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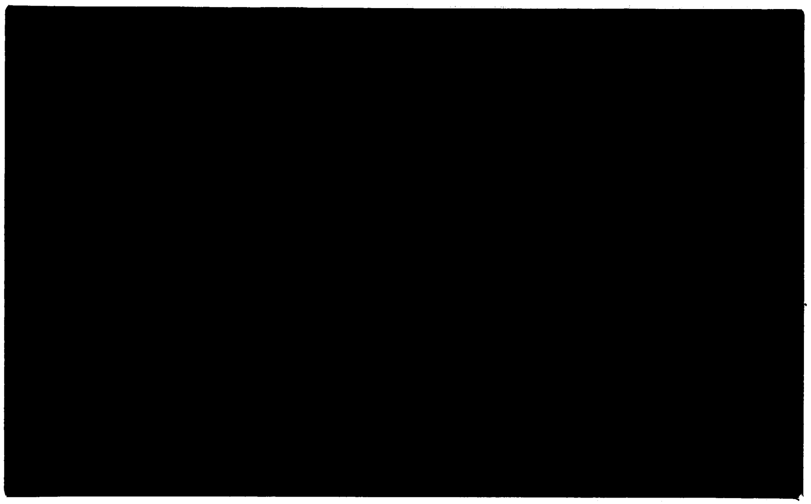

New Techniques for the Study of Stochastic Equilibrium Process

by

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Section 1.

The aim of this paper is to study stationary equilibria in a class of Markov Temporary Equilibrium models--models in which Temporary Equilibrium states propagate in a Markovian manner. Stationarity is a dynamic equilibrium concept. It is the appropriate generalization to an uncertain world of the steady-state equilibrium concept used in many dynamic models with certainty. Our purpose is to illustrate the usefulness of stationarity as an equilibrium concept by demonstrating that for most transition probabilities for Markov process of Temporary Equilibrium States (MTE's), the equilibrium is well-behaved.

Specifically, an equilibrium concept is good if it satisfies three criteria: for a given specification of the model, the equilibrium exists; is unique; and, in the natural dynamic posed by the model, we observe convergence to the equilibrium starting from any initial specification of values for the endogenously determined variables.

The use of stationarity as a dynamic equilibrium concept in general competitive analysis was introduced by Radner [1974]. Stationarity was first studied in a Temporary Equilibrium context by Grandmont and Hildenbrand [1974]. Grandmont and Hildenbrand use the Kryloff-Bogoliouboff Theorem to establish the existence of stationary equilibria and an ergodic decomposition for a class of MTE's arising from a version of the pure consumption loan model with stochastic endowments. This approach, however, has several drawbacks. First, it requires a strong continuity assumption on the Markov endowment process. Second, it requires a strong continuity assumption on the correspondence determining the set of equilibrium price vectors. Third, the ergodic decomposition of the space of Temporary Equilibrium states obtained by this method is not very sharp.

However, it can be seen that, with a natural topology for the space of transition probabilities for Markov processes on a complete separable space, an open and dense set of transition probabilites have a unique invariant probability; furthermore, convergence to this invariant distribution is geometric (in variation norm) from any initial distribution of states. This suggests that, in the absence of perverse behavior of the set of MTE's, a large class of MTE's should also have these ergodic properties.

In this paper we consider a pure exchange economy arising from either an overlapping generation model or an infinite horizon model. We state a sufficient condition on transition probabilities for the existence of a unique stable, equilibrium. We demonstrate that, in the natural topology on transition probabilities, the set of MTE's satisfying this sufficient condition is relatively open and dense in the set of all MTE's.

Section 2 presents some basic tools for the analysis of the asymptotic behavior of Markov processes. Intuition concerning the main theorem of this paper is also discussed. In section 3 we state our assumptions on agents' behavior. Section 4 analyses the agents' choice problems in both the overlapping generations and infinite horizon cases. Necessary results from the theory of dynamic programming are surveyed in an appendix. The existence of Temporary Equilibrium states is shown in section 5. In section 6 the transition probabilities for MTE's are constructed. Section 7 contains the statement and proof of the main theorems. Some extensions are discussed and conclusions drawn in section 8.

Section 2.

In this paper we will be concerned with those discrete-time Markov processes whose state spaces are (topologically) complete and separable. A Markov process is given by a four-tuple $(Z, \underset{\sim}{Z}, K, \mu)$. Z is the state space for the process, which we assume to be a complete, separable topological space. $\underset{\sim}{Z}$ is the σ-field of events in Z, which we will take to be the Borel σ-field. K is a transition probability, and μ is an initial distribution of states. In the sequel we will drop all references to Z and $\underset{\sim}{Z}$, and refer to a Markov process simply as (K, μ).

A transition probability is a map $K : Z \times \underset{\sim}{Z} \to [0,1]$. $K(z,A)$ is the probability of observing event A at time $t + 1$, given that the state occurring at time t is z.

Definition 2.1. A transition probability is a map $K : Z \times \underset{\sim}{Z} \to [0,1]$ satisfying two requirements:

i) For all $z \in Z$, $K(z,\cdot)$ is a probability on the measurable space $(Z, \underset{\sim}{Z})$.

ii) For all $A \in \underset{\sim}{Z}$, the map $K(\cdot,A) \to [0,1]$ is measurable.

The transition probability determines the stochastic evolution of the process.

Associated with every transition probability are two linear operators whose properties capture the Markov process' asymptotic behavior. Let $B(Z)$ be the Banach space of bounded, real valued functions measurable w.r.t. $\underset{\sim}{Z}$ with the sup norm. Its dual is $ba(Z)$, the set of all bounded, finitely additive set functions with domain $\underset{\sim}{Z}$. We topologize $ba(Z)$ with the variation norm.

Let $K(z,A)$ be a transition probability. Let $h \in B(Z)$. Define the linear operator $T : B(Z) \to B(Z)$ by

(1) $$(Th)(z) = \int h(x)K(z,dx).$$

The adjoint of T is the unique linear operator T^* on $ba(Z)$ satisfying

$$\int_Z \int_X h(x)K(z,dx)\lambda(dz) = \int_X \int_Z h(x)K(z,dx)\lambda(dz);$$

more briefly, $\langle Th,\mu\rangle = \langle h,T^*\mu\rangle$. Let $A \in \underset{\sim}{Z}$. T^* is defined by

(2) $$(T^*\lambda)(A) = \int K(z,A)\lambda(dz).$$

$Th(z)$ is the expected value of h in period $t + 1$ given that the state at time t is z_t. $T^*\lambda(A)$ is the probability of being in set A at time $t + 1$ if the distribution of states at time t is λ.

T and T^* have the following properties:

i) T is a continuous linear operator with $||T|| = 1$.

ii) $T \cdot 1 = 1$.

iii) $Tf \geq 0$ whenever $f \geq 0$.

iv) $T^*\delta_z$ is a probability (i.e., countably additive) for all point masses δ_z, $z \in Z$.

v) Any operator satisfying i) - iv) can be constructed with a transition probability in the manner of (1).

T and T^* are the Markov operators associated with the transition probability $K(z,A)$.

It is easy to see that the map from transition probabilities to Markov operators on $B(Z)$ is a bijection, and so we can topologize transition probabilities with the weakest topology making the map a homeomorphism.[1] This topology has the following interpretation.

Lemma 2.1. Let $\{K_n\}_{n=1}^{\infty}$, K be transition probabilities on a complete metric space Z with the Borel σ-field, and $\{T_n\}_{n=1}^{\infty}$, T be their associated Markov operators on $B(Z)$. $||T_n - T|| \to 0$ if and only if

$$\sup_{\substack{z\varepsilon Z \\ A\varepsilon \underset{\sim}{Z}}} | K_n(z,A) - K(z,A)| \to 0.$$

Proof. See Blume [1977] Lemma 3.1.

By Convention, $T^\circ = I$. We also have the identity $(T^n)^* = (T^*)^n$.

We denote by $MOP(Z)$ the space of all Markov operators on $B(Z)$. We also definte $MTE(Z)$ to be the subset of $MOP(Z)$ generated by MTE's.

We are interested in three asymptotic properties of Markov processes. The first of these is stationarity.

Definition 2.2. A Markov process (K,μ) is stationary if $T^*\mu = \mu$.

The importance of stationarity is twofold. First, as was discussed in Section 1, stationarity is a natural dynamic equilibrium concept. Second, for a large class of Markov operators (open dense, in fact), the average behavior of the system over the infinite time horizon is characterized by the probabilities on $\underset{\sim}{Z}$ invariant under T^*. Define $A_n^* = \frac{1}{n} \sum_{i=0}^{n-1} T^{*i}$. The study of the average behavior of the process (μ,K) is the study of $\lim_{n\to\infty} A_n^*\mu$. For most Markov operators, the operator $A^* = \lim_{n\to\infty} A_n^*$ exists, and it is easily seen that if $\nu \in \text{Im } A^*$, then $T^*\nu = \nu$.[2] Of course, if the invariant probability of ν is unique, then the stationary Markov process (ν,K) is ergodic, and further characterizations of the time average of economic variables may be obtained.

Recall from section 1 that we are also interested in the stability properties of Markov processes. Intuitively, stability requires that the period t distributions of states converge, as t gets large, to an invariant probability distribution.

Definition 2.3. A transition probability (Markov operator) $K(T)$ is stable if, for the associated Markov operator (adjoint) T^*, the sequence of probabilities $\{T^{*n}\mu\}$ converges in norm, for any μ, to an invariant probability.

Finally, we are interested in uniqueness of the invariant probability measure. With a slight abuse of notation, we shall call transition probabilities having this property ergodic.[3]

Definition 2.4. A transition probability (Markov operator) $K(T)$ is said to be ergodic of its associated Markov operator (adjoint) T^* admits a unique invariant probability measure.

The primary result of this paper is that there exists a (relatively) open and dense set of MTE's having these properties. This is to say, a relatively open and dense set of transition probabilities for MTE's have nicely behaved dynamic equilibria.

This result is not surprising. Consider first the case of Markov processes on finite state spaces. A transition probability is given by a matrix $A = [a_{ij}]$. The element a_{ij} is the probability of observing state a_i at time $t + 1$ given that the process is in state j at time t. Letting $n = \text{card}(z)$, we have the following properties for A:

i) $\forall\ i,j \in Z,\ a_{ij} \geq 0.$

ii) $\Sigma_{i=1}^{n} a_{ij} = 1.$

A probability measure on Z is a vector v in the positive $n - 1$ dimensional unit simplex. The operator T^* is given by the equation $v_{t+1} = Av_t$. The following theorem is well known:

Theorem A. If A has strictly positive entries, then properties 2.2-2.4 hold.

It is clear that the set of all A with strictly positive endowments is relatively open and dense in the set of all Markov matrices. This leads one to conjecture a similar result on $MOP(Z)$, when Z is a complete, separable topological space. Here the appropriate subset of $MOP(Z)$ to consider is the set of all operators T on $B(Z)$ having for some finite iterate T^n a contraction mapping. Call this set M.

Lemma 2.2. If $T \varepsilon M$, then properties 2.2-2.4 hold.

Proof. Suppose that T^n is a contraction. Then so is T^{*n}. Furthermore, cap(Z) is invariant under T^{*n}. Applying the Banach fixed point theorem, we observe that T^{*n} has a unique fixed point $\nu \varepsilon \text{cap}(Z)$. We have $T^{*n}T^*\nu = T^*T^{*n}\nu = T^*\nu$, and so $T^*\nu$ is also a fixed point of T^{*n}. Thus $\nu = T^*\nu$, and so ν is a fixed point for T^*.

Q.E.D.

Lemma 2.3. M is open and dense in MOP(Z).

Proof. Let $M_n = \{T \varepsilon M : T^n$ is a contraction$\}$. $M = U_n M_n$. $M_n = \text{MOP}(Z) \cap \{$Linear operators T on $B(Z) : T^n$ is a contraction$\}$. The set to the right of $\cap$ is open, and so M_n is relatively open in MOP(Z). Thus so is M.

To see density, choose $T \varepsilon \text{MOP}(Z)$ and $V \varepsilon M$. Let n be such that V^n is a contraction. Choose $0 < \alpha < 1$ and define $T_\alpha = \alpha V + (1-\alpha)T$. $T_\alpha{}^n = (\alpha V + (1-\alpha)T)^n = \alpha^n V^n + T'$, $T' = \Sigma_{k=1}^n \binom{n}{k} \alpha^{n-k}(1-\alpha)^k\, {}^{n-k}T^k$. Computing, $||T_\alpha^n|| \leq \alpha^n||V^n|| + \Sigma_{k=1}^n \binom{n}{n} \alpha^{n-k}(1-\alpha)^k < 1$. Thus $T_\alpha{}^n$ is in M. Since $\lim_{\alpha \to 0} T_\alpha = T$, M is dense in MOP(Z).

Q.E.D.

Corollary 2.1. The set of all $T \varepsilon \text{MTE}(Z)$ satisfying properties 2.2-2.4 contains an open set.

The proof of the corollary is immediate, and it is reasonable to conjecture that if MTE is a nice set, then $M \cap \text{MTE}$ will also be dense. In fact, our method of proof for our primary theorem is to first recall Corollary 2.1 and then construct a dense subset of $M \cap \text{MTE}$.

Section 3.

We consider an exchange economy with N agents and L goods. The economy has no contingent claims markets. Thus, the equilibrium price in any period will be a random variable. There are also no futures markets and no markets for financial assets. All goods are perishable, and so wealth cannot be transferred between time periods. The intertemporal link in our model is provided by the structure of agents' preferences. We postulate the existence of intertemporal consumption externalities. Marginal rates of substitution in period t+1 depend upon consumption in period t. A subsequent section will discuss extensions of our results to monetary economies.

Uncertainty enters the model through the random behavior of agents' endowments. We assume that the stochastic process of endowment allocations is Markovian.

When the market in our world first meets, each trader must specify his current consumption of each good. The consumption externalities guarantee that the consequences of this decision will depend upon those price vectors and endowment allocations prevailing in the future. Thus, once we specify how agents' expectations about the future evolve, and if we assume agents are expected utility maximizers, then each agent can solve his choice problem by applying the tools of dynamic programming.

We begin with a description of possible endowments.

A.1. The aggregate endowment is $\bar{e} \in R^L$, $\bar{e} >> 0$. $\bar{e}$ is non-random and independent of t.

Let $W = \{e \in R^{NL}, e = (e_i, \ldots, e_N), e_i \varepsilon R^L$ for $i = 1, \ldots, N$: $e_i \geq 0$ and $\sum_{i=1}^N e_i = \bar{e}\}$. W is the set of possible endowment allocations. $W_i = \{e_i \varepsilon R^L: 0 \leq e_i \leq \bar{e}\}$ is the set of possible endowment vectors for agent i. We write $e_{i,j,t}$ for the i-th agent's endowment of commodity j in period t.

A.2. The consumption set for agent i is $X_i = \{x \varepsilon R^L: 0 \leq x \leq a\}$. We assume $\bar{e} << a < +\infty$. The vector a, and therefore X_i, is independent of t.

Each agent has a compact consumption set which includes the aggregate endowment as an interior point.

Agent i has a utility function over consumption streams $\bar{x} = \{x_t\}_{t=0}^{T}$ of the form $\bar{U}_i(\bar{x}) = \sum_{t=0}^{k-T-1} \beta_i^t U_i(x_t, x_{t+1})$. Consumption in the last period has a direct effect on current period consumption. Our results generalize to include direct consumption externalities over any fixed, finite number of periods. In this specification we include the infinite horizon case $T = \infty$. $U_i(x_t, x_{t+1})$ is the i'th agent's one period utility function, and β_i is his discount.rate.

A.3. U_i is continuous, strictly concave and strictly monotone.

A.4. $0 \leq \beta_i < 1$.

Prices are elements in the unit simplex Δ^{L-1}.

The Temporary Equilibrium assumption is that first, only current spot markets clear, and therefore agents' plans need not be mutually consistent. Second, agents do not have perfect foresight. Instead, their expectations for the future evolve in light of what they observe about the state of the economy. We assume that agents condition their expections upon the current period price vector and upon their current endowments. Agent i must predict, based upon information in period t, p_{t+1} and $e_{i,t+1}$. Let $M(\Delta^{L-1}\times W_i)$ be the set of probability measures on $\Delta^{L-1}\times W_i$ with the topology of weak convergence.[4]

Definition 3.1. An expectation function for agent i is a map

$$\Psi_i: \Delta^{L-1}\times W_i \to M(\Delta^{L-1}\times W_i).$$

If agent i observers $(p_t, e_{i,t})$ in period t, then he believes that $(p_{t+1}, e_{i,t+1})$ will be distributed according to the probability distribution $\Psi_i(p_t, e_{i,t})$.

A.5 $\Psi_i: \Delta^{L-1}\times W_i \to M(\Delta^{L-1}\times W_i)$ is continuous, and for all $(p, e_i)\in\Delta^{L-1}\times W_i$

$$\Psi_i(p_i, e_i)(\partial\Delta^{L-1}\times W_i) = 0.$$

Each agent assigns probability 0 to any event in which the price of any good is 0. Th is is a minimal rationality requirement. Together with A.3 it implies that all equilibrium prices will be strictly positive.

Section 4.

Here we formulate and analyze the individual agent's dynamic programming problem. We consider both finite and infinite horizon problems. In the finite horizon case, we can use the familiar backward induction arguments. In the infinite horizon case, we can rely on Bellman's Optimality Equation. The ideas in this section are all well-known. The infinite horizon analysis is a straightforward application of the ideas of Christiansen [1975], who was the first to apply the contraction mapping methods of Denardo [1967] to the study of Temporary Equilibrium.

There is, however, one small difficulty in our analysis. We wish to allow agents to have zero endowment. We want to allow for the fact that the stochastic process governing the evolution of endowment allocations may award the endowment vector 0 to some agents with positive probability. This situation leads to possible discontinuities of the budget correspondence on the boundary of the price simplex, and so we have to ensure that the resulting discontinuity in demand behavior leads to no serious problems. In our model this will be the case since, by assumption, all agents assign probability 0 to the event of observing a price vector on the boundary.

To begin the analysis, we must establish some continuity properties of the budget correspondence.

Definition 4.1. Let $\gamma(p,e_i) = \{x\varepsilon X_i: p\cdot x \leq p\cdot e_i\}$

Proposition 4.1. The correspondence $\gamma(p,e_i)$ is continuous on $\text{Int}\Delta^{L-1}\times W_i$. Furthermore for $k = 1, 2, \ldots, L$; let $\Delta_k^{L-1} = \{p\varepsilon\Delta^{L-1}: p_k > 0\}$. Let $W_i^k = \{e_i\varepsilon W_i: e_{i,k} > 0\}$. Let $\Delta_{j,k}^{L-1}$, $W_i^{j,k}$, and so forth over all pairs of n-tuples, be defined analogously. Then $\gamma(p,e_i)$ is continuous on $\Delta^{L-1}_{j,k\ldots} \times W_i^{j,k,\ldots}$.

Proof. Upper hemi-continuity is standard. Lower hemi-continuity follows from the fact that in each case, for all p and e_i, $p\cdot e_i > \min\{p\cdot x;\ x\varepsilon W_i\}$.

Q.E.D.

Next we study the maximand for the agent's consumer choice problem. A key tool in both finite and infinite horizon settings is the valuation V defined on $B(X_i\times\Delta^{L-1}\times W_i)$, the set of bounded real valued functions on $X_i\times\Delta^{L-1}\times W_i$ measurable with respect to the Borel σ-field. V is given by

(1) $$Vw(x_0,p_1,e_1) = \sup\{U(x_0,x_1) + \beta\int w(x_1,p_2,e_2)\Psi(dp_2\times de_2\times dq_2);$$
$$x_1 \varepsilon \gamma(p_1,e_1)\}$$

We endow $B(X_i\times\Delta^{L-1}\times W_i)$ with the topology of uniform convergence.

In finite horizon problems, iterates of V applied to the function that is identically 0 give the agent objective functions to maximize. Suppose our agent is faced with an n period choice problem. Then in period k he maximizes

(2) $$U(x_{n-1},\ x_n) + \beta\int V^{n-k}0(x_k,\ p_{k+1},\ e_{k+1})\Psi(p_k,\ e_k)(dp_{k+1}\times de_{k+1})$$

with the convention that V^0 is the identity operator. This can be verified by the familiar backward induction argument of dynamic programming. Consider a problem in which n=2, with given initial consumption x_0. According to (2), the agent's optimal plan in the second period is given by maximizing

$$U(x_1,\ x_2) + \beta\int V^0 0\Psi(p_2,\ e_2)(dp_3\times de_3) = U(x_1,\ x_2)$$

when x_1 is a given first period consumption. The indirect utility function is given by $V\,0(x_1, p_2, e_2)$. Thus, in the first period, the agent maximizes

$$U(x_0, x_1) + \beta\int V'\,0(x_1, p_2, e_2)\ (p_1, e_1)(dp_2 \times de_2)\ ,$$

which is equation (2) when n=2 and k=1.

The applicability of the operator V to infinite horizon dynamic programming was first noted by Denardo [1967]. He observed that V is a contraction mapping on $B(X\times\Delta^{L-1}\times W)$, and so the sequence of iterates V^n0 converges to the unique fixed point of V. This fixed point provides a valuation function whose maximization subject to the budget constraint can be used to solve the agent's first period problem. Let U* be the fixed point of V. Then the maximand is

$$U(x_0, x_1) + \beta\int U^*(x_1, p_2, e_2)\Psi(p_1, e_1)(dp_2\times de_2)$$

Additional assumptions can be given ensuring the stationarity of the optimal plan. This is to say, we can ensure that the optimal action x_t at time t is a function $f_t(x_{t-1}, p_t, e_t)$, and $f_t = f_{t'}$ for all t, t'. This is discussed in Appendix 1.

We now proceed to study the operator V. Define $S = X_i\times\Delta^{L-1}\times W_i$ $s_t = (x_{t-1}, p_t, e_t)$. Let $r(s_t, x_t) = U(x_{t-1}, x_t)$. Let $\gamma(p_t, e_t) = \gamma(s_t)$. Let $\hat{\Psi}(s_t, x_t) = \Psi(\bar{p}_t, e_t) \times \delta_{x_t}$. δ_{x_t} is the point mass at x_t, and so $\hat{\Psi}(s_t, x_t)$ is the product measure on S×X induced by $\Psi(p_t, e_t)$ on S and δ_{x_1} on X. We can then rewrite (1) as

$$\text{(3)} \qquad Vw(s_t) = \sup\{r(s_t, x_t) + \beta\int w(s_{t+1})\Psi(s_t, x_t)(ds_{t+1});\ x_t \in \gamma(s_t)\}.$$

Define $S' = X_i \times \mathrm{Int}\Delta^{L-1} \times W_i$. Let $B'(S)$ be the set of all bounded measurable functions from S to R continuous on S', monotone and concave in x for each (p, e_i). $B'(S)$ is a closed subset of $B(S)$. We also define $S^k = X_i \times \Delta_k^{L-1} \times W_i^k$.

Proposition 4.2. Assume A.1-A.5. Then $B'(S)$ is invariant under V.

Proof. We will prove invariance of each property of $B'(S)$ separately as a lemma.

Lemma 4.1. The objective function

$$r(s_1, x_1) + \beta\int w(s_2)\hat{\Psi}(s_1, x_1)(ds_2) \tag{4}$$

is continuous on $S \times X$ if $w \varepsilon B'(S)$.

Proof. Since $r(s_1, x_1)$ is assumed to be continuous, it suffices to show that the integral term is continuous. Let $(s_{1,n}, x_{1,n}) \to (s_1, x_1) \varepsilon S \times X_i$, and let $w \varepsilon B'(S)$. Let $\nu_n = \hat{\Psi}(s_{1,n}, x_{1,n})$, $\nu = \hat{\Psi}(s_1, x_1)$. By A.5, $\nu_n \to \nu$. w is bounded and discontinuous only on a set of ν-measure zero by hypothesis. Thus $\int w(s_2)\nu_n(ds_2) \to \int w(s_2)\nu(ds_2)$. (See Billingsley [1968], Theorem 5.2.)

Q.E.D.

Lemma 4.2. Vw is continuous on S'.

Proof. $Vw(s_1)$ is the suppression of the objective function on the set $\gamma(s_1)$. γ is continuous on S', and so the result follows from the Maximum Theorem (Hildenbrand [1974], pg. 30).

Q.E.D.

Lemma 4.3. Vw is bounded and measurable.

Proof. Since $S \times X$ is compact, r is bounded, say by c. Then $||Vw|| \leq c + \beta||w||$.

To demonstrate measurability we shall find a finite partition of S into Borel sets such that on each set $V_\psi w$ is continuous. Let $\partial S_{k,1} = \{s \in S\colon\ p_k = 0,\ p_j > 0$ for $j \neq k$, and for some $j \neq k$ $e_j > 0\}$, $\partial S_{k,2} = \{s \in S\colon\ p_k = 0,\ p_j > 0$ for $j \neq k$, and for all $j \neq k$ $e_j = 0\}$, $\partial_{j,k,1} = \{s \in S\colon\ p_j = 0,\ p_k = 0,\ p_\ell > 0$ for $\ell \neq j,k$, and for some $\ell \neq j,k$; $e_\ell > 0\}$, $\partial_{j,k,2} = \{s \in S\colon\ p_j = 0,\ p_k = 0,\ p_\ell > 0$ for $\ell \neq j,k$, and for all $\ell \neq j,k$ $e_\ell = 0\}$, and so forth over all triples, quadruplets, etc., of prices. The union of all these sets with S' is S.

Note that on each set with a subscript 2, the only commodity bundle in the budget set is the 0 vector. Thus on each set the budget correspondence is continuous, and therefore on each set Vw is continuous by the Maximum Theorem.

Q.E.D.

Lemma 4.4 Vw is monotone in x for each (p,e).

Proof of lemma. From (1) we write Vw as

$$w(x_0,p_1,e_1) = \sup\{U(x_0,x_1) + \beta \int w(x_1,p_2,e_2)\Psi(p_1,e_1)(dp_2 \times de_2);$$

$$x_1 \in \gamma(p_1,e_1)\}$$

We want to show that for all (p_1,e_1), Vw is monotone in x_0. Let $x_0' \geq x_0''$. Let $x_1' = x_1(x_0',p_1,e_1)$ and $x_1'' = x_1(x_0'',p_1,e_1,)$; i.e. x_1' is optimal for (x_0',p_1,e_1), and similarly for x_1''. Then we have $U(x_0',x_1') + \beta \int w(x_1',p_2,e_2)$ $\Psi(p_1,e_2)(dp_2 \times de_2) \geq U(x_0',x_1'') + \beta \int w(x_1'',p_2,e_2)\Psi(p_1,e_1)(dp_2 \times de_2)$. The first term is $Vw(x_0',p_1,e_1)$ and the third is $Vw(x_0'',p_1,e_1)$, and so the lemma is proved.

Lemma 4.5. Vw is concave.

Proof of lemma. Let $0 < t < 1$. Denote by $x(t)$ the convex combination $tx_0' + (1-t)x_0''$. Again, let $x(x_0,p_1,e_1)$ be an optimal action. Then $tVw(x_0',p_1,e_1) + (1-t)Vw(x_0'',p_1,e_1,) = t\{U(x_0',x(x_0',p_1,e_1,)) + \beta \int w(x(x_0', p_1,e_1),p_2,e_2)\Psi(p_1,e_1)dp_2{\times}de_2)\} + (1-t)\{U(x_0'',x(x_0'',p_1,e_1)) + \beta \int w(x(x_0'', p_1,e_1),p_2,e_2)\Psi(p_1,e_1)(dp_2{\times}de_2)\} \leq U(x(t),tx(x_0',p_1,e_1) + (1-t)x(x_0'',p_1, e_1)) + \beta \int \{tw(x(x_0',p_1,e_1,), p_2,e_2) + (1-t)w(x(x_0'',p_1,e_1),p_2,e_2)\}\Psi(p_1,e_1) (dp_2{\times}de_2) \leq U(x(t), tx(x_0',p_1,e_1) + (1-t)x(x_0'',p_1,e_1)) + \beta \int w(tx(x_0',p_1, e_1) + (1-t)x(x_0'',p_1,e_1,),p_2,e_2)\Psi(p_1,e_1)dp_2{\times}de_2) \leq U(x(t), x(x(t), p_1,e_1) + \beta \int w(x(x(t),p_1,e_1),p_2,e_2)\Psi(p_1,e_1,)(dp_2{\times}de_2) = Vw(x(t),p_1,e_1)$. The first inequality follows from A.3. The second follows from hypothesis on w. The third follows from the optimality of $x(x(t),p_1,e_1)$ and the feasibility of $tx(x_0',p_1,e_1) + (1-t)x(x_0'',p_1,e_1)$. This proves the lemma, and the proposition.

Q.E.D.

Propositions 4.1 and 4.2 give sufficient monotonicity, convexity and continuity requirements for the finite horizon problem. The required results for the infinite horizon problem are contained in the next proposition.

Proposition 4.3. V has a unique fixed point $U^*\varepsilon B'(S)$.

Proof. We know that $B'(S)$ is an invariant subspace for V (Prop. 4.2), and that $B'(S)$ is complete. Thus it suffices to show that V is a contraction mapping. Computing,

$\|Vw-Vw'\| = \sup\{|Vw(s)-Vw'(s)|;\ s \in S\} = \sup\{|\sup\{r(s_1,x_1) + \beta \int w(s_2)\hat{\Psi}(s_1,x_1)(ds_2);\ x_1 \in \gamma(s_1)\} - \sup\{r(s_1,x_1) + \beta \int w'(s_2)\hat{\Psi}(s_1,x_1)(ds_2);\ x_1 \in \gamma(s_1)\}|;\ s_1 \in S\} \leq \sup\{\beta|\int w(s_2)\hat{\Psi}(s_1,x_1)(ds) - \int w'(s_2)\hat{\Psi}(s_1,x_1)(ds_2)|;\ x_1 \in \gamma(s_1),\ s_1 \in S\} \leq \beta \sup\{\int |w(s_2)-w'(s_2)|\hat{\Psi}(s_1,x_1)(ds_2);\ x_1 \in \gamma(s_1),\ s_1 \in S\} \leq \beta \sup\{|w(s_2)-w'(s_2)|,\ s_2 \in S\} = \beta\|w-w'\|$. Assumption A.4 says that $\beta < 1$, so V is a contraction.

Q.E.D.

Now we can construct our agent's demand function. We start with the finite horizon case. Time is indexed by the subscript t, and the horizon is T periods, $T < \infty$. We recall from our discussion of the backward induction for finite horizon programming that in period t the agent's optimal plan is given by solving the following maximization problem:

$$\text{(4)} \qquad \max U(x_{t-1},x_t) + \beta\int V^{T-t}O(x_t,p_{t+1},e_{t+1})\Psi(q_t)(dp_{t+1}\times de_{t+1})$$

$$\text{subject to } x_t \in \gamma(p_t,e_t).$$

Note that if the agent conditions on prices, then prices enter the objective function. By backwards induction, (4) is continuous for all $t \leq T$.

<u>Theorem 4.1</u>. Assume A.1-A.5, and assume that $T < \infty$. Then the agent's optimal plan in period t is a function $x_t(x_{t-1},p_t,e_t)$ continuous on S' and S^k, $k = 1,\ldots,L$.

Proof. Consider the maximand (4). For all $t \leq T$ and for all $(p_{t+1},e_{t+1}) \in \Delta^{L-1} \times W$, the function $V^{T-t}O(x_t,p_{t+1},e_{t+1})$ is monotone and concave in x_t, as can be seen by repeated applications of

Proposition 4.2.

The integral

$$\int V^{T-t}(x_t,p_{t+1},e_{t+1})\Psi(P_t,e_t)(dp_{t+1}\times de_{t+1}).$$

is monotone and concave in x_t. For all x_{t-1} we have from A.4 that $U(x_{t-1},x_t)$ is strictly concave and strictly monotone in x_t. Thus the optimand is strictly concave and strictly monotone in x_t for each value of $x_{t-1} \varepsilon X$. The optimand is also continuous on S (Proposition 4.2 and Lemma 4.1). From Proposition 4.1, the budget correspondence is continuous on S' and S^k for all k. The Maximum Theorem now implies that demand is an upper hemi-continuous correspondence on S' and S^k. Because the maximand is strictly concave, demand is also single valued, and thus a continuous function.

Q.E.D.

The results for the infinite horizon problem are similar. It can be shown that with assumptions A.1-A.6, the agent's best action in period t is found by solving

$$(5) \qquad \max U(x_{t-1},x_t) + \beta\int U^*(x_t,p_{t+1},e_{t+1})\Psi(p_t,e_t)(dp_{t+1}\times de_{t+1})$$

$$\text{subject to } x_t \ \varepsilon \ \gamma(p_t,e_t).$$

This is done in Appendix 1.

<u>Theorem 4.2</u>. Assume A.1-A.5, and assume that $T = \infty$. For all $t < \infty$ the agent's optimal plan in period t is a function $x(x_{t-1},p_t,e_t)$, which is continuous on S', continuous on S^k for $k = 1,\ldots,L$. The result follows from the Maximum Theorem, as before.

Q.E.D.

Remark: Note that we have demonstrated in both the finite and infinite horizon case that the agent's objective function is strictly monotone in x_t for each value of $(x_{t-1}, p_t, e_t) \in X \times \Delta^{L-1} \times W$. This fact will be used in the next section to establish a boundary condition on aggregate excess demand.

Section 5.

The first step in defining MTE's is to demonstrate that the state space is not vacuous. In this section we prove that Temporary Equilibrium states exist. As in Section 4, we consider both infinite horizon and overlapping generations economies. To make our discussion of overlapping generations economies more concrete, we assume that there are $N = QT$ agents, Q agents of each age with a lifespan of T years. As each agent dies, he is replaced with another of the same type.

We write $x_{i,j,t}(x_{i,t-1},p_t,e_{i,t})$ for the i'th agent's demand for good j in period t. We define the i'th agent's excess demand for good j in period t by

$$f_{i,j,t}(x_{i,t-1},p_t,e_{i,t}) = x_{i,j,t}(x_{i,t-1},p_t,e_{i,t}) - e_{i,j,t}.$$

In this section and in the sequel we write $x_t = (x_{1,t},\ldots,x_{n,t})$ for the consumption allocation in period t, and $e_t = (e_{i,t},\ldots,e_{n,t})$ for the endowment allocation in period t. Aggregate excess demand is then given by

$$F(x_{t-1},p_t,e_t) = \Sigma_{i=1}^{N} f_{i,t}(x_{i,t-1},p_t,e_{i,t})$$

Note that the functional form F is independent of t. This follows in infinite horizon economies because $x_{i,j,t}(\cdot)$ is independent of t. In overlapping generations economies, we identify an agent as a utility function (type) and an age). Independence of F and t then follow from the invariance over time of the joint distribution of age and types (in any period, one agent of each type in each age bracket).

Definition 5.1. A Temporary Equilibrium (TE) state is a vector $(x_{t-1},p_t,e_t) \varepsilon X \times \Delta^{L-1} \times W$ such that $F(x_{t-1},p_t,e_t) = 0$.

The fundamental result of this section is:

Theorem 5.1. Assume A.1-A.5. Then for every $(x_0, e_1,) \in X \times W$ there exists a $p_1 \in \text{Int } \Delta^{L-1}$ such that (x_0, p_1, e_1) is a TE state.

This theorem is true for both overlapping generations and infinite horizon economies.

We will prove this theorem by appealing to the following well-known result.

Proposition 5.1. Let F': Int $\Delta^{L-1} \to R^L$ be a continuous function satisfying three conditions.

i) For all $p \in \text{Int } \Delta^{L-1}$, $p \cdot F'(p) = 0$,

ii) Z is bounded from below on Int Δ^{L-1}

iii) Let $\{p_n\} \in \text{Int } \Delta^{L-1}$ be a sequence with limit $p \in \partial\Delta^{L-1}$. Then there exists a good j such that $F_j'(p_n) \to k > 0$.

Then there exists a $p^* \in \text{Int } \Delta^{L-1}$ such that $F'(p^*) = 0$.

We cannot use the standard boundary condition that the norm of excess demand tends to infinity as price approaches the boundary of the price simplex, because our consumption sets are compact. It will be seen that the point of having each trader consider consumption bundles strictly larger than the aggregate endowment vector (A.2) is to verify iii) for aggregate excess demand.

Proof of Theorem 5.1. We will demonstrate that for every $(x_0, e_1) \in X \times W$ the aggregate excess demand function F: Int $\Delta^{L-1} \to R^L$ given by $F(x_0, \cdot, e_1)$ satisfies the three conditions of Proposition 5.1. Note that the continuity of F follows from Theorems 5.1 and 5.2.

Lemma 5.1. For all $(x_0, p_1, e_1) \varepsilon X \times \mathrm{Int}\, \Delta^{i-1} \times W$, $p_1 \cdot F(x_0 p_1, e_1) = 0$.

Proof of Lemma 5.1. It follows from the strict monotonicity of the objective functions in both the infinite horizon and overlapping generations economies that for each agent $p_1 \cdot x_i(x_{i,0}, p_1, e_{i,1}) = p_1 \cdot e_i$. The result now follows by substracting the r.h.s. from both sides of the equation and adding over agents.

Lemma 5.2. F is bounded from below on $X \times \mathrm{Int}\, \Delta^{L-1} \times W$.

Proof of Lemma 5.2. Each agent's excess demand is bounded from below by $-e_i$. Thus F is bounded from below by $-\Sigma_i e_i = -\bar{e}$.

Lemma 5.3. Let $\{x_{0,n}, p_{1,n}, e_{1,n}\} \varepsilon X \times \Delta^{L-1} \times W$. Then $\lim_{n\to\infty} F_j(x_{0,n}, p_{1,n}, e_{1,n}) = k > 0$.

Proof of Lemma 5.3. Recall that $\Delta_\ell^{L-1} = \{p \,\varepsilon\, \Delta^{L-1}\colon p_\ell > 0\}$. Now for some ℓ, $p_1 \,\varepsilon\, \Delta_\ell^{L-1}$. From A.1 there exists an agent i such that $e_{i,1} \,\varepsilon\, W_i^\ell$ $(e_{i,\ell 1} > 0)$. By either Theorem 4.1 or Theorem 4.2 as the case may be, agent i's demand is continuous at $(x_{i,0}, p_1, e_{i,1})$. Let m be a commodity with $p_{m,1} = 0$. From the strict monotonicity of the agent's objective function (remark following Theorem 4.2) we have $x_{i,m}(x_{i,0}, p_1, e_{i,1}) = a_m$. Thus $f_{i,m}(x_{i,0}, p_1, e_{i,1}) = a_m - e_{i,m,1}$. It follows that $\lim_{n\to\infty} f_{i,m}(x_{i,0,n}, p_{1,n}, e_{i,1,n}) = -a_m - e_{i,m,1}$. For each $j \neq i$ the sequence $\{f_{j,m}(x_{j,0,n}, p_{1,n}, e_{m,1,n})\}$ is bounded below by $-e_{j,m,1,n}$. Thus the limit is bounded below by $-e_{j,m,1}$. Adding, we find $\lim_n F_m(x_{0,n}, p_{1,n}, e_{1,n}) \geq a_m - \Sigma_{i=1}^N e_{i,m,1} = a_m - \bar{e}_m > 0$ (by A.2). This proves the lemma.

Now, fix $(x_0, e_1) \,\varepsilon\, X \times W$, and apply Proposition 5.1.

Q.E.D.

Since aggregate excess demand is continuous on $X \times \text{Int } \Delta^{L-1} \times W$, we can demonstrate upper hemi-continuity of th equilibrium price correspondence.

Definition 5.2. Let $E: X \times W \to \text{Int } \Delta^{L-1}$ be given by $p \in E(x_0, e_1)$ iff $F(x_0, p, e_1) = 0$.

E is the equilibrium price correspondence.

Corollary 5.1. Assume A.1-A.5. Then the correspondence E is upper hemi-continuous.

Proof. It suffices to show that G(E), the graph of E, is closed. Let $\{(x_{0,n}, p_{1,n}, e_{1,n})\} \subset G(E)$ and let this sequence have limit (x_0, p_1, e_1). If $p_1 \notin \partial\Delta^{L-1}$, then $\lim_{n\to\infty} F(x_{0,n}, p_{1,n}, e_{1,n}) = F(x_0, p_1, e_1)$, and so $F(x_0, p_1, e_1) = 0$. Thus $(x_0, p_1, e_1) \notin G(E)$. Therefore, in order to show that G(E) is closed, it suffices to show that $p_1 \notin \partial\Delta^{L-1}$. Then by application of lemma 5.3, $\lim_{n\to\infty} \| F(x_{0,n}, p_{1,n}, e_{1,n}) \| > 0$. Thus, for n large enough, $F(x_{0,n}, p_{1,n}, e_{1,n}) \neq 0$, contradicting the fact that $(x_{0,n}, p_{1,n}, e_{1,n}) \in G(E)$. Thus, $p \in \partial\Delta^{L-1}$, as was to be shown.

Q.E.D.

Section 6.

In this section we begin our development of the theory of Markov processes of TE states. The study of these processes was initiated by Grandmont and Hildenbrand [1974], who studied an overlapping generations economy with money. In their paper, Grandmont and Hildenbrand use the so-called Kryloff-Bogoliouboff Theory for stable Markov operators to demonstrate stationarity of the stochastic process of TE states. Our approach is to apply the contraction mapping theorem to the appropriate Markov operator on the space of probability measures on the state space with the norm topology given by the variation norm. This approach is a special case of the operator theoretic treatment developed by Yosida and Kakutani [1944].

Now and in the sequel, $\underset{\sim}{Y}$ will denote the Borel σ-field of the topological space Y.

Define $Z = X \times \Delta^{L-1} \times W$; $z_t = (x_{t-1}, p_t, e_t)$. Z is the state space for the stochastic TE process. The sequence of TE states $\{z_t\}$ evolves in the following manner: Suppose the economy is in state z_t at time t. x_{t-1}, p_t, and e_t will determine, through the agents' demand functions, the consumption allocation x_t. e_{t+1} will be randomly drawn from W, according to the endowment allocation process. x_t and e_{t+1} will then determine p_{t+1} through the equilibrium price correspondence.

The first problem to be dealt with is that we need a unique choice of price p_t from the correspondence E.

Proposition 6.1. There exists a function $p: X \times W \to \text{Int } \Delta^{L-1}$ that is measurable with respect to the Borel σ-field $\underset{\sim}{X \times W}$ and, for all $(x_{t-1}, e_t) \in X \times W$, $p(x_{t-1}, e_t) \in E(x_{t-1}, e_t)$.

Proof. It follows from Corollary 4.1 that the correspondence E is upper hemi-continuous. Thus $\{(x,e): E(x,e) \cap F = \emptyset\}$ is closed for every set $F \subset \Delta^{L-1}$. Now we can apply the theorem of Kuratowski and Ryll-Nardzewski [1965] which asserts the existence of a map p with the required properties.

Q.E.D.

We define the map $h: Z \times W \to Z$ by

$$h(z_{t-1}, e_t) = (x(x_{t-2}, p_{t-1}, e_{t-1}),\ p(x(x_{t-2}, p_{t-1}, e_{t-1}), e_t),\ e_t)$$
$$= z_t;$$

where x is the map whose i'th coordinate map is the i'th agent's demand function. h is easily seen to be measurable with respect to the Borel σ-field $Z \underset{\sim}{\times} W$, since it is the composition of measurable functions. The function h contains all the economic theory has to say about the evolution of TE states from period t to period t+1, given the endowment allocation e_{t+1}. The function h describes the "deterministic" evolution of TE states. We will use h to define a transition law for the Markov process of TE states. First, however, we need to specify the stochastic evolution of the exogenous random parameter e.

A.6. The stochastic process $\{e_t\}$ is a Markov process with transition law $Q: W \times \underset{\sim}{W} \to [0,1]$.

Let $\pi_W: Z \to W$ be the projection map. We define the transition law $K: Z \times \underset{\sim}{Z} \to [0,1]$ by

(1) $$K(z_t, A) = Q(\pi_W(z_t), \{e_{t+1}: h(z_t, e_{t+1}) \in A\});\ A \in \underset{\sim}{Z}.$$

Intuitively, $K(z_t, A)$ is the probability of the economy being in the subset A of Z at time t+1 given that the economy is in state z_t at time t.

Formally, a transition law must satisfy two requirements:

i) For all $z \in Z$, $K(z,\cdot)$ is a probability on the measurable space $(Z,\underset{\sim}{Z})$.

ii) For all $A \in \underset{\sim}{Z}$, $K(\cdot,A)$ is a measurable function from $(Z,\underset{\sim}{Z})$ into $[0,1]$.

<u>Proposition 6.2</u>. $K: Z \times \underset{\sim}{Z} \to [0,1]$ is a transition law.

Proof. $K(z_t,\cdot)$ is the distribution of the random variable $h(z_t,\cdot)$ with respect to the probability $Q(\pi_w(z_t),\cdot)$, and is therefore itself a probability. It remains only to show condition ii).

Let $\underset{\sim}{Z} \times \underset{\sim}{W}$ be the smallest σ-algebra of $Z \times W$ containing all sets of the form $A \times B$ with $A \in \underset{\sim}{Z}$, $B \in \underset{\sim}{W}$. Since Z and W are separable, $\underset{\sim}{Z} \times \underset{\sim}{W} = Z \underset{\sim}{\times} W$. Let $A \in \underset{\sim}{Z} \times \underset{\sim}{W}$, and define

$$A_z = \{e\colon (z,e) \in A\}.$$

We establish first the following lemma:

<u>Lemma 6.1.</u> For any $A \in \underset{\sim}{Z} \times \underset{\sim}{W}$, $Q(e,A_z)$ is $\underset{\sim}{Z} \times \underset{\sim}{W}$ measurable.

Proof. Let $\underset{\sim}{A}$ be the subset of $\underset{\sim}{Z} \times \underset{\sim}{W}$ consiting of those sets A for which the lemma is true. We will show that $\underset{\sim}{A}$ contains the Boolean algebra of rectangles, and also that $\underset{\sim}{A}$ is a monotone class. The monotone class lemma then implies that $\underset{\sim}{A}$ is a σ-field containing $\underset{\sim}{Z} \times \underset{\sim}{W}$. Since $\underset{\sim}{A}$ is a subset of $\underset{\sim}{Z} \times \underset{\sim}{W}$, $\underset{\sim}{A} = \underset{\sim}{Z} \times \underset{\sim}{W}$, proving the lemma.

We first show that $\underset{\sim}{A}$ contains the Boolean algebra generated by the rectangles; i.e., that it contains all sets of the form $A = B \times C$, $B \in \underset{\sim}{Z}$ and $C \in \underset{\sim}{W}$, and is closed under finite unions and intersections, and under complementation.

Let $A = B \times C$ be a rectangle. Then

$$Q(e,A_z) = \begin{cases} 0 & \text{if } e \notin B \\ Q(e,C) & \text{if } e \in B. \end{cases}$$

This function is clearly measurable. Now, let $\{A_1,\ldots,A_n\}$ be a finite collection of rectangles. We wish to show that $Q(e,\{\bigcap_{i=1}^{n} A_i\}_z)$ is measurable. Note that the operation of taking z-sections commutes with all the usual set theoretic operations of union, intersection, and complementation. The union of a finite number of rectangles is a rectangle, and so $Q(e,\{\bigcap_{i=1}^{n} A_i\}_z)$ is $\underset{\sim}{Z} \times \underset{\sim}{W}$-measurable. To show that $Q(e,\{\bigcup_{i=1}^{n} A_i\}_z)$ is measurable, we proceed by induction. We have already proved the assertion for $n = 1$. Suppose it to be true for $n-1$. We can write

$$Q(e,\{\bigcup_{i=1}^{n} A_i\}_z) = Q(e,\{\bigcup_{i=1}^{n-1} A_i\}_z) + Q(e,\{A_n\}_z) - Q(e,\{\bigcup_{i=1}^{n-1} (A_i \cap A_n)\}_z).$$

This gives $Q(e,\{\bigcup_{i=1}^{n} A_i\}_z)$ as the sum of functions which, by the induction hypothesis, are measurable. Finally, observe that the complement of a rectangle is a finite union of rectangles. Thus if A^- is a rectangle, $Q(e,A_z^c)$ is measurable. These results and the laws for combining set theoretic operations imply that $\underset{\sim}{A}$ contains the Boolean algebra generated by the rectangles.

We next show that $\underset{\sim}{A}$ is a monotone class. It suffices to show that if $A_1 \subseteq A_2 \subseteq \ldots$ is a countable increasing sequence of sets in $\underset{\sim}{A}$, then $A = \bigcup_{i=1}^{\infty} A_i$ is in $\underset{\sim}{A}$. Define $B_n = \bigcup_{i=1}^{n} A_i$. For any fixed e, $\lim_{n\to\infty} Q(e,\{B_n\}_z) = Q(e,A_z)$ because taking z-sections commutes with unions, and because $Q(e,\cdot)$ is a probability. Thus $Q(e,A_z)$ is the pointwise limit of measurable functions, and therefore itself measurable. $\underset{\sim}{A}$ is then a monotone class, and the proof of the lemma is complete.

If $A \varepsilon \underset{\sim}{Z}$, then $h^{-1}(A) \varepsilon \underset{\sim}{Z} \times \underset{\sim}{W}$. Recalling (1),

$$K(z,A) = Q(\pi(z), h^{-1}(A)_z).$$

$K(z,A)$ is thus the composition of the map $Q(e,h^{-1}(A)_z)$, which is measurable by the lemma, and the map $Z \to Z \times W$ given by $z \to (z,\pi_w(z))$, which is continuous. Therefore, $K(z,A)$ is measurable with respect to. $\underset{\sim}{Z}$.

Q.E.D.

Section 7.

In this section we state and prove our main result.

Theorem 7.1. There exists a relatively open and dense set of transition probabilities for MTE's that are both ergodic and stable.

We begin with some notation. MOP(Z) is, as before, the set of Markov operators on B(Z). We let MTE(Z) be the subset of all Markov operators on B(Z) that arise from Markov Temporary Equilibrium processes. Denote by $B(Z\times W,Z)$ the set of measurable maps from $Z \times W$ into Z. Let TE be the subset of all maps in $B(Z\times W,Z)$ that describe the deterministic evolution of an economy satisfying A.1-A.5. Let MOP(W) be the set of Markov operators on W. Both MOP(W) and MOP(Z) are endowed with the relative strong operator topology. We define a map $\lambda : \mathrm{MOP}(W) \times B(Z\times W,Z) \to \mathrm{MOP}(Z)$. The map λ is defined by the construction of section 5. Choose $T \in \mathrm{MOP}(W)$, $f \in B(Z\times W,Z)$. Identify T with its transition probability Q. Now define the transition probability K on $Z \times \underset{\sim}{Z}$ by

$$K(z,A) = Q(\pi_W(z),\ f_z^{-1}(A));\ f_z(e) = f(z,e).$$

Then identify K with its associated Markov operator. With these definitions it is clear that $\mathrm{MTE}(Z) = \lambda(\mathrm{MOP}(W) \times \mathrm{TE})$. Finally, we define the map $\lambda_f : \mathrm{MOP}(W) \to \mathrm{MOP}(Z)$ by the equation

$$\lambda_f(T) = \lambda(T,f).$$

The set we will show to be relatively open and dense is the set $\mathrm{MTE}(Z) \cap M$. Recall from section 2 that M is the set of all linear operators on B(Z) having for some iterate a contraction mapping. In section 2, it was shown that $\mathrm{MTE}(Z) \cap M$ is open. It remains only to show that $MTE(Z) \cap M$ contains a dense subset. In fact, we will construct such a set. Using what we know about the set TE, we can construct a dense subset D of $MOP(W)$ with the property that for all $h \in TE$, $\lambda(D \times \{h\})$ $MTE(Z) \cap M$. Our result then follows from the fact that λ_n is a continuous map.

First we construct the set D. Our construction makes use of the "no trade" equilibrium that arises when agent 1 receives the entire aggregate endowment, and all other agents receive nothing. We define the endowment allocation $e^* = (\bar{e},0,\ldots,0)$, and can now identify D.

<u>Definition 7.1</u>. Let $D = \{T \in MOP(W)$: for all $\nu \in cap(W)$, $(T^*\nu)(\{e^*\}) \geq k^{1/2}$ for some $k > 0\}$.

In other words, $Q(e,\{e^*\}) \geq k^{1/2}$ for all $e \in W$.

Proof of Theorem 7.1.

We prove the theorem in a sequence of lemmas following the outline given above.

<u>Lemma 7.1</u>. Let $T \in \lambda(D \times TE)$. Then T^2 is a contraction with norm $(1-k)$.

Proof. Choose an arbitrary initial state $z_0 = (x_{-1},p_0,e_0)$. x_0 is then determined by the agents' demand functions. Now suppose in period one that endowment allocation e^* is observed. The equilibrium price will be $p(x_0,e^*) \in \text{Int } \Delta^{L-1}$. Then $z_1 = (x_0,p(x_0,e^*),e^*)$. In this state, because all prices are positive and because only one agent has positive wealth, $x_1 = e^*$. Now suppose endowment allocation e^* is again observed. $p_2 = p(e^*,e^*)$, and so $z_2 = (e^*,p(e^*,e^*),e^*) = z^*$. We have shown that, independent of the initial state, the state z^* will be observed in period 2 if the endowment allocation e^* is observed twice in succession.

Now suppose that $T = \lambda(V,h)$, $(V,h) \in D \times TE$, and let k be the bound assoicated with V. We have then shown that $K^2(z,\{z^*\}) \geq k$. ($K^2(z,A)$ is the transition probability which assigns a probability to observing the event A in period $t + 2$ given that state z occurs in period t. K^2 is defined by the Chapman-Kolmogorov equation, and thus will determine T^2 in the same manner in which K determines T.) We now show that the Markov operator associated with K^2 on $ba(z)$, T^{*2}, is a contraction mapping.

We have for all $\nu \in cap(z)$, $(T^{*2}\nu)(\{z^*\}) = \int K^2(z,\{z^*\})\ \nu\ (dz) \geq k$. Let L^* be the Markov operator on $ba(Z)$ associated with the transition probability $K'(z,\{z^*\}) = 1$. Define a map $K'' : Z \times \underset{\sim}{Z} \to \mathbb{R}$ by

$$K''(z,A) = (1-k)^{-1}(K^2(z,A) - kK'(z,A)); \quad A \in \underset{\sim}{Z}$$

For each $A \in \underset{\sim}{Z}$, $K''(\cdot,A)$ is measurable. If $z^* \in A$, then $kK'(z,A) = k$ and $K^2(z,A) \geq k$. Thus $K''(z,A) \geq k$. If $z^* \notin A$, then $K'(z,A) = 0$, and $K^2(z,A) \geq 0$. Thus $K''(z,A) \geq 0$. $K''(z,Z) = (1-k)^{-1}(K^2(z,Z) - kK'(z,Z)) = (1-k)^{-1}(1-k) = 1$. Thus $K''(z,\cdot)$ is a probability distribution, and so K'' is a transition probability. Let M^* be its associated Markov operator on $ba(Z)$. We have $T^{*2} = kL^* + (1-k)M^*$. Let $\mu,\eta \in ba(Z)$. $||T^*\mu - T^*\eta|| \leq k||L^*\mu - L^*\eta|| + (1-k)||M^*\mu - M^*\eta|| \leq (1-k)||M^*||\,||\mu-\eta|| = (1-k)||\mu-\eta||$. The second inequality holds because $L^*\mu = L^*\eta = \delta_{z^*}$. Since $k > 0$, T^{*2} is a contraction mapping.

Q.E.D.

Thus $\lambda(D \times TE) \subset MTE(Z) \cap M$.

Lemma 7.2. D is dense in $MOP(W)$.

Proof. Let $T \in MOP(W)$ and $V \in D$. Thus $(V\nu)(\{e^*\}) \geq K^{1/2}$ for some $k > 0$ and all $\nu \in cap(W)$. Define $T_\alpha = \alpha V + (1-\alpha)T$.
$(T_\alpha \nu)(\{e^*\}) = \alpha(V\nu)\{e^*\} + (1-\alpha)(T\nu)\{e^*\} \leq \alpha k^{1/2}$ for all $\nu \in cap(W)$. Thus, for $\alpha > 0$, $T_\alpha \in D$. Density is then established by noting that $\lim_{\alpha \to 0} T_\alpha = T$.

Q.E.D.

<u>Lemma 7.3</u>. The map $\lambda_h : MOP(W) \to MOP(Z)$ is continuous for all $h \in B(Z\times W, Z)$.

Proof. Lemma 2.1 equates norm convergence of Markov operators with uniform convergence of transition probabilities. Thus we can work directly with transition probabilities. Let $\{T_n\}$ MOP(W) and $T_n \to T$. Let $\{K_n\}$, K be the transition probabilities associated with $\{\lambda_h(T_n)\}$, $\lambda_h(T)$. Then

$$0 \leq \sup_{\substack{z\in Z \\ A\in \underset{\sim}{Z}}} | K_n(z,A) - K(z,A)| = \sup_{\substack{z\in Z \\ A\in \underset{\sim}{Z}}} | Q_n(\pi_W(z),h^{-1}(A)) - Q(\pi_W(z),h_z^{-1}(A))|$$
$$\leq \sup_{\substack{e\in W \\ B\in \underset{\sim}{W}}} | Q_n(e,B) - Q(e,B)|.$$

$T_n \to T$ implies that the last term in the inequality converges to 0.
Thus $\lambda(T_n,h) \to \lambda(T,h)$.

Q.E.D.

Now to prove the theorem. Openness of $MTE(Z) \cap M$ was established in Corollary 2.1. To establish that $MTE(Z) \cap M$ is dense in $MTE(Z)$, choose $V \in MTE(Z)$. Then $V = \lambda(T,h)$ for some $(T,h) \in MOP(W) \times TE$. Since D is dense in MOP(W) we can choose a sequence $\{T_n\} \subset D$, $T_n \to T$. Letting $V_n = \lambda_h(T_n)$, it follows from Lemma 7.3 that $V_n \to V$. But $\{V_n\} \subset \lambda(D \times TE) \subset MTE(Z) \cap M$.

Q.E.D.

This result is very encouraging. It says that stability and ergodicity are generic properties of Markov Temporary Equilibrium Processes derived under assumptions A.1-A.5 on agents' behavior. A word of caution is in order here. This does not say that an open and dense set of the stochastic processes of TE states derivable under our assumptions has these properties. Our theorem is only a statement about transition laws. It seems that no inference can be made in this direction, since the map taking transition laws and initial distributions of states into stochastic processes is not continuous when the set of stochastic processes is topologized in the usual manner.

This is not surprising, and does not affect the interest of our results. The observable object that our theory constructs is a transition law. Economic theory has no more to say about the process than what is embodied in the transition law, since it does not provide a theory of the initial distribution of states. However, the transition law is the appropriate tool for studying the asymptotic behavior of any Markov process, and this is what we are concerned with when we attempt to justify stationarity as an equilibrium concept.

Finally, one other aspect of Theorem 7.1 should be noted. The result has implications concerning the set of admissable agents's behaviors, even though the map from preferences into transition probabilities has no continuity properties.

Corollary 7.1. For every set of agent's utility functions, expectations, consumption sets and discount rates satisfying A.2-A.5, there exists an open dense set of the transition probabilities generated by these characteristics of agents that is contined in MTE'.

Proof. Apply Lemmas 7.2 and 7.3 as in the proof of Theorem 7.1.

Q.E.D.

Given any transition probability in MTE(Z), we can approximate it by a transition probability in MTE' that is generated by the same behavior on the part of agents. Only the transition probability for endowment allocations need be varied. This is to say, pathologies are caused by the endowment process, and not by agents' behaviors.

Section 8.

The model introduced in Section 2 can be extended to include money in the manner of Grandmont and Hildenbrand [1977]. In these models there is an $L + 1$'st good, money, whose sole purpose is to act as a means of transferring wealth between market periods. Denote money by m, and its price by s. Normalize prices such that $(p,s) \in \Delta^L$.

Consider, for convenience of exposition, an overlapping generations economy in which each agent lives for two periods. In any time period there is one young and one old agent of each type. Old agents solve the following consumer choice problem:

$$\max U(x_{t-1}, x_t)$$
$$\text{s.t. } p_t x_t \le p_t e_t + s_t m_{t-1}.$$

m_{t-1} is the money holdings purchased in period $t-1$. Demand is given by $x_t = x(x_{t-1}, p_t, e_t, s_t, m_{t-1})$. The indirect utility function is $V(x_{t-1}, m_{t-1}, p_t, s_t, e_t)$.

The young agent has expectations $\psi : \Delta^2 \times W \to M(\Delta^2 \times W)$. Again, $\psi(p,s,e)(\partial\Delta^2 \times W) = 0$. This will ensure that money will be demanded when current wealth is strictly positive. The young agent's optimization problem is:

$$\max_{x_t, m_t} \int V(x_t, m_t, p_{t+1}, s_{t+1}, e_t)\ \psi(p_t, s_t, e_t)(dp_{t+1} \times ds_{t+1} \times de_{t+1})$$

$$\text{s.t.} \quad p_t x_t + s_t m_t \le p_t e_t.$$

Note that old agents will never demand money because to them the marginal utility of further wealth is 0. Fix the money supply at M^*. It can now be seen that this model is formally identical to the model of Section 2. The analogs of all results in Sections 2-5 are true here.

When the price of money is 0, there is ambiguity about how M^* should be distributed among young agents. We will exploit this ambiguity to create an "absorbing state" for the MTE.

Define $z_t = (x_{t-1}, m_{t-1}, p_t, s_t, e_t)$ to be an element of the state space Z. We can define a state correspondence $H : Z \times W \to Z$, by applying the equilibrium price correspondent to the demand functions. H has the property that $s_{t+1} = 0$ iff $e_{i,t+1} = 0$ for all young agents. Furthermore, when $e_{i,t+1} = 0$, $m_{i,t+1} = 0$ for all young agents. At equilibrium we have an excess supply of money. We modify the correspondence by assuming that when $s_{t+1} = 0$, each agent inherits the money stock of the old agent of his type. Let $\mathring{W} = \{e \in W | e_i > 0$ for some young agent$\}$. H is u.h.c. on $Z \times \mathring{W}$, and also on $Z \times W/\mathring{W}$. Each set is Borel. Thus, by applying the selection theorem of Kuratowski and Ryll-Nardzewski to each set separately, and observing that the selections can be pasted together in a measurable manner, it follows that we can choose a selection $h : Z \times W \to W$ that is measurable and with the property that when $s_t = 0$, each young agent inherits his parent's money stock.

Let us first assume that the measurable selection choosen is such that if the price of money is 0, then each young agent inherits the money supply of the old agent of his type. (It can easily be shown that this allocation is in the correspondence.) Suppose that there are J types of agents, one young and one old of each type. (Types are distinguished by utility functions.)

Suppose that we are in state z_0, and that e_1 gives total endowment $\bar{e}$ to the young agent of type k. Assume that for all young agents, $\mu \in \operatorname{Im} \Psi_i$ implies that μ assigns positive probability to the event $s > 0$. Then each young agent's preference for (x_t, m_t) is strictly monotone for all (p_t, s_t, e_{it}). Then each young agent other than type k conserves $(x_t, m_t) = (0,0)$. The young agent of type m receives money M^*. Suppose in period 2 the old agent of type n receives $\bar{e}$. Then the price of money is 0, and all other prices are positive. All young agents conserve $(0,0)$, except that of type k conserves $(0, M^*)$. All old agents conserve $(0,0)$, except that of type k, who conserves $(\bar{e}, 0)$. We have equilibrium prices p^*, s^*, and it can be seen that p^*, s^* are independent of z_0. Define z^* to be this state. Let e_1^*, e_2^* be the allocations that give $\bar{e}$ to the young and old agents of type m, respectively. If for all $e \in W$, $Q(e, \{e_i^*\}) \geq k$, $i = 1, 2$; then for all $z \in Z$ $\inf \Pr(z_2 = z^* | z_0 = z) \geq k^2$. Thus there is a unique stable stationary equilibrium.

We can show that this set of transition probabilities for endowment allocations is dense. Thus, Theorem 7.1 is true in this setting as well. This analysis points out that intergenerational transfers, such as bequests, may play a stabilizing role in the economy. This question deserves serious study.

The results in Section 7 depend crucially on the assumption that only current period data is used to form expectations. If lagged endogenous variables are used in forming expectations, the situation is more complex. Consider expectations of the form $\Psi_i : \Delta^{L-1} \times W_i \times \Delta^{L-1} \times W_i \to M(\Delta^{L-1} \times W_i)$,

$\mu = \Psi(p_{t-1}, e_{i,t-1}, p_t, e_{i,t})$. An additional assumption is required to ensure the results of Theorem 7.1.

A.8. $\Psi_i(p_{t-1}, \bar{e}_0, p_t, \bar{e}) = \Psi_i(p'_{t-1}, \bar{e}, p_t, \bar{e})$ for all
$p, p' \in \Delta^{L-1}$, $i = 1, \ldots, n$.

This assumption says that an agent who receives endowment vector $\bar{e}$ twice in succession believes that next period's distribution of prices and endowments will be independent of p_{t-1}. We can tell a plausible story around the assumption: if the agent receives $\bar{e}$ twice in succession, he believes that with probability one he will receive $\bar{e}$ again. In this case the equilibrium price vector will be

$$\nabla u(\bar{e}, \bar{e}) / ||\nabla u(\bar{e}, \bar{e})||$$

which is independent of p_{t-1}. This story is not at all unreasonable, and is related to assumptions commonly employed to require the existence of steady states in Temporary Equilibrium analysis with certain endowments.

In this paper we have treated preferences as non-random. However, we could easily introduce stochastic preferences without materially charging the results. Doing so could lead to an overlapping generations model in which the length of agents lives are probablistic, but distributed according to a stationary process.

We conclude with one observation on technique. A natural conjecture given recent results on the set of possible excess demand functions consistent with utility maximization is that the set of MTE's is large. If this set is convex, Theorem 7.1 could be obtained immediately by finding one example of an economy with the properties stated in the conclusion of the theorem. Using either this method of proof or the method of Sections 6-7, it is hoped that powerful uniqueness and stability results can be obtained for many of the Markov processes arising in different economic contexts.

Appendix

In this appendix the basic results concerning the application of discounted dynamic programming to the agent's choice problem will be discussed. These results are well known, and the results here are only a slight generalization of those of Christiansen [1975].

A dynamic programming problem has six components:

Definition 1. A state space S is any non-empty Borel subset of some Euclidean space.

S is the set of states of the system. $s_t \in S$ describes the environment of the agent at time t. In our model, $S = X \times \Delta^{L-1} \times W_1$.

Definition 2. An action set X is any non-empty Borel subset of some Euclidean space.

X is the set of possible responses an agent can make. In our model, X is the agent's consumption set.

Definition 3. A feasibility constraint γ is a correspondence from the state space S to the action space X.

$\gamma(s_t)$ is the set of feasible responses to the environment s_t. γ in our model is the budget constraint.

Definition 4. A law of motion $\hat{\Psi}$ is a function from $S \times X \rightarrow M(S)$.

$\hat{\Psi}$ is the agent's perception of how the system evolves. In time period t in environment s_t, if the agent chooses action x_t, then $\hat{\Psi}(s_t,x_t)$ is a probability measure representing the agent's beliefs about s_{t+1}. In our model, $\hat{\Psi}(s_t,x_t) = \Psi(q_t) \times \delta_{s_t}$. $\Psi(q_t)$ represents the agent's beliefs about $\Delta^{L-1} \times W_i$ in time $t+1$, based on information $(A,e_{i,t})$ observed at time t. δ_{x_t} states that in time $t+1$, last period's consumption will be x_t, the action chosen at time t.

Definition 5. A return function r maps $S \times X$ into R.

Definition 6. A discount factor β is a real number, $0 \leq \beta < 1$.

The function r measures the return to the agent when, in state s_t, he chooses action x_t. In our model, $r(s_t,x_t) = U(x_{t-1},x_t)$. The behavioral assumption of our model is that agents maximize the sum of expected discounted utilities.

In choosing actions, an agent follows a plan. A plan is any rule that tells the agent how, on the basis of information the agent has, he should choose a feasible action. We formalize this action in the following manner: The set of histories of the system at time t is

$$H_t = (\Pi_{n=1}^{t-1} (S \times X)) \times S;\ h_t = (s_1,x_1,s_2,\ldots,s_{t-1},x_{t-1},s_t).$$

At time t, each agent knows h_t.

Definition 7. A plan is a sequence of maps $\pi = (\pi_1, \pi_2, \ldots)$; where each of the maps $\pi_t: H_t \to X$ satisfies $\pi(s_1, \ldots, x_{t-1}, s_t) \in \gamma(s_t)$.

Π is the set of all Borel-measurable plans.

If each π_t depends only on the t'th state, i.e. $\pi_t(h_t) = f_t(s_t)$ for all $(s_1, \ldots, x_{t-1}) \in H_{t-1} \times X$, then the plan π is Markov. If π is Markov and $f_t = f$ for all $t = 1, 2, \ldots$, then π is stationary, and we write $\pi = (f^{(\infty)})$.

To calculate the discounted sum of expected returns from plan π in initial state s_1, we compute

$$\begin{aligned} I(\pi, s_1) = {} & r(s_1, \pi(s_1)) + \beta \int r(s_2, \pi_2(s_1, \pi_1(s_1), s_2)) \hat{\Psi}(ds_2, s_1, \pi_1(s_1)) + \\ & \beta^2 \iint r(s_3, \pi_3(s_1, \pi_1(s_1), s_2(\pi_2(s_1, \pi_1(s_1), s_2), s_3)) \\ & \hat{\Psi}(ds_3, s_2, \pi_2(s_1, \pi_1(s_1), s_2)) \hat{\Psi}(ds_2, s_1, \pi_1(s_1)) + \ldots . \end{aligned}$$

Definition 8. The optimal return function $v^*: S \to \mathbb{R}$ is given by $v^*(s_1) = \sup_{\pi \in \Pi} I(\pi, s_1)$.

Definition 9. A plan π^* is optional if $I(\pi^*, s_1) = v^*(s_1)$.

This formulation is too general to ensure the existence of an optional plan. We give assumptions, satisfied by our infinite horizon model, that guarantee the existence of a stationary optimal plan.

Assumption 1.a) The correspondence γ is continuous and compact valued except on a set S' such that $\hat{\psi}(s,x)(S') = 0$ for all $(s,x) \quad S \times X$.

b) The function $g(s) = \sup_{x\varepsilon\gamma(s)} f(x)$ is Borel-measurable if f is continuous, and the maximizer correspondence has measurable graph.

In our model, S' is the set of all states containing a zero price for some commodity. Assumption 1b) is a consequence of the argument of Lemma 3.3.

Assumption 2. If $(s_n,x_n) \to (s,x)$, then $\hat{\Psi}(s_n,x_n) \to \hat{\Psi}(s,x)$ in the topology of weak convergence.

In our model the validity of Assumption 2 is guaranteed by A.4.

Assumption 3. r is bounded and continuous.

Assumption 3 is a consequence of A.2 and A.3.

First we demonstrate the continuity properties of the optimal return function. Let $B(S)$ be the space of bounded measurable functions from S to $\mathbb{R}$, with the topology of uniform convergence. $B'(S)$ is the subspace of $B(S)$ continuous on the complement of S' in S.

Note that $B'(S)$ is complete, since $B(S)$ is complete and $B'(S)$ is closed in $B(S)$.

Define the operator $T: B(S) \to B(S)$ by

$$Tw(s_1) = \sup\{r(s_1,x_1) + \beta\int w(s_2)\hat{\Psi}(ds_2,s_1,x_1);\ x_1 \in \gamma(s_1)\} .$$

We begin by demonstrating that $B'(S)$ is an invariant subspace under T .

Proposition 1. $T: B'(S) \to B'(S)$

Proof. Let $w \in B'(S)$. Let $g(s_1,x_1) = r(s_1,x_1) + \beta\int w(s_2)\hat{\Psi}(ds_2,s_1,x_1)$. Suppose that g is continuous. Assumption 1b) guarantees then that $Tw: S \to \mathbb{R}$ is measurable, and the maximum theorem guarantees that Tw is continuous on S' . Thus it suffices to show that g is continuous. Let $(s_{1n},x_{1n}) \to (s_1,x_1)$. From Assumption 3, $r(s_{1n},x_{1n}) \to r(s_1,x_1)$. From Assumption 2, $\hat{\Psi}(s_{1n},x_{1n}) \to \hat{\Psi}(s_1,x_1)$. w is continuous except on a set of $\hat{\Psi}(s_1,x_1)$-measure 0 , so, applying Billingsley [1968] Theorem 5.2, iii), $\int w(s_2)\hat{\Psi}(s_{1n},x_{1n}) \to \int w(s_2)\hat{\Psi}(s_1,x_1)$. Thus g is continuous.

Q.E.D.

Proposition 2. T is a contraction mapping.

Proof. Apply the argument of Proposition 4.3.

Q.E.D.

Since T is a contraction and $B'(S)$ is complete, T has a unique fixed point in $B'(S)$. We shall show that the optimal return function r^* is this fixed point.

Proposition 3. $Tv^* = v^*$.

Proof. Our assumptions are sufficient to guarantee that v^* is measurable. (See Hinderer [1970], section 13.) Let $\pi \in \Pi$. $I(\pi, s_1) = r(s_1, \pi_1(s_1)) + \beta\int W(\pi, s_2)\hat{\Psi}(ds_2, s_1, x_1)$, where $W(\pi, s_2)$ is the expected return from time 2 onward given that plan π is followed and that the state at time 2 is s_2. $W(\pi, s_2) \leqslant v^*(s_2)$ for all s_2. Thus $I(\pi, s_2) \leqq r(s_1, \pi_1(s_1)) + \beta\int v^*(s_2)\hat{\Psi}(ds_2, s_1, \pi_1(s_1)) \leqslant \sup\{r(s_1, x_1) + \beta\int v^*(s_2)\hat{\Psi}(ds_2, s_1, x_1);\ x_1 \in \gamma(s_1)\} = Tv^*(s_1)$. Since the choice of was arbitrary; $v^*(s_2) \leqq Tv^*(s_1)$.

We next show that $v^* \geqslant Tv^*$. Fix $s_1 \in S$ and $\varepsilon > 0$. Choose x_1^* such that $r(s_1, x_1^*) + \beta\int v^*(s_2)\hat{\Psi}(ds_2, s_1, x_1^*) \geqslant \sup\{r(s_1, x_1) + \beta\int v^*(s_2)\hat{\Psi}(s_1, x_1);\ x_1 \in \gamma(s_1)\} - \varepsilon$. Let π^* be a plan which chooses x_1^* at time 1, and is within ε of maximizing expected total discounted returns from time 2 onward, where the expectation of the optimal return function in period 2 is taken with respect to $\hat{\Psi}(s_1, x_1^*)$. Such a plan always exists. (See Blackwell [1965], Theorem 6b.) Then $I(\pi, s_1) = r(s_1, x_1^*) + \beta\int I(\pi^*, s_2)\hat{\Psi}(ds_2, s_1, x_1^*) \geqq r(s_1, x_1^*) + \beta\int v^*(s_2)\hat{\Psi}(ds_2, s_1, x_1) - \beta\varepsilon$. $v^*(s_1) \geqslant I(\pi, s_1)$, and so $v^*(s_1) \geqslant r(s_1, x_1^*) + \beta\int v^*(s_2)\hat{\Psi}(ds_2, s_1, x_1^*) - \beta\varepsilon$. From the way x_1^* was chosen, it follows that $v^*(s_1) \geq \sup\{r(s_1,x_1) + \beta\int v^*(s_2) - (1+\beta)\varepsilon;\ x_1 \in \gamma(s_1)\}$. Since ε is arbitrary, this gives $v^*(s_1) \geq Tv^*(s_1)$. Therefore $v^* = Tv^*$.

Q.E.D.

Now we can demonstrate the existence of a stationary optimal plan. For every $f \in B(S)$, define the operator $T_f\colon B(S) \to B(S)$ by

$$T_f w(s_1) = r(s_1,f(s_1)) + \beta\int w(s_2)\hat{\Psi}(ds_2,s_1,f(s_1))\ ;$$

Proposition 4. T_f is a contraction mapping on $B(S)$ with fixed point $I(f^{(\infty)})$.

Proof. To see that $T_f w$ is measurable, note that $f(s_1,f(s_1))$ is the composition of a continuous function and a measurable function, and hence measurable. Suppose now that w were an indicator function for a Borel set A. Then $\int w(s_2)\hat{\Psi}(ds_2,s_1,f(s_1)) = \hat{\Psi}(s_1,f(s_1))(A)$. The evaluation map $\lambda_A\colon M(S) \to [0,1]$ given by $\lambda_A(\mu) = \mu(A)$ is measurable for each Borel set A. The map $\hat{\Psi}\colon S \to M(S)$ given by $\hat{\Psi}(s_1,f(s_1))$ is measurable, as the composition of a measurable and a continuous function. Thus the map $\hat{\Psi}(s_1,f(s_1))(A)$ is the composition of two measurable maps, and hence measurable. Finally, note that the characteristic functions form a basis for $B(S)$. Any $w \in B(S)$ can be expressed as the uniform limit of finite linear combinations of characteristic functions; and thus, for all $w \in B(S)$, $\int w(s_2)\hat{\Psi}(ds_2,s_1,f(s_1))$ is measurable, and therefore in $B(S)$.

To see that T_f is a contraction, calculate $\|T_f w - T_f v\| = \sup_{s_1} |T_f w(s_1) - T_f v(s_1)| \leq \sup_{s_1} \beta \int |w(s_2) - v(s_2)| \hat{\Psi}(ds_2, s_1, f(s_1)) \leq \beta \sup_{s_2} |w(s_2) - v(s_2)| = \beta \|w - v\|$. Since $\beta < 1$, T_f is a contraction mapping. Since $B(S)$ is complete, it has a unique fixed point.

$T_f I(f^{(\infty)}, s_1) = r(s_1, f(s_1)) + \beta \int I(f^{(\infty)}, s_2) \hat{\Psi}(ds_2, s_1, f(s_1)) = I(f^{(\infty)}, s_1)$, and so $I(f^{(\infty)})$ is the unique fixed point.

Q.E.D.

Proposition 4 suggests that an optimal stationary plan is given by any f such that $f(s) \in \gamma(s)$ for all s and $T_f v^* = v^*$.

<u>Proposition 5</u>. There exists an optimal stationary plan $(f^{(\infty)})$.

Proof. Let $g(s_1, x_1) = r(s_1, x_1) + \beta \int v^*(s_2) \hat{\Psi}(ds_2, s_1, x_1)$. Since $v^* \in B'(S)$, g is continuous. Let $F(s_1) = \{x_1^* \in \gamma(s_1) : g(s_1, x_1^*) = \sup_{x_1 \in \gamma(s_1)} g(s_1, x_1)\}$. By Assumption 1b), F has measurable graph, and it is closed valued. Thus, applying the selection theorem of Kuratowski and Ryll-Nardzewski [1965], we may choose a measurable selection $f: S \to X$ satisfying $f(s_1) \in \gamma(s_1)$, and $r(s_1, f(s_1)) + \beta \int v^*(s_2) \hat{\Psi}(ds_2, s_1, f(s_1)) = Tv^*$. The expression on the left is $T_f v^*$, so $T_f v^* = Tv^*$. From Proposition 3, $Tv^* = v^*$. Thus $T_f v^* = v^*$. $T_f v^*$ has a unique fixed point $I(f^{(\infty)})$. Thus $I(f^{(\infty)}) = v^*$, and $f^{(\infty)}$ is an optimal plan.

Q.E.D.

1. This is the same topology as that given by the bijection to ba(Z), since the map $T \to T^*$ is an isometric isomorphism. We make free use of this fact in the sequel.

2. $$||T^*\nu-\nu|| = \lim_{n\to\infty} ||\frac{1}{n}\ \Sigma_{i=1}^{n}\ T^{*i}\nu - \frac{1}{n}\ \Sigma_{i=0}^{n-1}\ T^{*i}\nu||$$

$$\leq \lim_{n\to\infty}\ \frac{1}{n}\ ||T^{*n}\nu - \nu||$$

$$\leq \lim_{n\to\infty}\ \frac{2}{n}||\nu|| = 0.$$

3. Let K be a transition probability, and let I be the set of probability distributions invariant under K. A Markov process (K,μ) is stationay iff $\mu \varepsilon I$. The set I is convex (T^* is linear). A Markov process (k,μ) is ergodic iff μ is an extreme point of I, Our definition is justified by noting that if $I = \{\mu\}$, then every stationary process with transition probability K must be ergodic.

4. See Billingsley [1968] for a discussion of this topology.

BIBLIOGRAPHY

Blackwell, David, "Discounted Dynamic Programming," Annals of Mathematical Statistics 36 (1965), 226-235.

Blume, Lawrence, "On the Stationarity of Markov Processes of Temporary Equilibrium," unpublished Ph.D. thesis, University of California (Berkeley), 1977.

Christiansen, Daniel, "Temporary Equilibrium: A Stochastic Dynamic Programming Approach. Part I: The Stationary Case," Working Paper 75-17, Department of Economics, University of Rochester, August 1975.

Denardo, Eric V., "Contraction Mappings in the Theory Underlying Dynamic Programming," SIAM Review 9 (1967), 155-177.

Grandmont, Jean Michel, and Werner Hildenbrand, "Stochastic Processes of Temporary Equilibrium," Journal of Mathematical Economics 1 (1974), 247-278.

Hildenbrand, Werner, Core and Equilibria of a Large Economy, Princeton, 1974.

Hinderer, K., Foundations of Non-Stationary Dynamic Programming with Discrete Time Parameter, Berlin, 1970.

Kuratowski, C., and C. Ryll-Nardzewski, "A General Theorem on Selectors," Bulletin of the Polish Academy of Sciences 13 (1965), 397.

Radner, Roy, "Optimal Stationary Consumption with Stochastic Production and Resources," Journal of Economic Theory 6 (1973), 68-90.

Yosida, ˇsaku, and Shizuo Kakutani, "Operator Theoretical Treatment of Markov Process and Mean Ergodic Theorem," Annals of Mathematics 2 (1941), 188-228.