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Abstract. In this paper, we provide simple geometrical proofs of various results from the public-good literature using the Kolm triangle. We also present a new result concerning the subsidization of private contributions.

Keywords. Public Goods, Nash Equilibrium, Stackelberg Equilibrium, Kolm triangle, Redistribution, Subsidies

JEL Classification System. H41, H20

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1. Introduction

The Kolm triangle is the analogue of the Edgeworth box for an economy with two agents, one private good and one pure public good. It was first introduced in the literature by Malinvaud (1971) who refers to unpublished manuscripts by S.Ch. Kolm. Schlesinger (1989) describes it in some detail and illustrates its use in analyzing Lindahl and Nash equilibria.

Despite its potential, the Kolm triangle hardly appears in the literature. A search made on the March 1993 *EconLit* disc (covering the *Journal of Economic Literature* since 1969) for entries containing 'Kolm' and 'triangle' returned only Schlesinger (1989). Laffont (1988)'s textbook displays a few diagrams of this tool, but he just barely refers to them in the text.

In this paper, we provide simple geometrical proofs of various results from the public-goods literature using the Kolm triangle. We also use it to present a new result concerning the subsidization of private contributions. We want to show that the Kolm triangle is a powerful tool for analyzing public-goods problems.

Our reference framework will be the model of private contributions to public goods used by Bergstrom, Blume and Varian (1986). With the Kolm triangle, we can easily study the existence and uniqueness of Nash equilibria, the effects of redistribution of the initial wealth, the level of provision in Stackelberg equilibria, and the effects of subsidizing private contributions.

2. The Model

We have two agents, $i = 1, 2$, each of whom consumes one private good, x_i, and one shared public good, G. Agent i has a preference

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ordering over the pairs (x_i, G) that can be represented by a quasi-concave utility function, $U_i(x_i, G)$. We assume that the public good can be produced at a constant marginal cost. Choosing a suitable choice of units, we can then normalize all prices to be 1. Finally, let (w_1, w_2) be the agents' initial endowments of private goods.

The agents choose their private contributions, g_i, to the public good. The total amount of public good provided is determined by the sum of the individual contributions, $G = g_1 + g_2$. Each agent i solves

$$\begin{aligned} \max_{x_i, g_i} \quad & U_i(x_i, g_1 + g_2) \\ \text{s.t.} \quad & x_i + g_i = w_i \\ & x_i, g_i \geq 0. \end{aligned}$$

We can use the budget constraint to eliminate x_i and write the individual's optimization problem more compactly as

$$\begin{aligned} \max_{g_i} \quad & U_i(w_i - g_i, g_1 + g_2) \\ \text{s.t.} \quad & 0 \leq g_i \leq w_i. \end{aligned} \tag{1}$$

A more general version of this model, with any number of agents, has been extensively studied by Bergstrom, Blume and Varian (1986).

3. The Kolm Triangle

Figure 1 shows a Kolm triangle for our model economy. The height of the triangle is given by the total amount of resources available, $w_1 + w_2$. Since the sum of the distances to the sides is constant and equals the height of the triangle, then, for any point inside the triangle, we have

$$x_1 + x_2 + G = w_1 + w_2.$$

Therefore, any point inside the triangle is associated with a feasible allocation. In any allocation, z, agent i's private consumption is given by the distance from z to O_iO_0. The amount of public good, G, associated with z is simply given by the distance from z to the base of the triangle, O_1O_2.

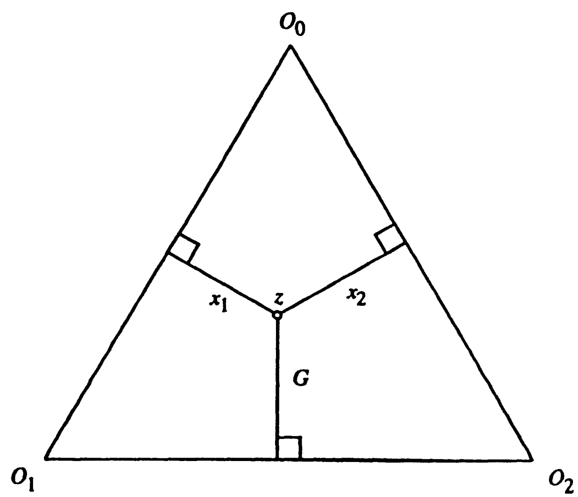

Figure 1. A feasible allocation in a Kolm triangle.

In figure 2 we represent the agents' indifference maps. We start from a given allocation, z. Any other allocation in the set B must be better than z for agent one since in B she gets more of both goods than in z. In W, on the other hand, agent one gets less from both goods so she must be worse off. The direction of the preferences is shown in figure 2. Agent i's indifference curves are convex to his origin, O_i, whenever his preferences are quasiconcave.

Since along O_1O_2 we have that $G = 0$, then any point along O_1O_2 is associated with an initial allocation, w. If we normalize the length of O_1O_2 to be 1, then the distance of w from O_1 will be

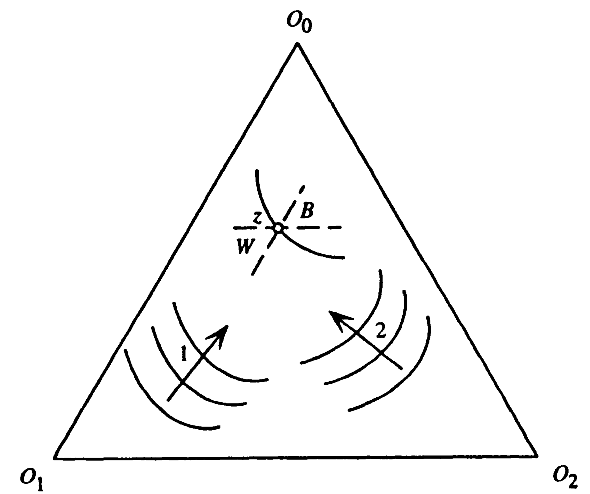

Figure 2. The indifference maps in a Kolm triangle.

given by $w_1/(w_1 + w_2)$.

4. Nash Equilibrium

A Nash equilibrium in this model is a vector of contributions (g_1^*, g_2^*) which solves the two agents' following optimization programs:

$$\begin{aligned} &\max \quad U_1(w_1 - g_1, g_1 + g_2^*) \\ &\text{s.t.} \quad 0 \leq g_1 \leq w_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} &\max \quad U_2(w_2 - g_2, g_1^* + g_2) \\ &\text{s.t.} \quad 0 \leq g_2 \leq w_2. \end{aligned}$$

Figure 3 shows a Nash equilibrium, denoted by E. Let $A = (w_1, w_2)$ represent the initial allocation. When $g_2 = 0$, agent one's opportunity locus is given by the segment AC which is parallel to O_2O_0—*i.e.*, along AC we have that $x_2 = w_2$. When agent two is contributing $g_2 = g_2^* = A'J$, agent one's opportunity locus shifts to $A'C'$. At the Nash equilibrium, E, agent one is optimized on his 'budget line' $A'C'$ where she contributes $g_1 = g_1^* = A''I$. (She consumes $EH = w_1 - g_1^*$ of the private good.) When agent one contributes g_1^*, agent two's opportunity locus shifts from AB (where $g_1 = 0$) to $A''B''$. (Note that AB and $A'B'$ are parallel to O_1O_0.) In $A''B''$, agent two's most preferred point is E, where he contributes g_2^*. Since the agents' indifference curves cross through E, the Nash equilibrium is not Pareto optimal. (A Pareto optimal Nash equilibrium is a possibility only at the endowment point, A.)

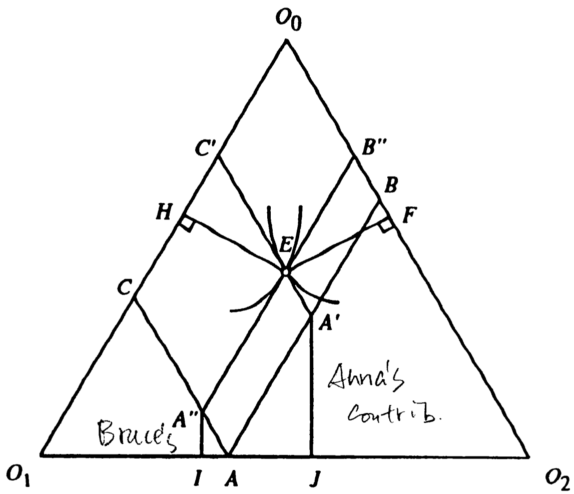

Figure 3. A Nash equilibrium.

Let's denote by $g_1(g_2)$ and $g_2(g_1)$ agent one's and agent two's optimal solutions to (1) as functions of the other agent's gift. Thus, $g_1(g_2)$ and $g_2(g_1)$ are the agents' reaction functions. Then, if (g_1^*, g_2^*) is a Nash equilibrium, we must have $g_1^* = g_1(g_2^*)$, and

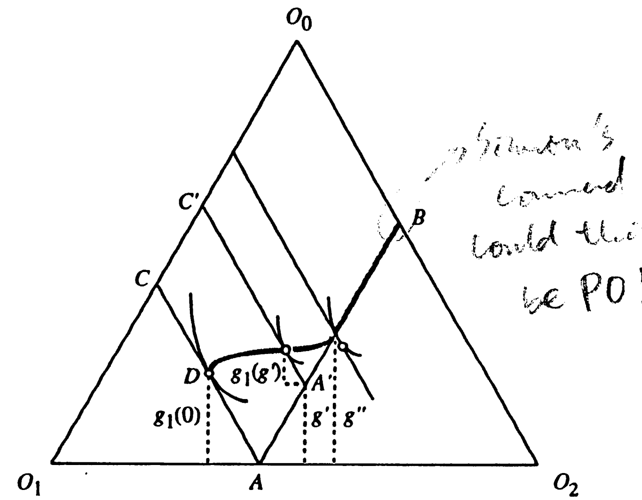

Figure 4. Agent one's reaction function.

$g_2^* = g_2(g_1^*)$.

We can represent agent one's reaction function in a Kolm triangle, see figure 4. Again, let $A = (w_1, w_2)$ represent the initial allocation. When $g_2 = 0$, agent one's opportunity locus is given by the segment AC. Given this constraint, agent one would choose to contribute $g_1(0)$. When $g_2 = g'$, the opportunity locus will shift to $A'C'$, and agent one will choose $g_1(g')$ for a total amount of G given by $g_1(g') + g'$. When $g_2 \geq g''$, we have that $g_1(g_2) = 0$.

If both goods are normal goods, the reaction function $g_1(g_2)$ cannot be steeper than O_1O_0 (since that would imply a smaller demand of x_1 as income increases) and it cannot be flatter than O_1O_2 (since that would imply a smaller demand of G as income increases). As a result, once agent one's reaction function hits

AB it has to stay in AB since AB is parallel to O_1O_0—of course, the reaction function $g_1(g_2)$ doesn't need to ever hit AB. Said another way, once agent one contributes nothing to the public good, bigger contributions by agent two will only induce agent one to keep contributing nothing. The curve DB in figure 4 represents agent one's reaction function. (A similar derivation for agent two will tell us that $g_2(g_1)$ has to be flatter than O_2O_0 and steeper than O_1O_2.)

4.1. Existence of Nash Equilibrium

Given an initial distribution of wealth, we can plot $g_1(g_2)$ and $g_2(g_1)$. The existence of Nash equilibrium (Theorem 2 in Bergstrom, Blume and Varian (1986)) will be proved if we can show that the graphs of the reaction functions cross inside the triangle. Refer to figure 5. We have that $g_1(g_2)$ must start out somewhere in AC and must reach the segment BO_0. Agent two's reaction function, $g_2(g_1)$, must go from AB to CO_0. Both reaction functions must always stay inside the romboid ACO_0B, and by the assumptions made about the preferences, they both have unbroken graphs. Thus, the existence of Nash equilibrium is established.

4.2. Uniqueness of Nash Equilibrium

Theorem 3 in Bergstrom, Blume and Varian (1986) says that "there is a unique Nash equilibrium with a unique quantity of public good and a unique set of contributing consumers." Here the uniqueness follows from the bounds imposed by the strict normality assumption (Bergstrom, Blume and Varian (1986), page 32) on the slope of the reaction functions. The first panel in figure 5 shows a unique Nash equilibrium. The second panel gives an example of multiple equilibria when G is an inferior good.

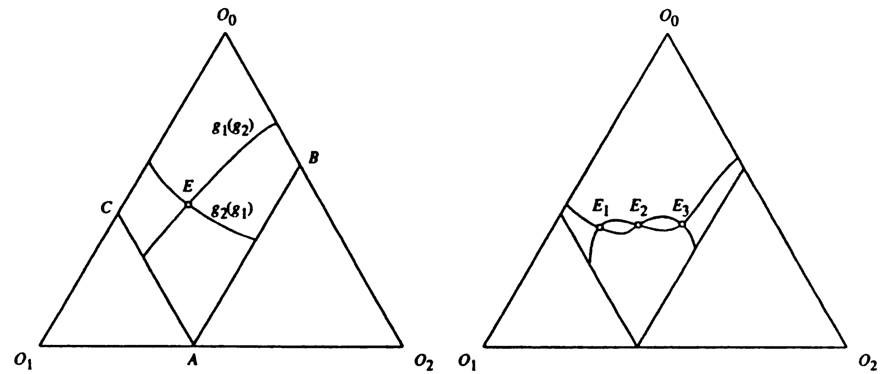

Figure 5. Existence and uniqueness of Nash equilibrium.

5. Redistribution of Income

Warr (1983) discovered an interesting neutrality theorem that was later extended by Bergstrom, Blume and Varian (1986). Assume that we have a Nash equilibrium, (g_1^*, g_2^*). If income is redistributed among contributing consumers in such a way that none of them loses more income than his original distribution, then there is a new Nash equilibrium, (g_1^{**}, g_2^{**}), where $g_1^{**} + g_2^{**} = g_1^* + g_2^*$, and $x_i^{**} = x_i^* = w_i - g_i^*$. That is, the same amount of public good is provided and each agent consumes the same amount of private goods that in the original equilibrium—*i.e.*, every consumer changes the amount of his gift by precisely the amount of the income transfer.

Figure 6 shows the effect of a redistribution of income from agent one to agent two that shifts the initial point from A to A'. The diagram shows the agents' reaction functions whose intersection determines the Nash equilibrium E. The portion of $g_1(g_2)$ between AC and $A'C'$ is no longer relevant after the redistribution. On the other hand, agent two's reaction function, $g_2(g_1)$, gains an

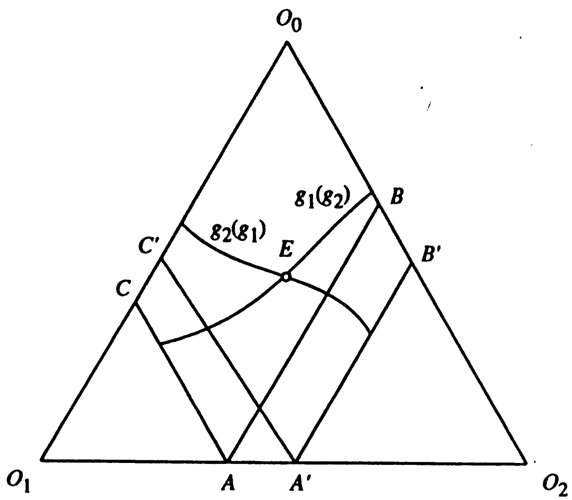

Figure 6. Redistribution of Income.

additional portion between AB and $A'B'$ after the redistribution. However, the old Nash equilibrium is the Nash equilibrum of the new game. The agents' consumptions remain unchanged.

Figure 7 shows the bounds on income redistribution. In A' we have taken away from agent two an amount of income equal to his gift in the initial Nash equilibrium. This is the maximum amount that we can take away from him and still get the same equilibrium level of public good and private consumptions. The maximum redistribution from agent one to agent two—that will leave the equilibrium amount of G and (x_1, x_2) unchanged—will move the endowment to A''.

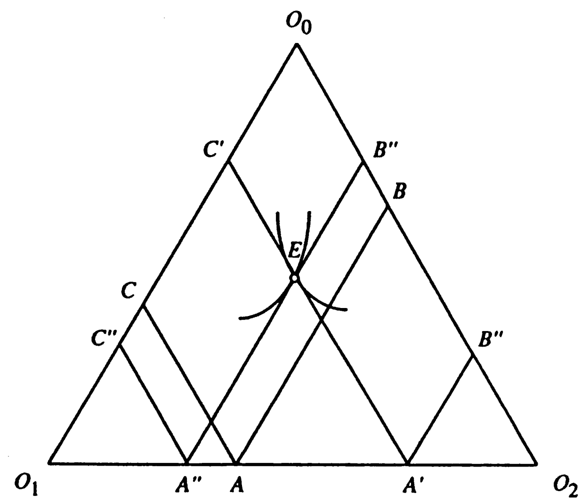

Figure 7. Bounds on the income redistribution.

6. Stackelberg Equilibrium

Varian (1993) studies sequential contributions to public goods. The Kolm triangle is a useful tool to gain further insights into his results. Let agent one be the leader and agent two be the follower. Then, the Stackelberg equilibrium will be determined by agent one choosing her most preferred point in agent two's reaction function. That is, agent one solves

$$\max_{g_1} \quad U_1(w_1 - g_1, g_1 + g_2(g_1))$$
$$\text{s.t.} \quad 0 \leq g_1 \leq w_1.$$

where $g_2(g_1)$ is agent two's reaction function—*i.e.*, the solution to (1) for agent two.

Varian (1993)'s main result (theorem 2) states that the leader's contribution at the Stackelberg equilibrium is bounded above by her contribution at the Nash equilibrium. As a corollary, the total amount of the public good in the Stackelberg equilibrium is never bigger than the total amount provided in the Nash equilibrium. Figure 8 shows these results.

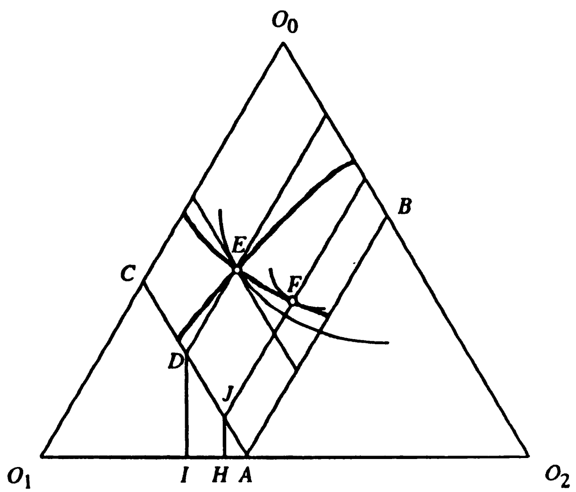

Figure 8. Stackelberg equilibrium: Agent one is leader.

Figure 8 shows Nash and Stackelberg equilibria. The Nash equilibrium, E, is determined by the crossing of the reaction functions. The Stackelberg equilibrium, F, is given by agent one's most preferred point in agent two's reaction function. We have drawn agent one's indifference curves through those equilibria. Looking at the indifference curve through E, we see that agent one will move to points of $g_2(g_1)$ to the right of E. Since $g_2(g_1)$ has a negative slope,

this movement necessarily implies less G in the Stackelberg equilibrium (Varian (1993), corollary to theorem 3). We can also easily see why agent one's contribution at the Stackelberg equilibrium can be no larger than her contribution at the Nash equilibrium. Since the Stackelberg equilibrium cannot lie to the left of the Nash equilibrium, it implies that agent one's contribution will be smaller. In figure 8, agent one contributes $g_1^* = DI$ at the Nash equilibrium and $g_1^s = JH$ at the Stackelberg equilibrium.

7. Subsidizing Contributions

Back in a Nash model, Roberts (1987) discovered the puzzling result that rich people might be made worse off when their contributions are subsidized at a higher rate than poor people—*i.e.*, when the contributions are tax-deductible in a system of progressive income taxation. This issue has been examined by Bergstrom (1989) who shows that if we have two *identical* individuals contributing to a public good, each will prefer to face a price higher than the price faced by the other individual.[1] In Roberts (1987) and Bergstrom (1989) the subsidy is paid by a lump-sum tax on both agents. Varian (1993) shows that each agent will prefer to subsidize the other agent even if he must pay the entire amount of the subsidy himself. In Varian's model, agents have *quasi-linear utility* functions.

[1] There is a small problem in Bergstrom (1989)'s argument in page 172 where he writes: "But since both are consuming the same amount of public good and since we have assumed disminishing marginal rate of substitution, the only way this can happen is that Plato has more private good than Aristotle." The concavity of the utility function only implies that the marginal rate of substitution disminishes along a *given* indiference curve and in the argument, both individuals end up at different utility levels. It is possible to construct a counterexample where Aristotle is better off than Plato after receiving the subsidy. However, assuming that the public good is a normal good then Bergstrom's claim holds true. In that case, holding the amount of public good fixed, a smaller marginal rate of substitution must necessarily imply a higher utility level.

We shall show that for two *non-identical* individuals with *general* quasi-concave utility functions, when both goods are normal, an agent will always want to subsidize the other agent's contributions even if he must pay the entire amount of the subsidy himself. We only analyze here the case where we have interior Nash equilibria before the subsidy.

In the subsidy game, agent one will subsidize agent two at the rate s in $(0,1)$. Agent two solves

$$\max_{g_2} \quad U_2(w_2-(1-s)g_2, g_1+g_2)$$
$$\text{s.t.} \quad 0 \leq (1-s)g_2 \leq w_2.$$

and agent one's problem is

$$\max_{g_1} \quad U_2(w_1-T-g_1, g_1+g_2)$$
$$\text{s.t.} \quad 0 \leq g_1 \leq w_1-T.$$

where $T = sg_2$. Given the subsidy rate, s, a Nash equilibrium is a vector of contributions $(g_1^*(s), g_2^*(s))$ which solves both agent one's problem,

$$\max_{g_1} \quad U_1(w_1-sg_2^*(s)-g_1, g_1+g_2^*(s))$$
$$\text{s.t.} \quad 0 \leq g_1 \leq w_1-sg_2^*(s)$$

and agent two's problem,

$$\max_{g_2} \quad U_2(w_2-(1-s)g_2, g_1^*(s)+g_2)$$
$$\text{s.t.} \quad 0 \leq (1-s)g_2 \leq w_2.$$

We claim that—provided that both agents are contributing at the initial Nash equilibrium where $s=0$—there always exists a

subsidy rate, s, such that agent one—who pays the subsidy—is better off at the resulting Nash equilibrium, *i.e.*,

$$U_1(w_1-sg_2^*(s)-g_1^*(s), g_1^*(s)+g_2^*(s)) > U_1(w_1-g_1^*(0), g_1^*(0)+g_2^*(0)).$$

Further, agent two—who is being subsidized—is worse off than before the subsidy, *i.e.*,

$$U_2(w_2-(1-s)g_2^*(s), g_1^*(s)+g_2^*(s)) < U_2(w_1-g_2^*(0), g_1^*(0)+g_2^*(0)).$$

7.1. The Geometry of a Subsidy

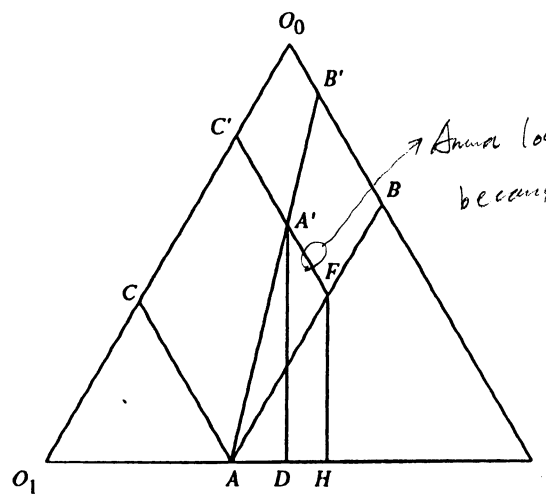

Figure 9. The Geometry of a Subsidy.

Figure 9 shows the geometry of a subsidy. The subsidy changes the slope of agent two's opportunity locii. In particular, if α measures the angle, in radians, of AB' with respect to O_1O_2, then

$$s = \frac{\sin(\alpha - \pi/3)}{\sin\alpha}.$$

When $\alpha = \pi/3$ so that AB' is parallel to O_1O_0, then $s = 0$. At the other extreme, when $\alpha = 2\pi/3$ so that AB' is parallel to O_2O_0 then $s = 1$. Since agent one has to pay for the subsidy, her opportunity locii will be also affected. In figure 9, when $g_2 = A'D$, agent one has to pay $T = sg_2 = A'D - FH$. This shifts her opportunity locus to $A'C'$.

7.2. Subsidizing the other Agent

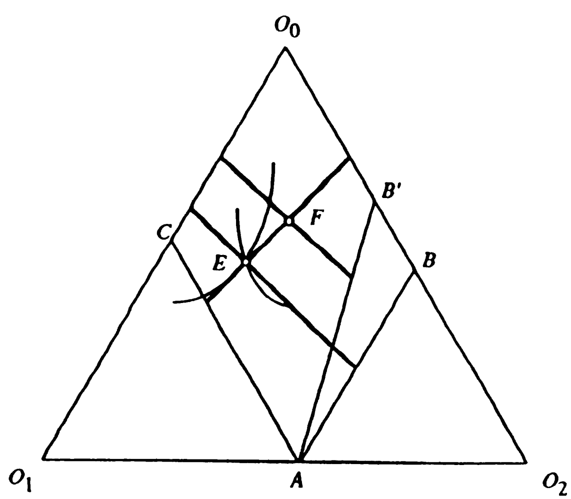

Figure 10. Agent one subsidizes agent two.

Figure 10 shows the effects of a subsidy from agent one to agent two. We display the initial Nash equilibrium, E, where the agents' reaction functions cross. We have also represented the agents' indifference curves through E. When agent one subsidizes agent two, agent two's new reaction function has to be above his old reaction function. At the new equilibrium, F, agent one is clearly better off than at E. What about agent two? Since the slope of agent two's indifference curve through E is parallel to O_1O_0 which is the upper bound for the slope of agent one's reaction function, $g_1(g_2)$, it follows that F must lie below agent two's indifference curve through F.

7.3. Corner Solutions

If agent two was not contributing towards the public good at the initial Nash equilibrium, the same results hold provided that there is a subsidy that induces him to contribute a positive amount. The other corner solution, where agent one was not contributing initially, can lead to anything. It may or may not be possible to improve agent one's welfare with the subsidy; and, in either case, agent two might end up better or worse off.

8. Conclusion

We have used the Kolm triangle to show various results from the literature that deals with private contributions to pure public goods. The geometrical proofs are quite simple and very useful to develop the intuition behind the results.

We have also generalized a result concerning the subsidization of the private contributions using this graphical tool. We have showed that for two *non-identical* individuals with *general* quasi-concave utility functions, when both goods are normal, an agent will generally want to subsidize the other agent's contributions even if she must pay the entire amount of the subsidy himself. Only when an agent perceived that there was 'too much' public good initially—*i.e.*, she is at a corner solution contributing nothing—, she might not be better off subsidizing the other agent's contribution.

We conclude that the Kolm triangle is a powerful tool that can be productively used as a research and pedagogic device in public-goods problems.

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