C-16

Center for Research on Economic and Social Theory
Research Seminar in Quantitative Economics

Discussion Paper

DEPARTMENT OF ECONOMICS

University of Michigan

Ann Arbor, Michigan 48109

[redacted]

"RATIONAL" DUOPOLY EQUILIBRIA

by

John P. Laitner

C-16

"RATIONAL" DUOPOLY EQUILIBRIA[1]

Consider a simple duopoly problem of the Cournot quantity-adjustment variety. Assume that both firms want to maximize their own profits and that cooperation agreements, side-payment contracts, and interfirm communications (other than notifications of the intent to market certain quantities) are ruled out. Then there are two possible analytical approaches in the static case. One we might label the "nonconjectural" model. In it the duopolists make their output decisions in isolation -- locked in separate rooms perhaps. Only after both have fixed their plans does each learn his rival's choice, and at that point no further output changes can take place.[2] The Cournot equilibrium is a natural concept for this type of model: Given sufficiently well behaved cost and demand functions, each duopolist can develop a "best-reply" function showing his profit-maximizing production level for each fixed output of the other firm. In a "nonconjectural" problem, each firm will want to choose a point on its best-reply function curve. A Cournot equilibrium has the desirable feature that both firms are choosing best-reply output levels at the same time.

We call the second possible model "conjectural." In this model firms make their output decisions simultaneously (as above), but each knows as it makes its decision what output its rival will choose.[3] Thus each firm should take account of the fact that its output decision will affect its rival's behavior. So, conjectural variation terms, functions showing the (immediate) reactions from their rivals that firms expect as responses to changes in their outputs, play an important role. We can think of the following story. At time $t - \varepsilon$, $\varepsilon > 0$, both duopolists announce their output decisions for the period to begin at time t, each firm attempting to maximize its profits in light of its conjectural variation function. After the first announcement both firms reevaluate their decisions and issue a (simultaneous) announcement of corrections. The time is $t - \delta$, $\varepsilon > \delta > 0$. The process continues until the corrected output bundle satisfies both firms. Each firm's chosen output at time t will reflect, therefore, a knowledge of its rival's output plans. Although most empirical analyses use price in place of quantity as a decision variable for producers, existing data suggest the conjectural model may be realistic, at least some cases (see Blair [2, chapter 19] and Kaplan, Dirlam, and Lanzillotti [10, pages 31-36], for example).[4]

In this paper we examine conjectural models. In particular, we study various types of "conjectural equilibria" -- stopping points for the iterative process outlined in the preceding paragraph. Such outcomes are Nash equilibria conditional on expectation functions: at equilibrium neither duopolist perceives any advantage to changing his output. In Section I we find, as might be imagined, that if we do not require that expectations correspond to reality, the set of possible "conjectural equilibrium" outcome points may well be enormous.

In Section II we attempt to remedy the overabundance of solutions by requiring that expectations ("conjectural variations") be "rational" -- in other words, we try demanding that the reactions each firm expects from its rival be the reactions which actually would occur according to the overall model. This constraint seems consistent with the "perfect information assumption" of traditional microeconomic theory. Surprisingly, we can show that the set of output pairs qualifying (with the correct choice of conjectural variation functions) as "rational conjectural equilibria" for a given problem may still be very large.

Section III extends the analysis to a dynamic model, which permits a wider "rationality" requirement. Although an extreme multiplicity of equilibria continues to prevail, we are able to define "rationality" in a more satisfactory way than is the case for the static model.

I. Conjectural Equilibria

First we lay out a simple model. Then we present our definition of a "conjectural equilibrium" and prove that a large set of output combinations can qualify.

The model has the following elements. Two firms produce a single homogeneous product which cannot be obtained elsewhere. If $Q = X + Y$ is the sum of their outputs, $P(Q)$ gives the market price. The firms' total cost functions are $\Phi(X)$ and $\Psi(Y)$, respectively. Each firm attempts to maximize its own profits. Although the firms make their output decisions simultaneously, plan changes are always possible before production begins. So, each firm is aware that its choice of a production level will affect its rival's behavior. If at any prospective output point (X,Y) with each firm aware of how much its rival plans to produce the first firm believes that an infinitesimal change Δ in its output level will elicit a reaction $\Delta \cdot (\partial Y/\partial X)^e\big|_{(X,Y)}$ in the planned output of its competitor, we define $F(X,Y) = (\partial Y/\partial X)^e\big|_{(X,Y)}$. Similarly we let the second firm's "expectation function" be $G(X,Y) = (\partial X/\partial Y)^e\big|_{(X,Y)}$.[5]

Provided second-order conditions are satisfied, a solution of the system

$$P(Q) + X \cdot P'(Q) \cdot (1 + F(X,Y)) - \Phi'(X) = 0, \tag{1}$$

$$P(Q) + Y \cdot P'(Q) \cdot (1 + G(X,Y)) - \Psi'(Y) = 0, \tag{2}$$

is an equilibrium for the recontracting process outlined in the introduction to this paper. To calculate a Cournot equilibrium we solve the system after setting $F(X,Y) = 0 = G(X,Y)$ all (X,Y). With our conjectural model there is no reason to require that a firm should expect that a change in its output will cause absolutely no change in its competitor's production level, however. Any outcome satisfying the following definition, therefore, has just as much appeal as Cournot's in terms of firms' expectations.

D1) A "Conjectural Equilibrium" is a tuple $(X^*, Y^*, F^*(\cdot), G^*(\cdot))$ such that if $F^*(\cdot)$ and $G^*(\cdot)$ give the expectation functions of firms 1 and 2, respectively, then at $(X,Y) = (X^*,Y^*)$ neither perceives that it can increase its profits by changing its output.

Each conjectural equilibrium $(X^*, Y^*, F^*(\cdot), G^*(\cdot))$ has the same game-theoretic "stability" property as an ordinary Cournot equilibrium: given $F^*(\cdot)$ and $G^*(\cdot)$, neither duopolist has any incentive to move away from (X^*,Y^*).

On the other hand, as in the case of conventional Cournot equilibria, an equilibrium $(X^*, Y^*, F^*(\cdot), G^*(\cdot))$ will not usually have to attribute "rationality". For, definition D1 does not require any correspondence between the actual reaction of the first firm to changes in Y, $\partial X/\partial Y$, and the second firm's expectations, $G^*(X,Y)$, even at (X^*,Y^*). Similarly for $\partial Y/\partial X$ and $F^*(X,Y)$. Yet, without some theory of expectations there seems to be no reason to prefer one equilibrium tuple to another.

What is more, with proper choices of $F^*(\cdot)$ and $G^*(\cdot)$ a great variety of output combinations (X^*,Y^*) can be made to satisfy D1, as the following proposition shows.

Proposition I: Suppose $\Phi(\cdot)$ and $\Psi(\cdot)$ are nondecreasing and strictly convex, $\Phi(0) = \Psi(0) = 0$, and $\Phi'(0) = \Psi'(0) = 0$. Let the market demand curve be downward sloping. Suppose $P(Q) > 0$ and $P''(Q) < 0$ all $Q > 0$. Then with the correct choice of $F^*(\cdot)$ and $G^*(\cdot)$ we can incorporate any point $(X^*,Y^*) > (0,0)$ with $P(X^* + Y^*) - \Phi'(X^*) \geq 0$ and $P(X^* + Y^*) - \Psi'(Y^*) \geq 0$ into a conjectural equi-

<u>librium tuple</u> $(X^*,Y^*,F^*(\cdot),G^*(\cdot))$.[6]

The appendix at the end of this paper supplies a proof. The proposition shows that if cost and demand functions meet the specified hypotheses, then any positive output combinations up to and including competitive-market outcomes can be made to constitute conjectural equilibria.[7] For example, fixing $F^*(X,Y) = -1 = G^*(X,Y)$ all (X,Y), any pair of competitive-market outputs (X^*,Y^*) solves system (1) - (2). Suppose $(X^c,Y^c) > (0,0)$ is the cartel outcome. Then setting $F^*(X,Y) = Y^c/X^c$ and $G^*(X,Y) = X^c/Y^c$ all (X,Y), (X^c,Y^c) solves the system. In each case we have a "stable" (in the game-theoretic sense) equilibrium provided the hypotheses of the proposition hold.

II. Rational Expectations

Our conjectural model simultaneously allows equilibrium output pairs as varied as the competitive-market and cartel outcomes. In this section we attempt to make the model more selective by requiring rational expectations. We will say the first firm's expectations are "rational" at a point (X,Y) if the firm can accurately predict the response of its rival to an infinitesimal change in X.[8/] Similarly for the second firm's expectations.

One issue immediately arises: in a static model we can only require rational expectations for each firm on a restricted set. For, according to our model each firm chooses its output to maximize its (perceived) profit function. If a firm is not maximizing (perceived) profits at (X,Y), it should immediately adjust its output, not waiting for its competitor to make any changes. If, for example, the first firm is not maximizing its profits at (X,Y), $(\partial X/\partial Y)\big|_{(X,Y)}$, the firm's actual variation, should equal $+$ or $-\infty$, and the second firm should not ask what the first's response to an infinitesimal change in Y will be until X has reached a profit-maximizing level through independent adjustments. Define

K(F(·)) = {(X,Y): firm 1 perceives X to be profit maximizing (for itself) given expectation function F(·) and an output Y for its rival}. (3)

Then if F(·) is the first firm's expectation function, we can only require that the second firm's expectations be "rational" at $(X,Y) \in K(F(\cdot))$. Similarly, if the second firm's expectation function is G(·), we can only require rational expectations for the first firm on

$L(G(\cdot)) = \{(X,Y)$: firm 2 perceives Y to be profit maximizing given X and expectation function $G(\cdot)\}$. (4)

Using these ideas, we make the following definition:

D2) <u>A "Rational Conjectural Equilibrium" is a point</u> (X^*,Y^*) <u>and a pair of expectation functions</u> $(F^*(\cdot),G^*(\cdot))$ <u>such that (i)</u> $(X^*,Y^*) \in K(F^*(\cdot)) \cap L(G^*(\cdot))$ <u>(i.e., at</u> (X^*,Y^*) <u>each firm perceives its output to be profit maximizing);</u> <u>(ii) the actual variation</u> $(\partial X/\partial Y)\big|_{(X,Y)}$ <u>equals</u> $G^*(X,Y)$ <u>at all</u> $(X,Y) \in U \cap K(F^*(\cdot))$ <u>where</u> U <u>is some open neighborhood of</u> (X^*,Y^*)<u>; and, (iii)</u> $(\partial Y/\partial X)\big|_{(X,Y)} = F^*(X,Y)$ <u>all</u> $(X,Y) \in V \cap L(G^*(\cdot))$ <u>where</u> V <u>is some open neighborhood of</u> (X^*,Y^*).

This definition requires rationality of expectations for each firm at all possible points in the immediate vicinity of (X^*,Y^*).[9] We apparently achieve a greater degree of restrictiveness than with D1 at the expense only of assuming away non-rationality.

Actually, however, in terms of points (X,Y) which can qualify (with the correct choices of $F^*(\cdot)$ and $G^*(\cdot)$) as equilibria the new definition does not turn out to be much stronger.

<u>Proposition II</u>: <u>Suppose</u> $P(\cdot)$, $\Phi(\cdot)$, <u>and</u> $\Psi(\cdot)$ <u>satisfy the hypotheses of Proposition I</u>. <u>Then if</u> (X^*,Y^*) <u>also satisfies the hypotheses, we can construct</u> $F^*(\cdot)$ <u>and</u> $G^*(\cdot)$ <u>such that</u> $(X^*,Y^*,F^*(\cdot),G^*(\cdot))$ <u>satisfies D2 provided</u> $|Y^*P'(Q^*)/(P(Q^*) - \Psi'(Y^*) + Y^*P'(Q^*))| \cdot |X^*P'(Q^*)/(P(Q^*) - \Phi'(X^*) + X^*P'(Q^*))| > 1$ <u>where</u> $Q^* = X^* + Y^*$.

In the proof, which is given in the appendix, we construct $K(F^*(\cdot))$ and $L(G^*(\cdot))$ to be straight lines. The new assumption at the end of the proposition requires that the former line be steeper than the latter. This is analogous to requiring "stable" reaction functions in the case of the standard Cournot model. While the new assumption rules out competitive-market outcomes (for which the product of absolute values is 1), it does not exclude some outcomes with X + Y infinitesimally smaller than the market-solution sum. Similarly, it rules out the cartel outcome (for which the product of absolute values is again 1), but it does allow some outcomes for which X + Y is infinitesimally larger than the cartel sum. In fact, if (X^c,Y^c) and (X^m,Y^m) are the cartel and competitive-market outcomes, respectively, the new assumption does not exclude any output point (X^*,Y^*) with $(X^c,Y^c) < (X^*,Y^*) < (X^m,Y^m)$ from being a component of an equilibrium tuple $(X^*,Y^*,F^*(\cdot),G^*(\cdot))$.[10]

Thus, although imposing "rationality" limits the set of equilibrium tuples $(X^*,Y^*,F^*(\cdot),G^*(\cdot))$ a great deal, it leaves a large set of points (X,Y) which can fit into such tuples (at least given the hypotheses of Proposition II). Furthermore, the concept of rationality embodied in definition D2 is far from satisfying. For, consider any point $(X^{**},Y^{**}) \in V \cap (L(G^*(\cdot)) - K(F^*(\cdot)))$. Assume $(X^*,Y^*,F^*(\cdot),G^*(\cdot))$ satisfies D2. Then the first firm's conjectures about its rival's reactions to changes in X are accurate at (X^{**},Y^{**}). Yet, the second firm's reactions are based on its conjectures at (X^{**},Y^{**}). The latter conjectures are meaningless at (X^{**},Y^{**}), however, since $(X^{**},Y^{**}) \notin K(F^*(\cdot))$. In a sense, therefore, we have succeeded in imposing "rationality" at one, and only one, additional level of sophistication.

Even at (X^*,Y^*) the last argument is true. For if firm 1 varies X slightly, the response of firm 2 will coincide with the former firm's expectations. Since firm 2 is reacting to a non-profit-maximizing action on the part of firm 1, however, its change in Y cannot be based on "rational" expectations.

III. A Dynamic Model

In this section we stop asking that one firm's expectations about its competitor's behavior be rational even if the competitor's behavior is based on expectations which are themselves unfounded. We switch to a discrete-time dynamic model with a cost-of-adjustment function for each firm (see, for example, Gould [7]). We assume that each wants to maximize the present value of its future cash flow. The adjustment costs make initial conditions important and the evolution of $X(t)$ and $Y(t)$ nontrivial. In place of the concept of rationality of Section II, we require (i) that the first firm's expectation about $Y(t + 1) - Y(t)$ conditional on $X(t + 1) - X(t)$ be accurate if $X(t + 1) - X(t)$ is a change the first firm perceives to be consistent with its goal of maximizing its own profits; (ii) that the second firm's expectation about $X(t +1) - X(t)$ conditional on $Y(t + 1) - Y(t)$ be accurate if the latter adjustment is perceived to be profit maximizing; and, (iii) that if the first (the second) firm perceives $X(t + 1) - X(t)$ $(Y(t + 1) - Y(t))$ to be profit maximizing, the reaction it anticipates must be profit maximizing for its rival conditional on the rival's expectations. Thus, the dynamic-model analogues of firms' conjectural variation functions must be rational for all profit-maximizing changes initiated by either firm. The competitor's reaction in each case must not only be correctly anticipated, but also must itself be based on profit maximization and rational expectations. We will require this stringent variety of rationality for all points which could be initial conditions for our model. Proposition IV (below) and our examples show that a multiplicity of equilibria continues to be a problem.

In the new model the first firm determines its optimal output time path by solving

$$\underset{\{X(t)\geq 0\}}{\text{Max}} \sum_{t=s}^{\infty} (1 + r)^{-t}(X(t) \cdot P(X(t) + Y(t)) - \Phi(X(t),X(t + 1) - X(t))) \quad (5)$$

subject to:

$$Y(t + 1) = F(X(t),Y(t),X(t + 1)), \quad (6)$$

$$Y(s) = y,\ X(s) = x \quad (7)$$

where s is the time now, r is the interest rate, $\Phi(\cdot)$ includes costs of adjustment as well as conventional production costs, and (x,y) gives initial conditions. Line (6) replaces the firm's conjectural variation function of earlier sections.[11] The second firm uses a similar setup with $\Psi(\cdot)$ in place of $\Phi(\cdot)$ and $G(X(t),Y(t),Y(t + 1)) = X(t + 1)$ in place of (6).

The additivity and stationarity of criterion function (5) enable us to write down a recursion for the present value of all future profits at any time s. Let the value be a function $V(\cdot)$. Then $V(\cdot)$ does not depend on s. We have

$$V(X(s),Y(s)) = \underset{\{X(s+1)>0\}}{\text{Max}} \{X(s)P(X(s) + Y(s)) - \Phi(X(s),X(s + 1) - X(s))$$

$$+ (1 + r)^{-1} V(X(s + 1),F(X(s),Y(s),X(s + 1)))\}. \quad (8)$$

If a finite valuation function exists, it will satisfy line (8). If the finite $V(\cdot)$ is nondecreasing and concave and $F(\cdot)$ is concave in $X(t + 1)$ (or if $V(\cdot)$ is concave and $F(\cdot)$ is linear in $X(t + 1)$), and if $\Phi(\cdot)$ is convex in its second argument, then (8) will define a single-valued function $g(\cdot)$ assigning to each pair $(X(t),Y(t))$ the first firm's optimal level of output at t+1, $X(t+1) = g(X(t),Y(t))$. To show that the functional form of $g(\cdot)$ will depend on $F(\cdot)$

we write

$$g(\cdot) = \Gamma(F(\cdot)). \tag{9}$$

Notice that an open-loop solution of the dynamic optimization problem will enable us to determine $g(\cdot)$ independently of the initial conditions (x,y). For the second firm let

$$f(\cdot) = \Omega(G(\cdot)) \tag{10}$$

with $f(X(t),Y(t)) = Y(t + 1)$ giving the optimal level of $Y(t + 1)$ for all $X(t)$ and $Y(t)$.

Continuing to assume that $g(\cdot)$ and $f(\cdot)$ are single valued, for the new model

D3) A "Dynamic Conjectural Equilibrium" is a tuple of functions $(f^*(\cdot),g^*(\cdot), F^*(\cdot),G^*(\cdot))$ such that (i) if the expectation functions for firms 1 and 2 are $F^*(\cdot)$ and $G^*(\cdot)$, respectively, then $g^*(\cdot) = \Gamma(F^*(\cdot))$ and $f^*(\cdot) = \Omega(G^*(\cdot))$; and (ii) for any initial conditions $(X(0),Y(0)) = (x,y)$ the actual time path of $(X(t),Y(t))$, which is generated by $(X(t + 1),Y(t + 1)) = (g^*(X(t),Y(t)), f^*(X(t),Y(t)))$, is a solution path for the systems $(X(t + 1),Y(t + 1)) = (G^*(X(t),Y(t),Y(t + 1)),F^*(X(t),Y(t),X(t + 1))) = (G^*(X(t),Y(t),Y(t + 1)), f^*(X(t),Y(t))) = (g^*(X(t),Y(t)),F^*(X(t),Y(t),X(t + 1)))$.[12] A "Long-Run Equilibrium Point" is a pair (X^*,Y^*) such that $f^*(X^*,Y^*) = Y^*$ and $g^*(X^*,Y^*) = X^*$ with $(f^*(\cdot),g^*(\cdot),F^*(\cdot),G^*(\cdot))$ a dynamic conjectural equilibrium.

If $(f^*(\cdot),g^*(\cdot),F^*(\cdot),G^*(\cdot))$ is an equilibrium tuple , let $\{(X^*(t),Y^*(t))\}_{t\geq 0}$

with $X^*(t+1) = g^*(X^*(t),Y^*(t))$, $Y^*(t+1) = f^*(X^*(t),Y^*(t))$, $X^*(0) = x$, and $Y^*(0) = y$ be an "equilibrium path." Then at every point along such a path we will have game-theoretic "stability": each firm will be maximizing its (perceived) valuation function at every t. The same will be true at an equilibrium point (X^*,Y^*) -- neither duopolist will perceive any benefit to changing his output. In other words, $(f^*(\cdot),g^*(\cdot),F^*(\cdot),G^*(\cdot))$ can be viewed as a Nash equilibrium as follows. The function $f^*(\cdot)$ gives the optimal behavior of firm 2 conditional on expectation function $G^*(\cdot)$. The function $g^*(\cdot)$ gives the optimal behavior for firm 1 conditional on expectation function $F^*(\cdot)$. Yet, from any initial conditions $g^*(\cdot)$ and $f^*(\cdot)$ generate a time path consistent with that generated by $(G^*(\cdot),F^*(\cdot))$. So, each firm chooses a control function which maximizes its profits given the perceived, and actual, behavioral inclinations of its rival.[13]

In order to be able to establish the existence of a dynamic conjectural equilibrium we make very strong assumptions about functional forms:

$$P(Q) = \alpha + \beta \cdot Q,\ \alpha > 0,\ \beta < 0;\ \text{and,} \tag{11}$$

$$\Phi(Z,Z^*) = \Psi(Z,Z^*) = \sigma \cdot Z^2 + \sigma \cdot Z^{*2},\ \sigma > 0. \tag{12}$$

(Adding linear or quadratic terms to (12) will not change our proofs below.) This specification can perhaps be viewed as an approximation to a wide variety of more realistic problems. We have

<u>Proposition III</u>: <u>Suppose</u> $r > 0$. <u>Fix any</u> d <u>with</u> $|d| < 1$. <u>There exist</u> a, b, c <u>and</u> A, B, C <u>such that if</u> $Y(t+1) = F(X(t),Y(t),X(t+1)) = aX(t) + bY(t) + c + dX(t+1)$, $X(t+1) = G(X(t),Y(t),Y(t+1)) = aY(t) + bX(t) + c + dY(t+1)$, $X(t+1) = g(X(t),Y(t)) = AY(t) + BX(t) + C$, <u>and</u> $Y(t+1) = f(X(t),Y(t)) =$

AX(t) + BY(t) + C, then (f(·),g(·),F(·),G(·)) constitutes a "dynamic conjectural equilibrium" for our model. Furthermore, for our solution the linear system (g(·),f(·)) has eigenvalues of modulus less than 1.[14]

The proof is in the appendix.

We can also show that different choices of d in Proposition III will yield different equilibrium tuples.

Proposition IV: In the proof of Proposition III different values of d must yield different equilibrium tuples.

Thus, we have a repetition of the multiple-equilibrium phenomenon that Propositions I and II established. Although Proposition IV does not insure the same wide assortment of equilibria that prevailed in the static model must reappear, the following numerical example shows that great variety may well again be the typical case.

Setting $\alpha = 10$, $\beta = -1$, $\sigma = 2$, and $r = .1$, we calculated equilibrium tuples for different values of d, $|d| < 1$. Tables I and II present the results, including the coefficients of $V(X,Y) = V1 \cdot X^2 + V2 \cdot XY + V3 \cdot Y^2 + V4 \cdot X + V5 \cdot Y + V6$ for firm 1, the eigenvalues of the system (g(·),f(·)), and the long-run equilibrium point $(X^*,Y^*) = (X^*,X^*)$.

As we fixed different values of d, a familiar train of solutions appeared. As d approached 1, X* approached the long-run cartel solution (X = 1.25). When d equaled 0, X* approximately equaled the long-run Cournot equilibrium outcome (X = 1.43). As d approached -1, X* approached the competitive-market long-run equilibrium (X = 1.67). Demanding rational expectations for the dynamic model did not, therefore, lead to any more restricted

Table I

Coefficients for Dynamic Equilibria of a Symmetric Duopoly Problem[15]

Trial	V1	V2	V3	V4	V5	V6	a	b	c	d
1	-4.31	-1.12	.00	14.00	-.23	61.50	-.03	.34	.99	.00
2	-4.31	-1.11	.00	14.00	-.23	61.89	-.11	.34	.72	.25
3	-4.31	-1.12	.00	13.98	-.24	60.94	.06	.34	1.26	-.25
4	-4.32	-1.11	.00	14.00	-.22	62.13	-.19	.34	.47	.50
5	-4.31	-1.12	.01	13.96	-.24	60.18	.14	.34	1.55	-.50
6	-4.32	-1.11	.00	14.00	-.22	62.24	-.26	.34	.23	.75
7	-4.32	-1.13	.01	13.93	-.25	59.21	.24	.34	1.84	-.75
8	-4.32	-1.11	.00	14.00	-.21	62.24	-.31	.34	.09	.90
9	-4.32	-1.13	.01	13.91	-.25	58.51	.30	.34	2.04	-.90

Table II

Coefficients, Long-run Equilibria, and Eigenvalues (E1,E2) for g*(·)

Trial	A	B	C	X*	E1	E2
1	-.03	.34	.99	1.44	.31	.37
2	-.03	.33	.97	1.39	.30	.36
3	-.03	.35	1.01	1.48	.32	.38
4	-.03	.32	.94	1.35	.30	.35
5	-.03	.36	1.04	1.53	.32	.39
6	-.02	.32	.92	1.31	.29	.34
7	-.03	.36	1.06	1.58	.33	.40
8	-.02	.32	.91	1.28	.29	.34
9	-.04	.37	1.07	1.61	.33	.40

set of equilibria than we observed in Section II.

In closing this section, let us compare our dynamic model with Friedman's [6]. The differences are apparent immediately: Although Friedman's duopoly game is repeated over and over, his model is "nonconjectural" within each time period because (using our notation) his function $F(\cdot)$ does not depend on $X(t + 1)$ and his function $G(\cdot)$ does not depend on $Y(t + 1)$. At Friedman's equilibrium point (X^*,Y^*) each firm is maximizing its perceived profit function and each firm's expectations about its rival's reactions after a one-period time lag are rational. On the other hand, firms' expectations are not necessarily rational away from (X^*,Y^*). Furthermore, because Friedman's firms do not incur adjustment costs, the problem alluded to at the end of Section II reappears: Suppose firm 1 varies X infinitesimally starting at (X^*,Y^*). Then firm 1 will be able to correctly anticipate its rival's response. Firm 1 will not perceive its own change in X to be profit maximizing, however, so its expectations are not rational about any output adjustment it would actually want to make. The same argument shows that the second firm's response to the first firm's output change can not itself be based on rational expectations (see Section II). In this section, our model is contemporaneously "conjectural", rationality prevails everywhere,[16] and rationality is only defined in terms of output changes which firms perceive to be profit maximizing. Each firm's response to a profit-enhancing output change on the part of its rival will always itself be based on rational expectations. Our analysis also points out the problem of multiple equilibria.

IV. Conclusion

This paper appraises conjectural duopoly models as an alternative to nonconjectural ones. Working in Cournot's familiar quantity-adjustment framework, we first define the simplest of our equilibrium concepts, the "conjectural equilibrium," and show that many different output combinations can satisfy the definition if we make the correct choice of expectation functions.

We then attempt to require rational expectations for the static model. The results are disappointing for two reasons. First, an enormous variety of output combinations (X^*,Y^*) can still fit into equilibrium tuples $(X^*,Y^*,F^*(\cdot),G^*(\cdot))$ (see Proposition II). Second, we can only require that one firm's expectations be accurate if the other is profit maximizing. Yet, the latter firm's profit-maximizing behavior can not itself be founded on rational expectations unless both firms are profit maximizing. At a point (X^*,Y^*) where both are maximizing, a change in output by the first firm will be followed by a correctly anticipated reaction from its rival. However, the change by the first firm must be contrary to its desire to profit maximize and hence the rival can not come up with a "rational" response.[17] Thus, even at points with both firms maximizing profits we can only have one-sided rationality of expectations.

Our dynamic model in Section III is difficult to analyze. However, our numerical computations and Proposition IV show that an over-supply of equilibria, and even of long-term equilibrium points, continues to be a problem. On the other hand, the dynamic model allows a wider (starting from any initial conditions) and more satisfactory concept of rational expectations than our static model -- our dynamic conjectural equilibria require that any profit-maximizing output change by one firm be accompanied by a fully anticipated and profit-maximizing response grounded on rational expectations from its rival.

Our numerical examples produce long-term equilibrium points (X^*,Y^*) as well as tuples $(f^*(\cdot),g^*(\cdot),F^*(\cdot),G^*(\cdot))$. Although there is a large and diverse set of such points, each point is globally stable -- Proposition III indicated this would be the case. Because of our rationality requirements, the process of convergence in the stability analysis is much more credible than, for instance, the usual convergence story for the nonconjectural Cournot model.[18]

Perhaps the final conclusion that we should draw from this analysis is that although conjectural duopoly models are interesting and in some applications realistic, they are troubled with an overabundance of possible equilibrium output bundles in each case. If we employ such models, therefore, we should not expect particular outcomes in the space of output combinations. Instead, we should concentrate on investigating equilibrium tuples $(X^*,Y^*,F^*(\cdot),G^*(\cdot))$ (or $(f^*(\cdot),g^*(\cdot),F^*(\cdot),G^*(\cdot))$) -- assumptions about rationality do imply observable limits on the set of possible equilibria in the space of tuples.

Appendix

Proof of Proposition I: Fix any $(X^*,Y^*) > 0$ with $P(X^* + Y^*) - \Phi'(X^*) \geq 0$ and $P(X^* + Y^*) - \Psi'(Y^*) \geq 0$. Define $F^*(X,Y) = F^*$ and $G^*(X,Y) = G^*$ all (X,Y) with F^* and G^* constants chosen so that if $Q^* = X^* + Y^*$, then $P(Q^*) + X^*P'(Q^*) \cdot (1 + F^*) = \Phi'(X^*)$ and $P(Q^*) + Y^*P'(Q^*)(1 + G^*) = \Psi'(Y^*)$. The last step is possible since $X^*,Y^*,P'(Q^*) \neq 0$. We have $1 + F^* \geq 0$ and $1 + G^* \geq 0$ because $P'(Q^*) < 0$.

Both firms now satisfy their first-order conditions at (X^*,Y^*). We next show the first firm's second-order conditions hold globally. The argument for the second firm is analogous.

Differentiating the first firm's perceived marginal profit function and using the constraint $(\partial Y/\partial X)\big|_{(X,Y)} = F^*$, we have $2P'(Q)(1 + F^*) + XP''(Q)(1 + F^*)^2 - \Phi''(X) < 0$ all (X,Y) because each term is less than 0 and $\Phi''(X) > 0$. //

Proof of Proposition II: Fix any (X^*,Y^*) satisfying the proposition's hypotheses. Let $T = \{(X,Y) \geq (0,0) : P(X + Y) - \Phi'(X) < 0\}$. Let $T^{\#}$ be the complement of T.

Step 1: For constants A, B, A^*, and B^* define $\phi(X) = A + BX$, $\psi(Y) = A^* + B^*Y$, $K = \{(X,\phi(X)) : X > 0, \phi(X) > 0\}$, and $L = \{(\psi(Y),Y) : Y > 0, \psi(Y) > 0\}$. Choose the constants so that $\phi(X^*) = Y^*$, $\psi(Y^*) = X^*$, $P(Q^*) + X^*P'(Q^*)$ $(1 + (1/\psi'(Y^*))) - \Phi'(X^*) = 0$, and $P(Q^*) + Y^*P'(Q^*)(1 + (1/\phi'(X^*))) - \Psi'(Y^*) = 0$. Then $1 + (1/B^*) \geq 0$ and $1 + (1/B) \geq 0$. Also, $|B| \cdot |B^*| = |Y^*P'(Q^*)/(P(Q^*) - \Psi'(Y^*) + Y^*P'(Q^*))| \cdot |X^*P'(Q^*)/(P(Q^*) - \Phi'(X^*) + X^*P'(Q^*))| > 1$. So, $|B| = |\text{slope of } K| > |1/B^*| = |\text{slope of } L|$.

Step 2: Consider firm 1. The firm's perceived marginal profit function is

$MP1(X,Y) = P(Q) + XP'(Q)(1 + F(X,Y)) - \Phi'(X)$ where $Q = X + Y$. Along K let $F(X,Y)$ be such that $MP1(X,Y) = 0$. Then $1 + F(X,Y) \geq 0$ all $(X,Y) \in K \cap T^{\#}$.

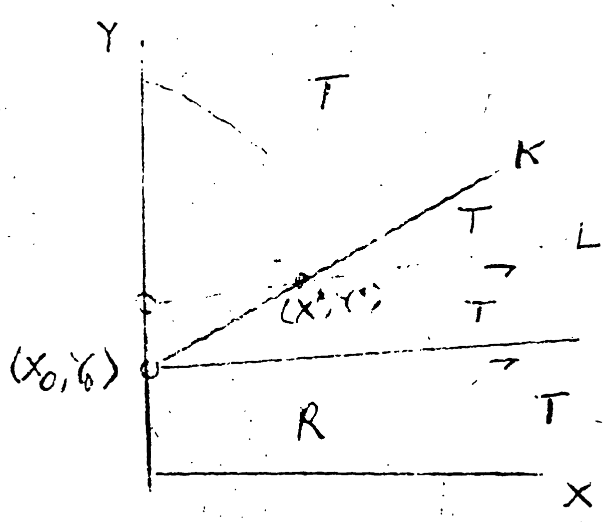
$F(X^*,Y^*) = 1/B^*$ = the slope of L. Thus, for some convex open set U containing (X^*,Y^*) we have $|F(X,Y)| < |B|$ = |the slope of K| all $(X,Y) \in U \cap K$. Let $L^* = \{(\psi(Y),Y) : Y \geq 0, \psi(Y) \geq 0\}$. Let $L^*(X,Y) = \{(X,Y)\} + (L^* - \{(X^*,Y^*)\})$ all $(X,Y) \in K$. Then for each $(X,Y) \in L^*(X',Y')$ with $(X',Y') \in K \cap U$ let $F(X,Y) = F(X',Y')$.

<u>Step 3</u>: Let $(X,Y) \in K \cap U$. Suppose $P(X,Y)$ is the path of points $(X'Y')$ firm 1 expects to encounter if starting from (X,Y) it varies X. Then at (X,Y) the graph of $P(X,Y)$ cuts K and has a smaller absolute slope than K.

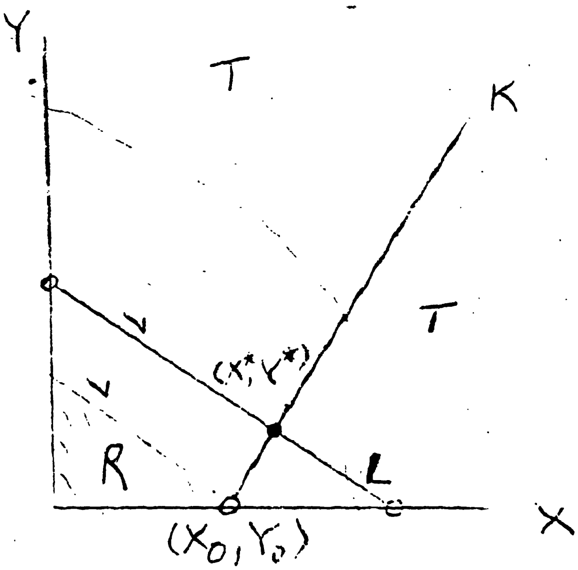
Let $S = \{(X,Y) : (X,Y) \in L^*(X',Y')$ some $(X',Y') \in K \cap U\}$. We now define $F(\cdot)$ on the complement of S. Set $F(X,Y) = -1$ all $(X,Y) \in T$. Let (X_L,Y_L) be the lower limit point of $K \cap U$ and let (X_U,Y_U) be the upper limit point. To begin, set $F(X,Y) = F(X',Y')$ all $(X,Y) \in L^*(X',Y')$ where $(X',Y') \in K$ and $(X,Y) \notin T$. If in the case of any of the diagrams below $X_0 > 0$, let $F(X',Y') = 1/B^*$ all $(X',Y') \in$ closure (R).

<u>Case A</u>: Suppose K has positive slope.

For $(X,Y) \in K \cap U$ consider $P(X,Y)$ to the right of K.



(i)



(ii)

In diagram (i) $F(X',Y') = \infty$ on the upper boundary of R. So, if $P(X,Y)$ falls as X increases, $P(X,Y)$ must exit into T above the upper boundary of R. In diagram (ii) set $F(X,Y) = 0$ along the X-axis to the right of X_0.

Suppose (X',Y') is to the right of K, $(X',Y') \notin T$, and $(X',Y') \in L^*(X'',Y'')$ with $(X'',Y'') \in K - T$ and $(X'',Y'') \geq (X_U,Y_U)$. Then if $F(X',Y') >$ slope of K, change $F(\cdot)$ so that $F(X',Y') =$ slope of K.

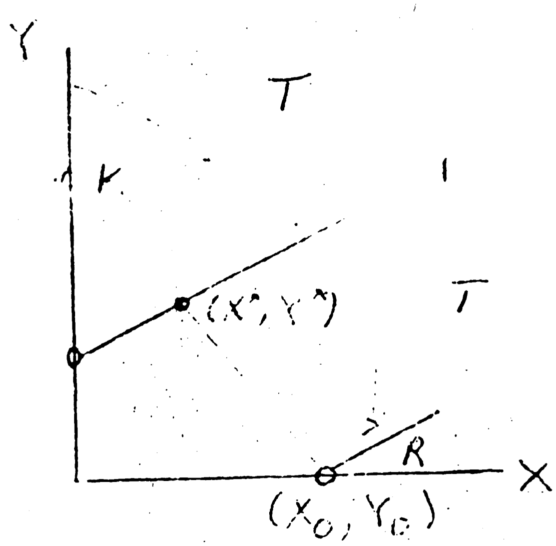
Thus, if $(X,Y) \in K \cap U$, $P(X,Y)$ will stay to the right of K as X is increased, finally exiting into T.

Consider (X',Y') to the left of K, $(X',Y') \geq (0,0)$, $(X',Y') \notin T$, and $(X',Y') \in L^*(X'',Y'')$ with $(X'',Y'') \in K - T$, $(X'',Y'') \leq (X_L,Y_L)$. Then if $F(X',Y') >$ slope of K, change $F(\cdot)$ so that $F(X',Y') =$ slope of K. For (X',Y') on the X-axis to the left of any insection point (X_0,Y_0) let $F(X',Y') = 0$.

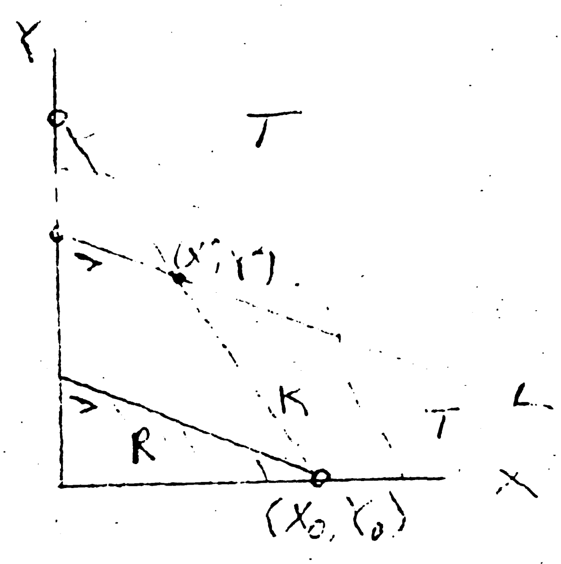
Thus, if $(X,Y) \in K \cap U$, $P(X,Y)$ will stay to the left of K as X is decreased, finally exiting into T or terminating on the Y-axis.

<u>Case B</u>: Suppose K has a negative slope. (Note that K cannot be horizontal since L must have a lower absolute slope.) Then $1 + (1/B) \geq 0$ and $B < 0$ implies $B \leq -1$. For $F(\cdot)$ unmodified in Case A above, $1 + F(X',Y') \geq 0$ all $(X',Y') \in K \cap T^{\#}$; so $F(X',Y') \geq -1$ all (X',Y').

Let $(X,Y) \in K \cap U$. Then to the right of K $P(X,Y)$ cannot fall faster



(i)



(ii)

than K as X is increased. Let $F(X',Y') = 0$ for (X',Y') on the X-axis to the right of (X_o,Y_o). Since $F(X',Y') = 1/B^*$ on the upper boundary of R, $P(X,Y)$ must stay to the right of K as X is increased, exiting into T.

For (X',Y') to the left of K with $(X',Y') > (0,0)$ and $(X',Y') \notin T$, let $F(X',Y') = 0$ if $Y' = 0$. No other modifications are needed -- if $P(X,Y)$ rises as X falls, $1 + F(X',Y') \geq 0$ insures that the rise cannot be as fast as that of K. $P(X,Y)$ will terminate on the Y-axis or in T.

<u>Step 4</u>: Let $(X,Y) \in U \cap K$. Suppose $dMP1/dX\big|_{L^*(X,Y)}$ gives the rate of change in perceived marginal profit as we move along $L^*(X,Y)$. Then if $(X',Y') \in L^*(X,Y)$, $dMP1(X',Y')/dX\big|_{L^*(X,Y)} = P'(Q')(1 + (1/B^*)) + P'(Q')(1 + F(X',Y')) + X'P''(Q')(1 + (1/B^*))(1 + F(X',Y')) + X'P'(Q')(F_X(X',Y') + F_Y(X',Y')/B^*)) - \Phi''(X')$ where $Q' = X' + Y'$ and $(X',Y') \in L^*(X,Y)$. If we are on a portion of $F(\cdot)$ unmodified in case A or B above, then $1 + (1/B^*) \geq 0$; $1 + F(X',Y') = 1 + F(X,Y) \geq 0$; $F_X(X',Y') + F_Y(X',Y')/B^* = dF(X',Y')/dX'$ along $L^*(X,Y)$, which $= 0$; and, $\Phi''(X') > 0$. So, $dMR1(X',Y')/dX\big|_{L^*(X,Y)} < 0$ in that situation.

Consider case A. Then $MR1(\cdot)$ is zero along K, so it is negative to the right of K where $F(\cdot)$ is unmodified. The modifications in R and along the X-axis do not affect this result. Modifications in the northeastern part of the graph do not affect the result either since holding X' and $F(\cdot)$ constant and increasing Y' lowers $MP1(X',Y')$. To the left of K $MP1(\cdot)$ is positive with the unmodified $F(\cdot)$ function. All possible modifications make $MP1(\cdot)$ even more positive by reducing the term multiplying $X'P'(Q')$, a nonpositive number.

Similar arguments hold for case B. Thus as firm 1 considers changing X starting at $(X,Y) \in K \cap U$, it perceives that its profit function would fall after any feasible increase or decrease in X. So, $K \cap U \subset K(F(\cdot))$.

We can repeat this analysis for firm 2, letting $MP2(\cdot)$ replace $MP1(\cdot)$,

$G(\cdot)$ replace $F(\cdot)$, and V replace U. Then $L \cap V \subset L(G(\cdot))$.

Step 5: Suppose we start at $(X,Y) \in K \cap U$, but firm 2 changes Y infinitesimally to $Y + dY$. Then $MP1(X,Y + dY) \gtrless 0$ depending on whether $(X,Y + dY)$ is to the left or right of K. In either case firm 1 will want to change X to $X + dX$ such that $MP1(X + dX, Y + dY) = 0$. That requires $(X + dX, Y + dY) \in K \cap U$. So, $dX = dY/\phi'(X) = dY/B$. But $G(\cdot)$ is level along K with $G(X,Y) = G(X^*,Y^*) = 1/\phi'(X^*) = 1/B$. So, $(\partial X/\partial Y)^e\big|_{(X,Y)} = 1/B = (\partial X/\partial Y)\big|_{(X,Y)}$, the actual response of firm 1. Similarly, $F(X,Y) = 1/B^* = (\partial Y/\partial X)\big|_{(X,Y)}$ all $(X,Y) \in L \cap V$. Thus expectations are rational for firm 1 along $L \cap V$ and for firm 2 along $K \cap U$.

So, $(X^*,Y^*,F(\cdot),G(\cdot))$ satisfies D2.//

Proof of Proposition III: Let $\pi(X,Y,X^*) = X \cdot (\alpha + \beta X + \beta Y) - \sigma X^2 - \sigma(X^* - X)^2$. Form $\pi^*(\cdot)$ from $\pi(\cdot)$ by dropping all nonquadratic terms. Given any A and B simulate the equation system

$$X(t + 1) = A \cdot Y(t) + B \cdot X(t),$$

$$Y(t + 1) = A \cdot X(t) + B \cdot Y(t)$$

forward from each $(X(0),Y(0)) = (X,Y)$ to form

$$V^*(X,Y,A,B) = \sum_{t=0}^{\infty} (1 + r)^{-t}\pi^*(X(t),Y(t),X(t + 1)).$$

Let $S = \{(A,B): |A| + |B| \leq 1\}$. Then for each $(A,B) \in S$ the equation system above has eigenvalues of modulus less than or equal to 1 and the function $V^*(X,Y,A,B)$ defined from simulations is finite, strictly concave in X, and quadratic (with no linear or constant terms) for each (X,Y).

Suppose

$$\psi(A,B) = \begin{pmatrix} 1 & -d \\ -d & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

Then if $(a,b) = \psi(A,B)$,

$$X(t + 1) = A \cdot Y(t) + B \cdot X(t),$$

$$Y(t + 1) = aX(t) + bY(t) + dX(t + 1)$$

will generate the same time path from any initial conditions as the system of the first paragraph. If we choose X^* to maximize

$$\pi^*(X,Y,X^*) + (1 + r)^{-1}V^*(X^*,Y^*,A,B)$$

subject to

$$Y^* = aX + bY + dX^*,$$

our control rule will have the form

$$X^* = A^*Y + B^*X$$

where A^* and B^* are independent of X and Y. The control rule will be unique. Let $\phi(\cdot)$ be such that

$$\phi(A,B) = (A^*,B^*).$$

For each (X,Y), $V^*(X,Y,A,B)$ varies continuously with (A,B). Thus, Berge's [1] "maximum theorem" shows $\phi(\cdot)$ is continuous on S. For $(A,B) \in S$ let $\phi^*(A,B) = \phi(A,B)$ if $\phi(A,B) \in S$, and let $\phi^*(A,B) = \gamma \cdot \phi(A,B)$ otherwise with $\gamma < 1$ such that $\gamma \cdot \phi(A,B)$ lies on the boundary of S closest to $\phi(A,B)$. Then $\phi^*\colon\ S \to S$ and $\phi^*(\cdot)$ is continuous on S. So, Brouwer's fixed-point theorem shows $\phi^*(\cdot)$ has at least one fixed point on S. Let (A,B) be any such fixed point.

Step 2: Since the equation system defined at the beginning of Step 1 has a symmetric matrix, its eigenvalues are real. We can easily see that the same is true for its eigenvectors. But, if (e1,e2) is such a vector, (e2,e1) is a second eigenvector for the same eigenvalue. Let (e3,e4) be an eigenvector for the second eigenvalue. Then (e4,e3) must work as well. (e1,e2) and (e2,e1) must both be orthogonal to both (e3,e4) and (e4,e3) (see Hadley [8]). The only way that can be true is if $|e1| = |e2|$ and $|e3| = |e4|$. In other words, we can assume $E1 = (e1,e2) = (1,1)$ and $E2 = (e3,e4) = (-1,1)$.

Suppose $(A,B) \in S$ is a fixed point for $\phi^*(\cdot)$ but not for $\phi(\cdot)$. Then $\phi(A,B) = \tau \cdot (A,B)$ where $\tau > 1$ and $(A,B) \in \partial S$. Suppose $A + B = 1$. Choose a very large $x > 0$, and set $(X,Y) = (x,x)$. Then the definition of $\phi(\cdot)$ shows the first firm's optimal behavior in the maximization problem outlined in Step 1 is to choose $X^* = \tau \cdot (A \cdot Y + B \cdot X) = \tau \cdot x \cdot (A + B) = \tau \cdot x = \tau \cdot X$ with $\tau > 1$. The definition of $\Phi(\cdot)$ and a graph of developments after the choice of X^* will show that for a large enough x, $X^* = \tau \cdot X$ cannot be optimal: $X^* = 0$ will dominate. This contradicts the definition of $\phi(\cdot)$.

Other cases of $(A,B) \in \partial S$ are similar. For example, let $A > 0$, $B < 0$, and $|A| + |B| = 1$. Suppose $\phi(A,B) = \tau \cdot (A,B)$, $\tau > 1$. Then $A - B = 1$. For large $x > 0$ let $(X,Y) = (-x,x)$. Then $X^* = \tau \cdot (A \cdot Y + B \cdot X) = \tau \cdot (A - B) \cdot x = \tau x$. A diagram will show that for a large enough x, $X^* = 0$ will dominate (from the first firm's point of view) the control rule defined by $\phi(\cdot)$. Again we have a contradiction of the definition of $\phi(\cdot)$. The same type of argument applies for any part of the boundary of S.

Thus, we can assume $\phi^*(A,B) = (A,B) \in S$ implies $\phi(A,B) = (A,B) \in S$. In fact, the same diagrammatic arguments which work above will show that $\phi(A,B) = (A,B) \in \text{int } S$ in each such case. $(A,B) \in \text{int } S$ implies the eigenvalues of our original equation system both have modulus less than one.

Step 3: Let $\phi(A,B) = (A,B) \in \text{int } S$. Then A and B define an optimal control rule $X^* = AY + BX$ for firm 1 given $Y^* = aX + bY + dX^*$ with $(a,b) = \psi(A,B)$; and, $Y^* = AX + BY$ defines the same for firm 2 given $X^* = aY + bX + dY^*$.

Now we switch from $\pi^*(\cdot)$ back to $\pi(\cdot)$. Let (A,B) and (a,b) be as generated above. For any c, the right choice of C will make $X^* = AY + BX + C$ give the optimal behavior for firm 1 given $Y^* = aX + bY + c + dX^*$. (Similarly, $Y^* = AX + BY + C$ will be optimal for firm 2 given $\pi(\cdot)$ and $X^* = aY + bX + c + dY^*$.) In fact, simple algebra will show the "right" choice of C will be given by

$$C = g \cdot c + h$$

where g and h depend only on exogenous constants and A and B, which we are no longer changing.

For a complete equilibrium tuple it remains only to show that

$$c/(1 - d) = g \cdot c + h$$

for some c. This will be true unless $g = 1/(1 - d)$ and $h \neq 0$. Suppose $g = 1/(1 - d)$ and $h \neq 0$. For any c let $c^* = (1 - d)(gc + h)$. Let $C = c/(1 - d)$ and $C^* = c^*/(1 - d)$.

For larger and larger values of c, (C^*,C) will become arbitrary close in relative terms to the 45-degree ray in the positive orthant. Set $(X,Y) = (0,0)$. Consider the evolution of the system,

$$X(t + 1) = A \cdot Y(t) + B \cdot X(t) + C^*,$$

$$Y(t + 1) = A \cdot X(t) + B \cdot Y(t) + C$$

from $(X(0),Y(0)) = (X,Y)$, and notice that for firm 1

$$V(X,Y,A,B,C,C^*) = \sum_{t=0}^{\infty} (1 + r)^{-t} \pi(X(t),Y(t),X(t + 1)).$$

For large enough c, a diagram will show that firm 1 is better off setting $A = B = C^* = 0$ than adopting C^* given $(X,Y) = (0,0)$ and any $(A,B) \in S$. This contradicts the definition of C^*. So, if $h \neq 0$, then $g \neq 1/(1 - d)$.

Thus, for any $(A,B) = \phi(A,B) \in \text{int } S$ we can find c such that $C = gc + h$ makes

$$X^* = AY + BX + C$$

optimal for firm 1 given

$$Y^* = aX + bY + c + dX^*$$

and $(a,b) = \psi(A,B)$. And,

$$Y^* = AX + BY + C$$

is optimal for firm 2 given

$$X^* = aY + bX + c + dY^*. \quad //$$

<u>Proof of Proposition IV</u>: Suppose $d \neq d'$, $|d|$ and $|d'| < 1$, yet we have the same equilibrium values A,B,C,a,b,c in both cases. By simulating the system $(g(\cdot),f(\cdot))$ forward from different initial conditions (x,y) and calculating the present value of each firm's profits in each case, we can determine $V(x,y)$ for each firm. Suppose we calculate $V(\cdot)$ for the second firm in this way. Then $f(x,y)$ must satisfy $2\sigma \cdot (f(x,y) - y) = (1 + r)^{-1}[V_2(g(x,y),f(x,y)) + d \cdot V_1(g(x,y),f(x,y))]$. But that cannot be true for both d and d' if $d \neq d'$ unless $V_1(g(x,y),f(x,y)) = 0$ all (x,y). An examination of $V(x^*,y^*)$ at different points (x^*,y^*) in the extremities of the positive orthant will show

$V_1(x^*,y^*)$ cannot be 0 everywhere. However, the correct choice of (x,y) will yield $(x^*,y^*) = (g(x,y)f(x,y))$ for any (x^*,y^*). So, for $d' \neq d$ we must have different set of parameters A', B', C', a', b', c'. //

Notes

1. I wish to thank Ted Bergstrom, John Cross, and Hal Varian for helpful comments on an earlier draft of this paper.

2. Cyert and de Groot [3] and Friedman [5, chapter 5], [6] deal with sophisticated variants of this model.

3. Marschak and Selten [11], [12] work with this type of model. Hahn [9] uses similar ideas in a different context.

4. In order to avoid the complications of having to incorporate product differentiation we work exclusively with quantity-adjustment models.

5. $F(\cdot)$ and $G(\ )$ represent an alternative to Marschak and Selten's [11], [12] approach, which is to use response functions. For example, $\phi((X,Y),X') = (X',Y')$ might show the response of firm 2 to a change in X. A benefit of our approach is that information requirements for firms and overall complexities are greatly reduced -- $F(X,Y)$ and $G(X,Y)$ are single numbers at each point (X,Y), while at (X,Y) $\phi((X,Y),X')$ is a function associating a number with each X'.

6. Our conventions are $(a',b') > (a,b)$ implies $a' > a$, $b' > b$ and $(a',b') \geq (a,b)$ implies $a' \geq a$, $b' \geq b$.

7. The hypotheses here provide a sufficient but not necessary set of conditions for our results. They are chosen to make the proofs in the appendix as straightforward as possible. No attempt is made to find the least restrictive set of sufficient conditions for the results of Proposition I (and Proposition II).

8. More specifically, suppose at the output point (X,Y) the second firm perceives itself to be profit maximizing. We say the first firm's expectations (given by $F(\cdot)$) are "rational" at (X,Y) if for any infinitesimal change Δ in X, $(X + \Delta,\ Y + \Delta \cdot F(X,Y))$ gives a new output bundle at which firm 2 perceives itself to be maximizing its profits -- i.e., a new point at which

the second firm is content to stop the tatonnement process outlined in the introduction.

9. Our definition of rationality here corresponds to Marschak and Selten's [11], [12] concept of a response function which is a "weak convolution."

10. For the cartel case, $P(Q) + QP'(Q) - \Phi'(X) = 0 = P(Q) + QP'(Q) - \Psi'(Y)$. Thus $(X^*,Y^*) > (X^c,Y^c)$ implies the denominators in the new assumption are less than $-X^*P'(Q^*)$, $-Y^*P'(Q^*)$, respectively. $(X^*,Y^*) < (X^m,Y^m)$ implies the denominators are greater than $Y^*P'(Q^*)$, $X^*P'(Q^*)$, respectively.

11. Note that although we are switching to a dynamic problem -- from which a sequence of points $\{(X(t),Y(t))\}_{t \geq 0}$ evolves such that at time t firm 1 perceives the output $X(t)$ to be profit maximizing given $X(t - 1)$, $Y(t - 1)$, and $Y(t)$, and firm 2 perceives $Y(t)$ to be profit maximizing given $X(t - 1)$, $Y(t - 1)$, and $X(t)$ -- our model is still "conjectural." In other words, given $X(t - 1)$ and $Y(t - 1)$ we can think of the point $(X(t),Y(t))$ as being determined as follows: At time $t - \varepsilon$, $\varepsilon > 0$, each firm selects a tentative output, $X(t)$ or $Y(t)$, for time t. If each firm behaves as the other expected, $(X(t),Y(t))$ is the output bundle for time t. Otherwise, at time $t - \delta$, $\varepsilon > \delta > 0$, the firms announce a corrected bundle $(X(t),Y(t))$. The "corrections" continue until both firms are satisfied -- all changes taking place before time t. The process recurs at each time t. In line (6), firm 1 reasons that $Y(t + 1)$ depends on $X(t + 1)$ because of this tatonnement: given $X(t)$ and $Y(t)$, firm 1 believes that if it announces output $X(t + 1)$ for time $t + 1$, its rival will choose $Y(t + 1) = F(X(t),Y(t),X(t + 1))$ and the tatonnement will stop immediately. The variable $X(t + 1)$ is an argument of $F(\cdot)$ because firm 1 may believe that if it announces $X^*(t + 1) \neq X(t + 1)$ instead of $X(t + 1)$, the tatonnement will not stop instantly -- rather it may converge to a point $(X^*(t + 1),Y^*(t + 1))$ with $Y^*(t + 1) \neq Y(t + 1)$.

12. Note that if the systems $(g^*(\cdot), f^*(\cdot))$ and $(G^*(\cdot),F^*(\cdot))$ always generate the same time paths, then $(g^*(\cdot),f^*(\cdot))$, $(G^*(\cdot),f^*(\cdot))$, and $(g^*(\cdot),F^*(\cdot))$ must always generate the same paths.

13. Suppose $\{(X^*(t),Y^*(t))\}_{t\geq 0}$ is an "equilibrium path." We do not require that if firm 1, for example, deviates from the path by choosing $X(t) \neq X^*(t)$ at some $t > 0$ that it necessarily be able to predict the second firm's responses (at all $s \geq t$) accurately: The first firm's deviation would be an irrational -- i.e., non-profit maximizing -- form of behavior, so it would contradict the second firm's expectations, assuming they were "rational." Thus, the second firm would have no transparently sensible response which the first firm, in turn, could "rationally" anticipate. Given a "dynamic conjectural equilibrium" such deviations should never occur.

14. In fact, we could prove that no solution exists for which the linear system $(g(\cdot),f(\cdot))$ has eigenvalues of modulus greater than or equal to 1.

15. The figures presented in both tables are rounded.

16. Note that we can specify any initial conditions $(X(0),Y(0)) = (x,y)$ for our quadratic model. Starting at any such point, the first firm's expectations of how its rival will react to an output change $X(1) - X(0)$ will be "rational" if firm 1 perceives $X(1) - X(0)$ to be a profit-maximizing change for itself. Similarly for the second firm.

17. Note that since K and L are never coincident in the proof of Proposition II, the first firm could never move away from (X^*,Y^*) expecting to have its profits remain unchanged.

18. See Fellner [4] and Friedman [5] for evaluations of stability analyses for the Cournot model.

References

1. Berge, C. Topological Spaces, trans. E.M. Patterson (Edinburgh: Oliver & Boyd, 1963).

2. Blair, J. Economic Concentration; Structure, Behavior and Public Policy (New York: Harcourt Brace Jovanovich, Inc., 1972).

3. Cyert, R.M. and de Groot, M. "Multiperiod Decision Models with Alternating Choice as a Solution to the Duopoly Problem," Quarterly Journal of Economics, 84 (1970), 410-429.

4. Fellner, W. Competition Among the Few (New York: Augustus M. Kelley, 1960).

5. Friedman, J.W. Oligopoly and the Theory of Games (Amsterdam: North-Holland Publishing Company, 1977).

6. ______________ "Reaction Functions and the Theory of Duopoly," Review of Economic Studies, 35 (1968), 257-272.

7. Gould, J. "Adjustment Costs in the Theory of Investment of the Firm," Review of Economic Studies, 35 (1968), 47-55.

8. Hadley, G. Linear Algebra (Reading, Massachusetts: Addison-Wesley, 1961).

9. Hahn, F. "On Non-Walrasian Equilibria," Review of Economic Studies, 45 (1978), 1-18.

10. Kaplan, A.D., Dirlam, J.B., and Lanzillotti, R.F. Pricing in Big Business: a Case Approach (Washington, D.C.: The Brookings Institution, 1958).

11. Marschak, T. and Selten, R. General Equilibrium with Price-Making Firms (Berlin: Springer-Verlag, 1974).

12. __________________ "Restabilizing Responses, Inertia Supergames, and Oligopolistic Equilibria," Quarterly Journal of Economics, 92 (1978), 71-94.

CREST Working Papers

C-1 The Ergodic Behavior of Stochastic Processes of Economic Equilibrium, by Lawrence E. Blume.

C-2 The Token Economy, Reinforcement and the Consumer Model, by John G. Cross.

C-3 Notes on a Theory of Learning to Search, by John G. Cross.

C-4 Redistributive Taxation as Social Insurance, by Hal R. Varian.

C-5 Catastrophe Theory and the Business Cycle, by Hal R. Varian.

C-6 Household Bequests, Perfect Expectations, and the National Distribution of Wealth, by John P. Laitner.

C-7 Inflationary Disequilibrium, by John G. Cross.

C-8 Ellet's Transportation Model of an Economy with Differentiated Commodities and Consumers, I: Generic Cumulative Demand Functions, by Carl P. Simon.

C-9 Cournot Equilibrium in Factor Markets, by Ted Bergstrom.

C-10 Is a Man's Life Worth More Than His Human Capital, by Ted Bergstrom.

C-11 A Model of Sales, by Hal R. Varian.

C-12 New Techniques for the Study of Stochastic Equilibrium Processes, by Lawrence E. Blume.

C-13 Consistent Expectations, by Lawrence E. Blume.

C-14 On Baubles and Bubbles, by John G. Cross.

C-15 When Does Majority Rule Supply Public Goods Efficiently?, by Ted Bergstrom.

C-16 "Rational" Duopoly Equilibria, by John P. Laitner.