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Abstract: Motivated by empirical evidence that age structure fluctuations affect relative wages across age groups, this paper asks whether there is a steady–state age distribution that maximizes the lifetime wages of a representative worker. The paper proves the surprising result that in a pure labor economy with any constant returns technology, a uniform age distribution (zero population growth) *minimizes* lifetime wages. The presence of other factors complicates, but does not necessarily reverse, this result. Effects of age structure on age–specific productivity are incorporated into overlapping generations models developed to analyze the economic effects of changes in population growth rates. Analogies of the effects of age structure on life–cycle wages with intergenerational transfer effects in consumption loan models are explored.

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1. Introduction

Empirical research on the effects of cohort size on wages suggests that age-specific wages are sensitive to the relative sizes of age groups.[1] Although these studies are interested in the effects of short-term fluctuations in cohort size, the results can also be interpreted as evidence that age-specific wages will be influenced by age structure in long-term demographic steady-states as well. Even if all other features of two economies are similar, for example, workers in a rapidly growing population with a young age distribution should have a significantly different wage profile than workers in a population with a low growth rate and a relatively uniform age distribution. Even though considerable attention has been given to the importance of age structure in models of the economic effects of population growth,[2] the effects of age structure on relative wages across age groups has been virtually ignored. This paper attempts to fill this gap by analyzing the relationship between population growth rates and life cycle wage profiles, explicitly recognizing the possibility of imperfect substitutability of workers of different ages.

In previous research on the economic effects of population growth, two basic economic forces drive the results. The first is the effect of population growth on capital-labor ratios, typified by Solow's (1956) neoclassical growth model. Higher rates of population growth in such a model unambiguously lower steady state per capita income and consumption due to a capital dilution effect analogous to an increased rate of capital depreciation. The second common effect of population growth is an intergenerational transfer effect typified by Samuelson's (1958) original consumption loan model. In its simplest version, increased population growth leads to unambiguously *higher* per capita lifetime utility through what amounts to a perpetually underfunded pay-as-you-go social security system. A number of authors have attempted to combine these two effects into a single model, beginning with Samuelson (1975).[3] Models with more general treatments of age structure have been developed by Arthur and McNicoll (1977, 1978), Lee (1980), and Willis (1982), but none of these models has considered the effects of age structure on age-specific labor productivity.

Section 2 of this paper analyzes the relationship between factor proportions and marginal products for the simple case of a pure labor economy in which labor can be divided into two types. The section proves that for any concave constant returns production function, the sum of the two marginal products is minimized when there are equal numbers of the two types of workers. The implications of non-labor factors are considered in section 3. Section 4 analyzes the effects of population growth rates on age-specific wages in a stable population with any number of age groups. A uniform age distribution is shown to minimize steady state lifetime wages in a pure labor economy with any constant returns technology. Effects of discounting are considered in section 5. Sections

[1] See, for example, Welch (1979), Freeman (1979), Stapleton and Young (1984), and Berger (1985).

[2] For example, Arthur and McNicoll (1977, 1978), Lee (1980), and Willis (1982).

[3] Samuelson's first order conditions for an optimum growth rate actually imply a welfare *minimum* for a large class of production and utility functions, as pointed out by Deardorff (1976). Deardorff proved, for example, that when both production and utility are Cobb-Douglas, the benefits from high capital labor ratios of negative growth rates and the "intergenerational chain letter" windfalls of positive growth rates are unbounded, and always offset the losses working in the opposite direction. No finite optimum population growth rate exists for such a model, a result explored in detail in Samuelson's (1976) reply to Deardorff.

6 and 7 incorporate effects of age structure on age–specific labor productivity into overlapping generations models of the effects of population growth on life cycle consumption profiles. Previous comparative steady state results on the effects of population growth on lifetime consumption are shown to generalize in a straightforward way when the conventional assumption that workers of different ages are perfect substitutes is replaced with a completely general production function. Section 8 concludes the paper and compares the results with the implications of Samuelson's original consumption loan model.

2. Factor Proportions and Factor Payments Under Constant Returns Technology

A fundamental but little recognized property of linearly homogeneous functions forms the foundation for many of the results developed below. The simplest form of the result can be demonstrated for an economy with two types of workers:

Proposition 1. *If L total workers are divided into two types, L_1 and L_2, and total output is given by a concave constant returns to scale production function $Y = F(L_1, L_2)$, the sum of the two marginal products $F_1 + F_2$ attains a global minimum when $L_1 = L_2$.*

To prove the result, consider the problem of choosing the fraction π_1, where $L_1 = \pi_1 L$ and $L_2 = (1 - \pi_1)L$. Assuming that workers of each type are paid their marginal products, what effect will the choice of π_1 have on the sum of the two wages $W = w_1 + w_2 = F_1 + F_2$? Noting that

$$\frac{\partial L_2}{\partial \pi_1} = -\frac{\partial L_1}{\partial \pi_1} = -L,$$

it follows that

$$\begin{aligned} \frac{\partial W}{\partial \pi_1} &= (F_{11} + F_{21})\frac{\partial L_1}{\partial \pi_1} + (F_{12} + F_{22})\frac{\partial L_2}{\partial \pi_1} \\ &= (F_{11} + F_{21} - F_{12} - F_{22})L \\ &= (F_{11} - F_{22})L. \end{aligned} \tag{1}$$

The sign of $F_{11} - F_{22}$ will be indeterminate in general, but becomes a simple function of π_1 under constant returns to scale. Under constant returns it will be analytically convenient to normalize by the total number of workers. Using lower case letters to denote per worker quantities, define the per worker production function $y = f(L_1, L_2) = F(\pi_1, \pi_2)$, where $\pi_i = L_i/L$. The properties of homogeneous functions require that the derivatives of the total production function F are related to the derivatives of the per worker production function f by the conditions $F_i = f_i$, $F_{ij} = f_{ij}L^{-1}$, and $F_{ijk} = f_{ijk}L^{-2}$, where $F_i = \partial F/\partial L_i$, $f_i = \partial f/\partial \pi_i$, etc. The condition on first derivatives implies that wages are independent of total population size, while the condition on second derivatives implies that (1) can be reduced to $\partial W/\partial \pi_1 = f_{11} - f_{22}$, a result which is independent of L. Given constant returns, Euler's theorem implies that $f_{11}\pi_1 + f_{21}\pi_2 = 0$. Using the fact that $f_{21} = f_{12}$, this condition implies that

$$f_{22} = f_{11}\left(\frac{\pi_1}{1 - \pi_1}\right)^2.$$

Substituting into (1), then

$$\frac{\partial W}{\partial \pi_1} = f_{11}\left[1 - \left(\frac{\pi_1}{1-\pi_1}\right)^2\right] = f_{11}\left[\frac{1-2\pi_1}{(1-\pi_1)^2}\right]. \tag{2}$$

By inspection, the derivative in (2) is equal to zero when $\pi_1 = .5$. To see whether this critical point is a minimum or maximum, differentiation of (2) gives

$$\frac{\partial^2 W}{\partial \pi_1^2} = (f_{111} - f_{112})\left[\frac{1-2\pi_1}{(1-\pi_1)^2}\right] - 2f_{11}(1-\pi_1)^{-2} + 2f_{11}\left[\frac{1-2\pi_1}{(1-\pi_1)^3}\right]. \tag{3}$$

The first and last terms equal zero when $\pi_1 = .5$, making the entire expression unambiguously positive if $f_{11} < 0$. Equal division of the workers therefore gives the global *minimum* W for any concave F. It is clear by inspection of (2) that the derivative is positive for all $\pi_1 > .5$, and is negative for all $\pi_1 < .5$ as long as $f_{11} < 0$. This establishes Proposition 1.[4] If L_1 and L_2 are the number of young and old workers respectively, then $(L_1/L_2) - 1$ is the labor force growth rate. A worker passing through the labor force with one period in each age group will earn W total lifetime wages. The result in (2), then, implies that for a pure labor economy with constant returns to scale, a stationary population produces the lifetime wage *minimizing* age structure. Either a positive or negative growth rate of the labor force will lead to greater lifetime wages for all workers. There is no finite growth rate that maximizes lifetime wages. As seen in (2), lifetime wages continually increase with increases in π_1 above .5 or with decreases in π_1 below .5

Proposition 1 holds for any concave constant returns production function in a pure labor economy, with no assumption about productivity differences or the elasticity of substitution between the two types of workers other than that implied by concavity. It is interesting to compare this result to the effect of the choice of π_1 on total *output*. Consider, for example, a CES production function $Y = [\beta L_1^\rho + (1-\beta)L_2^\rho]^{1/\rho}$. If the two types of workers have equal productivity parameters, i.e. $\beta = .5$, then total output will always be maximized when $L_1 = L_2$. If $\beta \neq .5$ then the division of labor that maximizes total output will depend on the elasticity of substitution, with any result possible in general. For all values of β and ρ, however, $w_1 + w_2$ always attains a global minimum when there are equal numbers of workers of each type.

If the age distribution of workers remains constant and each worker spends one period as type 1 and one period as type 2, these results imply the paradoxical condition that the distribution of workers that maximizes total output in each period may be the distribution that minimizes each worker's lifetime income. Total (or per worker) output in each period need not have any relation

[4] There is an interesting dual to Proposition 1, although the applications are less obvious. Note that the cost function $C(w_1, w_2, Y)$ corresponding to any production function $F(L_1, L_2)$ is homogeneous of degree one and the conditional factor demands $L_i(w_1, w_2, Y)$ are the first derivatives of C by Shephard's lemma. If there were a constraint on the sum of the wages $w_1 + w_2 = \bar{W}$, with wages set at $w_1 = \alpha_1 \bar{W}$, $w_2 = (1-\alpha_1)\bar{W}$, and if firms chose the cost minimizing labor inputs given those wages, then $\alpha_1 = .5$ would always generate the *minimum* total employment $L_1 + L_2$ for all possible values of α_1. Skewed wages would always generate higher employment than equal wages, subject to a constraint on the sum of the two wages.

to a representative worker's lifetime welfare. In the simple case in which workers seek to maximize lifetime income, a uniform age distribution is always the worst of all possible worlds in in this simple example of a pure labor economy.[5]

Consider a rapidly growing population, in which there are always many times more young workers than old workers. Young workers may have much lower wages than old workers, but Proposition 1 guarantees that $w_1 + w_2$ is greater than it would be if the same number of workers were rearranged so that there equal numbers at each age. The apparent inefficiency that total output in each period could be increased if there were equal numbers at each age is unimportant to the workers, as long as the positive growth rate can be sustained. If maximizing lifetime wages were an appropriate welfare criterion, any potential new worker would choose to enter the rapidly growing population rather than a stationary population as long as the growth rate persists throughout the worker's lifetime. The depressed ages earned while young are guaranteed to be more than offset by the higher wages earned while old. Similarities and differences between this result and results for a Samuelson type pure consumption loan economy will be discussed below.

3. Effects of Non–Labor Factors of Production

The effect of introducing a non–labor factor of production can be seen by augmenting the production function, $Y = F(L_1, L_2, K)$, where K can be thought of as a fixed resource, such as land, or as a reproducible factor, such as capital. (Distinctions between the two types on non–labor factors will be discussed below.) Assuming that K is exogenous and unaffected by the choice of π_1, the introduction of K has no effect on the derivation of (1), so under constant returns it is still true that $\partial W/\partial\pi_1 = (f_{11} - f_{22})$. Euler's theorem now implies that $f_{11}\pi_1 + f_{21}\pi_2 = -f_{k1}k$, where $k = K/L$, and therefore that

$$f_{11} = -f_{k1}\frac{k}{\pi_1} - f_{12}\frac{1-\pi_1}{\pi_1}.$$

Using the analogous expression for f_{22}, and substituting into (1), the derivative in (2) generalizes to

$$\frac{\partial W}{\partial \pi_1} = f_{11}\left[\frac{1-2\pi_1}{(1-\pi_1)^2}\right] + k\left[\frac{f_{k2}}{\pi_2} - \frac{f_{k1}}{\pi_1}\right]. \tag{4}$$

The first term in (4) is zero when $\pi_1 = .5$, is positive for all $\pi_1 > .5$, and is negative for all $\pi_1 < .5$. The sign of the second term in (4) is indeterminate in general, and depends on the complementarity between K and workers of different ages. Note, however, that if W attains a critical point at $\pi_1 = .5$, then it must be the case that $f_{k1} = f_{k2}$ at that point, since the first term

[5] Even if a worker could choose the steady state age distribution in the population, maximizing lifetime wages is not necessarily an appropriate objective function for the worker. The effect of age structure on lifetime wages provides an interesting baseline for the analysis, however. The effects of discounting lifetime wages are considered below, along with consideration of the effects of age structure on the possibilities for intertemporal consumption smoothing.

must equal zero. The second derivative is

$$\frac{\partial^2 W}{\partial {\pi_1}^2} = (f_{111} - f_{112})\left[\frac{1-2\pi_1}{(1-\pi_1)^2}\right] - 2f_{11}(1-\pi_1)^{-2} + 2f_{11}\left[\frac{1-2\pi_1}{(1-\pi_1)^3}\right] + k\left[\pi_2^{-1}(f_{k12} - f_{k22}) - \pi_2^{-2} f_{k2} + \pi_1^{-2} f_{k1} - \pi_1^{-1}(f_{k11} - f_{k12})\right]. \tag{5}$$

If W attains a critical point at $\pi_1 = .5$, implying that $\pi_1 = \pi_2$ and $f_{k1} = f_{k2}$, then (5) reduces to

$$\frac{\partial^2 W}{\partial {\pi_1}^2} = \frac{-f_{11}}{4} + 2k(f_{k12} - f_{k11} + f_{k12} - f_{k22}). \tag{6}$$

No general restrictions can be placed on the third derivatives in the second term that make the sign of the term unambiguous. We cannot rule out the possibility that the term is negative and large enough in absolute value to offset the first term, which will always be positive under concavity. It is possible, then, that in the presence of other factors a uniform distribution of workers will maximize lifetime wages. It is also possible that a uniform distribution continues to minimize lifetime wages, as it does in the absence of non–labor factors. Some obvious restrictions on the third derivatives will guarantee that the second derivative is positive when $W(\pi_1)$ attains a critical point at $\pi_1 = .5$. The simplest is that $f_{k12} = f_{k11} = f_{k22}$ when $\pi_1 = \pi_2$ and $f_{k1} = f_{k2}$. This implies a symmetry in the relationship between K and the two kinds of labor that is consistent with the requirement that $f_{k1} = f_{k2}$ when $\pi_1 = \pi_2$.[6]

If K represents reproducible capital, rather than a fixed resource like land, then there may be a direct effect of the age distribution on K. The effects of population growth and age structure on capital–labor ratios have been the principle focus of most models of the economic effects of population growth. Although a complete treatment of the relationship between population growth and capital accumulation is beyond the scope of this paper, the issue will be addressed below in analyzing golden rule steady states.

4. Population Growth, Age Structure, and Lifetime Wages

A more complete model of the relationship between age structure and wage profiles can be constructed by considering a stable population with a constant population growth rate. The number of persons aged i at time t, denoted $L_{i,t}$, is by definition $L_{i,t} = B_{t-i}p_i$, where B_t denotes births in period t and p_i denotes the probability of survival from birth to age i, assumed to be invariant over time. If age–specific fertility and mortality rates remain constant over time, then by well known ergodicity properties[7] the population will converge to a stable population with a constant proportional age distribution. Births and the size of every age group will grow at some constant growth rate g. Expressed in discrete time, $B_t = B_0(1+g)^t$, and therefore $L_{i,t} = B_t(1+g)^{-i}$. Total

[6] If the production function can be written as $F(K, G(L_1, L_2))$, then conditional on there being a critical point in $W(\pi_1)$ at $\pi_1 = .5$, that point will be a minimum if $G_{11} = G_{22}$ when $L_1 = L_2$.

[7] See Arthur (1981) for a recent restatement and proof of these results.

population size at time t will be $P_t = \sum_0^{\omega} L_{i,t}p_i = B_t \sum_0^{\omega}(1+g)^{-i}p_i$, where ω is the highest age in the population.

For demographic simplicity, assume all workers die at exactly age ω, so that mortality can be ignored.[8] Defining π_i as the (constant) proportion of workers aged i in the steady state, the effect of a fertility induced change in the population growth rate on this proportion is

$$\frac{\partial \pi_i}{\partial g} = \frac{\pi_i}{(1+g)}(\bar{i} - i),$$

where $\bar{i} = \sum_i \pi_i i$ is the mean age of the labor force. An increase in the population growth rate causes an increase in the steady state proportion of workers at all ages below the mean age and a decrease in the proportion of workers at all ages above the mean age. Assuming constant returns technology, the marginal products of workers are unaffected by total population size, and are therefore constant in the steady state, determined only by the relative sizes of age groups in a pure labor economy. If wages at each age are equal to marginal products then total lifetime wages are given by $W = \sum_i F_i = \sum_i f_i$. The effect of the population growth rate on lifetime wages is

$$\begin{aligned}\frac{\partial W}{\partial g} &= \sum_i \sum_j f_{ij} \frac{\partial \pi_j}{\partial g} \\ &= (1+g)^{-1} \sum_i \sum_j \pi_j (\bar{j} - j) f_{ij} \\ &= (1+g)^{-1} \sum_i \left[\bar{j} \sum_j \pi_j f_{ij} - \sum_j j \pi_j f_{ij} \right] .\end{aligned} \tag{7}$$

This result holds for any production function, as long as g affects only the sizes of age groups and not the relative quantity of other productive factors. In the case of a pure labor economy with constant returns, the following proposition holds:

Proposition 2. *In a pure labor economy with a concave constant returns production function $Y = F(L_1, L_2, \ldots, L_\omega)$, where $L_i = B_t(1+g)^{-i}$, with the labor force growing at a constant growth rate g, total lifetime wages $W = \sum_i F_i$ attain a global minimum when $g = 0$. No finite growth rate exists that maximizes lifetime wages.*

To prove Proposition 2, note that in the case of a pure labor economy with constant returns,

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[8] Equivalently, assume that all workers survive from the age of entry into the labor force until some retirement age ω. Mortality before and after working life can be ignored, with "births" referring to labor force entrants, and g representing the labor force growth rate.

Euler's theorem requires that $\sum_j \pi_j f_{ij} = 0$. This implies that

$$\begin{aligned}\frac{\partial W}{\partial g} &= -(1+g)^{-1}\sum_i\sum_j j\pi_j f_{ij} = -(1+g)^{-1}\sum_j j\pi_j \sum_i f_{ij} \\ &= -(1+g)^{-1}\sum_j j\pi_j\left[f_{jj} + \sum_{i\neq j} f_{ij}\right].\end{aligned} \tag{8}$$

Under the assumption of a pure labor economy with constant returns, $\pi_j f_{jj} = -\sum_{i\neq j}\pi_i f_{ij}$, so (8) can be simplified to

$$\begin{aligned}\frac{\partial W}{\partial g} =&(1+g)^{-1}\left[\sum_j j\sum_{i\neq j}\pi_i f_{ij} - \sum_j j\pi_j \sum_{i\neq j} f_{ij}\right] \\ =&(1+g)^{-1}\sum_j\sum_{i\neq j} j(\pi_i - \pi_j)f_{ij}.\end{aligned} \tag{9}$$

Note that $\sum_j\sum_{i\neq j} x_{ij} = \sum_j\sum_{i>j}(x_{ij}+x_{ji})$. Since $f_{ij} = f_{ji}$, it follows that

$$\sum_j\sum_{i\neq j} j(\pi_i-\pi_j)f_{ij} = \sum_j\sum_{i>j}[j(\pi_i-\pi_j)f_{ji} + i(\pi_j-\pi_i)f_{ij}] = \sum_j\sum_{i>j}(\pi_i-\pi_j)(j-i)f_{ij} \tag{10}$$

Substituting from (10), then, (9) can be rewritten as

$$\frac{\partial W}{\partial g} = (1+g)^{-1}\sum_j\sum_{i>j}(\pi_i-\pi_j)(j-i)f_{ij} \tag{11}$$

By (9) it is clear that $W(g)$ attains a critical point at $g=0$, since $\pi_i = \pi_j \ \forall\,(i,j)$ in a stationary population with no mortality before age ω. The second derivative is

$$\begin{aligned}\frac{\partial^2 W}{\partial g^2} =& -(1+g)^{-2}(1+g)\frac{\partial W}{\partial g} + (1+g)^{-1}\sum_j\sum_{i>j}\left[(j-i)f_{ij}\left(\frac{\partial \pi_i}{\partial g} - \frac{\partial \pi_j}{\partial g}\right)\right] \\ &+ (1+g)^{-1}\sum_j\sum_{i>j}\left[(\pi_i-\pi_j)(j-i)\left[f_{iji}\frac{\partial \pi_i}{\partial g} - f_{ijj}\frac{\partial \pi_j}{\partial g}\right]\right].\end{aligned} \tag{12}$$

At a uniform age distribution $\pi_i = \pi_j$, so the first term and last terms go to zero. The second term is unambiguously positive, since $\frac{\partial \pi_i}{\partial g} > \frac{\partial \pi_j}{\partial g}\ \forall i < j$. A uniform labor force age distribution thus gives the unambiguous global minimum lifetime wages for any constant returns production function in a pure labor economy. This establishes Proposition 2.

In a pure labor economy under constant returns the absolute size of the population has no effect on wages— only the relative sizes of age groups matter. If non–labor factors are introduced then it is necessary to explicitly model the effect of population growth on those factors. If the non–labor factors are fixed resources, such as land, then the relative supply of the factor must necessarily decrease with population size and there is no steady state level of output per worker. If the non–labor factor is capital then a formal model of capital accumulation is required. Assuming that a steady state capital–labor ratio exists, some model is required to analyze the effect of population growth on the capital–labor ratio. As pointed out above, this issue has been the principal focus of most previous literature on the economic effects of population growth. It is beyond the scope of this paper to offer any new approaches to the relationship between population growth and capital accumulation. One standard approach, the assumption that golden–rule savings is achieved in every steady state, will be used below. More generally, the effect of population growth on capital–labor ratios can simply be thought of as an additional term affecting lifetime wages in some unknown direction. If this term is strongly negative, as implied by most previous models, then it may dominate the tendency for higher rates of population growth to increase lifetime wages in a pure labor economy.

Assuming a steady state capital–labor ratio k always exists, (7) continues to hold when a non–labor factor K is introduced, but Euler's theorem now implies that $\sum_j \pi_j f_{ij} = -k f_{ik}$. Substituting into (7) and allowing an effect of g on k, the result is

$$\begin{aligned}\frac{\partial W}{\partial g} &= (1+g)^{-1} \sum_i \bar{j} k f_{ik} - \sum_i \sum_j j \pi_j f_{ij} + \frac{\partial k}{\partial g} \sum_i f_{ik} \\ &= (1+g)^{-1} \left[\sum_j \sum_{i>j} (\pi_i - \pi_j)(j-i) f_{ij} + k \left(\sum_i f_{ik}(i - \bar{i}) \right) \right] + \frac{\partial k}{\partial g} \sum_i f_{ik}.\end{aligned} \tag{13}$$

If it is assumed that $\frac{\partial k}{\partial g} = 0$, and if $\frac{\partial W}{\partial g} = 0$ at a uniform age distribution, then $\sum_i f_{ik}(i - \bar{i}) = 0$, since the first term in (13) has already been shown to be zero when $\pi_i = \pi_j \ \forall (i,j)$. If this is true, however, then it must be the case that $f_{ik} = f_{jk} \ \forall (i,j)$ when $\pi_i = \pi_j$. Even ignoring the final term in (13), there are no general restrictions on the third cross–partial derivatives to establish whether this critical point is a minimum or maximum, as discussed above in the case of two types of workers. If an increase in the population growth rate decreases the capital–labor ratio, as it does in a simple neoclassical growth model, then the last term further modifies the result, implying a negative effect of population growth on lifetime wages as long as $f_{ik} > 0$ for all i.

5. Effects of Discounting Lifetime Wages

It is straightforward to introduce a discount rate into the wage stream. Returning to the pure labor economy, if we redefine W as discounted lifetime wages $W = \sum_i F_i(1+r)^{-i}$, and analyze the effects of the population growth rate on W, the result in (9) generalizes to

$$\frac{\partial W}{\partial g} = (1+g)^{-1} \sum_j \sum_{i \neq j} j \left[\pi_i (1+r)^{-j} - \pi_j (1+r)^{-i} \right] f_{ij} - \frac{\partial r}{\partial g} (1+r)^{-1} \sum_i i w_i (1+r)^{-i}. \tag{14}$$

The second term in (14) allows the possibility that the discount rate r is itself a function of g. Assuming for the moment that r is exogenous, making the second term in (14) equal to zero, the result implies that there will be a critical point at the growth rate which sets $\pi_i(1+r)^{-j} = \pi_j(1+r)^{-i}$ for all (i,j). Continuing to abstract from mortality, this will occur when $g = r$, the population growth rate equals the discount rate. As in the case when $r = 0$, it is easy to show that this critical point always gives the *minimum* discounted value of lifetime wages. For any exogenous discount rate r, discounted lifetime wages attain a global minimum when the population growth rate equals the discount rate. Evaluated at a point at which $g < r$ (e.g. $g = .01$ and $r = .10$), an increase in g will decrease the discounted value of lifetime wages.

The result is more complicated if r is a function of g. One important special case is when $r = g$ at all values of g, corresponding to Samuelson's (1958) "biological interest rate." This case has some intuitive appeal, since it captures the relationship between age structure and the possibilities for intergenerational borrowing.[9] The value of earning high old–age wages to offset low young–age wages in a rapidly growing population depends on the ability to borrow against old age wages. If the young can only borrow from the old, then the interest rate must increase as the population growth rate increases. In the case when $r = g$ at all values of g, the result in (14) can be rewritten by substituting g for r, noting that $\frac{\partial r}{\partial g} = 1$ and $(1+g)^{-i} = \pi_i \sum_j (1+g)^{-j}$:

$$\begin{aligned} \frac{\partial W}{\partial g} &= \sum_i (1+g)^{-i-1}\left[\sum_i \pi_i \sum_j j\pi_j f_{ij} - \sum_i i\pi_i F_i\right] \\ &= \sum_i (1+g)^{-i-1}\left[\sum_j j\pi_j \sum_i \pi_i f_{ij} - y\bar{i}_w\right] \\ &= -\, y\bar{i}_w \sum_i (1+g)^{-i-1}, \end{aligned} \tag{15}$$

where y is per worker output and $\bar{i}_w = \sum_i i\pi_i w_i / \sum_i \pi_i w_i$ is a weighted mean age in which each age is weighted by the proportion of total wages earned at that age. The result in (15) is unambiguously negative, implying that if the discount rate r is always equal to the population growth rate g, then increases in the growth rate must reduce the discounted value of lifetime wages. Although lifetime wages will always increase with increases in the population growth rate, in accordance with Proposition 2, they can never increase at the same rate as the population growth rate itself. In this biological interest rate case, then, the discounted value of lifetime wages is maximized at the most negative feasible population growth rate.

Although discounting of lifetime wages may be appropriate, especially in the biological interest rate regime, discounted lifetime wages may still not be an appropriate objective function for the representative worker. The discounted cost of any given lifetime consumption stream also declines

[9] See Willis (1982) for an excellent treatment of the relationship between age structure and intergenerational debt.

as the population growth rate increases. This means that a decline in discounted lifetime wages does not necessarily imply a decline in lifetime utility. The following sections clarify this point by incorporating the effects of age structure on wages with the effects on lifetime consumption possibilities.

6. Age Structure and Social Budget Constraints with Overlapping Generations

The role of population growth and age structure in overlapping generations models has been clarified by Arthur and McNicoll (1977, 1978) and Lee (1980) using the tools of formal demography. These tools also provide insights into the results proven above on the effects of age structure on age-specific labor productivity.

Before introducing effects of age structure on age-specific productivity it is worth reviewing the comparative steady state results for models in which workers at different ages are perfect substitutes. Looking at a continuous time version of the stable population analyzed above, if age-specific fertility and mortality are constant then births and the size of every age group grow at some constant exponential growth rate g, with $B_t = B_0 e^{-gt}$, and $N_{a,t} = B_t e^{-ga}$. Total population size at time t is $P_t = \int_0^{\omega} N_{a,t} p_a \, da = B_t \int_0^{\omega} e^{-ga} p_a \, da$, where ω is highest age in the population.

To see the basic structure of these models, assume that consumption at age a is given by c_a,[10] labor supply is given by l_a, and period t production is described by a concave constant returns production function $F(K_t, L_t)$, where L_t is the effective labor force at time t. Total labor L_t is a linear aggregation of the number of workers at each age, $L_t = B_t \int_0^{\omega} e^{-ga} p_a l_a \, da$.[11] The social budget constraint at time t is $F(K_t, L_t) = C_t + \dot{K}_t$, where C is total consumption and $\dot{K}$ is the time derivative of the capital stock. Generalizing Solow's growth model without age structure, this age structured economy can be imagined to have an economic–demographic steady state growth path with a constant proportional age distribution, a constant capital–labor ratio k and constant age-specific consumption levels c_a. In the steady state $\dot{K}_t = gK$, so normalizing by B_t the social budget constraint can be rewritten as

$$[f(k) - gk] \int_0^{\omega} e^{-ga} p_a l_a \, da = \int_0^{\omega} e^{-ga} p_a c_a \, da, \tag{16}$$

Following Arthur and McNicoll (1977, 1978), the simplest case is to assume that capital is accu-

[10] Age-specific consumption may be chosen to maximize lifetime utility as in Arthur and McNicoll (1977). As shown by Lee (1980), however, the principal insights of the comparative steady state results come from simply differentiating the social budget constraint to find the change in lifetime consumption possibilities. Details of how age-specific consumption is altered in response to a change in population growth add little additional information.

[11] In most pure consumption loan models, workers at each age receive exogenous endowments which are unaffected by population growth (see, for example, Samuelson, 1958, and Willis, 1982). In models with capital, wages are affected by capital-labor ratios, and may also be affected by age-specific weights such as the l_a terms used here. Arthur and McNicoll (1978) and Lee (1980) differ somewhat in their interpretation of the weights l_a in the definition of L_t. Arthur and McNicoll define them as age-specific labor force participation rates. Lee also includes differences in productivity as reflected in wage differentials. These productivity differences are exogenous with respect to age structure, however, and therefore implicitly assume that workers of all ages are perfect substitutes.

mulated by a golden–rule steady state savings rate,[12] implying that $f_k = g$. Drop the t subscripts to denote the time invariant normalized total quantities $L = L_t/B_t$, $C = C_t/B_t$, and define the time invariant proportions $\pi_a = e^{-ga}p_a / \int_0^\omega e^{-ga}p_a\, da$, $\pi_a^c = e^{-ga}p_a c_a / \int_0^\omega e^{-ga}p_a c_a\, da$, and $\pi_a^l = e^{-ga}p_a l_a / \int_0^\omega e^{-ga}p_a l_a\, da$, where π_a is the proportion of the population that is age a, π_a^c is the proportion of total consumption consumed by persons aged a, and π_a^l is the proportion of the labor force that is age a. The fundamental comparative steady state result is derived by differentiating (16) with respect to g, imposing the golden rule savings condition that $f_k = g$, and noting that $(y - gk)L = C$ by the budget constraint:[13]

$$-kL + C\int_0^\omega \pi_a^l \frac{\partial \ln l_a}{\partial g}\, da - C\int_0^\omega a\pi_a^l\, da = C\int_0^\omega \pi_a^c \frac{\partial \ln c_a}{\partial g}\, da - C\int_0^\omega a\pi_a^c\, da. \tag{17}$$

This result can be simplified to

$$\int_0^\omega \pi_a^c \frac{\partial \ln c_a}{\partial g}\, da = \bar{a}_c - \bar{a}_l - \frac{k}{c} + \int_0^\omega \pi_a^l \frac{\partial \ln l_a}{\partial g}\, da. \tag{18}$$

Two weighted mean ages appear in (18). $\bar{a}_c$ is the mean age of the cross–section population when each age is weighted by its share in total consumption. $\bar{a}_l$ is simply the mean age of the labor force.[14] The interpretation of (18) can be seen with a simple special case. If all changes in consumption and labor supply implied by (18) are absorbed by constant proportional adjustments at each age, so that $\partial \ln c_a/\partial g = \gamma$ for all a and $\partial \ln l_a/\partial g = \lambda$ for all a, then (18) reduces to

$$\gamma = (\bar{a}_c - \bar{a}_l) - \frac{k}{c} + \lambda. \tag{19}$$

The proportional change in consumption at each age in response to an increase in the steady state population growth rate will be equal to the "average age of consumption" minus the "average age of production" minus the capital–consumption ratio plus the proportional change in labor effort at each age. If consumption occurs on average at older ages than production, as in Samuelson's consumption loan model in which there is a period of work followed by a period of retirement, then the average age terms imply a positive intergenerational transfer effect. If childhood consumption is included in the model, then the sign of the intergenerational transfer effect is ambiguous, and Arthur and McNicoll (1978) suggest that it can easily be negative. Whatever the sign of this effect, there will be an unambiguously negative capital dilution effect of minus the capital consumption ratio. Arthur and McNicoll (1978) and Lee (1980) suggest that this effect is likely to swamp any

[12] The golden rule assumption implies that a new optimal savings rate which maximizes per capita consumption is chosen whenever the population growth rate changes.

[13] See Arthur and McNicoll (1978: 244) and Lee (1980: 1145).

[14] It is a standard result in mathematical demography that the effects of changes in the population growth rate on cross–section population aggregates are described by mean ages and weighted mean ages. See Coale (1972), Keyfitz (1977), and Preston (1982). Lam (1984) generalizes the result to higher moments of distributions of population characteristics.

plausible positive intergenerational transfer effect. Finally, population growth may lead to increases or decreases in age–specific labor force participation rates.

The simple neoclassical growth model and consumption loan model can be seen to be special cases of (19). In a growth model with no age structure, such as Solow (1956), consumption and production always take place at the same age, so that the intergenerational transfer effect disappears. In the absence of an effect of population growth on labor effort, this leaves the standard capital dilution effect of population growth for golden rule steady states. In a simple consumption loan model there is no capital, leaving only the intergenerational transfer effect.

7. Age Structure and Labor Productivity

The continuous age structure models of Arthur and McNicoll and Lee, like the related overlapping generations models of Samuelson (1958, 1976), Deardorff (1976), and Willis (1982), make very simple assumptions about age–specific labor productivity. In the pure consumption loan model of Samuelson (1958), life cycle wage profiles are constant, and are simply modeled as age–specific endowments. The advantages of higher rates of population growth result from the relationship between age structure and the ability to make intergenerational consumption loans, not from any effect of age structure on age–specific productivity. When capital is introduced in Samuelson's later work (1975, 1976) and in the work of Arthur and McNicoll (1977, 1978) and Lee (1980), wages are affected, but labor is homogeneous except for exogenous variations across age in labor supply or effort. In other words, labor is is simply a linear aggregation of workers of all ages. Wages at all ages rise and fall together as the capital–labor ratio responds to varying population growth rates. Wages vary across ages only because of the exogenous difference in age–specific labor supply or effort which are built into the models.

Given the theoretical results presented above and the empirical evidence on the effects of relative age group size on age–specific wages, it is instructive to consider how models of the effects of population growth are affected by explicitly modeling age–specific productivity as a function of the relative sizes of all age groups. A more realistic model, for example, will have wages of young workers moving in the opposite direction from wages of old workers in response to an increase in the population growth rate.

More realistically, then, modify the model above to let output depend explicitly on the number of workers at each age $Y_t = F(K_t, L_{0,t}, \ldots, L_{a,t}, \ldots, L_{\omega,t})$, where $L_{a,t}$ is the number of workers aged a in period t. The model can be generalized in this way and yet kept quite tractable by continuing to assume that the production function is constant returns to scale. Continue to let l_a represent the labor force participation rate of workers aged a. Let w_a denote the marginal product of workers aged a, with w_a having a natural interpretation as the competitive wage, though a model of wage determination need not be specified for most of the results which follow.

It is still true that the budget constraint $Y_t = C_t - \dot{K}_t$ must be satisfied in every period, and that $\dot{K} = gK$ in the steady state. Normalizing by labor force size $L_t = B_t \int_0^{\omega} e^{-ga} p_a l_a \, da$, the linearly homogeneous production function can be normalized to $y = Y_t/L_t = f(k, \pi_0^l, \ldots, \pi_a^l, \ldots, \pi_\omega^l)$, where π_a^l, as defined above, is the proportion of the total labor force made up of workers aged a. The only modification to the budget constraint in (16), then, is to generalize per worker output, implying that

$$[f(k,\pi_0^l,\ldots,\pi_a^l,\ldots,\pi_\omega^l)-gk]\int_0^\omega e^{-ga}p_a l_a\,da=\int_0^\omega e^{-ga}p_a c_a\,da. \tag{20}$$

Differentiating (20) with respect to g,

$$L\left[\frac{\partial k}{\partial g}(f_k-g)-k+\int_0^\omega w_a\frac{\partial \pi_a^l}{\partial g}\,da\right]+(y-gk)\left[\int_0^\omega e^{-ga}p_a\frac{\partial l_a}{\partial g}\,da-\int_0^\omega ae^{-ga}p_a l_a\,da\right]$$
$$=\int_0^\omega e^{-ga}p_a\frac{\partial c_a}{\partial g}\,da-\int_0^\omega ae^{-ga}p_a c_a\,da. \tag{21}$$

Imposing the golden rule savings condition that $f_k=g$ and noting that $(y-gk)L=C$ by the budget constraint, (21) can be rewritten as

$$L\int_0^\omega w_a\frac{\partial \pi_a^l}{\partial g}\,da-kL+C\int_0^\omega \pi_a^l\frac{\partial \ln l_a}{\partial g}\,da-C\bar{a}_l=C\int_0^\omega \pi_a^c\frac{\partial \ln c_a}{\partial g}\,da-C\bar{a}_c. \tag{22}$$

The result looks identical to that in (17) except for the first term involving the marginal product of each age worker. In the previous model, note that all workers have the same marginal product $f_l=y-f_k k$. Substituting this into (22), the first term becomes $L(y-f_k k)\int_0^\omega \frac{\partial \pi_a^l}{\partial g}\,da$, which vanishes since $\int_0^\omega \frac{\partial \pi_a^l}{\partial g}\,da=0$. In the more general case, it is straightforward to show that

$$\frac{\partial \pi_{\hat{a}}^l}{\partial g}=\pi_{\hat{a}}^l\left[\bar{a}_l-\hat{a}+\frac{\partial \ln l_{\hat{a}}}{\partial g}-\int_0^\omega \pi_a^l\frac{\partial \ln l_a}{\partial g}\,da\right],$$

and therefore

$$\begin{aligned}\int_0^\omega w_a\frac{\partial \pi_a^l}{\partial g}\,da&=\bar{w}\left[\bar{a}_l-\int_0^\omega \pi_a^l\frac{\partial \ln l_a}{\partial g}da\right]+\int_0^\omega \pi_a^l w_a\frac{\partial \ln l_a}{\partial g}da-\int_0^\omega a\pi_a^l w_a\,da\\&=\bar{w}\left[\bar{a}_l-\bar{a}_w+\int_0^\omega \pi_a^w\frac{\partial \ln l_a}{\partial g}da-\int_0^\omega \pi_a^l\frac{\partial \ln l_a}{\partial g}da\right],\end{aligned} \tag{23}$$

where $\pi_a^w=e^{-ga}p_a l_a w_a/\int_0^\omega e^{-ga}p_a l_a w_a\,da$, the proportion of total wages earned by workers age a, and $\bar{w}=\int_0^\omega \pi_a^w w_a\,da$, the mean wage in the working population.

Substituting (23) into (22), imposing the requirement that $\bar{w}L=C$ in golden rule steady states, and rearranging terms, the effect of a change in the population growth rate on steady state lifetime consumption can be summarized as

$$\int_0^\omega \pi_a^c\frac{\partial \ln c_a}{\partial g}\,da=\bar{a}_c-\bar{a}_w-\frac{k}{c}+\int_0^\omega \pi_a^w\frac{\partial \ln l_a}{\partial g}\,da. \tag{24}$$

The only difference between the comparative steady state result in (24) and the result in (18) is that the mean age of the labor force and the integral of changes in labor supply are weighted by age–specific wages in the new result. The similarity is surprising, since (24) describes the effect of population growth for any general constant returns production function with every age worker considered as a separate factor of production. No assumption has been made about the elasticity of substitution between workers of different ages. The result is completely general in this respect, allowing, for example, workers close in age to be substitutes while workers farther apart in age are complements, or alternatively allowing workers of all ages to be either substitutes or complements.

How is it that (24) summarizes the effects of population growth on per capita lifetime consumption possibilities for any constant returns production function without any specification of the substitutability of workers of different ages? Surely there is some basis to the intuition that the effects of a change in age structure in an economy where workers of all ages are perfect substitutes will be very different than in an economy where there is very limited substitutability across ages. The answer is that all information about the elasticities of substitution across age groups of workers and between labor and capital is already captured in the weighted mean ages in (24). The result is less surprising when it is recalled that the result in (18) describes the effects of population growth for any constant returns function of capital and labor without any specification of the elasticity of substitution between capital and labor. In both cases it is not that the elasticity of substitution does not matter. It is rather that its effects are entirely captured in the mean age terms that summarize the comparative steady state result.

8. Conclusions

In Samuelson's original overlapping generations economy the lifetime incomes of workers are constant, modeled as exogenous age–specific endowments. Lifetime utility of workers is affected by changes in age structure not because of changes in income profiles but because of changes in the potential for intergenerational borrowing and lending. If a pay–as–you–go social security system is maintained, for example, higher rates of population growth lead to higher lifetime utility because of the increased ratio of contributors to dependents.

This paper analyzes a different mechanism through which age structure affects the lifetime welfare of a representative worker. Exogenous age–specific endowments are replaced with age–specific wages, where the wages are marginal products from a concave constant returns production function. For a pure labor economy, the closest analog to Samuelson's pure consumption loan economy, the paper proves that lifetime wages attain a global minimum when there is a uniform age distribution. Persistent positive or negative population growth rates, assuming they can be maintained, always generate higher lifetime wages than those in a stationary population. The results imply that even in the absence of intergenerational transfers, increases in the population growth rate would increase the utility of workers if lifetime utility were an increasing function of lifetime wages. The surprising result that lifetime wages tend to decrease as the age structure moves closer to uniformity does not appear to have been previously recognized in theoretical analysis of age–earnings profiles or in models of the economic effects of population growth.

If lifetime wages are discounted at some constant rate, the result generalizes to the condition that discounted lifetime wages attain a global minimum when the population growth rate equals the discount rate. This result may not hold if the discount rate is itself a function of the population growth rate. In the special case in which the discount rate is always equal to the population growth rate, corresponding to Samuelson's "biological interest rate," discounted lifetime wages unambiguously fall with increases in the population growth rate.

The presence of non–labor factors complicates, but does not necessarily reverse, the tendency for uniformity in the age distribution to minimize lifetime wages. If the population growth rate directly affects capital–labor ratios, as argued in most previous models of the economic effects of population growth, capital dilution effects may overcome the effects of population growth on lifetime wage profiles.

Lifetime wages, discounted or not, cannot describe lifetime utility without consideration of lifetime consumption profiles. In order to capture these effects and to consider the role of capital accumulation, the paper incorporates the effects of age structure on age– specific labor productivity into models which analyze cross–section social budget constraints in golden rule economic–demographic steady states. It is proven that previous comparative steady state results based on the assumption that workers of different ages are perfect substitutes continue to hold for any assumption about the elasticity of substitution between workers of different ages, providing the "mean ages" which determine the result are appropriately defined. The surprising robustness of the previous results occurs not because the elasticities of substitution between workers of different ages do not matter, but because their effects are captured in the "wage–weighted mean age" of the labor force.

The results demonstrate a number of important and previously unrecognized effects of age structure on life–cycle wage profiles. The results should not necessarily be interpreted as providing new ammunition for debates over population policy. Issues of population policy are better analyzed by looking directly at the optimality of private fertility decisions, as in Nerlove et al. (1987) and Willis (1987). The effects of age structure on lifetime wages established in this paper do provide useful insights into the changes in wage profiles that will be observed as populations move closer to or farther away from uniform age distributions, and fill an important gap in previous models of the economic effects of changing age structure.

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