C-23

Center for Research on Economic and Social Theory
Research Seminar in Quantitative Economics

Discussion Paper

DEPARTMENT OF ECONOMICS

University of Michigan

Ann Arbor, Michigan 48109

[redacted]

National Debt, Social Security, and Bequests*

John Laitner
Department of Economics
The University of Michigan
Ann Arbor, Michigan 48109

C-23

October 6, 1980

*I owe thanks to my colleagues A. Drazen and G. Loury for helpful comments on an earlier draft of this paper.

Abstract

This paper examines the hypothesis that bequest saving will neutralize the effect of a government debt or an unfunded social security system on an economy's aggregate capital accumulation. We develop an infinite-time-horizon model in which family bequest behavior is based on utility maximization and in which a random component in the (labor) earning ability of each household plays a central role. We then demonstrate that in our framework a national debt or social security system can decrease a society's steady-state capital intensity -- with bequest behavior augmenting rather than counterbalancing the drop.

National Debt, Social Security,

and Bequests

Barro's (1974) influential article "Are Government Bonds Net Wealth?" shows that if each family's utility-maximization problem yields an "interior-solution" bequest, a government debt or unfunded social security system will tend not to affect aggregate capital accumulation. The same line of reasoning has potential applications in other contexts. For example, Calvo, Kotlikoff, and Rodriguez (1979) invoke Barro's ideas to cast doubt on Feldstein's (1977) conclusion that a tax on property rents may eventually be partly shifted.

The purpose of this paper is to attempt to show, however, that for an endless economy, Barro's arguments may be relevant only in the short run. Using the bequest theory developed in Laitner (1979a, b, c), we construct a decentralized growth model. We then prove that the model's components compel the existence of "corner-solution" bequests for a positive fraction of the families in each cohort of a steady state. If the fraction is tiny, the transition interval between steady states -- following a change in the national debt, for instance -- might be very long, and during the adjustment's early stages Barro's model might supply fairly accurate predictions about actual behavior. Nevertheless, we show that for a comparative-steady-state analysis, the difference between all "interior-solution" bequests and even a minuscule number of "corner solutions" can be very great. (In other words, steady-state policy results can change "discontinuously" as we are forced to deviate from Barro's assumption of all "interior solutions.") Our model -- which focuses exclusively on steady states -- demonstrates, in fact, that bequest behavior may well augment, rather than neutralize, the long-run consequences of national debt or social security.

The organization of this paper is as follows. Section I sets up our household bequest model, and Section II develops sufficient conditions for the existence of a unique steady-state equilibrium. Sections III and IV then introduce government debt and an unfunded social security system. Section V concludes the analysis.

I. Family Bequests

In this section we present our model of household behavior. Section II incorporates the model into a simple general equilibrium framework.

Assumptions

Our household model has three basic elements.

The first is a direct utility function, which all households employ, and which each seeks to maximize. For a given generation-0 family, the arguments of the function include the family's own lifetime consumption, say, C_{01}; the lifetime consumption figures for its n offspring, say, $C_{11}, \ldots, C_{1n}$; the lifetime consumption figures for its $n \cdot n$ second-generation descendants, say, $C_{21}, \ldots, C_{2,n \cdot n}$; and so on. As is standard in microeconomic analysis, only consumption variables directly affect the household's utility.[1/]

We assume that the direct utility function is additively separable and that all descendants in any given cohort receive equivalent treatment. Thus,

$$U = \sum_{i=0}^{\infty} \lambda_i \cdot \sum_{j=1}^{n^i} u_i(C_{ij}).$$

Letting Koopmans' (1960) "stationarity postulate" apply for sequences of cohorts (see note 2 of Laitner (1979a)),

$$U = \sum_{i=1}^{\infty} h^i \cdot \sum_{j=1}^{n^i} u(C_{ij})$$

$$\text{some } h \varepsilon (0,1). \tag{1}$$

These restrictions plus conventional concavity and monotonicity properties for $u(\cdot)$ suffice for the consumption-loan analysis in Laitner (1979c). To accomodate production and technical progress here, however, we assume that in line (1)

$$u(C) = C^{\beta}/\beta \text{ for } \beta < 1,\ \beta \neq 0,$$

$$\text{or } u(C) = \ln(C).^{2/} \tag{2}$$

We sometimes refer to the logarithmic specification as the "$\beta = 0$ case."

The second element of our family bequest model is an exogenously given random variable $\tilde{Z}$. For the time being (see Section IV), suppose each family lives a single period. Let Y_{ij} be the labor income of the C_{ij}-family above. Let w_i be the wage rate in period i. Then we assume Y_{ij} is the product of w_i and an independent sampling, say, Z_{ij}, from $\tilde{Z}$: $Y_{ij} = w_i Z_{ij}$. We assume $\tilde{Z}$ has a given density function $p(Z)$ such that $p(\cdot)$ is continuous on an interval $[Z_L, Z_U] \subset (0,\infty)$; $p(Z) = 0$ off of the interval; and for some $p_L > 0$, $p(Z) \geq p_L$ all $Z \varepsilon [Z_L, Z_U]$. The random variable $\tilde{Z}$ reflects intra-cohort differences in motivation, ability, etc.

For the sake of simplicity we let "descent" be traced only through male offspring. Thus, only males can inherit from their parents and, for example, the C_{1j}, $j = 1,\ldots,n$ figures of line (1) refer solely to the consumption of families of male offspring. Each household can then receive at most one inheritance, say, I_{ij}. (See Laitner (1979c) for generalizations to dual inheritances with random mating patterns.) We call W_{ij} with

$$W_{ij} = Y_{ij} + I_{ij}$$

the "total wealth" of family (i, j). Normalizing,

$$X_{ij} = Z_{ij} + I_{ij}/w_i$$

with $w_i \cdot X_{ij} = W_{ij}$. The exogenous distribution of $\tilde{Z}$ together with utility-maximizing household behavior will determine the distribution of $\tilde{X}$.

In addition to affecting the distribution of $\tilde{X}$ directly, the variable $\tilde{Z}$ plays a basic role as follows. Suppose one plus the rate of labor-augmenting technical change is $\gamma \geq 1$, and suppose we have a steady state with $w_{i+1} = \gamma w_i$ all i. Then if $Z_L = Z_U$, there will be no positive bequests in the steady state because $h < 1$ in line (1).[3/] If $Z_L \leq \gamma Z_L < Z_U$, on the other hand, diminishing marginal utility may cause families with very favorable samplings from $\tilde{Z}$ to share part of their good luck with their "descendants." Even generation-t families with only moderate labor incomes may want to leave bequests to partially "insure" their "descendants" against very unlucky samplings from $w_{t+s}\tilde{Z}$ all $s \geq 1$. To allow the possibility of some positive bequests, we henceforth assume $\gamma Z_L < Z_U$.

The third key element of our household model is a nonnegativity requirement for all bequests.[4/] This can be viewed as an institutionally imposed constraint: sons are not legally bound to pay the debts of their fathers in our society.[5/] Some such restriction is necessary to make family budget constraints meaningful: If there is no lower bound on bequests, a time-0 family can set $C_{01} = \infty$ and bequeath an infinite debt. The family will know its n first-generation "descendants" can do likewise, setting $C_{1j} = \infty$ all $j = 1,\ldots,n$ and bequeathing their debts and their parents'. Future "descendants" can do the same. With an infinite time horizon for the economy, no generation of "descendants" will actually have to settle its debts.

The indirect utility function

Let W be the "total wealth" and $V(\cdot)$ the indirect utility function of a given generation-0 family. Suppose the economy is in a steady state with $w_i = \gamma^i \cdot w$ all i for some known w and with 1 plus the interest rate on bequests passed from period i to i + 1 equaling a known number r for all i. Let the (exogenous) distribution of $\tilde{Z}$, D_Z, be known as well. Our generation-0 family must perceive C_{ij} all $i > 0$, which depend on Y_{ij} all $i > 0$, to be random variables. Assume utility-maximizing behavior prevails for families in all generations and $V(W)$ equals the maximized expected value of U -- see line (1) -- given W. Then if $\gamma = 1$,

$$V(W) = \max_{B \geq 0}\{u(W - B) + n \cdot h \cdot E[V(w\tilde{Z} + rB/n)]\}$$

where $E[\cdot]$ is the expected value operator and $u(\cdot)$ is the function of line (2). We will generally not include w, r, and D_Z as arguments of $V(\cdot)$ in the interests of simplicity. We do not need to sum separate terms for the n first-generation male offspring on the right-hand side of the equation because each Z_{ij} is an independent sampling from the same distribution.

Suppose $\gamma > 1$. Let $V(\cdot)$ be the indirect utility function for a generation-0 household, and let ${}_tV(\cdot)$ apply for time-t households. Then we expect ${}_tV(W) > V(W)$ all W if $t \geq 1$. In fact, our isoelastic direct utility function yields

$${}_1V(W) = \gamma^{\beta} \cdot V(W/\gamma)$$

(or, if $\beta = 0$, ${}_1V(W) = V(W/\gamma) + \ln(\gamma)/(1-h)$). Thus, our functional equation for $\gamma = 1$ can be generalized to

$$V(W) = \underset{B \geq 0}{\mathrm{Max}}\{u(W - B) + nh\gamma^{\beta} \cdot E[V(w\tilde{Z} + rB/(n\gamma))]\} \tag{3}$$

(or

$$V(W) = \underset{B \geq 0}{\mathrm{Max}}\{\ln(W - B) + [nh/(1-h)] \cdot \ln(\gamma) + nh \cdot E[V(w\tilde{Z} + rB/n\gamma)]\}.)$$

The last term on the right-hand side of line (3) equals

$$nh \cdot E[{}_1V(w\gamma\tilde{Z} + rB/n)].$$

When we use line (3) to determine $V(\cdot)$ below, for any t

$${}_tV(W) = \gamma^{\beta t}V(W/\gamma^t) \tag{4}$$

(or, if $\beta = 0$,

$${}_tV(W) = [t/(1-h)] \cdot \ln(\gamma) + V(W/\gamma^t)).$$

Let

$$V_L(W) \equiv \sum_{i=0}^{\infty} (nh)^i \cdot u(w\gamma^i Z_L).$$

Define

$$M_0(W) = W,$$

$$M_{i+1}(W) = (r/n) \cdot M_i(W) + w\gamma^{i+1} Z_U \quad \text{all } i \geq 0.$$

Let

$$V_U(W) = \sum_{i=0}^{\infty} (nh)^i \cdot u(M_i(W)).$$

Then it is not difficult to see that we must have

$$V_L(W) \leq V(W) \leq V_U(W)$$

$$\text{if } W \geq wZ_L.^{6/} \qquad (5)$$

Thus, lines (3) and (5) are "necessary conditions" which $V(\cdot)$ must obey.

In fact, they are also sufficient to uniquely define $V(\cdot)$ if we assume

$$0 < nh\gamma^{\beta} < 1 \qquad (A)$$

and

$$0 < nh(r/n)^{\beta} < 1 \quad \text{when} \quad \beta \varepsilon (0,1). \qquad (A')$$

(Line (5) shows that when $u(w\gamma^t Z_L) > 0$ some t, condition (A) is required for a finite $V(\cdot)$. Line (5) also shows that if (A) and (A') both hold, $V(\cdot)$ will be finite.7/) We have

Proposition 1: If assumptions (A) and (A') hold, there exists a unique, finite, increasing, strictly concave function $V(\cdot)$ satisfying lines (3) and (5) for all $W \geq wZ_L$.

Laitner (1979a) contains a proof.

Utility-maximizing bequests

If $V(\cdot)$ is nondecreasing and concave, the right-hand side of equation (3) uniquely defines a function $B(W)$ (or $B(W,D_{wZ})$, $B(W,D_{wZ},r)$, etc.) giving the utility-maximizing bequest of a generation-0 family with "total wealth" W. If $w_t = \gamma^t w$, time-t families use $\gamma^t \cdot B(W/\gamma^t, D_{wZ})$ rather than $B(W, D_{wZ})$ -- and the former can be developed from the right-hand side of line (3).[8/] Laitner (1979a, b, and c) develop a number of characterizations of $B(\cdot)$. In this paper we need only

Proposition 2: *Suppose* (A) *and* (A') *hold. Let* $W \geq wZ_L$. *Then*

(i) $B(\cdot)$ *is continuous and nondecreasing in* W;

(ii) $B(xW, D_{xwZ}) = x \cdot B(W, D_{wZ})$ *any* $x > 0$;

(iii) *if* $c = c(r) = (hr)^{1/(1-\beta)}/\gamma < 1$, *and if* $Y^* \equiv wZ_L/c$, $\text{Max}\{0, c \cdot (W - Y^*)\} \geq rB(W)/(n\gamma)$; *and*,

(iv) *if* $\beta \geq 0$, $B(W, D_{wZ}, r)$ *is nondecreasing in* r.

Laitner (1979a) contains proofs of parts (i) - (iii); the appendix of this paper proves part (iv).

II. A Complete Economy

In this section we combine our model of family behavior and an aggregate production function. There is no government sector yet. Although Laitner (1979a) provides a proof of the existence of a steady state for such a model if there are no restrictions on β (see line (2)) or on the precise form of the production function, we assume the function is Cobb-Douglas and $\beta \geq 0$. These constraints enable us to establish uniqueness as well as existence. Uniqueness, in turn, leads to unambiguous comparative-static outcomes in Sections III and IV.

The national distribution of wealth

We first examine the evolution of the national distribution of "wealth."

Let D_{W_t} give the national distribution of "total wealth" figures for families at any time t. Then if w and r are fixed and $w_t = \gamma^t w$ all t,

$$\tilde{W}_1 \sim (r/n) \cdot B(\tilde{W}_0, D_{wZ}) + w\gamma\tilde{Z}.$$

For time t, using part (ii) of Proposition 2,

$$\tilde{W}_t \sim (r/n) \cdot B(\tilde{W}_{t-1}, D_{w\gamma^{t-1}Z}) + w\gamma^t\tilde{Z}. \qquad (6)$$

We will say the economy has reached a steady state if w and r are constant (and known), w_t equals the marginal revenue product of labor at time t (where labor is measured in natural units), $w_t = \gamma^t w$ all t, r - 1 equals the marginal revenue product of capital at all times, and $\tilde{W}_{t+1} \sim \gamma\tilde{W}_t$ all t. Thus, if

$$\tilde{W}_0 = w\tilde{X}$$

in a steady state,

$$\tilde{W}_t \sim w\gamma^t\tilde{X} \quad \text{all } t.$$

Returning to line (6), in a steady state we will have

$$w\gamma^t\tilde{X} \sim (r/n) \cdot B(w\gamma^{t-1}\tilde{X}, D_{w\gamma^{t-1}Z}) + w\gamma^t\tilde{Z}$$

for some random variable $\tilde{X}$. Using part (ii) of Proposition 2, we can divide through by $w\gamma^t$ and obtain

$$\tilde{X} \sim (r/n\gamma) \cdot B(\tilde{X}, D_Z) + \tilde{Z}. \tag{7}$$

To prove the existence of a steady state, therefore, we must prove the existence of a pair (D_X, r) satisfying line (7) and consistent with production conditions.

Recall (from Proposition 2) that

$$c = c(r) \equiv (hr)^{1/(1-\beta)}/\gamma.$$

If assumptions (A) and (A') hold and r is such that $c(r) < 1$, there will be a unique distribution $D_{X(r)}$ such that $(D_{X(r)}, r)$ satisfies line (7).

<u>Proposition 3</u>: <u>Suppose (A) and (A') hold and</u> $c(r) < 1$. <u>Then there exists a unique distribution</u> $D_{X(r)}$ <u>with</u> $\tilde{X}(r)$ <u>satisfying line (7). There are no cycling quasi-stationary distributions that satisfy (7). All realizations of</u> $\tilde{X}(r)$ <u>fall in</u> $[Z_L, X_U(r)]$ <u>where</u> $X_U(r) \equiv Z_U/[1 - c(r)]$.

Laitner (1979a) contains a proof of this proposition.

Let the number of families started at time t be P_t. We know from Section I that the population growth rate is $n - 1$. So

$$P_t = P_0 n^t. \tag{8}$$

Since each family (i,j) supplies Z_{ij} units of labor, we assume the aggregate natural labor supply at time t is

$$L_t \equiv P_t \cdot E[\tilde{Z}]. \tag{9}$$

(Note that although each family line is affected by the randomness of $\tilde{Z}$, L_t is non-stochastic, as, in consequence, is the evolution of the economy as a whole.) Letting E_t be the aggregate supply measured in "augmented" units,

$$E_t = \gamma^t L_t = \gamma^t n^t \cdot P_0 \cdot E[\tilde{Z}]. \tag{10}$$

Supposing assumptions (A) and (A') hold and the economy is in a steady state, let $a(r,t)$ give the following ratio: the numerator is the aggregate value -- measured in time-t wage units -- of bequests carried within family lines from period t to t + 1; the denominator is the population at time t. The bounds of Proposition 3 show we can appeal to the law of large numbers to treat $a(\cdot)$ as a (unique) non-stochastic function --

$$a(r,0) = E[B(\tilde{X}(r),D_Z,r)], \tag{11}$$

$$a(r,t) = a(r,0) \quad \text{all } t. \tag{12}$$

Since all bequests are nonnegative,

$$a(r,t) \geq 0 \text{ all } t.$$

Propositions 2 and 3 enable us to prove that $a(\cdot)$ is continuous in r and that it has an asymptote at $r = \hat{r}$ where $\hat{r}$ is such that

$$c(\hat{r}) = 1.$$

Proposition 4: *Suppose (A) holds, (A') holds, and* $r \in [1,\hat{r})$. *Then* $a(r,t)$ *all* t *and* $E[\tilde{X}(r)]$ *are continuous in* r *and* $\lim_{r \uparrow \hat{r}} a(r,t) = \infty$ *any* t.

Propositions 6 and 7 of Laitner (1979a) (and line (7)) provide a proof.

To establish the uniqueness of a steady state below, we also need

Proposition 5: *Suppose (A) holds, (A') holds,* $r \in [1,\hat{r})$, *and* $\beta \geq 0$. *Then* $a(r,t)$ *is a nondecreasing function of* r. *The same is true for* $E[\tilde{X}(r)]$.

A proof (based on part (iv) of Proposition 2) is given in the appendix of this paper.

Production

As stated, we adopt a Cobb-Douglas aggregate production function:

$$Q_t = F(K_t, E_t) = (K_t)^{\alpha}(E_t)^{1-\alpha},$$

$$\alpha \varepsilon (0,1) \tag{13}$$

where Q_t is the net national product and K_t is the aggregate capital stock (measured in the same units as Q_t). We assume net total investment and aggregate consumption sum to Q_t at each time t.

Normalizing the current price of a unit of net national product to 1 in each period, for a steady state we must have

$$r - 1 = \alpha \cdot (K_t/E_t)^{\alpha-1} \quad \text{all } t, \tag{14}$$

$$w = (1 - \alpha) \cdot (K_t/E_t)^{\alpha} \quad \text{all } t.\text{9/} \tag{15}$$

We must also have the value of assets carried by the household sector (as bequests) from each period t to t + 1 just equal to the physical capital stock at time t + 1:

$$K_{t+1} = a(r,t) \cdot w\gamma^t P_t.$$

Using line (12), this becomes

$$a(r,0) = K_{t+1}/(w\gamma^t P_t) = n\gamma E[\tilde{Z}] \cdot K_{t+1}/(wE_{t+1}) \quad \text{all } t. \tag{16}$$

Combining lines (15) and (16),

$$a(r,0) = [n\gamma E[\tilde{Z}]/(1 - \alpha)] \cdot (K_{t+1}/E_{t+1})^{1-\alpha}.$$

Looking at line (14), this gives

$$a(r,0) = [n\gamma E[\tilde{Z}]/(1 - \alpha)] \cdot [\alpha/(r - 1)] \equiv g(r). \tag{17}$$

Line (14) also shows

$$r - 1 > 0.$$

Thus, we can graph the right-hand side of line (17) as a single hyperbola in Figure 1.

Given assumptions (A) and (A'), Proposition 4 shows the graph of $a(r,0)$ will be continuous and will have an asymptote at $r = \hat{r}$. Proposition 5 shows that if $\beta \geq 0$, the graph of $a(r,0)$ will be nondecreasing as well.

Given assumption (A) and

$$n\gamma \geq 1, \tag{A*}$$

Figure 1 will then establish the existence of a steady state at some $r^* \varepsilon (1,\hat{r})$ if we can show (i) $\hat{r} > 1$ and (ii) assumption (A') holds for all $r \leq \hat{r}$. In fact, Laitner (1979a) shows that (A) and (A*) imply $\hat{r} > n\gamma \geq 1$. We have

$$nh(r/n)^{\beta} = nh(hr/hn)^{\beta} = (nh)^{1-\beta}(rh)^{\beta} = (nh\gamma^{\beta})^{1-\beta} \cdot$$
$$[rh/\gamma^{1-\beta}]^{\beta} = (nh\gamma^{\beta})^{1-\beta}[c(r)^{1-\beta}]^{\beta}.$$

So, (A) and (A*) also guarantee (ii). Proposition 8 of Laitner (1979a) shows steady-state values of r with $r > \hat{r}$ are impossible. Thus, Figure 1 indicates that if $\beta \geq 0$, the pictured steady-state $r^* \varepsilon (1,\hat{r})$ will be unique. This establishes

Proposition 6: Suppose assumptions (A) and (A*) hold. Then there exists a steady state with $r^* \varepsilon (1,\hat{r})$. If $\beta \geq 0$, the steady state is unique.

We have, therefore, proven the existence and, if $\beta \geq 0$, uniqueness of a steady state. Our discussion shows bequest behavior puts an upper limit, $\hat{r} - 1$, on the steady-state interest rate. The limit is above the golden-rule level $n\gamma - 1$.

More important for our purposes, however, is the fact that Proposition 2 enables us to establish the following result:

Proposition 7: Suppose (A) and (A*) hold and the economy has reached a steady state with $\tilde{W}_t \sim w\gamma^t\tilde{X}$ all t. Then part (iii) of Proposition 2 shows a generation-t family with "wealth" $W \leq w\gamma^t Z_L/c(r)$ will not wish to leave a positive bequest.[10/] Yet, for some $\zeta > 0$, $\Pr\{w\gamma^t\tilde{X} < w\gamma^t Z_L/c(r)\} \geq \zeta$ all t.

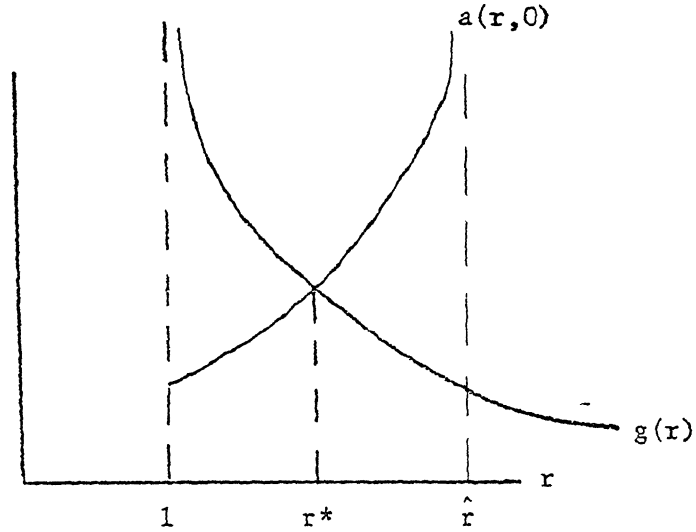

Figure 1: The supply and demand for assets

In other words, in a steady state not all of the families in any cohort will want to leave nonzero bequests (or will be on the "borderline" of doing so). Proposition 10 of Laitner (1979a) provides a proof.

An intertemporal manifestation of the same phenomenon is

Proposition 8: Suppose (A) and (A*) hold and the economy has reached a steady state at time 0. At each t we label the "total wealth" figures for all families with W_{tj}, $j = 1,\ldots,P_t$. We say the family with "wealth" W_{si} is an "ancestor" of a family with "wealth" W_{tj}, $t > s$, if the latter family is a "descendant" of the former. Let N_t be the expected at time 0 number of generation-t families such that $W_{tj} \geq Y_t^* \equiv w\gamma^t Z_L/c(r)$ and $W_{si} \geq Y_s^*$ for all "ancestors" with $s = 0,\ldots, t-1$. Let $N_t^* = N_t/P_t$. Then $\lim_{t\to\infty} N_t^* = 0$.

Proposition 8 is proven in the appendix to this paper. It shows that in a steady state, the percentage of cohort-t families which are connected to generation-0 households via "ancestral" sequences of positive or 0 but "borderline" bequests shrinks to 0 as $t \to \infty$.

We now add a government sector to our model.

III. National Debt

In this section we briefly discuss an infinite-time-horizon version of Barro's (1974) analysis of national debt and bequests. Then we modify the framework of Section II to include public debt and show that declines in steady-state average bequests (measured in wage units) and in the steady-state aggregate capital-to-"effective" labor ratio can result.

Barro's model

Barro argues that household bequest behavior will tend to eliminate any effect of government debt on aggregate (physical) capital accumulation. The reasoning in the case of an endless economy might run as follows.[11/] Suppose the government issues $1 worth of bonds at time 0, gives the sales proceeds to existing households, and warns the public to expect future tax increases sufficient to cover debt-service obligations.[12/] We can think of the bonds as consols. Suppose any redistributions stemming from disparities within lines of "descent" between the current government gift and the present value of new tax liabilities are of "second-order" importance. Finally, suppose that all families (in all cohorts) will desire to leave bequests in the absence of government actions, or will be on the "borderline" of doing so -- in Barro's terminology, that all bequest transfers are "operative." Then if a family receives $x from the government and learns that the present value of future tax liabilities within its line of "descent" will change by the same amount, all first-order conditions implicit in the definition of its $V(\cdot)$ will continue to hold after the bond issue if and only if the family sets aside the whole sum it receives to cover the new taxes. Thus savings will rise (now and in the future) by just enough to finance the government debt -- the physical capital stock remaining unchanged.

The steady-state consequences of national debt in our model

Propositions 7 and 8 show that in our model Barro's assumption of universally "operative bequest transfers" cannot hold. This result emerges because we require that bequests be nonnegative, we assume that there is a distribution of labor incomes within each cohort, and we place a minimum of $w_t Z_L$ with $Z_L > 0$ on any time-t family's labor earnings. Thus, a long sequence of unlucky samplings from $\tilde{Z}$ within a family line can lead to a household having such a low "total wealth" figure relative to the minimum labor income of each of its "descendants" that it will choose a "corner solution" bequest. Rather than merely reiterating Propositions 7 and 8, however, we now incorporate government debt into the model of Section II and investigate the changes which result.

Let D_t be the time-t government debt (measured in current dollars). Suppose the government first issued debt at time t = -T, transferred the debt-sales proceeds at time -T to the public, and has not engaged in any spending or transfer programs since. Suppose taxes for $t \geq -T$ have been used only to meet debt-interest obligations in the manner specified below. We assume the government pays the same rate of interest as private-sector participants. To facilitate the analysis of steady states, we also assume

$$D_t = (n\bar{\gamma})^t D_0 \quad \text{all } t \geq -T. \tag{18}$$

Suppose all tax revenues come from a proportional tax on all factor payments (recall note 12). Let the tax rate be

$$\tau \varepsilon [0,1).$$

The rate is set from a government budgetary constraint as follows. Government steady-state financial needs at time t will be

$$D_t \cdot (r - 1).$$

Given line (18), revenues from new bond issues used to keep D_t growing at the "natural rate" will be

$$(n\bar{\gamma} - 1) \cdot D_t.$$

Depending on the relationship of r and $n\gamma$, therefore, continuing bond sales may partially offset interest payments on the existing debt. Taxes must cover the remainder. So, τ must satisfy (in the steady state)

$$D_t \cdot (r - n\gamma) = \tau \cdot Q_t \text{ all } t. \tag{19}$$

Our household bequest model needs little modification. Letting

$$r^* = r \cdot (1 - \tau),$$

line (7) changes to

$$\tilde{X}(r^*,\tau) \sim (r^*/n\gamma) \cdot B(\tilde{X}(r^*,\tau), D_{(1-\tau)Z}, r^*) + (1 - \tau) \cdot \tilde{Z}. \tag{20}$$

Thus, part (ii) of Proposition 2 shows

$$\tilde{X}(r^*,\tau) \sim (1 - \tau) \cdot \tilde{X}(r^*,0).$$

Proposition 3 remains valid, as do Propositions 4 and 5 if we replace $a(\cdot)$ with

$$\begin{aligned} A(r^*,\tau,t) = A(r^*,\tau,0) &\equiv E[B(\tilde{X}(r^*,\tau), D_{(1-\tau)Z}, r^*)] \\ &= (1 - \tau) \cdot E[B(\tilde{X}(r^*,0), D_Z, r^*)] \\ &= (1 - \tau) \cdot A(r^*,0,0). \end{aligned} \tag{21}$$

Thus, if $\beta \geq 0$ and r is given, an increase in τ will lower $A(r^*,\tau,t)$ any t by lowering r^* -- hence (see Proposition 5), decreasing $A(r^*,0,0)$ -- and by raising the second argument of $A(\cdot)$ -- hence, decreasing $(1 - \tau) \cdot A(r^*,0,0)$ for any given r^*.

The new accounting equation for assets is

$$A(r^*,\tau,t) = (K_{t+1} + D_{t+1})/(w\gamma^t P_t) \quad --$$

household (bequest) savings must now finance both physical capital and government debt. Using lines (14), (15), (19), and (21), the equation becomes

$$A(r^*,\tau,0) = n\gamma(K_{t+1} + D_{t+1})/(w\gamma^{t+1}P_{t+1})$$

$$= n\gamma \cdot E[\tilde{Z}] \cdot \{[K_{t+1}/(wE_{t+1})] +$$

$$[D_{t+1}/(wE_{t+1})]\} = n\gamma \cdot E[\tilde{Z}] \cdot$$

$$\{[1/(1 - \alpha)] \cdot [\alpha/(r - 1)] +$$

$$[1/(1 - \alpha)] \cdot [\tau/(r - n\gamma)]\} \equiv G(r,\tau). \qquad (22)$$

For any given τ, a root $r > 1$ -- see line (14) -- of this equation determines a steady state for the new model.

Proposition 5 shows that given (A), (A*), and $\beta \geq 0$, $A(r,0,0)$ is nondecreasing in r. Proposition 4, therefore, establishes that assumption (A), assumption (A*), and $\beta \geq 0$ imply line (22) has a single root r^* with $r^* \geq n\gamma$ for each $\tau \varepsilon [0,1)$ -- see Figure 2. (Given the root and τ we can calculate D_0 from lines (13), (14), and (19).) These ideas suffice to prove

Proposition 9: Suppose $\beta \geq 0$ and assumptions (A) and (A*) hold. Then within the set $\{r: r \geq n\gamma\}$ there exists one and only one steady-state r -- $r = r^*$ in Figure 2. We have $r^* \varepsilon (n\gamma, \hat{r})$.

Figure 2 shows a second steady-state r, $r = r^{**} < n\gamma$, also exists for each $\tau > 0$ if $n\gamma > 1$. Besides requiring a capital-to-"effective" labor ratio above the golden-rule level, however, $r = r^{**}$ and $\tau > 0$ imply $D_0 < 0$ -- see line (19). We focus our attention exclusively on the root $r = r^* > n\gamma$.

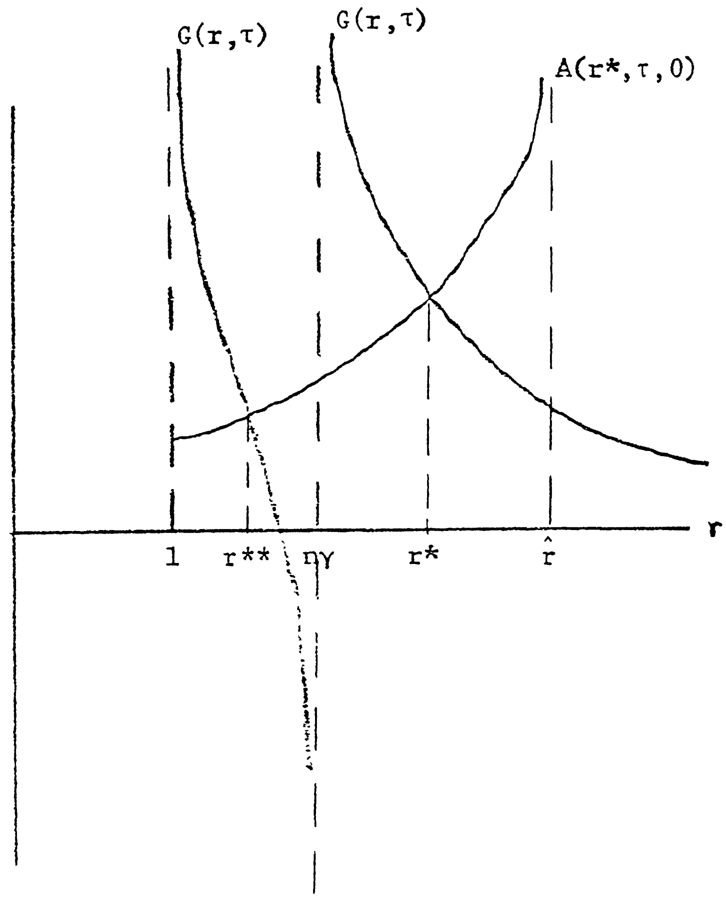

Figure 2: National debt and the steady-state interest rate

Consider an increase in τ. We have seen this will cause the graph of $A(r^*,\tau,0)$ in Figure 2 to fall. The graph of $G(r,\tau)$ will rise, on the other hand. Thus, r^* must increase. This establishes

Proposition 10: *Suppose* $\beta \geq 0$ *and assumptions* (A) *and* (A*) *hold. Let* $r^*(\tau)$ *give the unique value of* r^* *determined in Proposition 9 if* $\tau > 0$ *or in Proposition 6 if* $\tau = 0$. *Then* $r^*(\cdot)$ *is increasing. If* $D_0 = 0 = \tau$ *initially, an increase in* D_0 *(with* τ *endogenously determined by line (19)) must cause* r^* *to rise.*

Therefore, if two economies have $\beta \geq 0$ and identical parameters except that $D_0 = \tau = 0$ in one and D_0 and $\tau > 0$ in the other, the former will have a lower steady-state interest rate, a higher steady-state capital-to-"effective" labor ratio, a higher steady-state capital-to-output ratio, and a higher steady-state gross-of-tax wage rate.

Lines (19) and (22) show that provided $\beta \geq 0$ and $r > n\gamma$, each tax rate τ yields a unique steady-state D_0 and $D_0(\tau)$ is continuous. Unfortunately, we cannot guarantee that $D_0(\tau)$ is monotone nondecreasing. However, suppose that government chooses for each D_0 the lowest τ compatible with lines (19) and (22). Then τ and D_0 will have a nonnegative, monotone relationship, and Proposition 10 will show that larger choices for D_0 yield values for r^* which are larger or unchanged. The magnitude of D_0 may be a more natural measure of the size of the national debt that τ is.

We can examine in more detail the way Proposition 10 works. Let $\beta \geq 0$. Consider an economy which has no government debt and which has reached its unique (see Proposition 6) steady state. Then suppose the government issues bonds to finance one-time-only gift payments to f . Let the economy pass to its unique new steady state with $r^* > n\gamma$. Proposition 8 shows that when the state is reached, the effects of the one-time-only transfer payments other than the new steady-state tax liabilities and stock of public bonds must have completely disappeared. Recall that Barro's (1974) analysis relied on the transfer payments themselves to finance D_t all t. In our model, on the other hand, in the new steady state D_t must be financed entirely from household savings which would otherwise be available for supporting physical capital.

Not only do the government bonds compete with physical capital for private-sector financing, but also the taxes needed to service $D_t \cdot (r - n\gamma) > 0$ lower average steady-state bequest savings as measured in wage units. We can see this from line (21),

$$A(r^*,\tau,0) = (1 - \tau) \cdot A(r^*,0,0) <$$

$$A(r^*,0,0) < A(r,0,0) \text{ if } \tau > 0 \text{ and } \beta \geq 0.$$

Roughly speaking, the debt-induced increase in each cohort's current and "ancestral" taxes causes its bequests to fall (on average) more than the new tax burdens on "descendants" cause them to rise. The change from r to r* leads to further reductions. The consequence is a decrease in savings -- as measured in wage units -- just when the public debt expands. Proposition 10 shows an initial increase in public debt will also lower the amount of capital or debt each wage unit of bequest saving can finance.

IV. Social Security

This section introduces an unfunded social security system. Although Barro (1974) argues that such a system may have no effect on aggregate capital accumulation, we show that if $\beta \geq 0$, social security will tend to lower the steady-state capital-to-"effective" labor ratio in an overlapping-generations version of our model.

Two-period lives

We first modify our household model to allow a retirement stage of life.

Suppose every family now lives two periods and supplies labor only in the initial one. Let the steady-state labor income of a family started at time t be a random sampling from $w\gamma^t\tilde{Z}$. Assume we can think of each such a family as allocating its "total wealth," say, W_{tj}, in its youth among first-period-of-life consumption, ${}_1C_{tj}$; second-period-of-life consumption, ${}_2C_{tj}$; and a bequest. We assume cohorts overlap, so that a first-generation "descendant" receives his inheritance just as his parents enjoy their retirement-age consumption.[13/]

Consider a time-0 family with "wealth" W. Let B be its bequest, and let S be its saving at time 0 for retirement consumption at time 1. For a given value of B, let

$$u^*(W - B) \equiv \underset{S}{\text{Max}}\{u(W - B - S) + \Lambda \cdot u(rS)\} \tag{23}$$

be the family's lifetime utility -- where $u(\cdot)$ is as in line (2) and Λ is an exogenous constant. Then

$$u^*(W - B) = u^{**} \cdot u(W - B) \tag{24}$$

with

$$u^{**} \equiv u(x/(1 + x)) + \Lambda \cdot u(r/(1 + x)),$$

$$x \equiv r^{\beta/(\beta-1)}\Lambda^{1/(\beta-1)}.$$

(If $\beta = 0$,

$$u^*(W - B) = u^{**} + u(W - B).)$$

Replacing $u(\cdot)$ in lines (3) and (5) with $u^*(\cdot)$, we have necessary and sufficient conditions for defining a new version of $V(\cdot)$. Proposition 1 remains valid.

The simple form of $u^*(\cdot)$ implies that Propositions 2 and 3 carry over (without modification) as well. Let $A^*(r,t)$ be the normalized average of the sum of life-cycle and bequest saving for young families at time t. Using line (23), if $S(W - B)$ is a family's utility-maximizing S given W and B,

$$\begin{aligned} S(W - B) &= [1/(1 + r^{\beta/(\beta-1)}\Lambda^{1/(\beta-1)})] \cdot (W - B) \\ &\equiv s(r) \cdot (W - B). \end{aligned} \tag{25}$$

So,

$$\begin{aligned} A^*(r,t) = A^*(r,0) &= E[B(\tilde{X}(r),D_Z,r)] \\ &+ s(r) \cdot \{E[\tilde{X}(r)] - \\ &E[B(\tilde{X}(r),D_Z,r)]\} \text{ all } t. \end{aligned} \tag{26}$$

We can see that Proposition 4 remains valid. Proposition 5 carrys over as well: if $\beta \geq 0$, $s'(r) > 0$; $s(r) \varepsilon (0,1)$; so, if $r^* > r$ and $\beta \geq 0$, the old version Proposition 5 implies

$$\begin{aligned} A^*(r^*,0) &\geq E[B(\tilde{X}(r^*),D_Z,r^*)] + \\ &s(r) \cdot \{E[\tilde{X}(r^*)] - E[B(\tilde{X}(r^*),D_Z,r^*)]\} \\ &= [1 - s(r)] \cdot E[B(\tilde{X}(r^*),D_Z,r^*)] + \\ &s(r) \cdot E[\tilde{X}(r^*)] \geq \\ &[1 - s(r)] \cdot E[B(\tilde{X}(r),D_Z,r)] + \\ &s(r) \cdot E[\tilde{X}(r)] = A^*(r,0). \end{aligned}$$

Propositions 6 - 8 also continue to hold.

A social security system

Let a family initiated at time t and earning a labor income of Y_t have a social-security payroll tax obligation of

$$\sigma Y_t.$$

We assume

$$\sigma \varepsilon [0,1).$$

We also assume the family will receive a retirement benefit payment of

$$\sigma n \gamma Y_t.$$

The social security system will have no trust fund; provided the economy has reached a steady state, aggregate payroll tax receipts and benefit payments will just balance. For the sake of simplicity, we exclude national debt and all taxes other than σ from the model.

Steady-state growth

We now compare the steady states of economies with $\sigma = 0$ and $\sigma > 0$.

First consider family bequests. If a family's gross-of-tax labor income is Y_t, its after-tax labor earnings (plus social security benefits) will be

$$Y_t - \sigma \cdot [1 - (n\gamma/r)] \cdot Y_t \equiv \theta(r,\sigma) \cdot Y_t. \tag{27}$$

We have

$$r \geq n\gamma \text{ implies } \theta(r,\sigma)\varepsilon(0,1]$$

$$\text{and } \partial\theta(r,\sigma)/\partial\sigma \leq 0. \tag{28}$$

Suppose the family's inheritance is I_t. Then its net "total wealth" will be

$$W_t = \theta(r,\sigma) \cdot Y_t + I_t.$$

Its bequest will be

$$B_t = B(W_t, D_{\theta(r,\sigma)w\gamma^t Z}, r) \text{ --}$$

with $B(\cdot)$ as in Section I. When $r > n\gamma$, for a given r, w, and W_t, we could prove the family's bequest will be larger if $\sigma > 0$ than if $\sigma = 0$.

Next consider the stationary distribution of normalized "total wealth" values, however. Line (7) becomes

$$\tilde{X}(r,\sigma) \sim (r/n\gamma) \cdot B(\tilde{X}(r,\sigma), D_{\theta(r,\sigma)Z}, r) + \theta(r,\sigma)\tilde{Z}.$$

Part (ii) of Proposition 2 shows, therefore, that

$$\tilde{X}(r,\sigma) \sim \theta(r,\sigma) \cdot \tilde{X}(r,0).$$

So,

$$E[\tilde{X}(r,\sigma)] = \theta(r,\sigma) \cdot E[\tilde{X}(r,0)] \tag{29}$$

and, using Proposition 2 again,

$$E[B(\tilde{X}(r,\sigma), D_{\theta(r,\sigma)Z}, r)] = \theta(r,\sigma) \cdot E[B(\tilde{X}(r,0), D_Z, r)]. \tag{30}$$

Thus, for a given r, an unfunded social security program lowers the average normalized steady-state bequest if $r \geq n\gamma$.

Social security also affects life-cycle saving. Line (25) shows that if $\sigma = 0$, a time-t family with "wealth" W_t and bequest B_t would choose to save for life-cycle purposes

$$s(r) \cdot (W_t - B_t)$$

in its first period of life. From the family's perspective, however, a social security system both changes the net labor earnings component of W_t and transfers income from the first to the second period of life. Thus, if $S^*(W_t - B_t, Y_t, \sigma)$ is a family's first-period-of-life saving for retirement, and if W_t is net "total wealth,"

$$S^*(W_t - B_t, Y_t, \sigma) = s(r) \cdot (W_t - B_t)$$

$$- \sigma n\gamma Y_t / r \text{ --}$$

where s(•) is defined in line (25).

Let $A^{**}(r,\sigma,t)$ be average steady-state asset holdings carried from period t to t + 1 by young families and normalized by $w\gamma^t$ -- where is given in gross-of-tax terms. Then lines (26), (29), and (30) imply

$$\begin{aligned} A^{**}(r,\sigma,t) = A^{**}(r,\sigma,0) = \\ E[B(\tilde{X}(r,\sigma), D_{\theta(r,\sigma)Z}, r)] + \\ s(r) \cdot \{E[\tilde{X}(r,\sigma)] - \\ E[B(\tilde{X}(r,\sigma), D_{\theta(r,\sigma)Z}, r)]\} - \\ (\sigma n\gamma/r) \cdot E[\tilde{Z}] = \theta(r,\sigma) \cdot A^*(r,0) \\ - (\sigma n\gamma/r) \cdot E[\tilde{Z}]. \end{aligned} \tag{31}$$

We have already seen that Propositions 4 and 5 hold for A*(•).

As in Section II, the asset accounting equation for society,

$$A^{**}(r,\sigma,0) = K_{t+1}/(w\gamma^t P_t),$$

will determine the steady-state interest rate, r - 1. Using lines (14) and (15) the equation becomes

$$\begin{aligned} A^{**}(r,\sigma,0) &= n\gamma \cdot E[\tilde{Z}] \cdot [K_{t+1}/(w \cdot E_{t+1})] \\ &= [n\gamma \cdot E[\tilde{Z}]/(1 - \alpha)] \cdot [\alpha/(r - 1)]. \end{aligned}$$

So, line (31) yields

$$\begin{aligned} \theta(r,\sigma) \cdot A^*(r,0) = n\gamma \cdot E[\tilde{Z}] \cdot \\ \{[\alpha/(1 - \alpha)] \cdot [1/(r - 1)] + \\ (\sigma/r)\}. \end{aligned} \tag{32}$$

Let

$$\psi(r) \equiv r/(r - 1)$$

and

$$\theta^*(r,\sigma) \equiv r \cdot \theta(r,\sigma) = (1 - \sigma) \cdot r + n\gamma\sigma.$$

Then multiplying line (32) through by r,

$$\theta^*(r,\sigma) \cdot A^*(r,0) = (n\gamma \cdot E[\tilde{Z}]) \cdot \{[\alpha/(1 - \alpha)] \cdot \psi(r) + \sigma\}. \quad (33)$$

If $\beta \geq 0$, the left-hand side of (33) is finite, nonnegative, continuous in r, and nondecreasing in r for each $r \varepsilon [1,\hat{r})$. It diverges to ∞ as $r \uparrow \hat{r}$ and, as in Section II, is undefined for $r > \hat{r}$. Because $\psi'(r) < 0$ and $\psi(1) = \infty$, the right-hand side is finite, continuous, positive, and decreasing for all $r > 1$. It diverges to ∞ as $r \downarrow 1$. As shown in Figure 3, therefore, there will be a unique $r^* = r^*(\sigma)$:

Proposition 11: Assume conditions (A) and (A*) hold, $\beta \geq 0$, and $\sigma \varepsilon [0,1)$. Then our model has a unique steady-state $r = r^*$ with $r^* = r^*(\sigma) \varepsilon (1,\hat{r})$.

Suppose we also have $r^*(\sigma) \geq n\gamma$. Then line (28) shows that an increase in σ will lower (or leave unchanged) the graph of the left-hand side of line (33) in Figure 3. The graph of the right-hand side will rise. Thus, Figure 3 gives

Proposition 12: Suppose assumptions (A) and (A*) hold, $\sigma \varepsilon [0,1)$, and $\beta \geq 0$. Then the unique $r^* = r^*(\sigma)$ is an increasing function of σ if $r^*(\sigma) \geq n\gamma$.

Therefore, if the economy has not accumulated physical capital beyond the golden-rule level, and if $\beta \geq 0$, an enlargement of the role of the social security system (in the sense of an increase in σ) will raise the steady-state interest rate. Hence, the steady-state output-to-capital ratio will rise, the capital-to-"effective" labor ratio will fall, and the wage rate (in both net and gross terms) will fall.

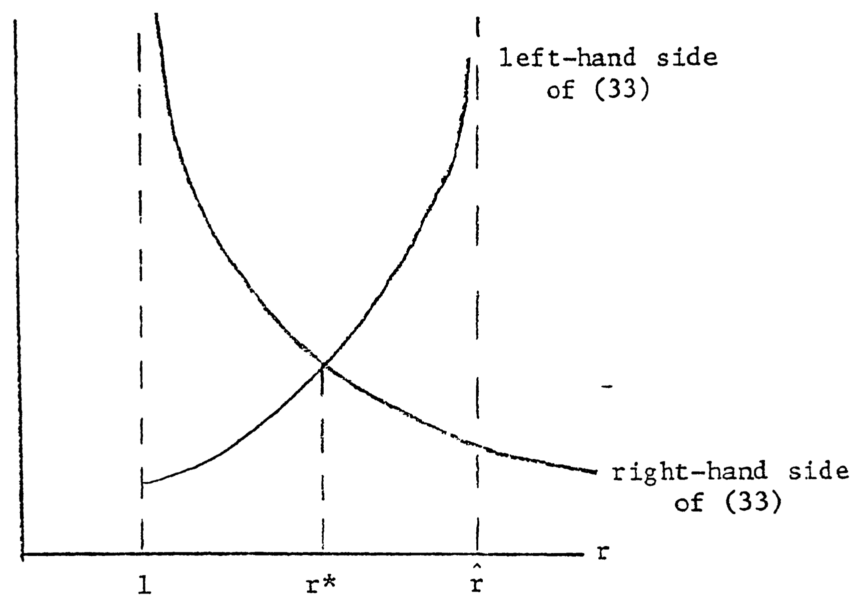

Figure 3: Steady-state outcomes with social security

The effects are generated through two channels: Suppose $\beta \geq 0$ and $r \geq n\gamma$. Then at any given (gross) wage rate social security reduces each family's net-of-tax lifetime labor income, thereby causing steady-state average bequests measured in (gross) wage units to decline; and, social security reduces family life-cycle saving. As in Section III, although second-period-of-life households receive a windfall at the time a social security system (with no trust fund) is created, Proposition 8 shows that the windfall has no effect on the new steady state of the economy.

V. Conclusion

We have presented a model of family bequest behavior with finite budget constraints and increasing, strictly concave indirect utility functions. We prove the existence of a steady state and present a parameter constraint, $\beta \geq 0$, sufficient to guarantee uniqueness in our simplest formulation. An important implication of the model is that in a steady state, some families in each cohort will choose not to leave bequests (and they will not be "borderline cases" in this regard).

Sections III and IV deal with applications of the basic model. Given $\beta \geq 0$ and interest rates above the golden-run level, Section III shows that moving from no government debt to a positive one will lower the steady-state capital-to-"effective" labor ratio. Section IV shows that in the same circumstances, an unfunded social security system will likewise reduce the economy's capital intensity. Part of the reason for each result is the phenomenon (in a different guise) mentioned at the end of the preceding paragraph -- once the economy has reached a steady state in our model, the windfall effects of instituting a government debt or social security system will have been completely damped out (see Proposition 8).

We find, in fact, that changes in steady-state bequest saving may serve to reinforce the reactions (to various policies) revealed in models with no inheritances -- see, for example, Diamond (1965), Phelps and Shell (1969), Samuelson (1975), and Kotlikoff (1979). Whether we can profitably study the long-run impact of public debt and social security with the latter type of model is an empirical question.

We have not ruled out the possibility that bequest behavior can almost neutralize the effects of additions to the national debt (or increases in social security benefits) in the short run -- the "short run" perhaps lasting for generations. A rigorous study of non-steady-state growth paths in the context of our model is, however, beyond the scope of this paper.

Appendix

I. Proof of part (iv) of Proposition 2.

We assume $\beta > 0$. The case with $\beta = 0$ is analogous.

Step 1: Let $V_0(W,r) \equiv u(W)$. Recursively define $V_{i+1}(W,r) = \underset{B \geq 0}{\mathrm{Max}}\{u(W - B) + nh\gamma^{\beta} \cdot E[V_i(\tilde{Y} + rB/(n\gamma),r)]\} \equiv u(W - B_{i+1}(W,r)) + nh\gamma^{\beta} \cdot E[V_i(\tilde{Y} + rB_{i+1}(W,r)/(n\gamma),r)]$ all $i \geq 0$ where $\tilde{Y} = w\tilde{Z}$. Then it is not difficult to see that each $B_i(\cdot)$ is uniquely defined and $\lim_{i\to\infty} B_i(W,r) = B(W,r)$ each W and r -- see page 472 of Laitner (1979b).

Step 2: Suppose $r^* > r$ and $B_k(W,r) \leq B_k(W,r^*)$ all W. Let $B_k^*(W,r)$ stand for $B_k(\cdot)$ with $(r/r^*) \cdot \tilde{Z}$ replacing $\tilde{Z}$ in Step 1. It is not difficult to see (using induction on k) that $B_k^*(W,r) \geq B_k(W,r)$ all W, r, and k.

The envelope theorem shows $\partial V_k(W,r)/\partial W = u'(W - B_k(W,r))$. A proof similar to the one used for part (i) of Proposition 2 shows $C_k(W,r) = W - B_k(W,r)$ and $C_k^*(W,r) = W - B_k^*(W,r)$ are increasing in W for all k and r. Similarly, we can easily extend part (ii) of Proposition 2 to apply for all $B_k(\cdot)$ and $B_k^*(\cdot)$.

Let $\Delta(r) \equiv rh\gamma^{\beta-1}$. Suppose $0 \leq B_{k+1}(W,r^*) < B_{k+1}(W,r)$. Then $u'(W - B_{k+1}(W,r)) > u'(W - B_{k+1}(W,r^*)) \geq \Delta(r^*) \cdot E[u'(\tilde{Y} + r^*B_{k+1}(W,r^*)/(n\gamma) - B_k(\tilde{Y} + r^*B_{k+1}(W,r^*)/(n\gamma),r^*))] = (r^*/r)^{\beta-1}\Delta(r^*) \cdot E[u'(\tilde{Y} \cdot (r/r^*) + rB_{k+1}(W,r^*)/(n\gamma) - B_k^*(\tilde{Y} \cdot (r/r^*) + rB_{k+1}(W,r^*)/(n\gamma),r^*))] \geq (r^*/r)^{\beta-1}\Delta(r^*) \cdot E[u'(\tilde{Y} + rB_{k+1}(W,r^*)/(n\gamma) - B_k^*(\tilde{Y} + rB_{k+1}(W,r^*)/(n\gamma),r^*))] = (r^*/r)^{\beta}\Delta(r) \cdot E[\cdot] > \Delta(r) \cdot E[\cdot] \geq \Delta(r) \cdot E[u'(\tilde{Y} + rB_{k+1}(W,r^*)/(n\gamma) - B_k(\tilde{Y} + rB_{k+1}(W,r^*)/(n\gamma),r^*))] \geq \Delta(r) \cdot E[u'(\tilde{Y} + rB_{k+1}(W,r^*)/(n\gamma) - B_k(\tilde{Y} + rB_{k+1}(W,r^*)/(n\gamma),r))] > \Delta(r) \cdot E[u'(\tilde{Y} + rB_{k+1}(W,r)/(n\gamma) - B_k(\tilde{Y} + rB_{k+1}(W,r)/(n\gamma),r))] = u'(W - B_{k+1}(W,r))$, which is impossible. Hence, $B_{k+1}(W,r^*) < B_{k+1}(W,r)$ is impossible any W.

Step 3: Setting $B_0(W,r) = 0$ all W and r, Step 2 inductively shows $r^* > r$ implies

$B_k(W,r^*) \geq B_k(W,r)$ all W and k. Thus, $B(W,r^*) = \lim_{k\to\infty} B_k(W,r^*) \geq \lim_{k\to\infty} B_k(W,r) = B(W,r)$ all W. //

II. Proof of Proposition 5.

Let $\tilde{X}_0(r) = \tilde{Z}$. Recursively define $\tilde{X}_{i+1}(r) = (r/n\gamma) \cdot B(\tilde{X}_i(r),D_Z,r) + \tilde{Z}$ all $i \geq 0$.

For any random variables $\tilde{R}$ and $\tilde{R}^*$ we say $D_R \geq D_{R^*}$ if $D_R(x) \leq D_{R^*}(x)$ all x. Then the nonnegativity of all bequests implies $D_{X_1} \geq D_{X_0}$. Suppose $\tilde{Z}_2 \sim \tilde{Z}_1 \sim \tilde{Z}_0 \sim \tilde{Z}$ and each is independent. Then $\tilde{X}_2 \sim (r/(n\gamma)) \cdot B[(r/(n\gamma)) \cdot B(\tilde{Z}_2,D_Z,r) + \tilde{Z}_1,D_Z,r] + \tilde{Z}_0$ and $\tilde{X}_1 \sim (r/(n\gamma)) \cdot B[\tilde{Z}_1,D_Z,r] + \tilde{Z}_0$. So, part (i) of Proposition 2 implies $D_{X_2} \geq D_{X_1}$. Continuing in this way, we can see that $D_{X_{i+1}} \geq D_{X_i}$ all $i \geq 0$. Thus, $D_{X_i} \to D_X$ monotonically -- with $\tilde{X}$ as in Proposition 3.

Let $r^* > r$. Then part (iv) of Proposition 2 shows $D_{X_1(r^*)} \geq D_{X_1(r)}$. Using induction on i, $D_{X_i(r^*)} \geq D_{X_i(r)}$ all i. So, $D_{X(r^*)} = \lim_{i\to\infty} D_{X_i(r^*)} \geq \lim_{i\to\infty} D_{X_i(r)} = D_{X(r)}$. Thus $E[\tilde{X}(r^*)] \geq E[\tilde{X}(r)]$, and $E[B(\tilde{X}(r^*),D_Z,r^*)] \geq E[B(\tilde{X}(r),D_Z,r^*)] \geq E[B(\tilde{X}(r),D_Z,r)]$. //

III. Proof of Proposition 8.

The proof of Proposition 9 in Laitner (1979a) shows that there exists a number $\zeta^* > 0$ and an integer $T > 0$ such that $N_{(m+1)T} \leq N_{mT} \cdot (1 - \zeta^*) \cdot n^T$. Thus, $N_{mT} \leq [(1 - \zeta^*)n^T]^m \cdot N_0$. So, $N^*_{mT} = (1 - \zeta^*)^m N_0/P_0$. Hence, $\lim_{m\to\infty} N^*_{mT} = 0$. The same reasoning yields $\lim_{m\to\infty} N^*_{mT+i} = 0$ any $i = 1,\ldots,T - 1$. //

References

Atkinson, A.B. "Capital Taxes, the Redistribution of Wealth and Individual Savings." Review of Economic Studies 38 (April 1971): 209-228.

Barro, R.J. "Are Government Bonds Net Wealth?" Journal of Political Economy 82 (November/December 1974): 1095-1117.

Becker, G.S., and Tomes, N. "An Equilibrium Theory of the Distribution of Income and Intergenerational Mobility." Journal of Political Economy 87 (December 1979): 1153-1189.

Blinder, A.S. "Distributional Effects and the Aggregate Consumption Function." Journal of Political Economy 83 (June 1975): 447-478.

Calvo, G.A., Kotlikoff, L.J., and Rodriguez, C.A. "The Incidence of a Tax on Pure Rent: A New(?) Reason for an Old Answer." Journal of Political Economy 87 (August 1979): 869-874.

Diamond, P. "National Debt in a Neoclassical Growth Model." American Economic Review 55 (December 1965): 1126-1150.

Feldstein, M. "The Surprising Incidence of a Tax on Pure Rent: A New Answer to an Old Question." Journal of Political Economy 85 (April 1977): 349-360.

Katzner, D.W. Static Demand Theory. New York: Macmillan, 1970.

Koopmans, T. "Stationary Ordinal Utility and Impatience." Econometrica 28 (April 1960): 287-309.

Kotlikoff, L.J. "Social Security and Equilibrium Capital Intensity." Quarterly Journal of Economics 93 (May 1979): 233-254.

Laitner, J. "Bequests, Golden-age Capital Accumulation and Government Debt." Economica 46 (November 1979a): 403-414.

__________. "Household Bequest Behavior and the National Distribution of Wealth." Review of Economic Studies 46 (July 1979b): 467-483.

__________. "Household Bequests, Perfect Expectations, and the National Distribution of Wealth." Econometrica 45 (September 1979c): 1175-1194.

Phelps, E.S., and Shell, K. "Public Debt, Taxation, and Capital Intensiveness." Journal of Economic Theory, 1 (December 1969): 330-346.

Samuelson, P.A. "Optimal Social Security in a Life Cycle Growth Model." International Economic Review 16 (October 1975): 539-544.

FOOTNOTES

1. This is not true in the bequest models of Atkinson (1971), Becker and Tomes (1979), and Blinder (1975), for example. The latter models will reproduce neither Barro's results nor our own, however.

2. Line (2) imposes homotheticity on the function of line (1) -- see Theorem 2.4-4 of Katzner (1970), for instance.

3. This could be rigorously established using part (iii) of Proposition 2 below.

4. Barro -- see page 1100 -- imposes this same constraint in his analysis.

5. See also note 1 of Laitner (1979a).

6. Use line (1). Page 405 of Laitner (1979a) supplies more details.

7. Page 470 of Laitner (1970b) presents more details.

8. To derive $\gamma^t \cdot B(W/\gamma^t, D_{wZ})$, replace $V(\cdot)$ on the right-hand side of line (3) with ${}_tV(\cdot)$ and then use lines (2) and (4).

9. Note that $w_t = \gamma^t w$ is the wage rate for a natural unit of labor at time t and w is the wage for an "effective" (or augmented) unit.

10. The bequest of a generation-t family with "wealth" W is $\gamma^t \cdot B(W/\gamma^t, D_{wZ})$. Part (ii) of Proposition 2 shows this equals $B(W, D_{w\gamma^t Z})$. Part (iii) of Proposition 2 carries over if we replace w with $w\gamma^t$.

11. Barro (1974) does not consider an endless economy in detail. See, however, his pages 1104-1106.

12. In this subsection we should think of all taxes as being collected from labor incomes (or, equivalently, as being lump-sum taxes). The outcome in the next subsection would hold for this type of taxes -- although results become even more emphatic with a general proportional tax on all factor payments.

13. Clearly there are other ways of arranging this model -- depending on the life-cycle timing of decisions and information availability.

CREST Working Papers

C-1 The Ergodic Behavior of Stochastic Processes of Economic Equilibrium, by Lawrence E. Blume.

C-2 The Token Economy, Reinforcement and the Consumer Model, by John G. Cross.

C-3 Notes on a Theory of Learning to Search, by John G. Cross.

C-4 Redistributive Taxation as Social Insurance, by Hal R. Varian.

C-5 Catastrophe Theory and the Business Cycle, by Hal R. Varian.

C-6 Household Bequests, Perfect Expectations, and the National Distribution of Wealth, by John P. Laitner.

C-7 Inflationary Disequilibrium, by John G. Cross.

C-8 Ellet's Transportation Model of an Economy with Differentiated Commodities and Consumers, I: Generic Cumulative Demand Functions, by Carl P. Simon.

C-9 Cournot Equilibrium in Factor Markets, by Ted Bergstrom.

C-10 Is a Man's Life Worth More Than His Human Capital, by Ted Bergstrom.

C-11 A Model of Sales, by Hal R. Varian.

C-12 New Techniques for the Study of Stochastic Equilibrium Processes, by Lawrence E. Blume.

C-13 Consistent Expectations, by Lawrence E. Blume.

C-14 On Baubles and Bubbles, by John G. Cross.

C-15 When Does Majority Rule Supply Public Goods Efficiently?, by Ted Bergstrom.

C-16 "Rational" Duopoly Equilibria, by John P. Laitner.

C-17 The Natural Rate of Unemployment, by John Laitner.

C-18 Learning to be Rational, by Lawrence E. Blume and David Easley.

C-19 Notes on Cost-Benefit Analysis, by Hal R. Varian.

C-20 On Capturing Oil Rents With a National Excise Tax, by Ted Bergstrom.

C-21 Cournot, Novshek and the Many Firms, by Ted Bergstrom.

C-22 The Nonparametric Approach to Demand Analysis, Hal R. Varian.

C-23 National Debt, Social Security, a Bequests*, by John Laitner

C-24 The Taking of Land: When Should pensation be Paid?, by Lawrence Blume, Daniel L. Rubinfeld and Perry Sha .