Center for Research on Economic and Social Theory

and

Department of Economics

Working Paper Series

Electoral Systems, Legislative Process and Income Taxation

Yan Chen

September, 1994
Number 94-08

DEPARTMENT OF ECONOMICS
University of Michigan
Ann Arbor, MI 48109-1220

Electoral Systems, Legislative Process and Income Taxation

Yan Chen

Department of Economics

University of Michigan

Ann Arbor, MI 48109-1220

JEL Classification Code: H21, D72, D78

September 21, 1994

[1] I would like to thank John Ledyard and Kim Border for all their advice and encouragement, Kay-Yut Chen, Glen George, Ehud Kalai, Weimin Lu, John Matsusaka, Richard McKelvey, Scott Page, Jay Wilson and Kwan Koo Yun for their comments and discussions, and Andrei Sherstyuk for his help on computer programming. A special note of thanks goes to Miguel Gouveia for his constructive criticism on an earlier version of the paper.

Abstract

We examine the effects that political institutions, i.e., electoral systems and legislative processes, have on income taxation and public good allocation. We characterize the equilibrium income tax schedules and the optimality conditions under two types of political institutions, a two party plurality system with a single district, and one with multiple districts where the tax policy is determined through a legislature. It is shown that the exogenous social welfare functions in the optimal taxation literature can be endogenously determined by explicitly modelling the political institutions, which put different welfare weights on different subsets of the population.

JEL Classification Code: H21, D72, D78

Key Words: equilibrium income tax schedule, legislative game, voting.

1 Introduction

The goal of this paper is to understand the effect of political institutions on income tax structures and the level of public goods provided and, in doing so, to merge the economics of the optimal income taxation approach with political science models of voting and legislative choice.

The optimal income taxation literature, starting from Mirrlees (1971), studies the features of income tax schedules, which arise when a social planner maximizes an exogenously given social welfare function, subject to incentive compatibility constraints and an exogenously given revenue requirement. These models have some good features: (1) they recognize that individuals have different productivity, or wage rates; (2) individual labor supply depends on the tax schedule, so incentive effects are taken into consideration; (3) most of them start with unrestricted tax schedules, without *a priori* limitations. The main shortcomings to these models are the neglect of institutional constraints and the exogeneity of the social welfare function. In practice, most public policies concerning income taxation and public goods provision are determined through political institutions, such as direct democracy or legislative processes. We will see that by incorporating these institutional features, social welfare functions can be endogenously determined.

There exists a relatively small literature (Roberts 1977, Kramer and Snyder 1988, Cukierman and Meltzer 1991, Berliant and Gouveia 1991, etc.) that models income tax schedules as the outcome of political processes. But all of these researchers only model simple majority rule. And nearly all results focus on the median voter. Due to

the nonexistence of majority rule equilibrium when the dimension of the issue space exceeds one[1], these models either start with a restricted set of tax schedules, such as a linear tax, or put restrictions on the environment. And most of them abstract from the economics and incentive problems inherent in the income tax problem.

This paper tries to combine the more realistic features of both literatures. Individuals in the economy have different productivity/wage rates; their labor supply depends on the tax schedule, and therefore incentive effects are incorporated. We do not restrict the class of tax schedules so that the tax schedule is the result of the forces caused by political institutions. We compare two types of political institutions: a two-party plurality system under single district, i.e., simple majority rule, and a two-party plurality system under multiple districts with a legislature deciding the final policy outcome.

By explicitly modelling the political institutions, we can characterize the equilibrium tax schedules and conditions under which they are optimal, and thereby endogenously determine the social welfare function. Under plurality rule, the equilibrium tax schedule of two candidate competition (the single district scenario) is compared with the equilibrium outcome from a legislative process when there are multiple districts. We establish that each equilibrium is equivalent to an optimal tax schedule for some welfare weights. Furthermore, we show the equilibrium which arises in a two-candidate, single-district competition puts equal welfare weight over the whole population, while the equilibrium tax schedule of the legislative process puts more weight on those subsets of the population whose legislators are in the majority coalition.

In Section 2 we construct a general equilibrium model where the amount of public

[1]See McKelvey 1979.

good level is endogenously determined. Section 3 includes a survey of voting models, with special emphasis on the probabilistic voting model and an extension of the equilibrium result to a general functional space. Section 4 presents a characterization of the equilibrium income tax schedules under two party plurality system for a single district, and that of a stochastic legislative game when there are multiple districts. Optimality conditions for these equilibria are also determined, thus establishing the relationship between these positive models and traditional optimal income taxation models. In Section 5 we present a numerical example of the equilibrium income tax under the two political systems. Section 6 concludes the paper.

2 The Model

A general equilibrium model is constructed in which the amount of public good level is endogenously determined. The general problem analysed in this section uses a framework similar to that of Mirrlees (1971), but includes a public good, financed by the tax revenue instead of having an exogenous revenue requirement[2]. This model serves as a building block for the latter part when we introduce the political institutions. It turns out that the two political institutions we consider will be special cases of the optimal tax model, in the sense that the equilibrium tax schedules from political processes are as if some social welfare functions are maximized.

Suppose individuals are identified by a single parameter, $\omega \in \Omega_0 = [\omega_0, \bar{\omega}] \subset R_{++}$, which can be interpreted as the wage rate or ability level of an individual. Assume

[2]Brito and Oakland (1977) also model a public good in their optimal income taxation model, but it is not explicitly financed by the tax revenue.

that $\omega \sim F(\cdot)$, and that ω has a density function $f(\omega)$, and $f(\omega) > 0$ a.s. on Ω_0. Call an individual whose ability-parameter is ω a ω-person. The individual parameter, ω, is private information, but its distribution is common knowledge. There are three commodities: a consumption good, $x \in R_+$, labor, $l \in [0,1)$, and a public good, $y \in R_+$. Let $I(\omega) = \omega l$ be the income of the ω-person. The utility function, $u(x,l,y)$ satisfies the following assumptions.

Assumption 1 *$u(x,l,y) = x + \nu(l,y)$, where $\nu(\cdot,\cdot)$ is concave, C^3, $u_2 = \nu_1 < 0$, $u_3 = \nu_2 > 0$, and satisfies the Inada conditions:*

$$\lim_{l\to 1} u_2(x,l,y) = -\infty; \quad \lim_{y\to 0} u_3(x,l,y) = \infty.$$

Assumption 2 *The marginal utility for private good consumption decreases with an increase of labor; the marginal utility of leisure is convex.*

$$u_{211} \leq 0; \quad u_{222} \leq 0.$$

Assumption 2 is introduced to avoid bunching of individuals when using the first order approach to solve the optimal taxation problem.

Let $I(\omega) = \omega l$ be the income of the ω-person. Then $I : \Omega \to \mathcal{I}$ is defined as the *income function*, where $\mathcal{I} \subset \mathcal{R}$ is the set of all possible incomes. Let $\mathcal{T} \subset \mathcal{R}$ be the set of all possible taxes. Define $T : \mathcal{I} \to \mathcal{T}$ as the *income tax function*.

Assumption 3 *The income tax function, $T(I)$, is lower semicontinuous.*

We use the revelation principle to analyse the general equilibrium optimal taxation problem[3].

Define a *revenue requirement function*, $\tau : \Omega \to \mathcal{T}$. The problem of taxation of income (the indirect mechanism) is transformed to the direct mechanism: an agent reports his type, ω, based on which he is required to have income, $I(\omega)$, and pay taxes, $\tau(\omega)$. We want to find a tax function T that implements τ in the sense that $T(I(\omega; T)) = \tau(\omega)$. The revenue requirement function satisfies the following assumption:

Assumption 4 *The revenue requirement function, $\tau : \Omega \to \mathcal{T}$, is lower semicontinuous, and bounded below, i.e.,*

$$\tau(\omega) > -\int_{\Omega} \omega dF(\omega).$$

In order to implement the revenue requirement function, $\tau(\omega)$, by means of an income tax, $T(I)$, we need a monotonicity condition.

Lemma 1 (Monotonicity) *Assumption 1 is sufficient to ensure $I(\omega)$ an increasing function, and therefore to implement $\tau(\omega)$ by means of an income tax.*

Proof. From Assumption 1, $u(x, y, l)$ is C^3, and

$$u(x, y, l) = \nu(l, y) + \omega l - \tau \equiv V(\omega, l, y) - \tau.$$

Therefore the Spence-Mirrlees Condition is satisfied, i.e.,

$$\frac{\partial^2 V}{\partial \omega \partial l}(\omega, l) = \frac{\partial}{\partial \omega}(\omega + \nu_l) = 1 > 0.$$

[3] I thank Miguel Gouveia for pointing out this approach to me. See Berliant and Gouveia (1992).

From Proposition 1 of Rochet '87, $l(\cdot)$ is rationalizable, i.e., $(l(\cdot), \tau(\cdot))$ is truthfully implementable in dominant strategies, if and only if $l(\cdot)$ is nondecreasing.

Since $I(\omega) = \omega l(\omega)$, and $l(\omega) \in [0, 1)$, $I(\omega)$ is increasing except possibly in the interval $[\omega_0, \underline{\omega}]$, where $I(\omega) = 0$. In this model, we treat the flat interval as one point, i.e., $\tau(\omega)$ is the same for all $\omega \in [\omega_0, \underline{\omega}]$. Therefore we can concentrate on the interval $[\underline{\omega}, \overline{\omega}] \equiv \Omega$, where $I(\omega)$ is increasing. Then we can invert the income function, $I(\omega)$, and get $\omega = \eta(I)$, and therefore, $T(I) = \tau(\eta(I))$, so we can implement a revenue requirement function by an income tax function. ∎

Lemma 1 shows that, in equilibrium, after all behavioral adjustments, income must be an increasing function of ability.

Given a revenue requirement function, $\tau(\omega)$, and an income function, $I(\omega)$, a ω-person chooses to report his type, ω', to maximize his utility,

$$\max_{\omega'} u(I(\omega') - \tau(\omega'), \frac{I(\omega')}{\omega}, y).$$

Solving this problem gives us an individual's optimal reported type, ω, and thus, his optimal amount of income, $I(\omega)$, his optimal supply of labor, $l(\omega)$, and the individual's private good consumption, $x(\omega) = I(\omega) - \tau(\omega)$. The total supply of labor adjusted for quality is $L^s = \int \omega l(\omega) dF(\omega)$, the aggregate demand for the private good is $X^d = \int_{\Omega} x(\omega) dF(\omega)$, and the total tax revenue is $\int_{\Omega} \tau(\omega) dF(\omega)$.

On the production side, assume that firms are price-takers. The input for the production of the private good is labor which, adjusted for quality, equals $L = \int_{\Omega} \omega l(\omega) dF(\omega)$. The public good is produced from the private good.

Assume that all firms are identical and that they maximize profit by choosing the optimal amount of labor input in the production of the private good and the public good. The production functions of the private good and the public good are assumed to be linear. The total amount of private good produced is $X^T = aL$, and the total amount of public good produced from the private good is $y = b(X^T - X)$. Normalize the price of the private good to 1, and let the price of the public good be p. The firm's problem can be expressed as the following,

$$\begin{aligned} \max \quad & X^s + py - L^d \\ s.t. \quad & X^T = aL^d \\ & y = b(X^T - X^s). \end{aligned}$$

In equilibrium, the firm's profit is zero, and demand equals supply in all markets. So we have

$$a(X^s + py) = X^s + \frac{y}{b},$$

and hence

$$a = 1, \; p = \frac{1}{b}.$$

The government uses the tax revenue to purchase the public good. Therefore, we have a balanced budget constraint, $py = \int_\Omega \tau(\omega))dF(\omega)$.

3 Properties of Voting Equilibria

We want to study the equilibria of two types of political institutions. This section lay a foundation for studying these political equilibria. We start with a survey of voting models for those who may be unfamiliar with that literature. Then we extend a result

from the probabilistic voting models to cover the case in which the policy belongs to a functional space, which is used later in characterizing the equilibrium tax policies.

3.1 Voting Models

In our problem of voting over the income tax schedules, we do not want to restrict the tax schedule *a priori* to one dimension. Existing voting models have different results when the issue space exceeds one dimension.

There are mainly three types of voting models, based on different behavioral assumptions. The first kind, used in most of the voting literature, is the **deterministic voting model**, which assumes no uncertainty. A voter votes for an alternative, T_j, if $u(T_j) \geq u(T_i)$, for any $T_i \neq T_j$. When the policy space is more than one dimension, a majority cycle usually prevails[4]. Equilibrium does not usually exist.

When we introduce uncertainty into voters' decision processes, which maybe a descriptively more accurate representation of the real decision processes, we can establish the existence of a voting equilibrium.

One approach in Ledyard (1984) uses the **Bayesian voting model**, where Bayesian equilibrium analysis is used, and voters can abstain. In the resulting equilibrium, both candidates adopt the same platform that maximizes a social welfare function. The analysis is based on an individual being pivotal in an election, which is not applicable when we have a continuum of voters/consumers.

An alternative way of modeling voting is the **probabalistic voting model**[5]. We

[4]See, e.g., McKelvey 1979.

[5]For a comprehensive treatment of this subject, see Coughlin 1992.

will briefly go over the underlying rationale for this approach. This approach can be understood as reflecting candidates' uncertainty about whom the individual voters will vote for. Assume that an individual's choice probabilities are "proportional to his strength of preferences" (Coughlin and Nitzan 1981).

Consider an electorate where everyone votes. In the two candidate case, this means that the probability with which an individual ω chooses candidate i, $P^i(T_1, T_2, \omega)$, satisfies

$$P^1(T_1, T_2, \omega) + P^2(T_1, T_2, \omega) = 1.$$

The individual-ω's utility from candidate i's platform is

$$\mu(T_i, \omega) = u(T_i, \omega)\exp(\epsilon_i), i = 1, 2.$$

Assuming that the error term, ϵ, is distributed logistically, we get the individual choice probabilities on any pair of platforms as[6]

$$P^i(T_1, T_2, \omega) = \frac{u(T_i, \omega)}{u(T_1, \omega) + u(T_2, \omega)}.$$

Therefore, a candidate's expected vote equals

$$Ev_i(T_i|T_{-i}) = \int_\Omega \frac{u(T_i, \omega)}{u(T_1, \omega) + u(T_2, \omega)} dF(\omega).$$

Assume that each party's objective function is to maximize expected plurality, which is equivalent to maximizing the probability of winning in a large electorate[7]. Define the expected plurality for party 1 as

$$EPl_1 = Ev_1 - Ev_2 = \int_\Omega \frac{u(T_1, \omega) - u(T_2, \omega)}{u(T_1, \omega) + u(T_2, \omega)} dF(\omega),$$

[6]See, e.g., Amemiya (1985), Chapter 9.

[7]See Ledyard (1984).

and the expected plurality for party 2 as $EPl_2 = -EPl_1$.

Notice that this game is two-person, symmetric and zero-sum. It satisfies the equivalence and interchangeability conditions, i.e., if $T_1^* \neq T_2^*$ in equilibrium, then both (T_1^*, T_1^*) and (T_2^*, T_2^*) are pure strategy equilibria as well.

Coughlin and Nitzan (1981) characterized an equilibrium when the policy set lies in Euclidean space.

Theorem 1 *(Coughlin, 1992, Theorem 6.3) If the policy space $X \subset R^m$ is compact, if voters vote probabilistically, and if $u(T)$ is concave in T, an alternative, $T^* \in X \subset R^m$, is an outcome of the electoral competition, if and only if $T^* \in argmax \int_\Omega \ln u(T, \omega) dF(\omega)$.*

We call $W = \int_\Omega \ln u(T, \omega) dF(\omega)$ the Nash social welfare function. In two party competition under plurality rule, the equilibrium policy outcome is the maximand of the Nash social welfare function.

3.2 Extension of Probabilistic Voting Results

Since we want to study the equilibrium tax structure, we need to extend the result to cover the case in which the policy belongs to a functional space. In this section we extend Theorem 1 to a functional space.

Lemma 2 *After tax consumption, $x(\omega, \omega_t)$, is nondecreasing in ω, where ω_t is his true type, and ω is his reported type.*

Proof: An individual's after tax consumption is $x(\omega, \omega_t) = \omega_t l(\omega) - \tau(\omega)$. He reports an optimal ω such that

$$x(\omega, \omega_t) + \nu(l(\omega), y) \geq x(\omega', \omega_t) + \nu(l(\omega'), y), \quad \forall \omega' \in \Omega.$$

Truthful revelation requires the above inequality holds for $\omega = \omega_t$, i.e.,

$$x(\omega_t, \omega_t) + \nu(l(\omega_t), y) \geq x(\omega', \omega_t) + \nu(l(\omega'), y), \quad \forall \omega' \in \Omega.$$

That is,

$$x(\omega_t, \omega_t) - x(\omega', \omega_t) \geq \nu(l(\omega'), y) - \nu(l(\omega_t), y).$$

If $\omega_t \geq \omega'$, by Lemma 1, we have $l(\omega_t) \geq l(\omega')$, and therefore, $\nu(l(\omega'), y) - \nu(l(\omega_t), y) \geq 0$. So

$$x(\omega_t, \omega_t) - x(\omega', \omega_t) \geq 0.$$

∎

Lemma 2 is used to put more structure on the revenue requirement function, as is shown in the following lemma.

Lemma 3 *$\tau(\omega)$ is of bounded variation.*

Proof: $\tau(\omega) = I(\omega) - x(\omega, \omega_t)$. From Lemma 1 and 2, we know that both $I(\omega)$ and $x(\omega, \omega_t)$ are nondecreasing in ω. So $\tau(\omega)$ is of bounded variation. ∎

Let $BV[a,b]$ denote the space of functions of bounded variation on $[a,b]$. Define $X_\tau = \{\tau$: lower semicontinuous and of $BV\}$, and $X_I = \{I$: nondecreasing$\}$. The policy space is therefore $X = \{(I \in X_I, \tau \in X_\tau) : I - \tau \geq 0; I.C.\}$, where $I.C.$ stands for the incentive compatibility constraint. To prove the existence of the electoral equilibrium, we need to show that X is compact.

Lemma 4 *The policy space X is compact.*

Proof: Since $l \in [0,1)$, we have $I \in [0, \bar{\omega})$. We know that I is nondecreasing. Therefore, I is of bounded variation, and variation norm bounded.

From Lemma 3, τ is of bounded variation. The feasibility constraint gives us $\tau \leq I$. From Assumption 4,

$$\tau(\omega) > -\int_\Omega \omega dF(\omega).$$

Therefore, τ is also variation norm bounded.

Let $M[a,b]$ be the set of all countably additive signed Borel measures on $[a,b]$. From Theorem 4.1 (Border 1991), the $\sigma(BV, M)$ topology and the topology of pointwise convergence coincide on the set $\{(I \in X_I, \tau \in X_\tau) : I - \tau \geq 0\}$.

Next, we show that adding the incentive compatibility constraint does not change pointwise convergence. The incentive compatibility constraint says

$$I_n(\omega) - \tau_n(\omega) + \nu(I_n(\omega)/\omega, y) \geq I_n(\omega') - \tau_n(\omega') + \nu(I_n(\omega')/\omega, y), \forall \omega' \in \Omega.$$

As $I_n(\omega) \to I(\omega)$, and $\tau_n(\omega) \to \tau(\omega)$, we have

$$I(\omega) - \tau(\omega) + \nu(I(\omega)/\omega, y) \geq I(\omega') - \tau(\omega') + \nu(I(\omega')/\omega, y), \forall \omega' \in \Omega.$$

So X is variation norm bounded and pointwise closed subset of BV, and therefore, from Theorem 4.1 (Border 1991), is $\sigma(BV, M)$-compact. ∎

Corollary 1 *In the policy space X, if voters vote probabilistically, and if $u(\cdot)$ is concave in (I, τ), then an equilibrium of the two party electoral competition exists; furthermore, (I^*, τ^*) is an equilibrium to the electoral competition if and only if*

$$I^*, \tau^* \in argmax \int_\Omega \ln u(I - \tau, I/\omega, y) dF(\omega).$$

Proof. Since $u(I-\tau, I/\omega, y)$ is concave in (I,τ), it follows that

$$EPl_i = \int_\Omega \frac{u(I_i-\tau_i, I_i/\omega, y) - u(I_{-i}-\tau_{-i}, I_{-i}/\omega, y)}{u(I_i-\tau_i, I_i/\omega, y) + u(I_{-i}-\tau_{-i}, I_{-i}/\omega, y)}$$

is concave in (I_i, τ_i), convex in (I_{-i}, τ_{-i}), and continuous in both (I_i,τ_i) and (I_{-i},τ_{-i}). From Lemma 4, X is compact. Therefore, an electoral equilibrium exists.

Next, we show that $(I,\tau) \in X$ is an electoral equilibrium to the electoral game, if and only if it is a global maximum of $EPl_i((I_i,\tau_i),(I,\tau))$, given that $(I_{-i},\tau_{-i}) = (I,\tau)$. This follows from the interchangeability condition for two-person, zero-sum games.

Let $W(I,\tau) = \int_\Omega \ln u(I-\tau, I/\omega, y)dF(\omega)$. We then show that $I^*, \tau^* \in argmaxW(I,\tau)$ is equivalent to $I^*, \tau^* \in argmaxEPl_i((I_i,\tau_i),(I,\tau))$, for $i = 1,2$. Since $\ln u(I-\tau, I/\omega, y)$ is a strictly monotone increasing concave function of $u(I-\tau, I/\omega, y)$, then $W(I,\tau)$ is concave in (I,τ). Therefore, every local maximum of $W(I,\tau)$ is also a global maximum. Similarly, since $EPl_i((I_i,\tau_i),(I,\tau))$ is concave in (I_i,τ_i), it follows that any of its local maxima are also a global maximum. So the first order conditions for the maximization problems are both necessary and sufficient. It suffices to show that the first order conditions of the two functions are equivalent. We prove this by using calculus of variation.

$$W(\tau+\epsilon h) = \int_\Omega \ln u(I-\tau-\epsilon h, I/\omega, \int_\Omega (\tau+\epsilon h)dF)dF(\omega).$$

Then,

$$\begin{aligned}
\delta W(\tau;h) &= \frac{d}{d\epsilon}W(\tau+\epsilon h)|_{\epsilon=0} \\
&= \int_\Omega [-\frac{1}{u} + \int_\Omega \frac{u_3}{u}dF]hdF(\omega) \\
&= 0, \text{ for all } h.
\end{aligned}$$

Therefore,

$$-\frac{1}{u} + \int_\Omega \frac{u_3}{u}dF = 0.$$

Similarly,

$$\begin{aligned}
\delta EPl_1(\tau_1;h)|_{\tau_1=\tau; I_1=I} &= \frac{d}{d\epsilon}EPl_1(\tau+\epsilon h)|_{\epsilon=0;\tau_1=\tau;I_1=I} \\
&= \int_\Omega \frac{2u[-h+u_3\int_\Omega hdF]}{(2u)^2} \\
&= \int_\Omega [-\frac{1}{2u} + \int_\Omega \frac{u_3}{2u}dF]hdF(\omega) \\
&= 0, \text{ for all } h.
\end{aligned}$$

Therefore,

$$-\frac{1}{u} + \int_\Omega \frac{u_3}{u}dF = 0.$$

It follows that

$$\delta W(\tau;h) = 2\cdot\delta EPl_1(\tau_1;h)|_{\tau_1=\tau;I_1=I},$$

so that $\delta W(\tau;h) \le 0$ if and only if $\delta EPl_1(\tau_1;h)|_{\tau_1=\tau;I_1=I} \le 0$. Similarly, we can prove that $\delta W(I;h) \le 0$ if and only if $\delta EPl_1(I_1;h)|_{\tau_1=\tau;I_1=I} \le 0$. ∎

Remark (Concavity): Notice that one of the critical assumptions for the characterization of the equilibrium in probabilistic voting is the concavity of the indirect utility function in the policy proposal which, in this case, is the tax function, τ. Let $V(\tau)$ denote the indirect utility function, then $V(\tau)$ is concave in τ, if and only if

$$V(\alpha\tau_1 + (1-\alpha)\tau_2) \ge \alpha V(\tau_1) + (1-\alpha)V(\tau_2),$$

for $\alpha \in [0,1]$. An example of a utility function whose indirect utility function is concave in τ is a quasilinear utility function, $u(x,l,y) = I - \tau(I) + \beta\ln(1-I/\omega) + (1-\beta)\ln y$

where $\beta \in [0, 1]$. For a general utility function where the indirect utility function cannot be solved explicitly, the sufficiency proof of Proposition 1 checks the concavity of the indirect utility function in τ, i.e., Assumption 2 guarantees the concavity of the indirect utility function in τ.

Corollary 1 establishes that the equilibrium tax schedule under a two party plurality system with a single district can be obtained as if we are solving an optimal tax problem, with the exogenously given social welfare function taking the form of the Nash social welfare function.

So far, we have not assumed differentiability of the revenue requirement function or the income function. The next corollary establishes that we can restrict our attention to the subset of differentiable functions.

Corollary 2 *If one party's equilibrium policy proposals are differentiable functions, (τ_i, I_i), then it is an equilibrium for the other party to propose the same differentiable functions.*

Proof: It follows from the interchangeability conditions of the symmetric, two-person, zero-sum game. ∎

From here on, we can restrict ourselves to differentiable revenue requirement functions, τ, income functions, I, and income tax functions, $T(I)$.

4 Characterization of Equilibrium Tax Functions and the Optimality Conditions

The results of Section 3 suggests that in equilibrium the outcome of political processes is as if some particular social welfare function is maximized. In the case of two party plurality system under a single district, the equilibrium tax policy maximizes a Nash social welfare function. In this section, we start with a general optimal taxation model, and then characterize the equilibria of the two political institutions and the optimality conditions of these equilibria, which suggest that they are special cases of the optimal taxation model. The first type is a two party plurality system under a single district, which can be viewed as a simplified version of implementing the platform from a presidential election or the outcome of a simple majority rule/referendum. For comparison, we study the equilibrium policy outcome of a legislative game under a two party plurality system with multiple districts.

4.1 The General Case: Optimal Taxation with Public Good

We use the revelation principle to analyse the general equilibrium optimal taxation problem. The following analysis uses the first order approach to solve the optimization problem.

Given a revenue requirement function, $\tau(\omega)$, and an income function, $I(\omega)$, a ω-person chooses to report his type, ω', to maximize his utility,

$$\max_{\omega'} u(I(\omega') - \tau(\omega'), \frac{I(\omega')}{\omega}, y).$$

The first order condition for this problem is

$$\frac{du}{d\omega'} = \frac{dI(\omega')}{d\omega'} - \frac{d\tau(\omega')}{d\omega'} + \frac{u_2}{\omega}\frac{dI(\omega')}{d\omega'} = 0.$$

Truthful revelation requires $\frac{du}{d\omega'}|_{\omega'=\omega} = 0$, i.e.,

$$\frac{du}{d\omega'}\Big|_{\omega'=\omega} = \frac{dI(\omega)}{d\omega} - \frac{d\tau(\omega)}{d\omega} + \frac{u_2}{\omega}\frac{dI(\omega)}{d\omega} = 0.$$

Using the shorthand, $I'(\omega)$, to stand for $\frac{dI(\omega)}{d\omega}$, and similarly for other variables, the incentive compatibility constraint becomes

$$I'(\omega) - \tau'(\omega) + \frac{u_2}{\omega}I'(\omega) = 0.$$

The optimal income tax problem is thus defined as

$$\begin{array}{lll} \max\limits_{\tau,I} & \int_\Omega A(u(I(\omega)-\tau(\omega), \frac{I(\omega)}{\omega}, b\int_\Omega \tau(\omega)dF(\omega))dF(\omega) & (1Op) \\ s.t. & I'(\omega) - \tau'(\omega) + \frac{u_2}{\omega}I'(\omega) = 0 & (IC) \\ & I(\omega) - \tau(\omega) \geq 0 & (F) \end{array}$$

where $A(u(\omega))$ is some exogenously given, strictly increasing, concave and differentiable welfare function. Equation (IC) is the incentive compatibility constraint. Equation (F) is the feasibility constraint.

Proposition 1 *The optimal tax schedule, $T(I)$, satisfies Equation (1), (IC) and (F).*

Proof: This is a calculus of variations problem. Define the function J as

$$\begin{aligned} J &= \int_\Omega \{A[u(I(\omega)-\tau(\omega), \frac{I(\omega)}{\omega}, b\int_\Omega \tau(\omega)dF(\omega)]f(\omega) \\ &\quad +\xi(\omega)[I'(\omega) - \tau'(\omega) + \frac{u_2}{\omega}I'(\omega)] + \theta(\omega)(I(\omega)-\tau(\omega))\}d\omega. \end{aligned}$$

Then,

$$\begin{aligned} J(\tau+\epsilon h) &= \int_\Omega \{A[u(I(\omega)-\tau(\omega)-\epsilon h, \frac{I(\omega)}{\omega}, b\int_\Omega [\tau(\omega)+\epsilon h]dF(\omega))]f(\omega) \\ &\quad +\xi(\omega)[I'(\omega) - \tau'(\omega) - \epsilon h' + \frac{u_2}{\omega}I'(\omega)] \\ &\quad +\theta(\omega)(I(\omega)-\tau(\omega)-\epsilon h)\}d\omega \end{aligned}$$

So,

$$\begin{aligned} \delta J(\tau,h) &= \frac{d}{d\epsilon}J(\tau+\epsilon h)|_{\epsilon=0} \\ &= \int_\Omega \{A'[-h + u_3 b\int_\Omega hf(\omega)d\omega]f(\omega) + \xi(\omega)(-h') - \theta(\omega)h\}d\omega \\ &= \int_\Omega \{[-A' + \int_\Omega bA'u_3 f(\omega)d\omega]f(\omega) + \xi' - \theta\}hd\omega \\ &= 0, \text{ for all } h, \end{aligned}$$

it follows that

$$[-A' + \int_\Omega bA'u_3 f(\omega)d\omega]f(\omega) + \xi' - \theta(\omega) = 0, \text{ or}$$
$$bf\int_\Omega A'u_3 dF(\omega) - (A'f - \xi') - \theta(\omega) = 0.$$

Define the function G as

$$\begin{aligned} G &= A[u(I(\omega)-\tau(\omega), \frac{I(\omega)}{\omega}, b\int_\Omega \tau(\omega)dF(\omega)]f(\omega) \\ &\quad +\xi(\omega)[I'(\omega) - \tau'(\omega) + \frac{u_2}{\omega}I'(\omega)] + \theta(\omega)(I(\omega)-\tau(\omega)). \end{aligned}$$

The Euler equation for I is

$$\frac{\partial G}{\partial I} = \frac{d(\frac{\partial G}{\partial I'})}{d\omega} \Longleftrightarrow$$
$$(1+\frac{u_2}{\omega})(A'f - \xi') + \frac{\xi}{\omega^2}(\frac{I}{\omega}u_{22} + u_2) + \theta(\omega) = 0.$$

Combining the two necessary conditions, we have

$$bf\int_\Omega A'u_3 dF(\omega) + \frac{u_2}{\omega}(A'f - \xi') + \frac{\epsilon^* u_2 \xi(\omega)}{\omega^2} = 0,$$

where $\varepsilon^* = 1 + lu_{22}/u_2$. From the inverse function theorem, we have $T' = \frac{\tau'}{I'} = 1 + \frac{u_2}{\omega}$. Then we have

$$(1-T')(A'f - \xi') = bf \int_\Omega A' u_3 dF(\omega) + \varepsilon^* \xi(\omega)/\omega^2. \quad (1)$$

Notice that T is also on the righthand side of Equation (1). Equation (1), (IC) and (F) are the necessary conditions for a solution of the optimal income tax problem. To prove sufficiency, we need to check the concavity of G. Since G is linear in I' and τ', the Legendre and Weierstrass conditions are trivially satisfied. We only need to check the concavity of G in I and τ, which requires the matrix of the second partial derivatives with respect to I and τ to be negative semi-definite. Since both $A()$ and u are concave in I and τ, we can decompose the matrix as a sum of two matrices where one of them is negative definite. Then the sufficient conditions are verified if the other matrix, derived from the incentive compatibility and feasibility constraint, is concave in I and τ. Using Assumption 1 that $u()$ is C^3, we require the matrix

$$D = \frac{I'}{\omega} \begin{pmatrix} u_{211} + \frac{u_{222}}{\omega} & -u_{211} \\ -u_{211} & u_{211} \end{pmatrix}$$

to be negative semi-definite.

We get $I' > 0$ from Lemma 1. Thus the sufficiency condition is reduced to requiring $u_{211} \leq 0$ and $u_{222} \leq 0$, which are satisfied from Assumption 2. Thus, the first order approach used in obtaining the necessary conditions for the optimal income tax is valid.

∎

Interpretations for the optimal tax schedule using a general social welfare function can be found in Atkinson and Stiglitz (1980). Our result is different from Mirrlees due to the

endogeneity of the public good and the additional feasibility constraint. The integral on the right-hand side of Equation (1) can be interpreted this way: suppose we reduce the utility of everyone by a marginal unit, then the gain in increased social welfare is $A'u_3$. Therefore, the integral summarizes the net gain of the marginal reduction of utility. The net gain depends on the form of the social welfare function, $A()$, which, as we demonstrate in the later sections, is determined by the political institutions.

Having characterized the optimal income tax schedule, we proceed to analyse how the social welfare functions are endogenously determined by political processes and show that political institutions endogenously determine the weight of the social welfare function.

4.2 Two Party Plurality System Under a Single District

From Corollary 1, the equilibrium tax schedule for two party plurality system under a singel district is the solution to the following optimization problem,

$$\begin{aligned} \max_{\tau,I} \quad & \int_\Omega \ln u(I(\omega) - \tau(\omega), \tfrac{I(\omega)}{\omega}, b\textstyle\int_\Omega \tau(\omega)dF(\omega)dF(\omega) \\ s.t. \quad & I'(\omega) - \tau'(\omega) + \tfrac{u_2}{\omega} I'(\omega) = 0 \qquad (IC) \\ & I(\omega) - \tau(\omega) \geq 0. \qquad (F) \end{aligned}$$

Solving the above problem, we get the following proposition.

Proposition 2 *(a) The equilibrium tax schedule under the single district, two party plurality system satisfies (IC), (F) and the following equation:*

$$(1-T')(f/u - \xi') = bf \int_\Omega u_3/u dF(\omega) + \varepsilon^* \xi(\omega)/\omega^2.$$

(b) It is optimal if the welfare function is $\int_\Omega A(u)dF(\omega) = \int_\Omega \ln u dF(\omega)$.

Proof. Substituting $\ln u$ for $A(u)$ in Equation 1, we get the above result. ∎

The above result can be interpreted either as the equilibrium outcome of a single district two party competition, or as the outcome of a national election, where the winning party/candidate implements his platform. A more complicated political institution involves a legislature where each legislator is elected by plurality rule, and the final policy is the result of a legislative bargaining game.

4.3 Multiple Districts – Legislative Process

An alternative mechanism for deciding the income tax schedule in two party plurality systems is through the election of a legislative body. In each legislative district, the voting game determines each legislator's equilibrium platform and his/her objective function in the legislature. Suppose voters are sophisticated in the sense that they know their legislator is not going to be a dictator in the legislature, that the policy outcome is through a complicated process according to some legislative rule, $\gamma(\cdot) : \{(I_j, \tau_j)\}_{j\in J} \to (I, \tau)$. Then the probability that a ω-person vote for the Incumbent from his district, given the Incumbent's platform, (I_j^I, τ_j^I), and the Challenger's platform, (I_j^C, τ_j^C), is

$$P((I_j^I, \tau_j^I), (I_j^C, \tau_j^C)) = \frac{u[\gamma((I_j^I, \tau_j^I), (I_{-j}, \tau_{-j}))]}{u[\gamma((I_j^I, \tau_j^I), (I_{-j}, \tau_{-j}))] + u[\gamma((I_j^C, \tau_j^C), (I_{-j}, \tau_{-j}))]}.$$

Then, applying Corollary 1 to each legislative district, maximizing expected plurality or the expected probability of winning is equivalent to maximizing the Nash social welfare function for the district in equilibrium, taking the legislative rules, $\gamma(\cdot)$, into consideration. Therefore, we get the following corollary for the equilibrium in each district.

Corollary 3 *In the voting game in district i, the equilibrium platform satisfies*

$$\begin{aligned} I_i^*, \tau_i^* \quad &\in argmax \int_{\Omega_i} \ln u[\gamma((I_i, \tau_i), (I_{-i}, \tau_{-i}))] dF_i(\omega) \\ s.t. \quad & I_i'(\omega) - \tau_i'(\omega) + \tfrac{u_2}{\omega} I_i'(\omega) = 0 \\ & I_i(\omega) - \tau_i(\omega) \geq 0 \end{aligned}$$

Although there are many different legislative processes, we consider a generalized version of the Baron-Ferejohn *random recognition* rule and model the legislative process as a *stochastic game*, $\Gamma^t = (S^t, \pi^t, \psi^t)$, where S^t is the set of *pure strategy n tuples*, where $\pi^t : S^t \to \mu(Z)$ is a *transition function* specifying for each $s^t \in S^t$ a probability distribution $\pi^t(s^t)$ on Z, the set of states that can be achieved in a game, and where $\psi^t : S^t \to X$ is an *outcome function* that specifies for each $s^t \in S^t$ an outcome $\psi^t(s^t) \in X$. Finally, we use $S = \prod_{t\in T} S^t$ to denote the collection of pure strategy n tuples, where $S^t = \prod_{i\in N} s_i^t$. Formally, $Z = R \cup P \cup V$ is the set of states. We use z to denote the possible states the game moves to. We use R to denote the *Recognition Game*, P to denote the *Proposal Game*, and V to denote the *Voting Game*.

At the beginning of period t, legislator j is recognized as a proposer with probability $p_j^t \in [0, 1]$, $\sum_{j\in J} p_i^t = 1, \forall t$. Whoever is recognized proposes a tax schedule, (I_j^t, τ_j^t), then every legislator votes yes or no simultaneously. If, under m-majority rule, the number who say "yes" is greater than or equal to m, (I_j^t, τ_j^t) becomes the new status quo and the game ends; otherwise, the game proceeds to period $t + 1$. If nothing gets passed forever, the payoff to the legislators is zero: $U_j(\phi) = 0$, for all $j \in J$.

In the legislative game, each legislator's objective function is to maximize the Nash social welfare function of his/her district, given the legislative rules, subject to the incentive compatibility constraint and the feasibility constraint, as stated in Corollary 3.

There are many equilibria to the stochastic game. The selection criteria we use is "simplicity" - we want to characterize the simplest equilibria involving no stage dominated strategies[8]. The simplest equilibrium can be described by an automaton of size 4, with one "rest" state (the Recognition Game), one "propose" state (the Proposal Game), and the "vote yes" and "vote no" state (the Voting Game), which gives us the simplest automaton. The resulting equilibria from the automaton are stationary equilibria[9]. The stationary equilibrium is characterized by a set of values $\{v_t\} \subseteq \mathcal{R}^n$ for each stage of the game, and a strategy profile $\sigma^* \in \Sigma$, such that

a) $\forall t$, σ^* is a Nash equilibrium with payoff function $G^t : \Sigma^t \to R^n$ defined by

$$\begin{aligned} G^t(\sigma^t; v) &= U(\psi^t(\sigma^t)) + \sum_{z \in Z} \pi^t(\sigma^t)(z) v^z \\ &= E_{\sigma^t}[U(\psi^t(s^t)) + \sum_{z \in Z} \pi^t(s^t)(z) v^z] \\ &= \sum_{s^t \in S^t} \sigma^t(s^t)[U(\psi^t(s^t)) + \sum_{z \in Z} \pi^t(s^t)(z) v^z]. \end{aligned}$$

b) $\forall t, v^t = U^t(\sigma^t; v)$.

We use the average payoff for each legislator's payoff for the entire stochastic game. So a legislator's payoff for the entire game is

$$U(\{I^t, \tau^t\}_t) = \lim_{N \to \infty} \sum_{t=1}^{N} U^t(\sigma^t; v).$$

Claim. *There exists a vector of continuation values,* $\overline{U} = (\overline{U}_1, \overline{U}_2, \cdots, \overline{U}_J)$, *where*

[8] See Baron and Kalai (1993) for an analysis of the simplest equilibrium in the majority rule divide-the-dollar game with a random recognition rule.

[9] For the existence and characterization of stationary equilibrium, see Sobel (1971).

$\overline{U}_i = \sum_{j \in J} p_j U_i(I_j, \tau_j)$, *representing the expected payoffs to player* i *at the beginning of each stage game.*

In the following proposition, we prove that one equilibrium strategy for legislator j is to vote yes with probability 1 if $U_j(I_i, \tau_i) \geq \overline{U}_j$, and to vote no otherwise. The simultaneous equilibrium strategy for any proposer is to maximize his own utility such that the "least expensive" $m-1$ members of the legislature would vote yes. Denote the set of legislators whose payoffs from the proposed tax schedule are greater than or equal to their continuation value as $M = \{k \in J : U_k(I_j, \tau_j) \geq \overline{U}_k\}$. Therefore, in equilibrium, proposer i proposes the tax schedule (I_i, τ_i) that maximizes the Nash social welfare of his own district subject to the constraint that at least $m-1$ other players also vote yes, and his proposal will be accepted. Baron (1993) characterizes similar equilibrium strategies with alternatives in the Euclidean space and presents a closed-form characterization of the equilibrium when the utility function is quadratic. Proposition 3 is a generalization of Baron's results when the set of alternatives lies in a functional space with generalized utility functions.

We introduce some more notations used in solving for the equilibrium tax schedule for the legislative game. We will use the indicator function,

$$\chi_i(\omega) = \begin{cases} 1 & \text{if } \omega \in \Omega_i, \\ 0 & \text{if } \omega \in \Omega - \Omega_i. \end{cases}$$

Proposition 3 *The following is a simplest subgame perfect stationary Nash equilibrium to the legislative game with stage undominated strategies:*

For $z \in P$ *and* $i = p$ *(Proposer* i*):*

$$I_i, \tau_i \quad \in argmax \int_\Omega \chi_i(\omega) \ln u(I_i(\omega) - \tau_i(\omega), \tfrac{I_i(\omega)}{\omega}, y) dF_i(\omega)$$

$$s.t. \qquad I_i'(\omega) - \tau_i'(\omega) + \frac{u_2}{\omega} I_i'(\omega) = 0$$

$$I_i(\omega) - \tau_i(\omega) \geq 0$$

$$|\{k \in J \backslash \{i\} : U_k(I_i, \tau_i) \geq \overline{U}_k\}| \geq m - 1$$

For $z \in V$ and $j \in J \backslash \{i\}$ (Voter j):

$$s_j(I_i, \tau_i) = \begin{cases} 1 & \text{if } U_j(I_i, \tau_i) \geq \overline{U}_j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We start by defining the strategy sets, transition functions and outcome functions for the game elements:

$$\begin{array}{ll} \text{For } z \in R: & S_i^t = \{0\}, \forall i \in J, \\ \text{(Recognition Game)} & \pi^t(s^t)(z) = 1, \text{if } z \in P \\ & \psi^t(s^t) = \phi, \forall s^t \in S^t. \end{array}$$

The Recognition Game is indexed by $z \in R$. The order of recognition is randomly decided according to some exogenously given probabilities; therefore, the strategy set of each player is $\{0\}$. The game proceeds to the Proposal Game with probability 1, and the null outcome prevails.

$$\begin{array}{ll} \text{For } z \in P: & S_i^t = \begin{cases} \{I_i, \tau_i\} & \text{if } i = p, \\ \{0\} & \text{if } i \in N - \{p\}, \end{cases} \\ \text{(Proposal Game)} & \pi^t(s^t)(z) = 1, \text{if } z \in V, \\ & \psi^t(s^t) = \phi, \forall s^t \in S^t. \end{array}$$

In the Proposal Game, we use p to denote the Proposer. The strategy set for the Proposer is the set of tax schedules $\{I_i, \tau_i\}$, while the strategy set for each voter is still $\{0\}$. The game proceeds to the Voting Game with probability one, and the null outcome prevails in this game.

$$\begin{array}{ll} \text{For } z \in V: & S_i^t = \{0, 1\}, \forall i \in J, \\ \text{(Voting Game)} & \pi^t(s^t)(z) = 1, z \in R, \\ & \psi^t(s^t) = \begin{cases} I_i^t, \tau_i^t & \text{if } \sum_{i \in J} S_i^t \geq m, \\ \phi & \text{otherwise.} \end{cases} \quad \forall s^t \in S^t. \end{array}$$

In the Voting Game, each player can vote either no or yes (0 or 1) to the proposed tax schedule. If the new proposal, (I_i, τ_i), is accepted by at least m of the legislators, it becomes the new status quo; otherwise, the null outcome prevails for this period and the game moves to a new round starting from the Recognition game with probability 1.

The main steps to prove Proposition 3 follow the definition of stationary Nash equilibrium. We first specify the values associated with the equilibrium strategies, and then show that these values are self-generating. The third step is to show that the strategies specified in the proposition are subgame perfect Nash equilibria.

The values of the games are defined below. The interpretations of these values go back to the definitions of each game element above.

For $z \in R$: $v_i^t = v_i^{(R)}, \forall i \in R.$

(Recognition Game)

For $z \in P$: $v_i^t(I_i, \tau_i) = U_i(I_i, \tau_i)$, for $i = p$,

(Proposal Game) $v_j^t(I_i, \tau_i) = \overline{U}_j$, for $j \in J - \{p\}$,

where

$I_i, \tau_i \in argmax \int_\Omega \chi_i(\omega) \ln u(I_i(\omega) - \tau_i(\omega), \frac{I_i(\omega)}{\omega}, y_i) dF_i(\omega)$

$s.t. I_i'(\omega) - \tau_i'(\omega) + \frac{u_2}{\omega} I_i'(\omega) = 0$

$I_i(\omega) - \tau_i(\omega) \geq 0$

$|\{k \in J \backslash \{i\} : U_k(I_i, \tau_i) \geq \overline{U}_k\}| \geq m - 1.$

For $z \in V$: $v_i^t = \alpha(|M|)U_j(I_i, \tau_i) + (1 - \alpha(|M|))\overline{U}_j(I_i, \tau_i),$

(Voting Game) $\forall j \in J$, where

$$\alpha(|M|) = \begin{cases} 1 & \text{if } |M| \geq m, \\ 0 & \text{otherwise.} \end{cases}$$

The next step is to verify that these values are self-generating, i.e., that they correspond to the payoffs under the equilibrium strategies. To do this, we plug the equilibrium strategies and other game elements into the definition of G, and show that they equal the corresponding values.

For $z \in R$: *(Recognition Game)*

$$\begin{aligned} G^t(\sigma^t, v^t) &= E_{\sigma^t}[U(\psi^t(s^t)) + \sum_{z \in Z} \pi^t(s^t)(z) v^z] \\ &= U(\phi) + \pi^t(s^t)(z) \cdot v^t \\ &= v^{(R)} = v^t. \end{aligned}$$

For $z \in P$: *(Proposal Game)*

For $i = p$ (Proposer i):

$$\begin{aligned} G_i^t(\sigma^t; v_i^t) &= E_{\sigma^t}[U(\psi^t(s^t)) + \sum_{z \in Z} \pi^t(s^t)(z) v^z] \\ &= U_i(\phi) + 1 \cdot U_i(I_i, \tau_i) \\ &= U_i(I_i, \tau_i) \\ &= v_i^t(I_i, \tau_i). \end{aligned}$$

For $j = J - \{p\}$ (Voter j):

$$\begin{aligned} G_j^t(\sigma^t; v_j^t) &= E_{\sigma^t}[U(\psi^t(s^t)) + \sum_{z \in Z} \pi^t(s^t)(z) v^z] \\ &= U_j(\phi) + 1 \cdot \overline{U}_j \\ &= \overline{U}_j \\ &= v_j^t(I_i, \tau_i). \end{aligned}$$

For $z \in V$: *(Voting Game)*

$$\begin{aligned} G^t(\sigma^t, v^t) &= E_{\sigma^t}[U(\psi^t(s^t)) + \sum_{z \in Z} \pi^t(s^t)(z) v^z] \\ &= \alpha(|M|)U_j(I_i, \tau_i) + (1 - \alpha(|M|))(U_j(\phi) + \overline{U}_j(I_i, \tau_i)) \\ &= v^t. \end{aligned}$$

Next, we verify that the strategies specified in Proposition 3 are subgame perfect Nash equilibrium strategies. Since the strategies are history-independent, it suffices to show that for each game element no player will benefit from a unilateral one-shot deviation.

For $z \in P$, we want to show that tax proposal (I_i, τ_i) is the equilibrium strategy for Proposer i, where

$$\begin{aligned} I_i, \tau_i \quad & \in argmax \int_{\Omega} \chi_i(\omega) \ln u(I_i(\omega) - \tau_i(\omega), \tfrac{I_i(\omega)}{\omega}, y_i) dF_i(\omega) \\ s.t. \quad & I_i'(\omega) - \tau_i'(\omega) + \tfrac{u_2}{\omega} I_i'(\omega) = 0 \\ & I_i(\omega) - \tau_i(\omega) \geq 0 \\ & |\{k \in J \backslash \{i\} : U_k(I_i, \tau_i) \geq \overline{U}_k\}| \geq m - 1. \end{aligned}$$

The corresponding payoff for Proposer i is

$$G_i^t(\sigma^t; v^t(I_i, \tau_i)) = U_i(I_i, \tau_i).$$

If the proposer defects to any other pure strategy $(I_i^0, \tau_i^0) \neq (I_i, \tau_i)$, $\forall i \in J$, there are two possible consequences:

(i) $U_i(I_i^0, \tau_i^0) \leq U_i(I_i, \tau_i)$:

in which case he is not better off by defection, so he will not defect in this case.

(ii) $U_i(I_i^0, \tau_i^0) > U_i(I_i, \tau_i)$:

in this case, if $|\{k \in J \backslash \{i\} : U_k(I_i^0, \tau_i^0) \geq \overline{U}_k\}| \geq m - 1$ still holds, $\forall j \neq i$, then

$$\begin{aligned} I_i, \tau_i \quad & \notin argmax \int_{\Omega} \chi_i(\omega) \ln u(I_i(\omega) - \tau_i(\omega), \tfrac{I_i(\omega)}{\omega}, y_i) dF_i(\omega) \\ s.t. \quad & I_i'(\omega) - \tau_i'(\omega) + \tfrac{u_2}{\omega} I_i'(\omega) = 0 \\ & I_i(\omega) - \tau_i(\omega) \geq 0 \\ & |\{k \in J \backslash \{i\} : U_k(I_i, \tau_i) \geq \overline{U}_k\}| \geq m - 1. \end{aligned}$$

but this contradicts the definition of (I_i, τ_i).

So the proposer has no positive incentive to defect unilaterally from his strategy

specified in Proposition 3, which means that it is a Nash equilibrium for the Proposer. Since it is history independent, it is also a subgame perfect equilibrium.

For $z \in V$, we want to check if voters' strategies specified in the proposition are Nash equilbrium strategies. We consider three cases:

(1) When $|M| > m$, no voter is pivotal, so they have no positive incentive to defect from their equilibrium strategies.

(2) When $|M| = m$, any voter $i \in M$ is pivotal. Since $G_i(s_i = 0, s_{-i}^*) - G_i(s_i = 1, s_{-i}^*) = \overline{U}_i - U_i(I_i, \tau_i) \leq 0$, i has no positive incentive to defect from his equilibrium strategy.

(3) When $|M| = m - 1$, any voter $i \in J \backslash M$ is pivotal. Since $G_i(s_i = 1, s_{-i}^*) - G_i(s_i = 0, s_{-i}^*) = U_i(I_i, \tau_i) - \overline{U}_i \leq 0$, i has no positive incentive to defect either.

Therefore, the voter strategies specified in the proposition are Nash equilibrium strategies. They are subgame perfect, since they are history independent. ∎

We use a three district example to solve the stationary equilibrium tax schedule for the legislative game. It can be easily extended to the J district case. In the following legislative game, $J = 3$, $m = 2$, and $p_i = 1/3$, for $i = 1, 2, 3$. The problem in Proposition 3 reduces to

$$\begin{aligned} \max_{\tau_i, I_i} \quad & \int_{\Omega} \chi_i(\omega) \ln u(I_i(\omega) - \tau_i(\omega), \tfrac{I_i(\omega)}{\omega}, y_i) dF_i(\omega) & \\ s.t. \quad & I_i'(\omega) - \tau_i'(\omega) + \tfrac{u_2}{\omega} I_i'(\omega) = 0 & (IC_i) \\ & I_i(\omega) - \tau_i(\omega) \geq 0 & (F_i) \\ & U_j(I_i, \tau_i) \geq \overline{U}_j & (MA_i) \end{aligned}$$

where j is i's coalition member. Equation (MA_i) can be expanded as

$$\int_\Omega \chi_j(\omega) \ln u(I_i(\omega) - \tau_i(\omega), \frac{I(\omega)}{\omega}, y) dF_j(\omega) \geq$$

$$\frac{1}{2} \int_\Omega \chi_j(\omega) [\ln u(I_j(\omega) - \tau_j(\omega), \frac{I_j(\omega)}{\omega}, y_j) + \ln u(I_k(\omega) - \tau_k(\omega), \frac{I_k(\omega)}{\omega}, y_k)] dF_j(\omega).$$

Apart from the usual incentive compatibility constraint and balanced budget constraint, the tax schedule has to pass a majority of the legislature. The last constraint, (MA), requires the payoff to legislator j to be greater than or equal to his continuation value.

Proposition 4 *(a) The equilibrium tax schedule, (I_i, τ_i), for the legislative game under a random recognition rule in the three district case, satisfies (IC_i) ,(F_i) and the following equation:*

$$\begin{aligned} & (1 - T_i')[(\chi_i(\omega) f_i(\omega) + \lambda \chi_j(\omega) f_j(\omega))/u - \xi'] \\ = \; & b \int_\Omega u_3 [\chi_i(\omega) f_i(\omega) + \lambda \chi_j(\omega) f_j(\omega)]/u dF(\omega) + \epsilon^* \xi(\omega)/\omega^2, \end{aligned}$$

where $\lambda \geq 0$, i is the proposer and j is the legislator in the majority coalition with i.

(b) It is optimal if the welfare function is

$$\int_\Omega A(u) dF(\omega) = \int_\Omega [\chi_i(\omega) f_i(\omega) + \lambda \chi_j(\omega) f_j(\omega)] \ln u d\omega.$$

Proof. Define the function J as

$$\begin{aligned} J \; = \; & \int_\Omega \{[\chi_i(\omega) f_i(\omega) + \lambda \chi_j(\omega) f_j(\omega)] \ln u \\ & + \xi(\omega)[I'(\omega) - \tau'(\omega) + \frac{u_2}{\omega} I_i'(\omega)] + \theta(\omega)(I_i(\omega) - \tau_i(\omega))\} d\omega. \end{aligned}$$

Let $g(I_i, \tau_i, I_{-i}, \tau_{-i}) = \int_\Omega \chi_j(\omega) \ln u(I_i(\omega) - \tau_i(\omega), \frac{I(\omega)}{\omega}, y) dF_j(\omega) - \frac{1}{2} \int_\Omega \chi_j(\omega) [\ln u(I_j(\omega) - \tau_j(\omega), \frac{I_j(\omega)}{\omega}, y_j) + \ln u(I_k(\omega) - \tau_k(\omega), \frac{I_k(\omega)}{\omega}, y_k)] dF_j(\omega)$. The complementary slackness

condition requires

$$\lambda(\omega) g(I_i, \tau_i, I_{-i}, \tau_{-i}) = 0, \quad \text{with } \lambda(\omega) \geq 0,$$

$$\theta(\omega)(I_i - \tau_i) = 0, \qquad \text{with } \theta(\omega) \geq 0.$$

The rest of the proof is similar to that of Proposition 1, with $A(u) = [\chi_i(\omega) f_i(\omega) + \lambda \chi_j(\omega) f_j(\omega)] \ln u$. ∎

Notice the equilibrium tax schedule of the legislative process is different from that of the two candidate competition. The difference comes from the specific forms of the social welfare functions. Therefore, the welfare weight of individuals in districts whose legislators are not in the majority coalition is zero, while the welfare of individuals whose legislators are in the majority coalition is taken into account when solving for the equilibrium income tax schedule. This confirms our conjecture that the welfare weights of the optimal income tax schedule are endogenously determined by the political processes.

We have characterized the *ex post* equilibrium income tax schedule. One question is if the *ex ante* result is the same as the single district case. To see that this is usually not the case, consider the following situation. Suppose we have three districts, and the distribution of types are such that if 1 is the proposer, he will form a coalition with 2; if 2 or 3 is the proposer, they will form a coalition with each other. Let the probability of i being recognized be p_i. Then the *ex ante* equilibrium tax schedule will be

$$\begin{aligned} I, \tau \in argmax \int_\Omega \; & [p_1(\chi_1(\omega) f_1(\omega) + \lambda_{12} \chi_2(\omega) f_2(\omega)) + p_2(\chi_2(\omega) f_2(\omega) \\ & + \lambda_{23} \chi_3(\omega) f_3(\omega)) + p_3(\chi_3(\omega) f_3(\omega) + \lambda_{32} \chi_2(\omega) f_2(\omega))] \ln u dF(\omega). \end{aligned}$$

The *ex ante* result will be the same as the single district case if and only if

$$
\begin{aligned}
p_1(\chi_1(\omega)f_1(\omega) + \lambda_{12}\chi_2(\omega)f_2(\omega)) + p_2(\chi_2(\omega)f_2(\omega) + \lambda_{23}\chi_3(\omega)f_3(\omega)) & \\
+p_3(\chi_3(\omega)f_3(\omega) + \lambda_{32}\chi_2(\omega)f_2(\omega)) &= 1.
\end{aligned}
$$

One special case is when all districts are identical, i.e., when $\lambda_{ij} = 0$ for all $i \neq j$, then the single district case has the same outcome with the multiple district case.

When the districts are hetereogeneous, however, the outcome of the legislative process in multiple districts will usually be different from that of single district. We illustrate this with an example.

5 An Example

We will use a simplified economy to show the difference in income tax structures under different political institutions. Suppose wage rate, ω, is uniformly distributed in the interval $[1,4]$. Individuals have quasilinear utility function of the form, $I - \tau + \ln(1 - I/\omega) + \ln y + e$, where y is the amount of public good produced, and e is the initial endowment.

Under a two-party, plurality system in a single district, the equilibrium tax structure, $\tau(\omega)$, maximizes the Nash social welfare function of the whole district subject to the incentive compatibility constraint and the feasibility constraint. This is a calculus of variation problem, which is set up as follows.

$$\max_{\tau(),I()} \quad \textstyle\int_1^4 \ln[I - \tau + \ln(1 - I/\omega) + \ln(\int_1^4 \tau/3d\omega) + e]d\omega \tag{2}$$

$$s.t. \qquad I' - \tau' = \tfrac{I'}{\omega - I} \tag{3}$$

$$I - \tau \geq 0 \tag{4}$$

Define

$$
\begin{aligned}
J &= \int_1^4 \{\ln[I - \tau + \ln(1 - I/\omega) + \ln(\int_1^4 \tau/3d\omega) + e] \\
&\quad + \xi(\omega)(I' - \tau' - \frac{I'}{\omega - I}) + \theta(\omega)(I - \tau)\}d\omega.
\end{aligned}
$$

$$
\begin{aligned}
\delta J(\tau, h) &= \frac{d}{d\epsilon}J(\tau + \epsilon h)|_{\epsilon=0} \\
&= \int_1^4 \frac{1}{u}\left[-h(\omega) + \frac{\int_1^4 h(\omega)d\omega}{\int_1^4 \tau(\omega)d\omega}\right] - \xi(\omega)h'(\omega) - \theta(\omega)hd\omega \\
&= \int_1^4 \left[-\frac{1}{u} + \frac{\int_1^4 \frac{1}{u}d\omega}{\int_1^4 \tau(\omega)d\omega} + \xi'(\omega) - \theta(\omega)\right] h(\omega)d\omega \\
&= 0, \text{ for all } h(\omega), \text{ which implies}
\end{aligned}
$$

$$-\frac{1}{u} + \frac{\int_1^4 \frac{1}{u}d\omega}{\int_1^4 \tau(\omega)d\omega} + \xi'(\omega) - \theta(\omega) = 0. \tag{5}$$

Define

$$F = \ln[I - \tau + \ln(1 - I/\omega) + \ln(\int_1^4 \tau/3d\omega) + e] + \xi(\omega)(I' - \tau' - \frac{I'}{\omega - I}) + \theta(\omega)(I - \tau).$$

Using Euler's equation, we get

$$\frac{\partial F}{\partial I} = \frac{d(\frac{\partial F}{\partial I'})}{d\omega} \iff$$

$$(\omega - I)(\omega - I - 1)(\frac{1}{u} - \xi'(\omega)) - \xi(\omega)I' + \theta(\omega)(\omega - I)^2 = 0. \tag{6}$$

There is no analytical solution to the set of equations (Equation 3, 4, 5, 6), so we resort to numerical solutions. For simplicity of calculation, we normalize $\tau(1) = 0$, and let $e = 5.0$.

Figure 1 shows the equilibrium income function, $I(\omega)$, and revenue requirement function, $\tau(\omega)$. Both the income function and the revenue requirement function are monotone increasing in ω, but $I''(\omega) > 0$ while $\tau'(\omega)$ starts from zero, increases, then decrease down to zero.

Figure 2 shows the income tax schedule. The marginal tax rate at both ends of income is zero. Tax is an increasing function of income.

The level of public goods provided in this case is 0.2892.

It is interesting to compare the outcome of single district case with that of the multiple district case. We consider the case when there are three districts, each with a uniform distribution of wage rates over the intervals, $[1,2)$, $[2,3)$ and $[3,4]$. Then there are eight cases of legislative coalition formation. We use the symbol, $\rightarrow$, to represent "propose to and form coalition with". The eight cases are $(1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1)$, $(1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 2)$, $(1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 1)$, $(1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 2)$, $(1 \rightarrow 3, 2 \rightarrow 3, 3 \rightarrow 1)$, $(1 \rightarrow 3, 2 \rightarrow 3, 3 \rightarrow 2)$, $(1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 1)$, $(1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2)$. As an example, the first case is set up as the following,

$$\max_{\tau_1, I_1} \quad \int_1^2 \ln[I_1 - \tau_1 + \ln(1 - I_1/\omega) + \ln(\textstyle\int_1^4 \tau_1/3d\omega) + e]d\omega$$

$$s.t. \quad I_1' - \tau_1' = \frac{I_1'}{\omega - I_1}$$

$$I_1 - \tau_1 \geq 0$$

$$\int_2^3 \ln[I_1 - \tau_1 + \ln(1 - I_1/\omega) + \ln(\textstyle\int_1^4 \tau_1/3d\omega) + e]d\omega \geq$$

$$\tfrac{1}{2}\{\int_2^3 \ln[I_2 - \tau_2 + \ln(1 - I_2/\omega) + \ln(\textstyle\int_1^4 \tau_2/3d\omega) + e]d\omega$$

$$+ \int_2^3 \ln[I_3 - \tau_3 + \ln(1 - I_3/\omega) + \ln(\textstyle\int_1^4 \tau_3/3d\omega) + e]d\omega\}$$

$$\max_{\tau_2, I_2} \quad \int_2^3 \ln[I_2 - \tau_2 + \ln(1 - I_2/\omega) + \ln(\textstyle\int_1^4 \tau_2/3d\omega) + e]d\omega$$

$$s.t. \quad I_2' - \tau_2' = \frac{I_2'}{\omega - I_2}$$

$$I_2 - \tau_2 \geq 0$$

$$\int_3^4 \ln[I_2 - \tau_2 + \ln(1 - I_2/\omega) + \ln(\textstyle\int_1^4 \tau_2/3d\omega) + e]d\omega \geq$$

$$\tfrac{1}{2}\{\int_3^4 \ln[I_3 - \tau_3 + \ln(1 - I_3/\omega) + \ln(\textstyle\int_1^4 \tau_3/3d\omega) + e]d\omega$$

$$+ \int_3^4 \ln[I_1 - \tau_1 + \ln(1 - I_1/\omega) + \ln(\textstyle\int_1^4 \tau_1/3d\omega) + e]d\omega\}$$

$$\max_{\tau_3, I_3} \quad \int_3^4 \ln[I_3 - \tau_3 + \ln(1 - I_3/\omega) + \ln(\textstyle\int_1^4 \tau_3/3d\omega) + e]d\omega$$

$$s.t. \quad I_3' - \tau_3' = \frac{I_3'}{\omega - I_3}$$

$$I_3 - \tau_3 \geq 0$$

$$\int_1^2 \ln[I_3 - \tau_3 + \ln(1 - I_3/\omega) + \ln(\textstyle\int_1^4 \tau_3/3d\omega) + e]d\omega \geq$$

$$\tfrac{1}{2}\{\int_1^2 \ln[I_1 - \tau_1 + \ln(1 - I_1/\omega) + \ln(\textstyle\int_1^4 \tau_1/3d\omega) + e]d\omega$$

$$+ \int_1^2 \ln[I_2 - \tau_2 + \ln(1 - I_2/\omega) + \ln(\textstyle\int_1^4 \tau_2/3d\omega)]d\omega\}$$

The equilibrium proposals of all three legislators can be calculated using numerical solutions. Figure 3 and 4 shows the numerical solutions to the three district case. It is interesting to observe that the equilibrium proposal of Legislator 2, the representative of the "middle productivity" district, coincides with the equilibrium proposal of the single district case. The public goods levels as outcomes of the three proposals are 0.3667, 0.2892, and 0.3272 respectively.

Although analytical solutions and comparative statics results are hard to obtain, we learned from the example that we can form some testable implications if the distribution of the wage rates are known and if we can parameterize the utility function somehow.

6 Conclusions

In this paper we address two shortcomings of the optimal income taxation literature, i.e., exogenous social welfare functions and the neglect of institutional constraints. We characterize the optimal income tax schedule using a general equilibrium model with a public good entering consumers' utility functions. We show that the social welfare functions can be determined endogenously by political processes, i.e., electoral systems and the legislative process. We characterize the equilibrium tax schedules under the two party plurality system, including the single-district case and the multiple-district case. It is shown that under the two party plurality system, the equilibrium income tax is equivalent to an optimal tax schedule which puts equal weight over the whole population when there is a single district; when there are multiple districts, however, in the simplest subgame perfect stationary equilibrium to the legislative game, the equilibrium is equivalent to an optimal tax schedule which puts more welfare weight on the subsets of the population whose legislators are in the winning coalition of the legislature. Thus we have shown that the political processes endogenously determine the welfare weights of the optimal income taxation problem.

The characterizations of the equilibrium tax schedules in this paper provide considerable insight into the factors influencing the equilibrium marginal tax rates under different political processes, and the way they interact. More general results are hard to obtain from these formulas. Given the distribution of the productivity levels, however, we can form some testable implications by parameterizing the utility functions to get the explicit equilibrium income tax schedules.

References

[1] Amemiya, Takeshi (1985), *Advanced Econometrics*, Harvard University Press: Cambridge.

[2] Baron, David (1993), "A Theory of Collective Choice for Government Programs," Stanford Research Paper No. 1240.

[3] Baron, David (1993), "A Sequential Choice Theory Perspective on Legislative Organization," Stanford Research Paper No. 1273.

[4] Baron, David and John Ferejohn (1989), "Bargaining in Legislatures," *American Political Science Review* 83:1181.

[5] Baron, David and Ehud Kalai (1993), "The Simplest Equilibrium in Majority Rule Division Game," *Journal of Economic Theory* 61.

[6] Berliant, Marcus and Miguel Gouveia (1991), "On the Political Economy of Income Taxation," University of Rochester memeo.

[7] Berliant, Marcus and Miguel Gouveia (1992), "Incentive Compatible Income Taxation, Individual Revenue Requirements and Welfare," University of Rochester memeo.

[8] Border, Kim C. (1991), "Functional Analytical Tools for Expected Utility Theory," in Aliprantis, Border and Luxemburg (Eds.) *Positive Operators, Riesz Spaces, and Economics*, Springer-Verlag.

[9] Brito, D. and W. Oakland (1977), "Some Properties of the Optimal Income Tax," *International Economic Review* 18: 407 - 423.

[10] Coughlin, Peter and Shmuel Nitzan (1981), "Electoral Outcomes with Probabilistic Voting and Nash Social Welfare Maxima," *Journal of Public Economics* 15: 113-121.

[11] Coughlin, Peter (1992), *Probabilistic Voting Theory*, Cambridge University Press.

[12] Cukierman, Alex and Allan H. Meltzer (1991), "A Political Theory of Progressive Income Taxation," in *Political Economy* by Allan H. Meltzer, Alex Cukierman and Scott F. Richard, Oxford University Press: New York.

[13] Ekeland, Ivar and Thomas Turnbull (1983), *Infinite Dimensional Optimization and Convexity*, University of Chicago Press: Chicago.

[14] Ledyard, John O. (1984), "The Pure Theory of Large Two-Candidate Elections," *Public Choice* 44: 7-41.

[15] McKelvey, Richard D. (1979), "General Conditions for Global Intransitivities in Formal Voting Models," *Econometrica*, Vol. 47, No. 5. (September 1979).

[16] Mirrlees, J.A. (1971), "An Exploration in the Theory of Optimal Income Taxation," *Review of Economic Studies* 38: 175 - 208.

[17] Mirrlees, J.A. (1976), "Optimal Tax Theory," *Journal of Public Economics* 6: 327 - 358.

[18] Mirrlees, J.A. (1986), "The Theory of Optimal Taxation," *Handbook of Mathematical Economics*, Vol. III, [ed] K.J. Arrow and M.D. Intrilligator, Elsevier Science Publisher B.V. (North-Holland).

[19] Piketty, T. (1993), "Dynamic Voting Equilibrium, Political Conservatism and Income Taxation," DELTA manuscript.

[20] Roberts, Kevin W. S. (1977), "Voting Over Income Tax Schedules," *Journal of Public Economics* 8:329 - 340.

[21] Rochet, J-C (1987), "A Necessary and Sufficient Condition for Rationalizability in a Quasi-linear Context," *Journal of Mathematical Economics* 16: 191-200.

[22] Romer, Thomas (1975), "Individual Welfare, Majority Voting, and the Properties of a Linear Income Tax," *Journal of Public Economics* 4: 163-185.

[23] Snyder, James M. and Gerald Kramer (1988), "Fairness, Self-Interest, and the Politics of the Progressive Income Tax," *Journal of Public Economics* 36: 197-230.

[24] Sobel, M.J. (1971), "Noncooperative Stochastic Games," *The Annals of Mathematical Statistics*, Vol. 42, No. 6, 1930-1935.

[25] Zangwill, Willard I. (1969), *Nonlinear Programming: A Unified Approach*, Prentice-Hall: New Jersey.

Figure 1: Revenue Requirement Function and Income Function: Single District

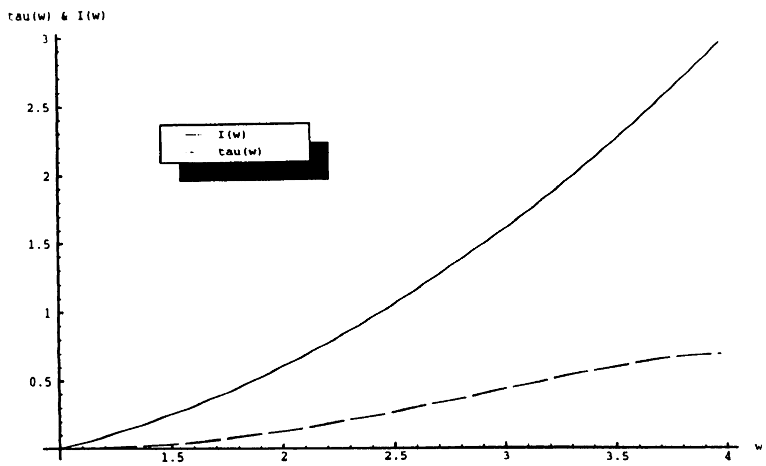

Figure 2: Income Tax Function: Single District

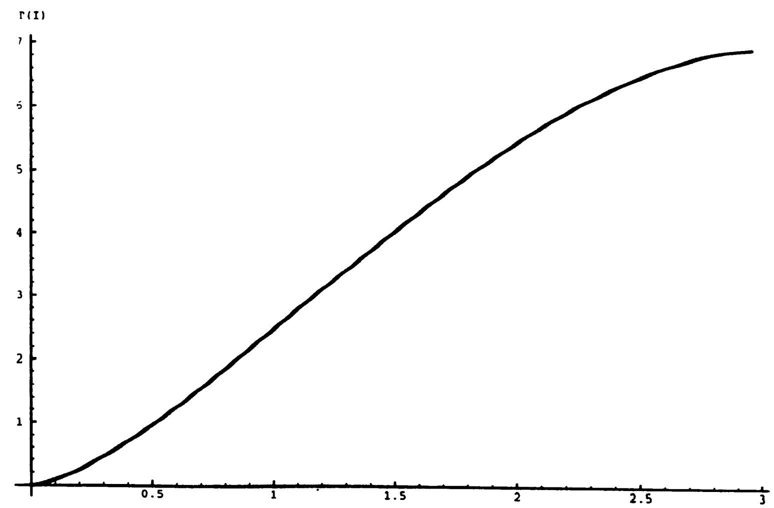

Figure 3: Revenue Requirement Functions and Income Functions: Three Districts

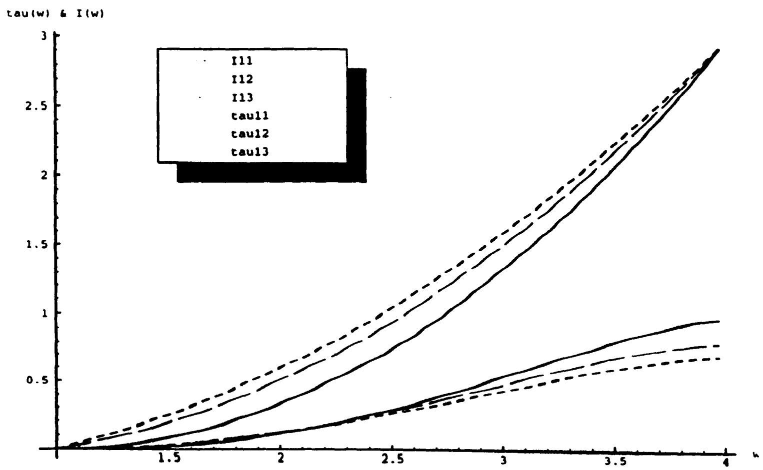

Figure 4: Income Tax Functions: Three Districts