of workers. The old schedule can be divided into segments where the assumptions of proposition 3 are satisfied, and segments where they are not. It should be fairly easy to figure out how large part of the population that is situated on segments where the assumptions of proposition 3 are satisfied. One would then have a lower bound estimate of the number of people that would decrease hours of work if the change in tax schedules were done. As an example, consider the tax schedules in figure 3, where the "old" tax schedule is given by g(h), and the "new" tax schedule by G(h). That is, the tax reform would widen the income brackets, lower the marginal tax for low incomes and raise it for high incomes. Using proposition 3, and assuming leisure is a normal good, it is easily established that individuals in income brackets corresponding to the intervals $c < h < d$ and $h > e$ will decrease their labor supply if $g(h)$ is replaced by $G(h)$.

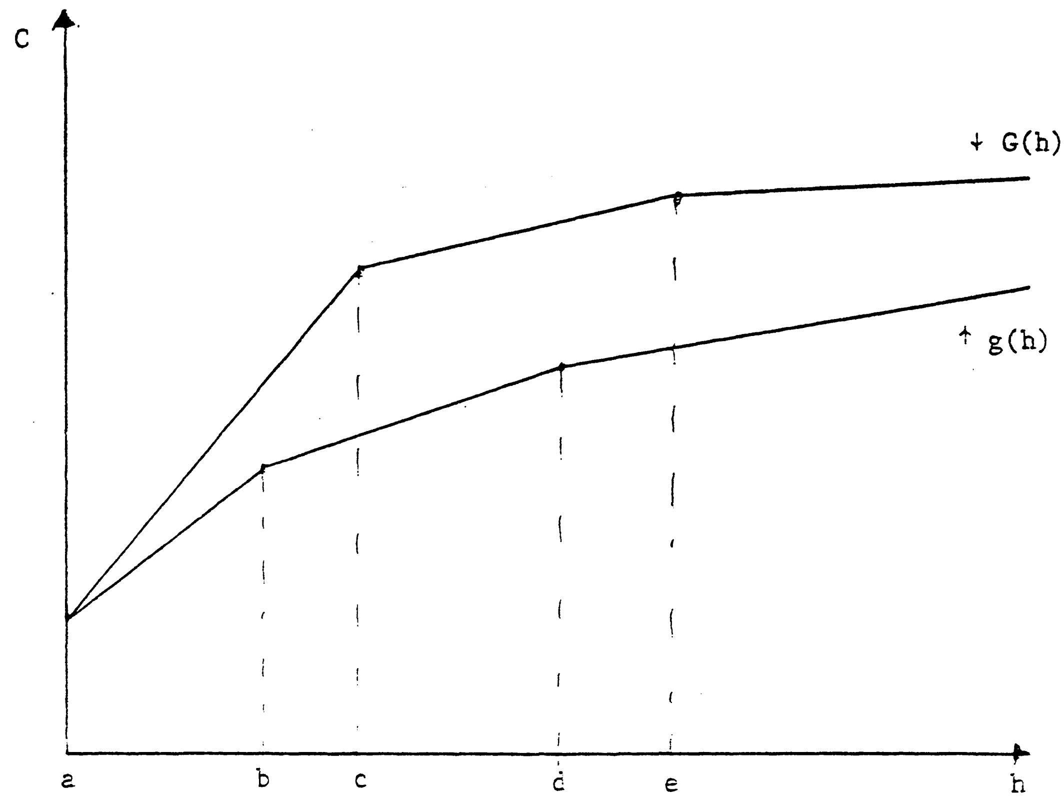


Figure 3

4. Parameter Changes in a Convex Piecewise Linear Tax Schedule

Almost all real world tax schedules are piecewise linear. In many instances the schedules also have the property that the marginal tax is non-decreasing. Actual tax reform often is of the form that one or more parameters of such a schedule are changed. It is hence of large practical interest to establish the effects on labor supply of variations in the parameters of a piecewise linear convex tax schedule. Below this is done by studying effects of changes where just one parameter is changed at a time.

Let B_t and B_{nt} be taxable and nontaxable nonlabor income respectively. Let w be the gross wage rate and h hours of work. If E denotes a general exemption, the taxable income is $x = wh + B_t - E$. Let the income tax be of the form

$$\mathrm{Tax}(x) = T + \int_0^x t(z)dz, \qquad (1)$$

where t is an increasing step function such that $t(z) = t_i$ for $A_{i-1} < z \leq A_i$, $i = 1, \ldots$. If the tax function has this form, then the budget constraint will have the general shape shown in figure 4.

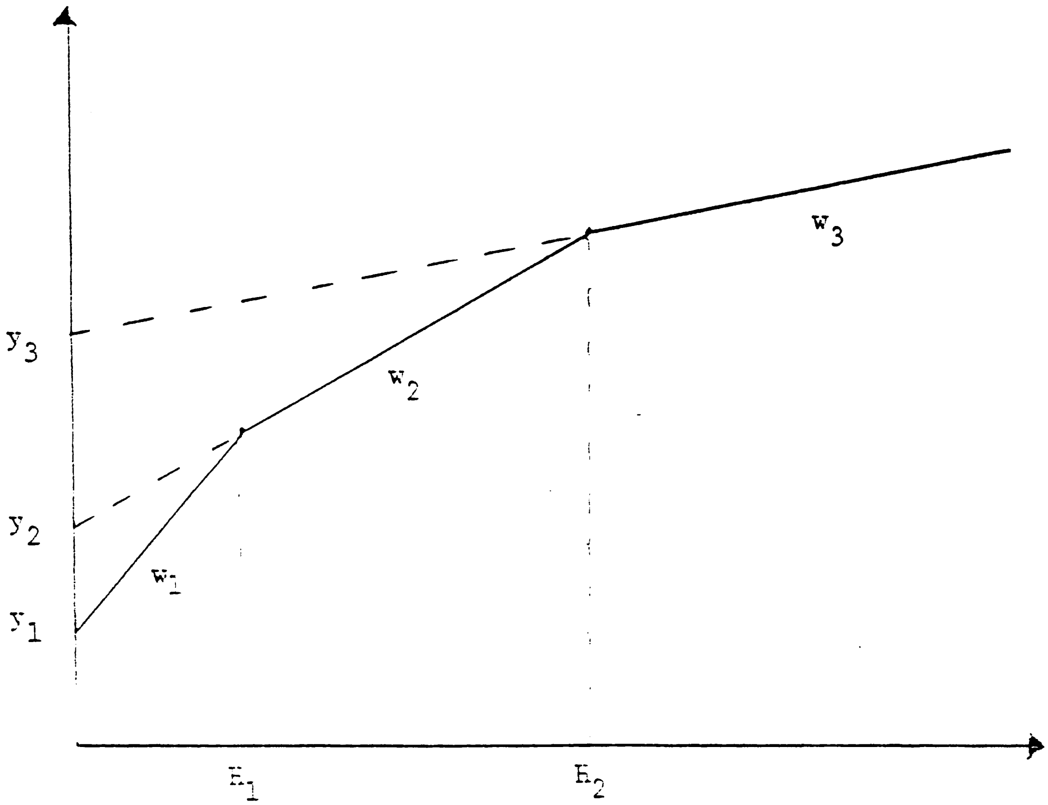


Figure 4

We assume $B_t > E$, but that $B_t - E < A_1$, then $y_1 = B_{nt} + E + (1-t_1)(B_t - E) - T$. The slope of the first segment of the budget constraint is $w_1 = w(1-t_1)$. The upper limit of the first interval is $H_1 = (A_1 + E - B_t)/w$. In general, the slope of the i:th segment is $w_i = w(1-t_i)$ and the upper limit of the corresponding interval on the h-axis is $H_i = (A_i + E - B_t)/w$. The intercepts on the

C-axis of the extended segments have come to be called virtual incomes. The virtual incomes can be calculated by the recursive formula $y_i = y_{i-1} + (w_{i-1} - w_i)H_{i-1} = y_{i-1} + (t_i - t_{i-1})(A_{i-1} + E - B_t)$, $i = 2,\ldots$

We will now study the effect on labor supply of variations in T, E, A_i, t_i, and w.

Change of the lumpsum tax

A change in T has exactly the same effect as a change in $-B_{nt}$, so what is said for changes in T applies also for changes in $-B_{nt}$. A decrease in T will shift the budget constraint upwards in a parallel fashion. It is easy to show that the following is true.

Proposition 4: If leisure is a normal good, then a decrease in T will decrease labor supply for individuals who do not have their optimum points at kink points. The new optimum might be on the same segment as before, or on a lower segment. The labor supply of individuals with the optimum at a kink point will either not change or decrease.

The proposition is obvious and needs no formal proof.

Change of the exemption level

An increase in the exemption level E will:

i) increase all y_i

ii) not change any w_i

iii) increase all the H_i.

The change in the budget restriction is illustrated in figure 5.

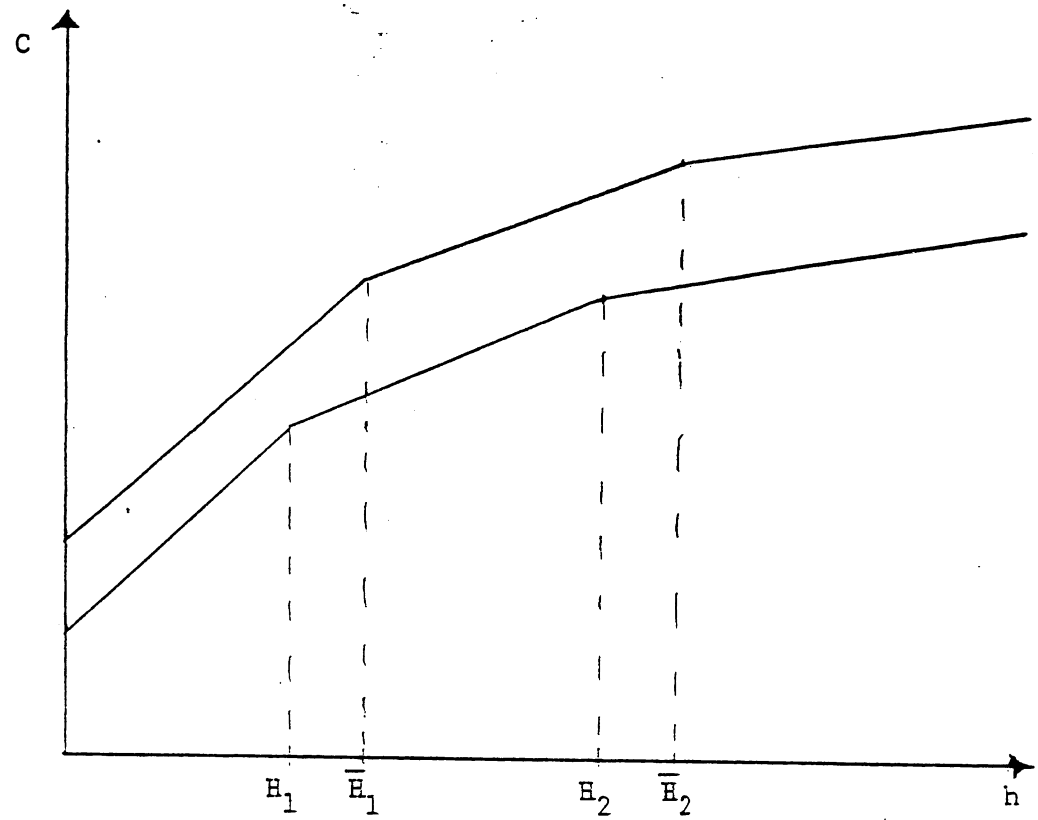


Figure 5

Proposition 5: If leisure is a normal good, then

i) individuals with an original optimum in the intervals $(0,H_1)$ and $(\overline{H}_i,H_{i+1})$, $i=1,\ldots,$ will decrease their labor supply.

ii) individuals with an original optimum in an interval $(H_i,\overline{H}_i)$, $i=1,\ldots,$ can either increase or decrease their labor supply. If labor supply increases it can at most increase up to the upper limit $\overline{H}_i$ of the original interval.

Part i of the proposition is easily proved by applying proposition 3. It is easily seen that part ii. is true by the following argument. Let the original optimum be on segment i in the interval $(H_i,\overline{H}_i)$. Suppose (falsely) that the new optimum is on segment i+1 on the new budget constraint. However, since leisure is a normal good this implies the old optimum must have been on segment i+1 or a segment to the right of segment i+1 of the old budget constraint. This is a contradiction. We hence conclude the new optimum must be less than or equal to $\overline{H}_i$.

From proposition 5 we see that some people might increase their labor supply if the exemption level is increased. Overall, however, there is a strong tendency for a decrease.

Change of tax bracket limit

An increase in the upper limit A_j of an income tax bracket will:

i) leave all lower segments unchanged

ii) increase the value of H_j by $\Delta A_j/w$

iii) increase the value of y_i for $i = j+1, j+2, \ldots$.

The change of the budget constraint is illustrated in figure 6.

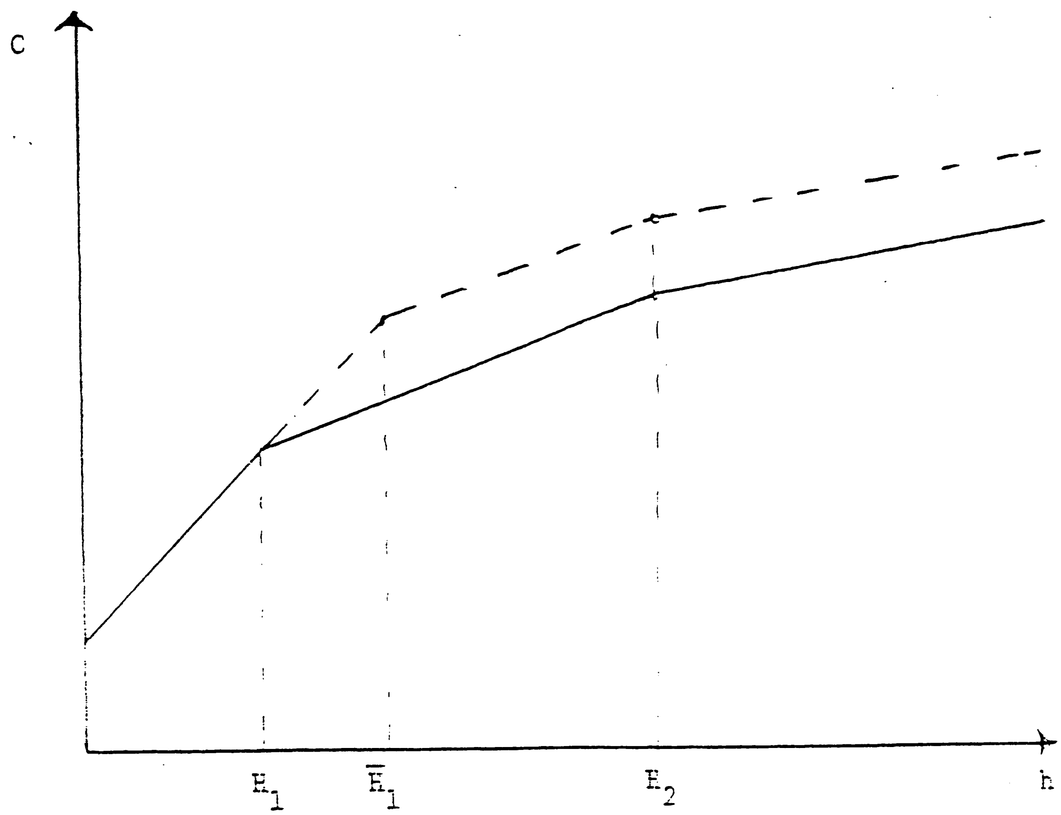


Figure 6

Proposition 6: If leisure is a normal good, then an increase in A_j will

i) not change the individual's optimum if the original optimum was on any of the segments $i = 1, \ldots, j$.

ii) if the original optimum was in the interval $(H_j, \overline{H}_j)$, then labor supply might either increase or decrease. If it increases, it can at most increase up to $\overline{H}_j$.

iii) if the original optimum was greater than $\overline{H}_j$, then labor supply will decrease.

Parts i. and iii. of the proposition can be proved by use of proposition 3. Part ii. can be proved using the argument used for part ii. of proposition 5.

<u>Change of the marginal tax rate</u>

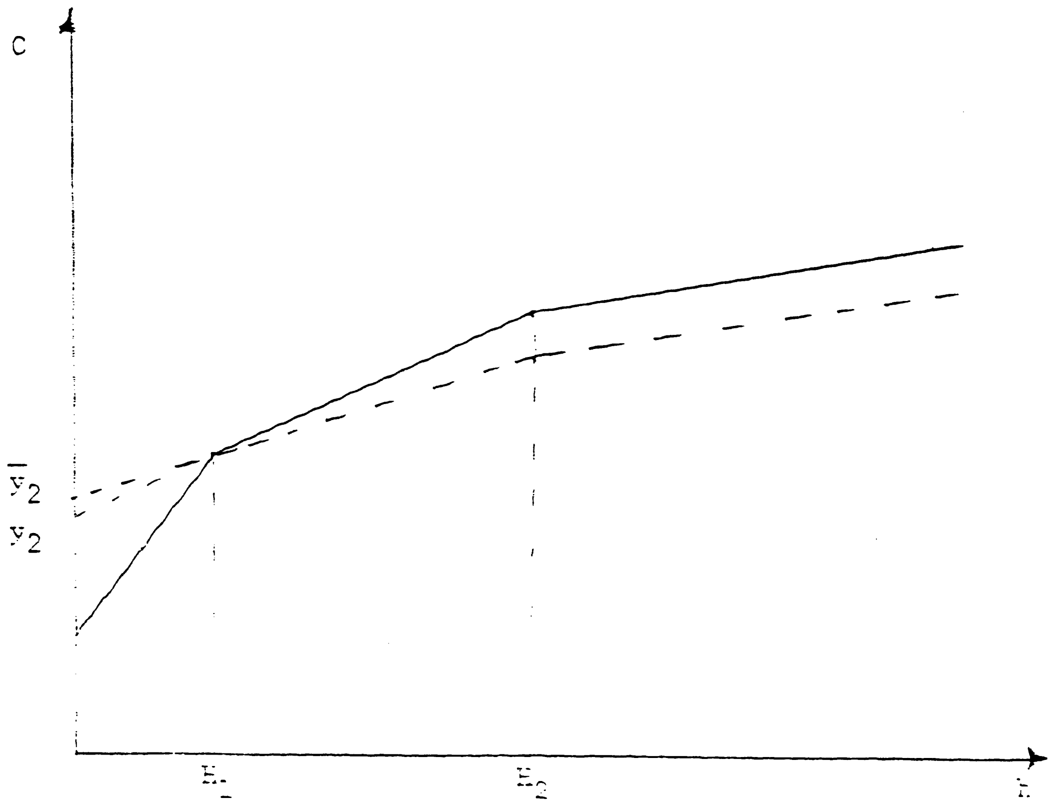
An increase of the marginal tax rate for tax bracket j will

i) leave segments i = 1,...,j-1 unchanged

ii) decrease the slope of segment j and increase the corresponding virtual income.

iii) decrease the virtual incomes of segments i = j + 1, j + 2,... .

The change in the budget constraint is illustrated in figure 7.



Figure

<u>Proposition 7</u>: An increase in the tax rate t_j, which might render the new budget constraint nonconvex, will:

i) not change labor supply if the original optimum was on segment $i = 1,\ldots,j - 1$.

ii) if the original optimum was on segment j labor supply might either increase or decrease. However, labor supply can not decrease more than down to H_{j-1}.

iii) if leisure is a normal good and the original optimum is greater than H_j, then labor supply will increase.

Part i. of the proposition is obvious. Part iii. can be proved along similar lines as when proposition 3 was proved. Hence I will only give a detailed proof of part ii. and only of the fact that labor supply cannot decrease below H_{j-1}.

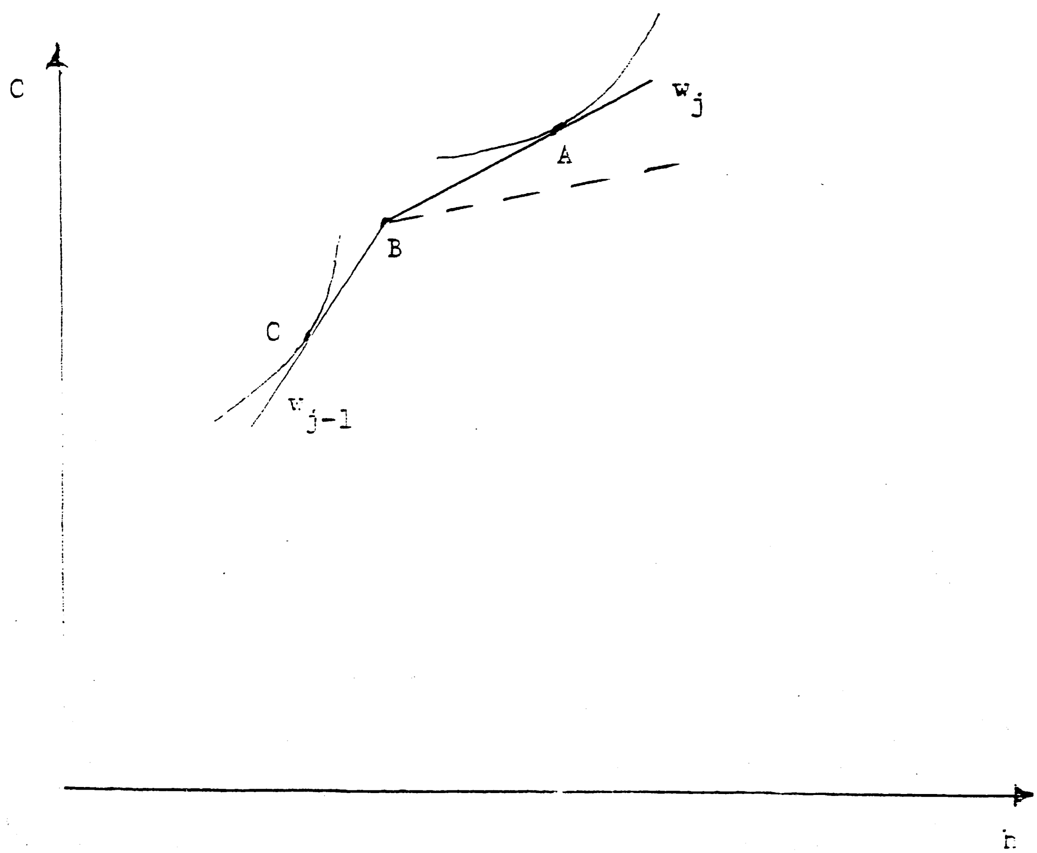


Figure 8

Consider figure 8. Let the original optimum be at A. This implies that the slope of the indifference curve at A is w_j. To the left of A the indifference curves cut the segment from above. Hence, the slope of the indifference curve passing through B has slope $I_B < w_j$. Now, we make the (false) assumption that the new optimum is on segment $j-1$ at C. The slope of the indifference curve at C is w_{j-1}. To the right of C the indifference curves cut the segment from below. Hence, the slope of the indifference curve passing through B has slope $I_B > w_{j-1} > w_j < I_B$. We obtain a contradiction and conclude that the new optimum cannot be to the left of B.

Change of the gross wage rate

An increase in the gross wage rate will

i) increase the slope of all segments

ii) decrease the upper limits H_i for all segments

iii) leave all virtual incomes unchanged. We illustrate the change in the budget constraint in figure 9.

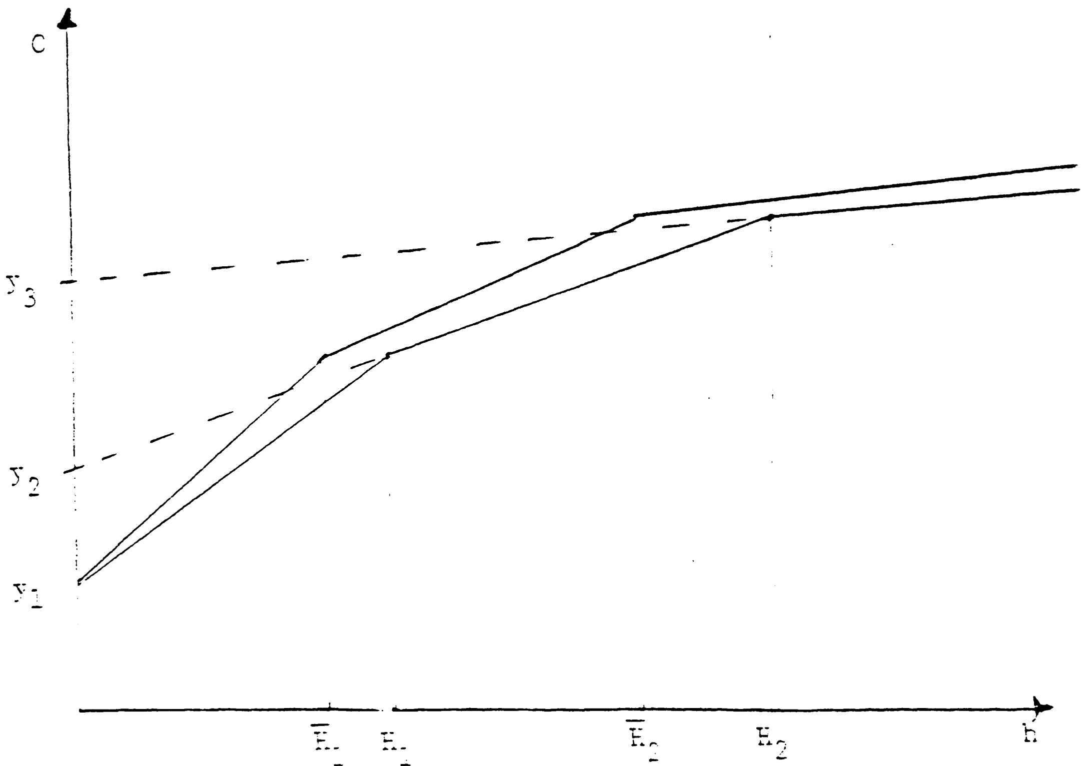


Figure 9

Let us denote the marginal wage rates on the old budget constraint by w_i^{old} and the marginal wage rates on the new budget constraint by w_i^{new}. We can then state:

Proposition 8: An increase of the gross wage rate w has the following effect

i) if leisure is a normal good, the original optimum is in an interval $(\bar{H}_i, H_i)$, and $w_{i+1}^{new} < w_i^{old}$, then labor supply decreases

ii) if the conditions in i) are not satisfied, then labor supply might either increase or decrease.

Part i of the proposition can be proved by using proposition 3. Part ii. is obvious.

5. Implications for the interpretation of estimated labor supply functions

The analysis above shows that for many types of changes in a nonlinear budget constraint we can determine whether hours of work decrease or increase, without having to use estimated labor supply functions. However, in some instances the theory above is not sufficient. In such cases and/or if we want to quantify the effect, we need empirically estimated labor supply functions. The analysis above have implications for how such empirical work should be done, and how to interpret the results of earlier empirical studies.

Assuming there exists a unique global solution to the individual's utility maximization problem we can always solve for hours of work as a function of the before tax wage rate. In the following I will call this the gross wage rate labor supply function. In some cases this function can be written in a closed analytic form, in other cases not. In the latter case it is always possible to solve for hours of work numerically given the value of the gross wage rate, and the function can be tabulated.

The functional form of the gross wage rate labor supply function will depend on the form of the preferences and the form of the tax schedule. It should be obvious, and is illustrated by propositions 4-7 above, that the parameters of the tax schedule acts as shift parameters in the gross wage rate function.

It is, of course, possible to linearize the gross wage rate function and estimate a labor supply function of the form

$$h = a + bw + cy. \qquad (2)$$

Many earlier empirical studies have used specifications like this. These specifications can be interpreted as linear approximations to nonlinear gross wage rate labor supply functions.

We can make three observations regarding estimations of functions like (2). Firstly, since tax systems have changed quite much in most countries during the last decades, the parameters a, b and c in (2) has changed over time.[1] Hence, it seems inappropriate to estimate functions like (2) on time series data. Secondly, the usefulness of a function like (2) estimated on cross-section data is quite limited, as they can only be used to predict the effect of changes in the gross wage rate given the tax system in force at the time data were collected. That is, the estimated functions cannot be used to predict the effect of changes in the gross wage rate given that another tax system is in force, nor can it be used to infer the effect of taxes on labor supply. A change in the tax schedule changes the values of the parameters a, b and c. However, in order to know how the parameters are affected we would have to know the underlying nonlinear function that includes the tax parameters explicitly. Thirdly, low values of the estimated gross wage rate elasticity can not be taken as evidence that taxes do not have a significant

[1] The inflation during the seventies resulted in large changes of the taxation of real income.

influence on labor supply. As shown below, an observed almost vertical gross wage rate function is consistent with a large effect of taxes on labor supply.

To illustrate the last point, let us assume individuals' preferences can be represented by a Cobb-Douglas utility function $U = C^{\alpha}(z-h)^{1-\alpha}$, where C is consumption, z total number of hours available and h hours of work. If this utility function is maximized subject to a linear budget constraint $C = wh + y$, we obtain the labor supply function

$$h = \alpha z - (1-\alpha)\frac{y}{w}. \tag{3}$$

Let us next derive the labor supply function when there is a nonlinear tax. For simplicity a smooth differentiable function is used. Let x denote taxable income and K after tax income. The tax function used is then implicitly defined by $K = \theta x^{\beta}$, $0 < \theta \leq 1$, $0 < \beta \leq 1$. If $x = wh + y$, then the labor supply function will have the form

$$h = \frac{\alpha\beta z}{1-\alpha+\alpha\beta} - \frac{(1-\alpha)y}{(1-\alpha+\alpha\beta)w} \qquad (4)\ (3)$$

If taxable income equals wh, and y is untaxed, it is not possible to write h as an explicit function of w. However, the inverse function have the form

$$w = \frac{h^{\frac{1-\beta}{\beta}}\, y^{1/\beta}}{\left[\frac{\alpha\theta\beta z}{1-\alpha} - h\theta\left(1+\frac{\alpha\theta\beta}{1-\alpha}\right)\right]^{1/\beta}} \tag{5}$$

In figure 10 these three labor supply functions are shown for the case where $y = 10{,}000$, $\alpha = 0.5$, $\theta = 0.9$ and $\beta = 0.9$. The figure reveals several interesting facts. First, the labor supply functions are highly nonlinear. If the gross wage rate labor supply function is estimated by a linear form, this implies that the results are extremely sensitive to the sample composition. A sample consisting of individuals with high wage rates would yield an almost vertical supply curve, while a sample with individuals with low wage rates would yield a supply curve with a large wage rate elasticity. Thus, if females and males had the same preferences, but females in general have lower wage rates than males, we would expect estimated gross wage rate elasticities to be considerably larger for females than for males. This is exactly what has been empirically observed.

Secondly, in our example, taxes has a considerable influence on hours of work. However, if linearized gross wage rate functions were estimated on a sample of individuals with wage rates in the interval 18-40, we would obtain almost vertical supply curves. The effect of the tax is to shift the gross wage rate labor supply function considerably to the left. The effect of the tax can not be inferred from the size of the gross wage rate coefficient.

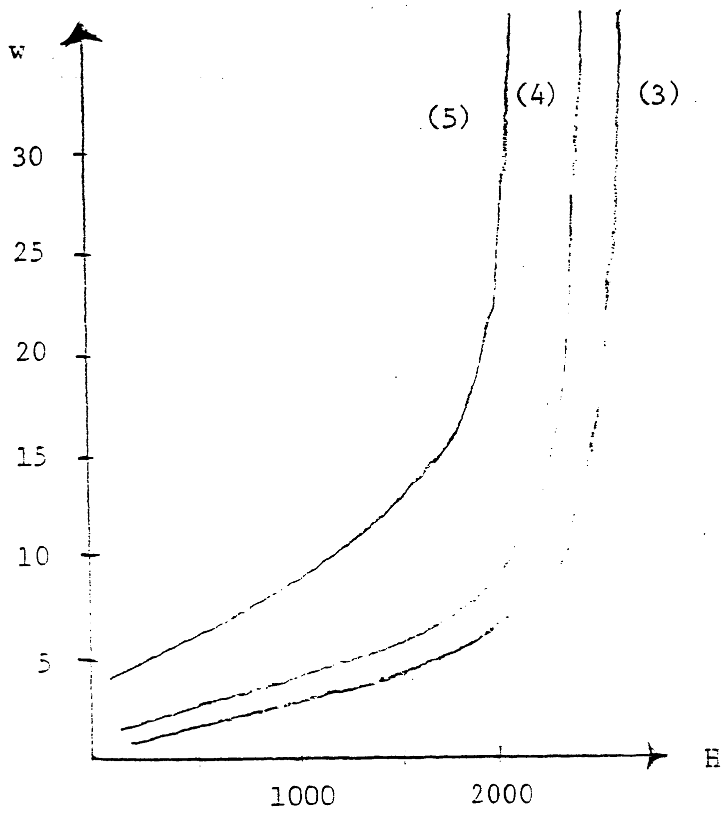


Figure 10

The example above points out the importance of estimating the structural labor supply functions, where the influence of taxes enter explicitly. How this can be done in the presence of piecewise linear taxes have been shown by, for example, Burtless and Hausman (1978), Wales and Woodland (1979) and Blomquist (1983).

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