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David D. Li
Francesca Cornelli

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Large Shareholders, Private Benefits of Control and Optimal Schemes for Privatization*

Francesca Cornelli
(London School of Economics)

David D. Li
(University of Michigan)

ABSTRACT

We analyze optimal schemes for privatization in a transitional economy. In many cases, established Western firms are good candidates for large shareholders of a local firm, since the sale of the shares can generate large amount of revenues and furthermore, in the future, the home country can free-ride on the efficiency improvement of the firm. However, not all Western firms are good owners. Some of them are more interested in the private benefit of control than the potential of efficiency improvement. Such Western firms are bad owners in the long run, although they may well be willing to pay a high price to obtain the control right. Assuming that the government cares about a convex combination of sales revenue and the future value of the firm, we show that the optimal scheme is dependent upon the magnitude of the control benefit. Moreover, we show that the number of shares sold is a crucial instrument to attract the most efficient company.

ADDRESS FOR CORRESPONDENCE: Francesca Cornelli, The London School of Economics & Political Science, Houghton Street, London WC2A 2AE, England.

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1. Introduction

Privatization of state-owned enterprises in formerly socialist countries is a difficult undertaking. The difficulty stems from the fact that ownership of the firm carries not only financial benefits but also the control right of the firm. Therefore the efficiency and character of the privatized firm will depend on who are the owners. This is to say that privatization is a much more complex problem than just to give away or sell shares.

One of the central issues in privatization is the forming of large shareholders: in Western economies they have proved to be indispensable to maintaining the efficiency of the firm. In many circumstances, established Western companies can be ideal large shareholders for the privatized firms, since they have access to advanced technology, well-established overseas financial markets and relatively competent and skillful management teams.[1] By holding the remaining shares of the firm, domestic shareholders can benefit from the value improvement that the foreign companies bring to the privatized firm. However, not all foreign companies are good large shareholders. Some are interested in the private benefit of control which cannot be shared by local shareholders. For example, they may want to deter future entry of other foreign companies in that new market or exploit some synergies. If this is the case, the local firm will not become more efficient after privatization.

In this paper, we are concerned with a situation where several Western firms want the control right of a local firm. The government does not know their intentions and has to choose the most appropriate buyer and how many shares of the firm should go with the control right. Unfortunately, giving the control to the company willing to pay more may not necessarily be a good choice, since the private benefits from control could be so high that the company offering more would be a less desirable one. In general, the government finds itself in a trade-off between trying to obtain the highest possible payment (the "revenue" objective) and identifying the company which will operate better in the future (the "efficiency" objective).

We show that the number of shares sold is an important instrument for the government to discriminate between different potential buyers. Given this result, we then show that it is often sub-optimal for the government to commit in advance to selling a given fraction of the shares. By making the number of shares sold depend on the actual offers, the government has more flexibility to discriminate between companies which are efficient for the firm and companies that only aim to high benefits from control.

We derive the optimal mechanism for the sale of control by solving an optimal auction design

[1]See, for example, Aghion and Burgess (1991).

problem. Compared with existing literature on optimal auction, our approach is more general, since in our problem the seller has one additional instrument (the number of shares) to utilize. This may be seen as an additional contribution of this paper.

In a heuristic way, the optimal scheme of privatization can be summarized as follows. When the ratio of private benefits of control to the total public value of the firm is small, the government should sell the minimum number of shares and sell them to the highest bidder, who will be the most efficient company. When this ratio is high, the government still should sell the minimum number of shares to the highest bidder, but in this case the highest bidder will be a company with high benefits from control. This is the case in which the benefits from control are so high that it is too costly to discriminate between different companies, therefore the government will choose simply to maximize revenues.

In between the above two cases there is a trade-off between revenues and efficiency. It is possible to discriminate between different firms at a cost of revenues. The main feature of the optimal mechanism we propose is that the government will give the control to the highest bidder, but offering more shares the higher is the winning bid. This last feature is the one that guarantees that the highest bidder will be the most efficient firm.

Section 2 describes the objectives of the government and of the foreign companies. Section 3 tries to convey the main intuition of the paper with some simple numerical examples. Section 4 describes the model and looks for a set of necessary conditions on the optimal scheme in the most general case. Section 5 then gives more detailed descriptions of the optimal privatization scheme for some common cases. Section 6 concludes.

2. Government and foreign companies

We consider a government which wants to sell a large firm. Since the value of the shares of the firm must be equal to the value of the whole firm minus the value of the debt, to the extent that the firm's debt is secured and riskless, we will not lose generality by assuming in the rest of the chapter that the firm is an all-equity firm. In order to better describe the model, we will first discuss the objective function of the government and the nature of the large bidders.

2.1. The Objective of Government

Two leading alternative goals of privatization stand out among others. The first is to maximize total revenues from the process of privatization. The second is to maximize the total value of the privatized firm under the control of the new owner. From an economic point of view, Maskin

(1992) argues forcefully that the government should maximize the value of the firm. He called this the efficiency objective. Kornai (1991) expressed the same view and justified it on the ground of his social philosophy. Judging from limited observation of the practice of various countries, it seems that this efficiency objective has been adopted by most governments.

On the other hand, revenues are also important to the government during the post-socialist transition period. The main reason is that Eastern European countries will have to go through a long and costly process of economic restructuring with major investments in infrastructural facilities. Bolton and Roland (1992) point out that major budgetary problems may arise as a larger share of the economy becomes private, since a greater fraction of the profits is not remitted anymore to the state. To raise all these badly needed financial resources through borrowing will be a burden too heavy for these countries. A second reason is due to privatization process *per se*. As was explained by Shleifer and Vishny (1992), there are many incumbent claimants to the control right of the firm and therefore the central government needs the revenues from the privatization to bribe these claimants. Otherwise, privatization is politically impossible. A third reason for maximizing revenues is also political. Public opinion is in general against selling assets abroad, since it is feared that foreign companies may take advantage of the situation to appropriate the already scarce capital of the country. High revenues may therefore be necessary in order to justify internally the decision to give part of the ownership to foreign firms.[2]

To take into account both aspects, we assume that the government maximizes a weighted sum of revenues and value. Since value maximization is most likely and economically most appropriate for governments in Eastern Europe and the former Soviet Union, we assume the government gives a heavier weight to value than to revenue maximization.

2.2. Large Bidders and Private Benefits of Control

The control right of a firm entitles the large shareholder to two kinds of benefits. Through its intervention in the operations of the firm, the value of the firm may change and the benefit to the large shareholder is proportional to the number of its shares. In addition, the large shareholder may enjoy the control of the firm *per se*. This is called the private benefit of control. By definition, this private benefit of control cannot be shared by other (minority) shareholders.

Private benefits of control can take many different forms. First, the controlling party can change the objective of the firm to its own benefit. For example, a father company in control of a son company can demand to get supplies at a price which is below the market price. Second, there may exist synergies between the controlling party and the firm. For example, the owner of

[2] See Cornelli (1993).

a local sports club can benefit from owning a local radio station which broadcasts the games of the club.

In the transitional economies, private benefits of control to the foreign companies are likely to be bigger than those in established market economies. In fact, in these economies the control right usually gives the controlling firm enormous market power. For example, recently Fiat, Mercedes-Benz and Volkswagen acquired majority stakes of several Eastern European car makers. GM also attempted to buy shares of these car makers, as a part of its European strategy to take full advantage of the new Pan-European market. These companies may not necessarily believe that the acquired factories *per se* have great potential value. However, these acquisitions have significant strategic value in allowing early entries to the East Europe car market. Sometimes, this is also a way to circumvent tariffs. For example, Heineken bought a majority stake in a Hungarian brewery, so that local production facilities will enable it to get around quotas on foreign beers.

Let the public values of the firm under n different large buyers be v_1, v_2, ..., v_n, respectively. The private benefits of control to each of the large shareholders are B_1, B_2, ..., B_n. Both v_i and B_i are private information to large buyer i. Let α_i be the proportion of shares that the large shareholder obtains. For the large shareholder to get the control of the firm, α_i must be at least as big as a certain threshold level $\underline{\alpha}$. If α is the number of shares that the large shareholder obtains with the control of the firm, then these α shares will be worth $\alpha v + B$ to the large bidder. The public value v is shared by the large and the small shareholders, while the private benefit B is solely enjoyed by the controlling large shareholder.

We will focus on a situation in which the problem to identify the most desirable large shareholder is particularly difficult. To show this, let us assume there are only two firms: firm 1 with values (v_1, B_1) and firm 2 with values (v_2, B_2). If $v_1 \geq v_2$ and $B_1 \geq B_2$, there is no trade-off between the two objectives of the government: the company which can more effectively increase the value of the domestic firm is also the one which is willing to pay more, whatever is the number of shares sold α. It is straightforward to prove that the optimal mechanism for the government would be to set up an auction to sell $\underline{\alpha}$ shares to the highest bidder, which is also the most efficient firm.

However, assume that $v_1 \geq v_2$, but $B_1 \leq B_2$. Then the company willing to pay more is not necessarily the most efficient one. In fact, it may be that for some α_1, $\alpha_1 v_1 + B_1 < \alpha_1 v_2 + B_2$, but for some other $\alpha_2 > \alpha_1$, $\alpha_2 v_1 + B_1 > \alpha_2 v_2 + B_2$. If the government sets up an auction to sell α_1 shares, the highest bidder will be the company with the lowest value v, which is not necessarily optimal from the point of view of the government. Therefore, in this case, there is a trade-off between the two objectives of the government. For the rest of the paper, we will focus on this case, i.e. one in which v_i and B_i are negatively related.

The parameters v_i and B_i can also be reinterpreted as characterizing foreign companies plans when trying to obtain the control of the domestic firm. In other words, foreign companies may want to obtain the control of the firm because they believe it has good opportunities to expand and they want to invest in it, in view of future profits, or because they just want to exploit the private benefits of control. Of course, the government does not know what are the plans of each company. In general, we expect that v_i and B_i are negatively related. The more a foreign company invests and increases the value of the firm, the less it will exploit the firm by abusing the control right. This is particularly true if the private benefits of control are obtained by buying the output of the domestic firm at a below-market price, or by diverting profits of the controlled firm into profits of the controlling firm. The negative relationship between v_i and B_i could therefore be interpreted also as a "possibility frontier" of the plans available to a foreign company.

2.3. The optimal α and the Voting Structure of Equity

Some remarks are due here about the number of shares α and the voting structure of the equity. In the paper we generally assume a given structure of the shares, i.e. one-share-one-vote. However, α can also be implemented by issuing two classes of equity: voting and non-voting. The number of shares of the two classes are designed so that 50% of the voting shares are exactly α percent of the *total* number of both classes of shares.[3] Grossman Since we assume $\underline{\alpha}$ exogenously given, one may ask whether the government may want to manipulate $\underline{\alpha}$ to its own benefit. As will be shown later, in many cases the optimal α is independent of $\underline{\alpha}$, hence changing $\underline{\alpha}$ is useless. Yet, in other cases, the optimal α is equal to $\underline{\alpha}$. In these situations, there is an incentive for the government to change $\underline{\alpha}$. Lowering $\underline{\alpha}$ enables the government to retain more shares after privatization. However, the controlling party may have less incentive to improve the efficiency of the firm, i.e. v_i may be lower. Thus, an optimal $\underline{\alpha}$ can be found. In this paper, we are not concerned with such issues and assume without loss of generality that $\underline{\alpha}$ is already optimally chosen by the government.

In the Introduction we state that the paper shows that the number of shares sold is a crucial instrument for the government. Another possible interpretation is that the voting structure of the shares is a crucial instrument. In this sense, the result of this paper can be reinterpreted as an extension of Grossman and Hart (1988) and Harris and Raviv (1988). They claim that when the benefits of control are zero, one-share-one-vote is the optimal voting structure. In this paper, we show how to optimally deviate from the one-share-one-vote rule, when the private benefits of control are positive.

[3] Naturally, 50% could be substituted by the minimum percentage of voting shares necessary to gain the control, according the charter of the firm, which is set by the government. In the extreme cases, it can be 100%.

In reality, we do observe changing of voting structures as a result of privatization. For example, the contract between Volkswagen and the government of Czechoslovakia implied the immediate sale of 50% of the existing shares of Skoda to Volkswagen and one year later an issuing of new shares. The amount of new shares that went to Volkswagen had been agreed so that in the end Volkswagen would have owned 70% of total shares. Such procedure would allow the Czechoslovakian government to modify the existing structure of the firm's equity by issuing new shares, .

2.4. The payoffs

Now that we have characterized both the government and the foreign companies' interests, we can express their payoff functions. Let us define p_i as the probability that company i obtains the control and t_i the payment it has to make to the government. Then a foreign company maximizes its expected payoff given by

$$E[(\alpha v_i + B_i)p_i - t_i].$$

The government, on the other hand, maximizes a weighted sum of revenues and efficiency. Following Maskin (1992) we say that the efficiency is obtained if the control is given to the company which maximizes the value of the firm, i.e. to the company with the highest v_i.

Since the most efficient firm is the one that maximizes the value of the shares remaining in the hands of domestic shareholders, if λ is the weight to the revenue objective, the government maximizes its expected payoff:

$$E[\lambda \sum_i t_i + (1-\lambda) \sum_i (1-\alpha) v_i p_i],$$

where $0 \leq \lambda \leq \frac{1}{2}$, since in general the government cares more about value (efficiency).[4]

Such an objective function is actually a simplified version of a general one. In general, the government cares about expected revenues and *total* value of the firm, since the total value is associated with employment and the country's economic development. However, selling abroad has a cost in terms of lost future dividends. The objective function of the government would then

[4] We are here implicitly assuming that if the firm remains in domestic hands its value will be 0. As a result, the government will always choose to sell the control to a foreign company. This assumption could easily be relaxed, introducing a minimum threshold necessary in order to induce the government to sell the control abroad. For simplicity, we ignore this issue in the paper.

be:

$$E\left\{\lambda \sum_i t_i + (1-\lambda) \sum_i [v_i - C(\alpha v_i)] p_i\right\},$$

where $C(\alpha v_i)$ is a cost function.

In this case, even if $\alpha = 1$ the government may still care about the expected v_i. If we assume that such costs are linear, i.e. if $C(\alpha v_i) = \gamma \alpha v_i$, with $0 \leq \gamma \leq 1$, then the objective function becomes:

$$E[\lambda \sum_i t_i + (1-\lambda) \sum_i (1-\gamma\alpha) v_i p_i]. \tag{1}$$

When $\gamma = 1$—i.e. the cost is the highest possible—we are back to the previous function. We use this simplified objective function in the following pages, since it is intuitive and most of the results obtained remain true in general.[5]

3. Some numerical examples

In this section we present some numerical examples to give the intuition of why the mechanism we suggest is optimal. We first assume that the number α of shares to be sold is chosen before the actual sale and we show why the government may sometimes want to sell a number of shares higher than the minimum necessary to have the control. As a second step, we show how the same result can be achieved at a lower cost when the government makes the number of shares sold contingent on the offers.

Let us assume that there are two potential buyers: buyer 1 has $v_1 = 100$ and $B_1 = 40$ and buyer 2 has $v_2 = 130$ and $B_2 = 20$. In other words, buyer 2 is more efficient than buyer 1. On the side of the government, we assume, for simplicity, that $\lambda = \frac{1}{2}$ and that the minimum number of shares necessary to obtain the control right is $\underline{\alpha} = 50\%$. Then the total willingness to pay of each company, w_i, for different values of α, are given in the table.

Suppose that the government sells the α shares through an English auction. Then, if the government decides to sell $\alpha = 50\%$ shares, company 1 will win the auction with a bid of 85. The payoff to the government is $R = \lambda 85 + (1-\lambda)(1-\alpha)100 = 67.5$. If the government sells $\alpha = 65\%$ shares, the buyer is again company 1 for a price of 104.5. The payoff to the government is $R = \frac{1}{2} \times 104.5 + \frac{1}{2} \times 35\% \times 100 = 69.75$. If the government sells 70% of the shares, company 2 wins the auction with a price of 110. The payoff is $R = \frac{1}{2} \times 110 + \frac{1}{2} \times 30\% \times 130 = 74.5$, which

[5] Notice that we are assuming that $\lambda \leq \frac{1}{2}$. Therefore, even if we set $\gamma = 1$, somehow minimizing the importance of v_i, we can always compensate this by reducing λ.

w_i: / α:	Company 1 $v_1 = 100, B_1 = 40$	Company 2 $v_2 = 130, B_2 = 20$
$\alpha = 50\%$	$w_1 = \alpha v_1 + B_1 = 90$	$w_2 = \alpha v_2 + B_2 = 85$
$\alpha = 65\%$	$w_1 = \alpha v_1 + B_1 = 105$	$w_2 = \alpha v_2 + B_2 = 104.5$
$\alpha = 70\%$	$w_1 = \alpha v_1 + B_1 = 110$	$w_2 = \alpha v_2 + B_2 = 111$

is the highest.

It is clear why the value of α influences who wins the auction: when α increases, the willingness to pay of company 2 increases more rapidly than that of company 1. By increasing the number of shares the government influences the final result of the auction. From the point of view of maximizing revenues, the auction is a very efficient way to sell the shares, since it allows the government to extract the highest possible surplus from the companies, but it may end up attributing the control to an inefficient company. By choosing both the selling procedure and the number of shares to be sold, the government is able to reach the right balance among the two objectives. As a result, the government has an incentive to sell more shares than the minimum in order to maximize its own payoff.

We can intuitively explain why in this example the government prefers to sell more shares up to 70%.[6] First of all, so long as the shares are less than 70%, the winner is always company 1. In these cases, given the rules of the English auction, the winner, company 1, has to pay company 2's willingness to pay. An increase in α by 1% makes company 2's willingness to pay increase by $1\% \times v_2 = 1.3$ and therefore, the winner (company 1) has to pay an extra of 1.3. Thus, the government's payoff will increase by $\frac{1}{2} \times 1.3$. However, there is a drawback of such an increase in α, i.e. the government will have 1% less shares remaining in its hands after the auction and the value of this is $1\% v_1 = 1\% \times 100 = 1$. Comparing benefits and costs of an 1% increase in α, the government clearly wants to increase it. By the same argument, the government would not increase α when $\alpha = 70\%$.

The next example brings out the key point of this paper, that is, the government can benefit by

[6]To be precise, the maximum is reached at $\frac{2}{3}$, where $w_1 = w_2$.

making α dependent on the bids. Until now we assumed the government first chooses the number of shares to sell and then tries to sell them to the best buyer. When choosing α, the government takes into account what happens in the second period, i.e. the result of the auction (or whatever mechanism is optimal). In the second part of the paper, we show that the government can do even better if it does not commit to selling a given α in advance but it sells a number of shares contingent on the offers made by the companies.

To show this, let us continue with the above set-up. We suggest the following mechanism. It is a sealed bid auction in which each bidder submits its bid secretly.[7] If all the bids are lower than 104.5, then the winner is randomly chosen among the bidders and has to pay a price of 89.5 and gets 50% of shares; if the highest bid is bigger or equal to 104.5, then the highest bidder wins, pays an amount equal to his bid and gets 65% of shares. Moreover, the loser gets a transfer $t = 0.5$.

Given this mechanism there exists a continuum of equilibria in which company 1 bids $b_1 \in [0, 104.5)$ and company 2 bids $b_2 = 104.5$. Let us first check that this is an equilibrium. If company 1 bids $b_1 \in [0, 104.5)$, given the strategy of company 2, it always loses and therefore it obtains a payoff of 0.5. If it deviates, the best it can do is to bid $b_1 = 104.5 + \varepsilon$. In such case it always wins 65% of the shares and pays its bid. The expected payoff is then $105 - b_1 < 0.5$. Thus, company 1 has no incentive to deviate from its original strategy.

Let us now check the strategy of company 2. Company 2 of course has no incentive to bid higher than 104.5, since it obtains a negative surplus if it wins. Moreover, if it bids $b_2 < 104.5$, with probability $\frac{1}{2}$ it wins and has a surplus of $85 - 89.5 = -4.5$, while with probability $\frac{1}{2}$ it loses and receives a transfer of 0.5. Therefore, the expected payoff is negative.

Given these strategies, the government sells 65% shares to company 2 and its total payoff is $\frac{1}{2} \times [104.5 - 0.5] + \frac{1}{2} \times 35\% \times 130 = 74.75$. The government is better off with this mechanism than the previous one of $\alpha \equiv 70\%$. Further, whenever $\lambda < \frac{1}{2}$—which is the most general case—the advantage of the second mechanism is even larger.

Three features of this new mechanism are worth noticing. First, when all the bids are low, the winner is randomly chosen and the payment is fixed. In the general mechanism, this corresponds to "bunching" for low v_is. When the highest bids are high enough, instead, the mechanism looks like a sealed bid auction. Second, the number of shares α to be sold is a non-decreasing function of the winning bid. Later, we will show that this feature is optimal for a class of situations.

[7]The choice of the two different mechanisms, one for the case where α is chosen *ex ante* and one for the case of an endogenous α, may seem arbitrary. However, in the general solution of the model we show that each of them is the best possible mechanism for its corresponding case.

Finally, notice that company 2 wins but has zero surplus, while company 1 loses and has a positive expected payoff. In the general solution of the model we show that the expected profits of the bidders decrease as v_i increases and we give the intuition of such results.

4. The model

In this section we introduce the formal model and characterize the government's problem. Until now, we simply assumed that v_i and B_i were negatively related. However, in order to make the model more tractable and to have a better intuition of the results, we will assume that there is a linear relationship between v_i and B_i, i.e.:[8]

$$B_i = \bar{B} - \beta v_i.^{9} \tag{2}$$

Thus, the willingness to pay of a buyer i who gets α shares can be re-written as

$$w_i = \alpha v_i + B_i = \bar{B} + (\alpha - \beta)v_i.$$

If $\alpha(v_i)$ is constant (or chosen *ex ante*), then $\frac{\partial w_i}{\partial v_i} = \alpha - \beta$. This means that if $\alpha < \beta$, then the willingness to pay decreases with v_i; if $\alpha > \beta$, then it increases with v_i; while if $\alpha = \beta$, then it is independent from v_i. It is easy to show that if α is chosen *ex ante*, the optimal α is either β^+ or $\underline{\alpha}$, depending on the value of the parameters and on whether the number of shares will be sold through a Vickrey or English auction. The intuition is that to increase α is costly. Therefore the government will do it only if in this way it will be able to attract a more efficient company. For any α lower than β the winner of the auction will always be the less efficient company, i.e. the one with the higher benefit of control. Then there is no incentive for the government to sell more than $\underline{\alpha}$ shares. To attract the most efficient company it is enough that α is slightly higher than β and the government has no incentive to increase it even more. In Section 5 and 6 a more complete analysis of the mechanism for the various cases will be done.

If instead the government has the option to make the number of shares sold α contingent on the declared value v_i, then $\frac{\partial w_i}{\partial v_i} = \alpha_i - \beta + \alpha_i' v_i$. Thus, the willingness to pay may be monotonically increasing in v_i, even if $\alpha < \beta$. This means that if the government wishes to attract the buyer who

[8] More precisely, we do not need a linear relationship, but simply a relationship $B_i = h(v_i)$, with $h' < 0$. For simplicity, we assume here that $h(\cdot)$ is linear. Moreover, in Section 5.4 we discuss what happens if the government does not know the relationship between v_i and B_i.

[9] In all the paper we assume that $\beta < 1$. Empirical studies (see Barclay and Holderness (1989)) show in fact that in general β is substantially lower than 1. In any case, the case of $\beta > 1$ would not add much to our analysis.

maximizes the value of the shares, the government does not have to set $\alpha \geq \beta$; it may be enough to let the number of shares sold increase with the winning bid. This is clearly an improvement: the government achieves the same result by selling less shares abroad.

4.1. The optimal selling procedure

The problem of the government is to design an optimal scheme which maximizes its objective function. By the Revelation Principle, we can restrict ourselves to studying only the direct revelation mechanism. Let $\tilde{v}$ be the vector of announced valuations, then $p_i(\tilde{v})$ is the probability that buyer i gets control; $\alpha_i(\tilde{v})$ is the proportion of shares buyer i obtains if he obtains the control and $t_i(\tilde{v})$ is the amount he has to pay.

The expected payoff to a large buyer i with value v_i is

$$U_i(v_i, \tilde{v}_i, \alpha, p, t) = \int_{V_{-i}} \left\{ \left[(\alpha_i(\tilde{v}_i, v_{-i}) - \beta)\, v_i + \bar{B} \right] p_i(\tilde{v}_i, v_{-i}) - t_i(\tilde{v}_i, v_{-i}) \right\} g_{-i}(v_{-i}) dv_{-i}. \tag{3}$$

The government's objective function is:

$$\int_V \left[\lambda \sum_i t_i(v) + (1-\lambda) \sum_i [1 - \alpha_i(v)] v_i p_i(v) \right] g(v) dv. \tag{4}$$

The government maximizes its objective function with respect to α, p and t subject to several constraints. The individual rationality constraint

$$U_i(v_i, v_i, \alpha, p, t) \geq 0, \ \forall i \in N, \ \forall v_i \in [0, \bar{v}]; \tag{5}$$

and the incentive compatibility constraint

$$U_i(v_i, v_i, \alpha, p, t) \geq U_i(v_i, \tilde{v}_i, \alpha, p, t), \ \forall \tilde{v}_i \in [0, \bar{v}], \ \forall i \in N, \ \forall v_i \in [0, \bar{v}]. \tag{6}$$

There are also other constraints. For $p_i(v)$, its sum across all bidders must be less than or equal to 1, since at most one bidder can obtain the control. Also, the number of shares to be sold together with the control right has to be at least as big as $\underline{\alpha}$ and cannot be bigger than 1. In other words:

$$\sum_i p_i(v) \leq 1; \tag{7}$$

$$\underline{\alpha} \leq \alpha_i(v) \leq 1. \tag{8}$$

The incentive compatibility condition, constraint (6), entails a first order condition which is necessary to guarantee that truth telling is optimal for all the bidders. Following Myerson (1981) and Maskin and Riley (1990), we can show that the first order conditions can be transformed into the following form:[10]

$$\int_v t_i(v)g(v)dv = \int_V \left\{ \; [(\alpha_i(v) - \beta)v_i + \bar{B}] \; - \; [\alpha_i(v) - \beta]g(v_i) \right\} p_i(v)g(v)dv - U_i(0,0) \tag{9}$$

where $q(v_i) \equiv \frac{1-F(v_i)}{f(v_i)}$ is the reciprocal of the hazard rate of the distribution $F(\cdot)$ and $U_i(0,0)$ is the expected utility of a bidder with $v_i = 0$ when it announces its true v_i.

This expression can be explained intuitively. The first term in the integral $(\alpha_i(v) - \beta)v_i + \bar{B}$ is the true evaluation of bidder i. However, the government cannot extract all of this from bidder i, since it is costly to induce the bidder to tell the truth. As a result, the bidder gets an informational rent, which is $U_i(0,0) + [\alpha_i(v) - \beta]q(v_i)$.[11]

Constraint (9) is not sufficient to guarantee the incentive compatibility condition. In searching for such a sufficient condition, we introduce a second order condition. The second order condition requires that the function $U_i(v_i, \tilde{v}_i, \alpha, p, t)$ is concave in $\tilde{v}_i$ at $\tilde{v}_i = v_i$.[12] This second order condition turns out to be:[13]

$$\int_{V_{-i}} \left\{ \frac{\partial \alpha_i(v)}{\partial v_i} p_i(v) + \left[\alpha_i(v) - \beta \right] \frac{\partial p_i(v)}{\partial v_i} \right\} g_{-i}(v_{-i})dv_{-i} \geq 0, \tag{10}$$
$$\forall i \in N, \; \forall v_i \in [0, \bar{v}].$$

Substituting equation (9) into (4), the objective function of the government can be transformed

[10] For a long and detailed derivation, see Appendix A1.

[11] Notice that the second term could be positive or negative.

[12] To be precise, the second order condition only guarantees a local maximum. However, we will show that with the optimal mechanisms the function $U_i(v_i, \tilde{v}_i, \alpha, p, t)$ is globally concave in $\tilde{v}_i$ for all v_i. This is enough to guarantee that we reach a global maximum.

[13] See Appendix A1.

into the following:

$$\int_V \left\{ \sum_i \left[v_i \left[\lambda(1-\beta) + (1-2\lambda)(1-\alpha_i(v)) \right] \right. \right. \tag{11}$$
$$\left. \left. + \lambda\bar{B} - \lambda\left(\alpha_i(v) - \beta\right) q(v_i) \right] p_i(v) \right\} g(v)dv - \sum_i U_i(0,0).$$

Since all the potential buyers are risk neutral, there is no advantage for the government to choose randomized outcomes.[14] This may be seen also in equation (11) where the p_is enter linearly. Therefore, each p_i will be equal to 0 or 1. In the same way it is possible to prove that α_i is independent of v_{-i}. In fact, α_i is the number of shares buyer i obtains if he wins. Hence, α_i is defined only with respect to those vectors v such that i is a winner. Then, making α_i dependent on v_{-i} is equivalent to adding a randomization to α_i. From now on, we assume that α_i depends only on v_i. To simplify the notation, we can then drop the subscript and write only $\alpha(v_i)$.

Moreover, we define

$$P(v_i) = \int_{V_{-i}} p_i(v) g_{-i}(v_{-i}) dv_{-i}$$

which is the (unconditional) probability that bidder i wins the control right, given that its valuation is v_i. The second order conditions can therefore be written as

$$\alpha'(v_i)P(v_i) + [\alpha(v_i) - \beta] P'(v_i) \geq 0, \tag{12}$$
$$\forall i \in N, \; \forall v_i \in [0, \bar{v}].$$

Looking for the optimal policy, we can at first ignore the individual rationality constraint (5) and solve the above problem. This will give us a solution pair α_i, p_i. Once we found such a solution, we can use the following procedure to modify the t_i's in order to satisfy the individual rationality constraint. First, compute

$$\underline{v} = \arg\min_{v_i} U(v_i, v_i). \tag{13}$$

Second, increase or decrease $t_i(v)$ by the same constant (this is like re-scaling $U_i(0,0)$, which is not necessarily the minimum in this context) so that $U_i(\underline{v}, \underline{v}) = 0$. Notice that such adjustment

[14] See Maskin and Riley (1990).

in $t_i(v)$ is always possible according to (9).

To summarize, we have Lemma 1.

Lemma 1 *The government's problem is to choose α and p_i in order to maximize the following objective function:*

$$\int_V \sum_i \phi(v_i, \alpha(v_i)) p_i(v) g(v) dv, \tag{14}$$

subject to the following constraints:

$$\alpha'(v_i)P(v_i) + [\alpha(v_i) - \beta]\, P'(v_i) \geq 0; \tag{15}$$

$$\sum_i p_i \leq 1, \quad 0 \leq p_i \leq 1; \tag{16}$$

$$\underline{\alpha} \leq \alpha \leq 1. \tag{17}$$

where

$$\phi(v_i, \alpha(v_i)) \equiv v_i[\lambda(1-\beta) + (1-2\lambda)(1-\alpha(v_i))] + \lambda\bar{B} - \lambda\,(\alpha(v_i) - \beta)\,\frac{1-F(v_i)}{f(v_i)} \tag{18}$$

and

$$P(v_i) \equiv \int_{V_{-i}} p_i(v) g_{-i}(v_{-i}) dv_{-i}. \tag{19}$$

Constraint (15) holds for all v_i at which $\alpha(v_i)$ and $P(v_i)$ are differentiable.

Some remarks about the role of α are in order here. If we compare the problem with a standard optimal auction (as in Myerson (1981), the government now has one more instrument (α) to induce the bidder to tell its true willingness to pay. An increase in α has both positive and negative effects on the objective function of the government. On the positive side, increasing the number of shares sold generates more revenues from the auction. Let us call this the "sales effect". Furthermore, such an increase in α may attract more efficient bidders (with higher v_i). This is called the "incentive effect". On the negative side, an increase in α reduces the value of the remaining (unsold) part of the firm. This is called the "value effect". However, we have

argued that in general, the government is more concerned with value than with the revenue (i.e. $\lambda \leq \frac{1}{2}$).[15] Therefore, if we ignore the incentive effect, increasing α from $\underline{\alpha}$ is not a good strategy for the government. These comments form the intuition of the following proposition.

Proposition 1 *If constraint (15) is not binding for all v_i, then the optimal α must be $\alpha(v_i) = \underline{\alpha}$, for all v_i. If instead constraint (15) is binding for all v_i and $\alpha(v_i)$ and $P(v_i)$ are differentiable, then the optimal α must be given by the differential equation*

$$\alpha'(v_i)P(v_i) + [\ \alpha(v_i) - \beta\ ]\ P'(v_i) = 0$$

Proof: We need to show that for the dynamic control problem defined in Lemma 1, $\alpha = \underline{\alpha}$ is the optimal choice for α if constraint (15) is ignored. From the definition of $\phi(v_i, \alpha)$, we have

$$\frac{\partial \phi}{\partial \alpha} = (2\lambda - 1)v_i - \lambda \frac{1 - F(v_i)}{f(v_i)} < 0, \tag{20}$$

since we assumed that $0 \leq \lambda \leq \frac{1}{2}$. Hence, the partial derivative of the objective function (14) of the dynamic control problem with respect to α, given the optimal rule p_i, is

$$\frac{\partial}{\partial \alpha}\left(\int_V \sum_i \phi(v_i, \alpha(v_i)) p_i(v) g(v) dv\right) = \int_V \sum_i \phi'_\alpha(v_i, \alpha(v_i)) p_i(v) g(v) dv \leq 0. \tag{21}$$

Thus, by the Maximum Principle, if constraint (15) on α is never binding, the optimal α is $\underline{\alpha}$. On the other hand, constraint (15) always binding means, that the formula in (15) is always satisfied with equality. Therefore we have the differential equation.

Q.E.D.

When constraint (15) is ignored, in addition to setting $\alpha = \underline{\alpha}$, the government would pick the winner by finding the highest $\phi(v_i, \underline{\alpha})$. If such a mechanism satisfies constraint (15), then the government will adopt it. In other words, the government first best would be to sell the minimum number of shares to the most efficient company. As we already showed, however, this is in general not feasible. In fact, if the number of shares sold is low, often the company willing to pay more will be the less efficient one. Only by selling a number of shares higher than the minimum the government will be able to discriminate among the different companies.

[15] Moreover, since the government does not know the private information v_i of each company, it will not be able to extract all the rents of the private companies. Therefore, as α increases, the value part of the objective decreases, while the revenues increase less than proportionally.

If the second order condition (15) is not satisfied by the above unconstrained solution, then the government has to find a different solution. In the standard literature on optimal auctions, if the unconstrained solution violates the second order condition, a "bunching" solution is proved to apply. This implies that instead of a regular auction, if all bidders announce their valuation in a certain range, the seller picks the winner randomly.[16] This is equivalent to say that the optimal $P(v_i)$ is modified (and set equal to a constant) in order to satisfy the second order condition. In our case, however, if the second order condition are violated, there are two instruments that the government can modify in order to satisfy them: $\alpha(v_i)$ and $P(v_i)$. Instead of modifying $P(v_i)$, the government can increase or decrease the number of shares offered if this is a less costly way to satisfy the constraint. In particular, there are many ways to modify the optimal mechanism: the government can make α contingent on v_i, or it can change the optimal rule P, or both. Finding the optimal selling procedure in this case implies finding the right balance between these two instruments.

5. Three Representative Cases

We will focus on three representative cases that are illustrative of the procedure the government should follow. These cases correspond to important economic environments and have clear interpretations. The first two cases are sub-cases of $\underline{\alpha} < \beta$, which means β is not too small. The third case is just the opposite. In the first case the government prefers not to attract the most efficient company. In the second and third case it does, but while in the second one the government has to offer more shares the higher is the bid, in the last one it is enough to offer the minimum number of shares.

5.1. The case of $\phi(v_i, \underline{\alpha})$ monotonically decreasing in v_i

In this case and the next one we assume that $\beta > \underline{\alpha}$, which means that if the government sells only the minimum number of shares in a normal auction the highest bidder will be the less efficient company. In other words, there is a trade-off between the revenue and the efficiency objectives.

The assumption that $\phi(v_i, \underline{\alpha})$ is monotonic decreasing in v_i is equivalent to assuming

$$q'(v_i) < \frac{\lambda(1-\beta) + (1-2\lambda)(1-\underline{\alpha})}{\lambda(\underline{\alpha}-\beta)},$$

where $q(.)$ is the reciprocal of the hazard rate of distribution $f(.)$. It can be easily checked that

[16] See Guesnerie and Laffont (1984) and Maskin and Riley (1991).

this condition is more easily satisfied the higher β and λ are.

The optimal selling procedure in this case is quite simple and the following proposition gives a full description.

Proposition 2 *If $\phi(v_i, \underline{\alpha})$ is a monotonically decreasing function of v_i, then the government will give $\underline{\alpha}$ shares to the firm declaring the lowest v_i.*

Proof: From Proposition 1, if we ignore constraint (15), the optimal choice is $\alpha = \underline{\alpha}$. Since $\phi(v_i, \underline{\alpha})$ is, by assumption, decreasing in v_i, setting $p_i = 1$ for the lowest v_i is the policy that maximizes the objective function. On the other hand, such a policy implies $P'(v_i) < 0$. Therefore, constraint (15) is indeed satisfied. ¿From (18), $\phi(v_i, \underline{\alpha})$ is always positive, so no reservation prices are needed.

Q.E.D.

A remark is due here. From the optimal choice of α and p_i, as described above, t_i can be determined from the first order condition (9), if $U_i(0,0)$ is specified. The specification of $U_i(0,0)$ is obtained from considering the individual incentive constraint (5) which has been ignored so far. We can find $U_i(0,0)$ in this case rather easily. From the incentive compatibility constraint (6), by the Envelope Theorem, we have:

$$\frac{dU_i(v_i, v_i)}{dv_i} = [\alpha_i(x) - \beta]P(x) \tag{22}$$

Therefore,

$$U_i(v_i, v_i) = \int_0^{v_i} [\alpha_i(x) - \beta]P(x)dx + U_i(0,0) \tag{23}$$

Given the optimal choice of p_i and α, the expected utility of a bidder is decreasing with v_i and $\overline{v}$ has the lowest expected payoff, which should be set to be 0 by the government. Therefore, if the government sets

$$U_i(0,0) = (\beta - \underline{\alpha}) \int_0^{\overline{v}} [1 - F(x)]^{n-1} dx,$$

then all bidders' utilities are bigger than 0 and the individual rationality constraint is satisfied.

The intuition of the above optimal scheme is rather simple. When λ is high, the government cares relatively less about value and more about revenues. When β is high, the bidder's willingness to pay for α shares, $\overline{B} + (\alpha - \beta)v_i$ is likely to decrease at a high rate as v_i increases. Thus, to be sure that the company which obtains the control has a high v_i, the government should offer a large α. This may prove to be too costly and the government may not find it worthwhile to attract bidders with high values and instead will prefer to sell the minimum number of shares. Thus, the government prefers to maximize the revenue component of its objective function instead of the value part.

A simple mechanism to implement this result is a Vickrey or an English auction where the government commits to selling $\underline{\alpha}$ shares to the highest bidder. Since $\phi(v_i, \underline{\alpha})$ is decreasing, the highest bidder will be the individual with the lowest v_i.

Clearly, if the government had chosen α ex-ante, the same mechanism would have been chosen. Therefore, in this case the government does not gain by making α dependent on the bid, i.e. it could have just pre-committed to sell a fixed number of shares before the auction starts.

5.2. The case of $\phi(v_i, \alpha)$ monotonically increasing in v_i for any α

The second case we consider is the opposite to the one above. This case is likely to arise when λ is low —the objective of revenue is less important — and β is low — the trade-off between benefits of control and value of the firm is not sharp. The case is defined by assuming that ϕ is monotonically increasing with v_i for any α which is constant in the range $[0, \overline{v}]$.[17] This is equivalent to assuming that for any constant α

$$q'(v_i) > \frac{\lambda(1-\beta) + (1-2\lambda)(1-\alpha)}{\lambda(\alpha-\beta)}.$$

Before analyzing the optimal selling scheme, in which α can vary with the winning bid, let us confine ourselves to schemes in which α is chosen *ex ante*.

Proposition 3 *Assume that $\phi(v_i, \alpha)$ is monotonically increasing in v_i. If the government has to commit to selling a fixed number of shares α before the auction starts, then it is optimal for the government either to commit to selling $\underline{\alpha}$ shares and then randomly choose the winner or to commit to selling β shares and then conduct an English auction.*

Proof: We first have to consider the second stage of the game and find the optimal way to sell the α shares. When α is constant, constraint (15) becomes

$$[\alpha - \beta] P'(v_i) \geq 0.$$

[17] Of course, this is only a sufficient condition and not a necessary one for the following mechanism to be optimal.

The unconstrained optimum would be to set $p_i(v) = 1$ for the company declaring the highest v_i, since $\phi(v_i, \alpha)$ is monotonically increasing with v_i. If the α chosen *ex ante* is higher or equal to β, then the constraint is satisfied and this is the optimal direct mechanism. However, if $\alpha < \beta$, then the solution violates the constraint. As in the standard literature of optimal auctions, it is not difficult to show that in this case the optimum would be to set $P'(v_i) = 0$ over all the interval, i.e. to give the control randomly.

Now let us consider the first stage, when α is chosen, and compare the overall payoffs to the government. From Lemma 1, it is easy to show that the expected payoff to the government is (see Appendix A2, equation (035)):

$$n \int_0^{\overline{v}} \phi(v_i) P(v_i) f(v_i) dv_i. \tag{24}$$

If $\alpha > \beta$, we just showed that $P(v_i) = F^{n-1}(v_i)$. The payoff then becomes:

$$[\lambda(1-\beta) + (1-2\lambda)(1-\alpha)] \int_0^{\overline{v}} v_i dF^n(v_i) + \lambda \overline{B} \int_0^{\overline{v}} dF^n(v_i) - \lambda(\alpha-\beta) \int_0^{\overline{v}} \frac{1-F(v_i)}{f(v_i)} dF^n(v_i).$$

Notice that the derivative of this payoff with respect to α is strictly negative. Therefore, the government would set $\alpha = \beta^+$ and the payoff becomes:

$$(1-\alpha)(1-\lambda)E(v_{(1)}) + \lambda \bar{B}, \tag{25}$$

where $v_{(1)}$ is the highest order statistics and $E(v_{(1)}) = \bar{v} - \int_0^{\bar{v}} [F(v_i)]^n dx$. If $\alpha < \beta$, $P(v_i) = \frac{1}{n}$ and the expected payoff is

$$[\lambda(1-\beta) + (1-2\lambda)(1-\alpha)] \int_0^{\overline{v}} v_i df(v_i) + \lambda \overline{B} \int_0^{\overline{v}} dF(v_i) - \lambda(\alpha-\beta) \int_0^{\overline{v}} \frac{1-F(v_i)}{f(v_i)} dF(v_i)$$

which turns out to be:[18]

$$(1-\alpha)(1-\lambda)E(v_i) + \lambda \bar{B}. \tag{26}$$

This is maximized by setting $\alpha = \underline{\alpha}$. Obviously, the optimal α is either β^+ or $\underline{\alpha}$. Depending

[18] Using the fact that $\int_0^{\overline{v}} [1-F(x)] dx = x[1-F(x)] |_0^{\overline{v}} + \int_0^{\overline{v}} x dF(x) = E(v_i)$.

on the value of the parameters both could be the best, since $E(v_{(1)}) > E(v_i)$ but $(1-\underline{\alpha}) > (1-\beta)$.

Q.E.D.

When α can vary with the bid, however, the government can do better by not committing to a fixed α and always succed in attarcting the most efficient company. In the following proposition we show that the government always sells strictly less shares if α is endogenously determined.

Proposition 4 *If $\phi(v_i, \alpha)$ is monotonic increasing with v_i for any α, then $\alpha^*(v_i) < \beta$ for any v_i.*

Proof: See Appendix A2.

We can now characterize the optimal $\alpha^*(v_i)$ and $p_i^*(v_i)$.

Proposition 5 *If $\phi(v_i, \alpha)$ is monotonically increasing in v_i for any α, the optimal scheme of privatization is to randomly assign the control right among the bidders if only low v_is are declared or, if there is some higher v_i, to give the control right to the company declaring the highest v_i. In the former case, the control right is won with $\underline{\alpha}$ number of shares. In the second case, the control right carries a share α which is positively dependent on the winner's declared valuation v_i.*

Proof: See Appendix A3.

The elaborate characterization of the optimal scheme is presented in Appendix A3. Here we want only to describe its main features. First of all, it is shown in Appendix A3 that the optimal $\alpha^*(v_i)$ and $P^*(v_i)$ are linked over the entire interval $[0, \bar{v}]$ by the relationship

$$\alpha^*(v_i) = \beta + \frac{c_0}{P^*(v_i)} \tag{27}$$

where c_0 is negative (since we just showed that $\alpha^*(v_i) < \beta$ everywhere) and is set so that $\alpha^*(0) = \underline{\alpha}$:

$$c_0 = (\underline{\alpha} - \beta)P^*(0). \tag{28}$$

Moreover, it is possible to show that there always exists a $\hat{v} > 0$ such that for any $v_i \in [0, \hat{v}]$ $\alpha^*(v_i) = \underline{\alpha}$ and $P^*(v_i)$ is constant (and determined by equation (27)), while for any $v_i \in [\hat{v}, \bar{v}]$ $P^*(v_i) = F^{n-1}(v_i)$ and $\alpha(v_i)$ is strictly higher than $\underline{\alpha}$ and given by equation (27).

A constant $P^*(v_i)$ over an interval $[0, \hat{v}]$ means that all bidders who declare a v_i in that interval have the same chance to win. On the other hand, $P(v_i) = F^{n-1}(v_i)$ is implemented by the rule

that gives the bidder the highest v_i the control right. The optimal scheme will therefore look like the one represented in Figure 1.

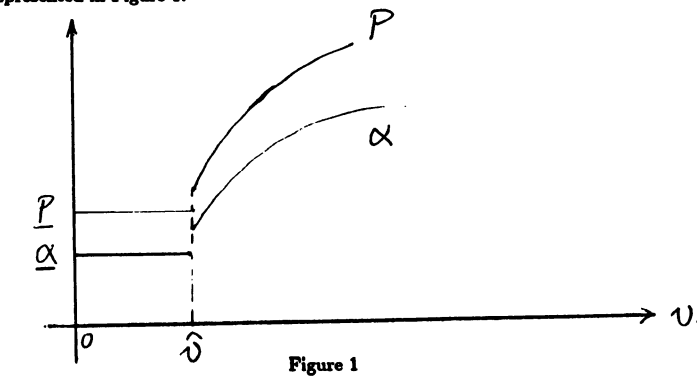

Figure 1

It is easy to show that an indirect mechanism implementing this optimum would be the following. All the foreign companies make a bid. If all the bids are below a predetermined level, then the minimum number of shares $\underline{\alpha}$ is given to one of them at random. If at least one bid is above that level, then α^* shares will be given to the highest bidder, where the actual α^* will depend on the winning bid (according to a function derived from the relationship (27)).

A major feature of this optimal scheme is the increasing α which serves as an important screening device to induce truth telling from the bidders so that the government is able to choose the most efficient company. If a company makes a high bid, it obtains more shares and a higher probability to win but the price also increases more than proportionally.[19] In this way, bidders with high v_i would not make a low bid, since they are attracted by the high α which increases their willingness to pay $\alpha v_i + B_i$. Also, they are attracted by the higher probability to win. At the same time, a bidder with low v_i would not bid high, since its low v_i cannot make its willingness to pay $\alpha v_i + B_i$ increase as rapidly as the price.

However, increasing α is costly and it is not always optimal to give the control to the most efficient firm. The government faces a trade off between attracting the most efficient company and minimizing the number of shares sold. If it gives the control randomly, the government can keep α constant. The loss is given by the fact that sometimes the winning company may not be the one with the highest value, while the gain is given by the low number of shares sold. The government will give the control randomly if all the v_is are below a certain amount. As

[19] You can see this in the example given in Section 3.

v_i increases, the cost from not separating different types becomes larger and larger, while the decrease in $\alpha^*(v_i)$ is reduced. Therefore the government will prefer to increase α and attribute the good to the highest bidder.

5.3. The Case with $\beta < \underline{\alpha}$

If $\beta < \underline{\alpha}$, the term $\alpha - \beta$ is always positive. This is the case in which the trade-off between benefits of control and to the total public value of the firm is low. Thus, by offering the minimum proportion of shares, the government can already attract the most efficient company. The standard optimal auction literature can apply here.

Proposition 6 *If $\beta < \underline{\alpha}$ and the hazard rate $q(v_i)$ is monotonic increasing with v_i, then the government gives $\underline{\alpha}$ shares to the firm with the highest v_i, if $v_i \geq v^*$ where v^* is such that $\phi(v^*, \underline{\alpha}) = 0$.*

Proof: The proof is standard, once we notice that in such a case $\phi(v_i, \underline{\alpha})$ is a monotonically increasing function of v_i and therefore constraint (15) is never binding.[20]

The government does not lose anything by pre-committing to selling a fixed number of shares. An optimal mechanism is for the government to sell $\underline{\alpha}$ shares in a Vickrey or an English auction, with the reservation price given by $\bar{B} + (\underline{\alpha} - \beta)v^*$. Therefore, the optimal mechanism is very simple and easy to use. Moreover, this case is particularly successful , since the government can attract the highest possible v_i with the minimum necessary shares. This is exactly because there is a small trade-off between the two objectives of revenue and efficiency.

5.4. Discussion

In Section 2 above we introduced a relationship between the v_is and the B_is of foreign companies and suggested that such relationship could also be interpreted as an "efficiency frontier". Later, to keep the exposition simple, we assumed that such relationship is linear. Moreover, throughout the paper we assumed that such efficiency frontier is common knowledge. One possible objection to the latter assumption, however, comes from the fact that the government may not know the shape of this frontier exactly. Therefore, in this section we ask how the mechanism we suggest would perform if the government does not know such relationship.

[20] Monotonicity of the hazard rate is a standard assumption in the literature of optimal auctions and it is satisfied by most of the typical distributions. Whenever it is not satisfied the usual problems of incentive compatibility arise. For simplicity, we ignore here such case.

The characterization of the optimal mechanism when the government does not have perfect information about such relationship is a formidable task, moreover the formal treatment would allow at most a very general characterization of the mechanism. In an attempt to obtain more precise predictions on the form of the optimal mechanism we will characterize such a mechanism under the assumption that the relationship is common knowledge and then will ask what is the robustness of the mechanism constructed.

For simplicity, we consider a particular case: the government knows that the relationship between v_i and B_i is linear, but β could take two values—$\beta_1 < \beta_2$—and the government does not know which one is the exact value of β. First of all, as long as the government construct the mechanism on the basis of a β higher than the true one, the incentive compatibility constraint is always satisfied. In fact, assume the true $\beta = \beta_1$ and the government chooses to sell according to the mechanism given in Proposition 5 with $\beta = \beta_2$. This means he would choose $P^*(v_i) = F^{n-1}(v_i)$ and

$$\alpha^*(v_i) = \beta_2 + \frac{c_0}{F^{n-1}(v_i)}.$$

Constraint (15) would therefore become:

$$(\beta_2 - \beta_1)P'(v_i) \geq 0$$

which is clearly satisfied. Therefore the mechanism would still work and attract the most efficient company, but the government would sell more shares than it is actually necessary. If p is the probability that $\beta = \beta_2$, then when the government chooses β_2, with probability p it choose the right β and the objective function is maximized. However, with probability $1 - p$, $\beta = \beta_1$. In such a case, it is possible to show that the value of the objective function is the same as in the first case, with the only difference that the government could have done even better. In other words, choosing a β too high does not have any effect on the efficient choice of the buyer.

The question is then whether in the presence of uncertainty the government could have done much better with another mechanism. For example, we can show that the mechanism proposed in Section 5 will always do better than a normal auction in which the government sells a fixed number of shares. If $\phi(v_i)$ is decreasing for both β_1 and β_2, then we showed in Proposition 2 that there is no difference between these two mechanisms. Assume instead that $\phi(v_i)$ is increasing for both β_1 and β_2 and the government has chosen to sell β_i shares through an auction. Then the mechanism proposed in Proposition 5 will do strictly better. In fact, if β_i is actually higher or equal to the true β we have already shown that the second mechanism is strictly better. If instead β_i is lower than the true one, then both mechanisms would end up attracting the less efficient firm, but while in the first case the government would still sell β_i shares, at least with the mechanism we suggested the government would sell only $\underline{\alpha}$ shares.[21] Finally, if $\phi(v_i)$ is decreasing for β_2 and increasing for β_1, a combination of the two arguments above would still prove that the mechanism given in Proposition 5 is strictly better.

The essence of such discussion is the following. Even when the government does not know β, the mechanism still performs quite well. In fact, even if we have not identified the optimal mechanism, the one we propose can still do better than other simple mechanisms. The cost for the government is that it may end up selling more shares than it is actually necessary. However, since the mechanism is specifically designed in order to sell the minimum possible number of shares, this cost is not too high.

6. Conclusions

The focus of this paper is the optimal scheme of privatization. In the privatization of formerly state-owned industrial firms, Western firms and investors are often good candidates to be large shareholders of the privatized firm, since they can increase the efficiency and value of the privatized firm. However, since the control right *per se* can be of great value, many foreign investors want to buy shares just because of the private benefit of control. Such investors are not necessarily suitable for the domestic firm.

The optimal privatization scheme can be much more complex than an ordinary auction. There are several reasons for this. First, unlike a simple auction where the seller only cares about the revenue, here the government cares about a combination of the revenue and the value of the firm. Second, compared with a standard auction, the government can vary the number of shares to be sold as an additional instrument to give incentives to large buyers to reveal their true values.

We are able to characterize such optimal schemes for several important cases. If the ratio of private benefit of control (B_i) to the total public value of the firm (v_i) is always smaller than $\underline{\alpha}$ (the minimum proportion of shares that entitles the buyer to the control right), then a simple English auction is optimal and the government should offer the minimum amount of shares. This implies that consideration of the private benefit of control does not have any effect on the optimal scheme of privatization. Outside the above scenario, we study the situation where the trade-off between v_i and B_i can be approximated by a linear function. We show that if the ratio $\frac{B_i}{v_i}$ is high, then in general the optimal scheme is to sell only the minimum amount of shares to the foreign bidder and to give the control right to the bidder with the lowest v_i. On the other hand, if the ratio $\frac{B_i}{v_i}$ is low (close to $\underline{\alpha}$), then the government should randomly assign the control right when

[21] From Proposition 1 we know it is never optimal to increase α.

all bids are low and give the control to the highest bidder when there are high enough bids. At the same time, in this case the government should gradually increase the number of shares sold with the announced v_i. We are also able to show that by not committing to sell a given number of shares *ex ante* the government is able to achieve a better outcome than otherwise.

The same approach could be used also for the privatization of public firms in Western economies. The basic points of this paper are also relevant in these countries, since the government cares relatively more about the future value of the privatized firm than revenue, and also private benefits of control are present. Moreover, governments involved in privatizations are very often concerned with problems of concentration or monopolization of an industry. It can therefore be argued also in this case that increasing α is costly for the government and our analysis would apply.

APPENDIX

A1. The Derivation of the First Order Condition and the Second Order Condition

(NOTE to Editor: Appendix A1 can be eliminated or reduced if space is a concern.)

The incentive compatibility constraint can be expressed as:

$$v_i = \arg\max_{\tilde{v}_i} \int_{V_{-i}} \left\{ \left[\alpha_i(\tilde{v}_i, v_{-i})v_i + \bar{B} - \beta v_i \right] p_i(\tilde{v}_i, v_{-i}) - t_i(\tilde{v}_i, v_{-i}) \right\} g_{-i}(v_{-i}) dv_{-i}. \tag{029}$$

Define

$$U_i(v_i, x_i) \equiv \int_{V_{-i}} \left\{ \left[\alpha_i(x_i, v_{-i})v_i + \bar{B} - \beta v_i \right] p_i(x_i, v_{-i}) - t_i(x_i, v_{-i}) \right\} g_{-i}(v_{-i}) dv_{-i}. \tag{030}$$

Assuming differentiability, the envelope theorem gives

$$\frac{dU_i}{dv_i}(v_i, x_i(v_i)) = \int_{V_{-i}} \left[\alpha_i(x_i(v_i), v_{-i}) - \beta \right] p_i(x_i(v_i), v_{-i}) g_{-i}(v_{-i}) dv_{-i}. \tag{031}$$

Since $x_i(v_i) = v_i$, this becomes

$$\frac{dU_i}{dv_i}(v_i, v_i) = \int_{V_{-i}} \left[\alpha_i(v_i, v_{-i}) - \beta \right] p_i(v_i, v_{-i}) g_{-i}(v_{-i}) dv_{-i}. \tag{032}$$

Re-integrating it, we get:

$$U_i(v_i, v_i) = \int_0^{v_i} \int_{V_{-i}} \left[\alpha_i(x, v_{-i}) - \beta \right] p_i(x, v_{-i}) g_{-i}(v_{-i}) dv_{-i} dx + U_i(0,0). \tag{033}$$

Comparing the above expression for $U_i(v_i, v_i)$ and its definition, we have:

$$\int_{V_{-i}} \left\{ [\alpha_i(v_i, v_{-i})v_i + \bar{B} - \beta v_i] p_i(v_i, v_{-i}) - t_i(v_i, v_{-i}) \right\} g_{-i}(v_{-i}) dv_{-i}$$
$$= \int_{V_{-i}} \left[\int_0^{v_i} [\alpha_i(x, v_{-i}) - \beta] p_i(x, v_{-i}) dx \right] g_{-i}(v_{-i}) dv_{-i} + U_i(0,0),$$

or

$$\int_{V_{-i}} t_i(v) g_{-i}(v_{-i}) dv_{-i} =$$
$$\int_{V_{-i}} \left\{ [\alpha_i(v)v_i + \bar{B} - \beta v_i] p_i(v_i, v_{-i}) - \int_0^{v_i} [\alpha_i(x, v_{-i}) - \beta] p_i(x, v_{-i}) dx \right\} g_{-i}(v_{-i}) dv_{-i}$$
$$-U_i(0,0), \qquad \forall i \in N,\ \forall v_i \in [0, \bar{v}].$$

Integrating further, we have:

$$\int_v t_i(v) g(v) dv = -U_i(0,0) +$$
$$\int_{v_i} g_i(v_i) \int_{v_{-i}} \left\{ [\alpha_i(v)v_i + \bar{B} - \beta v_i]\, p_i(v_i, v_{-i}) - \right.$$
$$\left. \int_0^{v_i} [\alpha_i(x, v_{-i}) - \beta]\, p_i(x, v_{-i}) dx \right\} g_{-i}(v_{-i}) dv_{-i} dv_i.$$

The double integration on the left-hand-side can be re-written as:

$$\int_V \left[\alpha_i(v)v_i + \bar{B} - \beta v_i \right] p_i(v) g(v) dv -$$
$$\int_{v_{-i}} g_{-i}(v_{-i}) \int_{v_i} g_i(v_i) \int_0^{v_i} \left[\alpha_i(x, v_{-i}) - \beta \right] p_i(x, v_{-i}) dx dv_i dv_{-i}.$$

The second integration in the expression above can be further simplified as the following.

$$\int_{v_{-i}} g_{-i}(v_{-i}) \int_{v_i} g_i(v_i) \int_0^{v_i} \left[\alpha_i(x, v_{-i}) - \beta \right] p_i(x, v_{-i}) dx dv_i dv_{-i}$$
$$= \int_{v_{-i}} g_{-i}(v_{-i}) \int_{v_i} dG_i(v_i) \int_0^{v_i} \left[\alpha_i(x, v_{-i}) - \beta \right] p_i(x, v_{-i}) dx dv_{-i}$$
$$= \int_{v_{-i}} g_{-i}(v_{-i}) \left\{ G(v_i) \int_0^{v_i} [\alpha_i(x, v_{-i}) - \beta] p_i(x, v_{-i}) dx \Big|_0^{\bar{v}} \right.$$
$$\left. - \int_{v_i} G_i(v_i) [\alpha_i(v_i, v_{-i}) + -\beta] p_i(v_i, v_{-i}) dv_i \right\} dv_{-i}$$
$$= \int_{v_{-i}} g_{-i}(v_{-i}) \left\{ \int_0^{\bar{v}} [\alpha_i(x, v_{-i}) - \beta] p_i(x, v_{-i}) dx - \right.$$
$$\left. \int_{v_i} G_i(v_i) [\alpha_i(v_i, v_{-i}) - \beta] p_i(v_i, v_{-i}) dv_i \right\} dv_{-i}$$
$$= \int_V \left[\alpha_i(v) - \beta \right] p_i(v) \frac{g(v)}{g_i(v_i)} dv - \int_V \left[\alpha_i(v) - \beta \right] p_i(v) \frac{G_i(v_i)}{g_i(v_i)} dv$$
$$= \int_V \left[\alpha_i(v) - \beta \right] \frac{1 - G_i(v_i)}{g_i(v_i)} p_i(v) g(v) dv.$$

Notice that in the above derivation, the independence of the distribution of v_i's is utilized. Also, by definition, $g_i(v_i) = f(v_i)$ and $G_i(v_i) = F(v_i)$.

Let

$$q(v_i) \equiv \frac{1 - G_i(v_i)}{g_i(v_i)} = \frac{1 - F_i(v_i)}{f_i(v_i)},$$

where the reciprocal of $q(.)$ is customarily called the hazard function of the distribution $f(.)$. To summarize, the expected payment from bidder i is:

$$\int_v t_i(v)g(v)dv = \int_V \left\{ \ [\alpha_i(v)v_i + \bar{B} - \beta v_i] \ - \ [\alpha_i(v) - \beta]q(v_i) \right\} p_i(v)g(v)dv. \tag{034}$$

This is the expression that was quoted in the paper.

As for the second order condition, it is frac

$$\frac{\partial^2 U_i(v_i, \tilde{v}_i)}{\partial \tilde{v}_i^2} \mid_{\tilde{v}_i = v_i} \leq 0.$$

But, according to the first order condition,

$$\frac{\partial U_i(v_i, \tilde{v}_i)}{\partial \tilde{v}_i} \mid_{\tilde{v}_i = v_i} \equiv 0.$$

Differentiating this first order condition on both sides with respect to $\tilde{v}_i$, we have

$$\frac{\partial^2 U_i(v_i, \tilde{v}_i)}{\partial v_i \partial \tilde{v}_i} \mid_{\tilde{v}_i = v_i} + \frac{\partial^2 U_i(v_i, \tilde{v}_i)}{\partial \tilde{v}_i^2} \mid_{\tilde{v}_i = v_i} = 0.$$

Therefore, a second order condition can be found as:

$$\frac{\partial^2 U_i(v_i, \tilde{v}_i)}{\partial v_i \partial \tilde{v}_i} \mid_{\tilde{v}_i = v_i} \geq 0.$$

This second order condition turns out to be

$$\int_{V_{-i}} \left\{ \frac{\partial \alpha_i(v)}{\partial v_i} p_i(v) + \left[\alpha_i(v) - \beta \right] \frac{\partial p_i(v)}{\partial v_i} \right\} g_{-i}(v_{-i}) dv_{-i} \geq 0,$$
$$\forall i \in N, \ \forall v_i \in [0, \bar{v}].$$

A2. Proof of Proposition 4

Step 1

In what follows it will be convenient to use a transformed form of the government objective function. Note that the distributions of v_i's are independent and α—and therefore ϕ—is a function of v_i only. Also, all players are, *ex ante*, symmetric. Thus, using the definition of $P(v_i)$ given in equation (19), the objective function of the government can be rewritten as

$$n \int_0^{\bar{v}} \phi(v_i, \alpha(v_i)) P(v_i) f(v_i) dv_i. \tag{035}$$

Step 2

The first problem that arises is that the optimal mechanism may have discontinuities. At each discon-

tinuous point, we cannot use condition (15) to check the incentive compatibility constraint. The following lemma gives us a more general condition which allows for discontinuities.

Lemma 02 *Suppose that $\alpha(v_i)$ and $P(v_i)$ are discontinuous at a point v_0, i.e. $\alpha(v_0^+) \neq \alpha(v_0^-)$ or $P(v_0^+) \neq P(v_0^-)$. Then the second order condition of the incentive compatibility constraint at the point v_0 becomes*

$$P(v_0^-)(\alpha(v_0^-) - \beta) - P(v_0^+)(\alpha(v_0^+) - \beta) \leq 0. \tag{036}$$

Proof: Notice that to the left and to the right of the point v_0, $\alpha(v_i)$ is continuous and therefore the usual incentive compatibility constraint holds in this region. Moreover, all the derivations in the last section will hold for these intervals of v_i, since there is no non-differentiability problem. In particular, the derivation for the optimal expected transfers is valid.

Define

$$U_i(v_i, x_i) \equiv \int_{V_{-i}} \left\{ \left[(\alpha_i(x_i, v_{-i}) - \beta) \, v_i + \bar{B} \right] p_i(x_i, v_{-i}) - t_i(x_i, v_{-i}) \right\} g_{-i}(v_{-i}) dv_{-i} \tag{037}$$

Combining equation (037) and the expression for the expected transfers, we get

$$U_i(v_i, x_i) = \int_{V_{-i}} \left\{ \left[(\alpha(x_i) - \beta) \, v_i + \bar{B} \right] p_i(x_i, v_{-i}) - t_i(x_i, v_{-i}) \right\} g_{-i}(v_{-i}) dv_{-i}$$

$$= \int_{V_{-i}} \left\{ \left[(\alpha(x_i) - \beta) \, v_i + \bar{B} \right] p_i(x_i, v_{-i}) \right\} g_{-i}(v_{-i}) dv_{-i}$$

$$- \int_{V_{-i}} \left\{ \left[(\alpha(x_i) - \beta) \, x_i + \bar{B} \right] p_i(x_i, v_{-i}) - \int_0^{x_i} [\alpha(s) - \beta] \, p_i(s, v_{-i}) ds \right\} g_{-i}(v_{-i}) dv_{-i}$$
$$+ U_i(0, 0).$$

Using the definition of $P(x)$ given in equation (19) we can rearrange the above expression as:

$$U_i(v_i, x_i) = P(x_i) \left[\alpha(x_i) - \beta \right] (v_i - x_i) + \int_0^{x_i} [\alpha(s) - \beta] \, P(s) ds + U_i(0, 0). \tag{038}$$

Suppose that an individual with valuation $v_i = v_0 - \epsilon$ declares a value $x_i = v_0 + \epsilon$. His gain in doing so is

$$G = U_i(v_0 - \epsilon, v_0 + \epsilon) - U_i(v_0 - \epsilon, v_0 - \epsilon)$$

$$= P(v_0 + \epsilon) \left[\alpha(v_0 + \epsilon) - \beta \right] (-2\epsilon) + \int_{v_0 - \epsilon}^{v_0 + \epsilon} [\alpha(s) - \beta] \, P(s) ds$$

The second term in the last expression can be re-written as

$$\int_{v_0-\epsilon}^{v_0} [\alpha(s) - \beta]\, P(s) ds + \int_{v_0}^{v_0+\epsilon} [\alpha(s) - \beta]\, P(s) ds \tag{039}$$

which by Cauchy theorem is equal to,

$$[\alpha(\xi_1) - \beta]\, P(\xi_1)\epsilon + [\alpha(\xi_2) - \beta]\, P(\xi_2)\epsilon, \tag{040}$$

where $v_0 - \epsilon \leq \xi_1 \leq v_0$ and $v_0 \leq \xi_2 \leq v_0 + \epsilon$. When $\epsilon \to 0$, $\xi_1 \to v_0^-$ and $\xi_2 \to v_0^+$. Therefore, if we define $P^+ \equiv P(v_0^+)$, $P^- \equiv P(v_0^-)$, $\alpha^+ \equiv \alpha(v_0^+)$ and $\alpha^- \equiv \alpha(v_0^-)$, then G/ϵ converges to

$$P^-(\alpha^- - \beta) - P^+(\alpha^+ - \beta). \tag{041}$$

Therefore, we need to require that the expression above is non positive. Notice that if ϵ is negative, then G/ϵ is non-positive and P^+ and α^+ should be written as P^- and α^-. Hence, the expression above holds whether an individual declares a lower type or a higher type.

Q.E.D.

From now on, whenever we want to check the incentive compatibility constraint, we will use condition (15) if there are no discontinuities or condition (036) if there is a discontinuity.

Step 3

We can now show that the government will never offer a number of shares higher than β.

Lemma 03 *$\alpha(v_i) > \beta$ for some v_i is never optimal.*

Proof: Let us suppose first that $\alpha(v_i)$ becomes higher than β only through a discontinuity, i.e. there does not exist an interval $[v^1, v^2]$ over which $\alpha(v_i)$ is continuous, $\alpha(v^1) < \beta$ and $\alpha(v^2) > \beta$ (or vice versa, $\alpha(v^1) > \beta$ and $\alpha(v^2) < \beta$). Then let us take the interval of v_is for which $\alpha(v_i) > \beta$ and give it $\alpha(v_i) = \beta$. To guarantee incentive compatibility it will be enough to give all of them the same P as before. In fact, from equation (036) it is evident that whenever there is a jump from $\alpha^- - \beta$ negative to $\alpha^+ - \beta$ positive, the incentive compatibility constraint is satisfied with a strict inequality for any P^+. Moreover, this is true also if $\alpha^+ - \beta = 0$. Therefore, if we give to these v_i the same P, the incentive compatibility at the discontinuity is satisfied. Moreover, in the interval of v_is we set at $\alpha(v_i) = \beta$, the incentive compatibility is satisfied too, since $\alpha' = 0$ and $\alpha - \beta = 0$ make the differential condition (15) always satisfied.

The new scheme is strictly better for the government, since the shares sold have decreased and the choice of the winner has not changed. To see this, consider equation (035). It is clear that with this scheme the government has lowered $\alpha(x)$ (and therefore increased $\phi(x, \alpha(x))$) for some values of x and it has not changed any P. Therefore, the government is strictly better off.

Now, let's verify that it is impossible for α to have a downward jump from higher than β to lower than β. From (036), so long as one of P^- and P^+ is positive[22], a jump from $\alpha^- - \beta$ positive to $\alpha^+ - \beta$ negative is never incentive compatible. Therefore, we can ignore downward jumps.

Finally, let us check the continuous case. Suppose that there exists an interval $[v^1, v^2]$ in which $\alpha(v_i)$ is continuous and $\alpha(v^i) > \beta$ for some v_is in that interval. This cannot be optimal, since the government can do better by setting $\alpha(v_i) = \beta$ for those v_i while keeping the old $P(v_i)$. The incentive compatibility constraint is still satisfied, because (15) holds strictly.

Q.E.D.

The proof of Lemma 2 shows that the government can always do better than setting $\alpha(v_i) > \beta$ since it can always set $\alpha(v_i) = \beta$ and leave the same $P(v_i)$, which remains incentive compatible.

Step 4

We are now left with only two possibilities: $\alpha = \beta$ or $\alpha < \beta$. To show that $\alpha < \beta$ for all v_i's, we need first to prove that $\phi(v_i, \alpha^*(v_i))$ is monotonic increasing in v_i.

Lemma 04 *If $\phi(v_i, \alpha)$ is monotonic increasing with v_i for any α constant, then $\phi(v_i, \alpha^*(v_i))$ is also monotonic increasing in v_i. Moreover, with $\alpha^*(v_i))$, constraint (15) (or (036)) always holds with an equality at all v_is.*

Proof: Let us only deal with the case where $\alpha(v_i)$ is continuous at v_i and the constraint (15). The same logic can be exactly applied to the discontinuous case and constraint (036).

First of all, notice that

$$\frac{d\phi}{dv_i} = \frac{\partial \phi}{\partial v_i} + \frac{\partial \phi}{\partial \alpha}\frac{d\alpha}{dv_i}.$$

The first term on the right hand side is always positive by assumption, while the partial derivative in the second term is always negative by formula (20). Therefore, $\frac{d\phi}{dv_i}$ can be negative only if $\frac{d\alpha}{dv_i} > 0$. Suppose $\frac{d\phi}{dv_i} < 0$, from constraint (15)

$$\alpha'^*(v_i) P(v_i) + (\alpha^*(v_i) - \beta) P'(v_i) \geq 0.$$

The first term on the left-hand-side is positive, since we are assuming that $\alpha'^*(v_i) > 0$. We know that $\alpha^*(v_i) \leq \beta$. Given that $\phi' < 0$, optimality requires that $P'(v_i) < 0$ (i.e. to give a higher ϕ a higher probability to win). Therefore, also the second term must be positive. In other words, the constraint is satisfied with a strict inequality. Therefore, the government can always do better by slightly reducing $\frac{d\alpha}{dv_i}$

[22]We ignore the case of both P^+ and P^- equal to 0, since the government would never give the control to such bidders.

so that it is still positive and the constrain is still not violated. This is a contradiction to the fact that α^* is optimal.

¿From the proof above, we have also verified the last claim in the lemma.

Q.E.D.

The proof of Lemma 3 shows that it is never optimal for the government to increase the number of shares sold (as v_i increases) so fast that the function ϕ becomes decreasing with v_i. Lemma 3 implies that constraint (15) (or (036) if there is a discontinuity) is always binding.

Step 5

We can now show that the optimal α is indeed always less than β.

Proof: First, in any intervals where α is continuous and differentiable, either $\alpha < \beta$ everywhere or $\alpha \equiv \beta$. From the last lemma, in each of such intervals, constraint (15) holds strictly, i.e.,

$$\alpha'(v_i)P(v_i) + (\alpha(v_i) - \beta)P'(v_i) = 0,$$

which gives

$$\alpha(v_i) = \beta + \frac{c_0}{P(v_i)}, \tag{042}$$

where c_0 is given by the initial condition. If in the beginning of the interval $\alpha(0) < \beta$, then c_0 is negative. But then, $\alpha(v_i)$ is bounded above by $\beta + c_0$.

Therefore, we are left with only two possibilities: either $\alpha = \beta$ for the whole interval $(0, \bar{v})$ or α reaches β via a discontinuous point. Let us consider this second possibility and call such a point v_0, where $\alpha(v_i) < \beta$ for $v_i < v_0$ and $\alpha(v_i) = \beta$ for $v_i > v_0$. Notice that $P(v_0^-)$ cannot be equal to 0, since this would imply, from (042), that $\alpha(v_0^-) = -\infty$. Therefore from equation (036) we obtain

$$P(v_0^+) \leq P^- \frac{\beta - \alpha^-}{\beta - \alpha^+}$$

Whatever is the original $P(v_0^+)$ we want to keep, we can always find an $\alpha^+ < \beta$ but near enough to β to make the above inequality hold. For the remaining points in this interval, we keep again the same P and we modify α following the solution to the differential equation, i.e.

$$\alpha = \beta + \frac{c}{P}$$

By the argument above, all these α will be below β. Moreover, the government is doing strictly better, since ϕ has been increased for some v_i and P has not changed.

Hence $\alpha = \beta$ is possible only if it is true over all the interval $[0, \bar{v}]$. But also in this case, the government can always set a lower initial condition and increase α so that the constraint is always satisfied. Therefore

$\alpha = \beta$ is never optimal.

Q.E.D.

A3. Proof of Proposition 5

Step 1

The first step is to transform the problem into a tractable optimal control set-up. The following lemma shows that the optimal $\alpha^*(v_i)$ and $P^*(v_i)$ are linked by the same relationship for all the interval $[0, \bar{v}]$.

Lemma 05 *Suppose that $\phi(v_i, \alpha)$ is monotonically increasing in v_i for any α constant and that $\alpha(v_i)$ and $P(v_i)$ represent the optimal choice of the government, then there exists a constant $c_0 < 0$ such that for all v_i,*

$$\alpha(v_i) = \beta + \frac{c_0}{P(v_i)}.$$

Proof: ¿From previous lemmas, we know that (042) must hold at all v_i where $\alpha(v_i)$ and $P(v_i)$ are differentiable. If in the whole range of $(0, \bar{v})$, $\alpha(v_i)$ and $P(v_i)$ are differentiable, then the second order condition will give the relationship in the lemma. What needs to be proved is the case where either $\alpha(v_i)$ or $P(v_i)$ are not differentiable at a point v_0. Without losing generality, let us only consider that such a v_0 is the largest of its kind.

In the neighborhoods of v_i to the left and the right of v_0 we know that the differential equation $\alpha' P + (\alpha - \beta)P' = 0$ holds. Therefore, there exist constants c_1 and c_2, such that

$$\alpha = \beta + \frac{c_1}{P}, \quad v_i < v_0$$

and

$$\alpha = \beta + \frac{c_2}{P}, \quad v_i > v_0.$$

These give the following:

$$P(v_0^-)(\alpha^- - \beta) = c_1;$$

$$P(v_0^+)(\alpha^+ - \beta) = c_2.$$

Thus, according to lemma 2, the incentive compatibility constraint becomes:

$$c_1 \leq c_2.$$

If $c_1 < c_2$, then the government could have done better by setting $c_2 = c_1$, since this way $\alpha(v_i)$ will be smaller to the right of v_0. Thus, $c_1 = c_2 \equiv c_0$. Therefore, (042) holds at all points.

Q.E.D.

Step 2

We now have to transform the constraint in order to write down the optimal control problem. According to Proposition 4, the optimal $\alpha(v_i)$ must be such that $\alpha(v_i) < \beta$. In other words, the constraint $\alpha \leq 1$ is never binding. We can ignore this constraint from now on. The constraint $\alpha \geq \underline{\alpha}$ becomes:

$$\beta + \frac{c_0}{P(v_i)} \geq \underline{\alpha}. \tag{043}$$

Moreover, there is another very important constraint on $P(v_i)$. $P(v_i)$ is the probability that bidder i wins the auction *unconditional* on other bidder's valuations. It is derived from the conditional probability $p_i(v)$. However, $p_i(v)$ does have a constraint, that is $\sum_i p_i(v) \leq 1$. Such a constraint should be translated into a constraint on $P(v_i)$. To put it another way, one has to put a constraint on $P(v_i)$ so that $P(v_i)$ can be integrated back to $p_i(v)$. Fortunately, the constraint on $P(v_i)$ is not too complicated. It has been found by Maskin and Riley (1984). To repeat their condition, here is lemma 5.

Lemma 06 *(Maskin and Riley (1984)) The necessary condition for a family of symmetric permutation functions $p_i(v)$ to exist, such that $\sum_i p_i(v) \leq 1$ and $P(v_i) = \int_{v_{-i}} p_i(v) g_i(v_{-i}) dv_{-i}$ is: for any $0 \leq y \leq \bar{v}$,*

$$\int_y^{\bar{v}} P(v_i) f(v_i) dv_i \leq \int_y^{\bar{v}} F^{n-1}(v_i) f(v_i) dv_i. \tag{044}$$

In order to utilize the Maskin-Riley condition, define a new function $s(\cdot)$:

$$\int_y^{\bar{v}} P(v_i) f(v_i) dv_i - \int_y^{\bar{v}} F^{n-1}(v_i)(v_i) f(v_i) dv_i + s(y) = 0, \tag{045}$$

$$s(y) \geq 0.$$

Equivalently,

$$s'(y) = [P(y) - F^{n-1}(y)] f(y), \tag{046}$$

$$s(y) \geq 0$$

since this is true for any y.

Step 3

To summarize, the problem of finding the optimal $\alpha(v_i)$ and $P(v_i)$ becomes

$$\max_{\alpha, P} n \int_{v_i} \phi(v_i, \alpha(v_i)) P(v_i) f(v_i) dv_i \tag{047}$$

$$s.t. \quad s'(v_i) = (P(v_i) - F^{n-1}(v_i)) f(v_i) \tag{048}$$

$$\alpha(v_i) = \beta + \frac{c_0}{P(v_i)} \tag{049}$$

$$s(v_i) \geq 0 \tag{050}$$

$$\beta + \frac{c_0}{P(v_i)} \geq \underline{\alpha} \tag{051}$$

This is not a typical optimal control problem, since in addition to the state equation, there are algebraic constraints on both the state variable w and the control variable P. To proceed, we use a mixture of Lagrangian and Pontryagin methods (see Takayama (1985), pp 646-651, for a formal exposition). First, define a generalized Hamiltonian (α is substituted away):

$$H(s, P, \lambda_1, \lambda_2, \lambda_3) = \phi(v_i) P(v_i) f(v_i) + \lambda_1 (P(v_i) - F^{n-1}(v_i)) f(v_i) + \lambda_2 s + \lambda_3 (\beta + \frac{c_0}{P(v_i)} - \underline{\alpha}),$$

where λ_1, λ_2, and λ_3 are the co-state variable, multiplier of constraint $w \geq 0$ and multiplier of constraint $\alpha \geq \underline{\alpha}$, respectively. A set of necessary conditions on the optimal $\alpha(v_i)$ and $P(v_i)$ is the following:

$$\frac{\partial H}{\partial P} = \frac{\partial \phi}{\partial P} P(v_i) f(v_i) + \phi(v_i) f(v_i) + \lambda_1 f(v_i) + \lambda_3 (-\frac{c_0}{P^2}) = 0; \tag{052}$$

$$\frac{\partial H}{\partial s} = \lambda_2 = -\lambda_1'; \tag{053}$$

$$\lambda_2 \geq 0 \quad and \quad \lambda_2 s = 0; \tag{054}$$

$$\lambda_3 \geq 0 \quad and \quad \lambda_3(\beta + \frac{c_0}{P(v_i)} - \underline{\alpha}) = 0. \tag{055}$$

An analysis of these first order conditions gives the following proposition.

Proposition 7 *Assume that $\phi(v_i, \alpha)$ is monotonically increasing in v_i for any α constant. If $(\alpha(v_i), P(v_i))$ is the optimal privatization scheme, then for any $v_i \in [0, \bar{v}]$, either $\alpha(v_i) = \underline{\alpha}$ or $P(v_i) = F^{n-1}(v_i)$. Furthermore, both cases cannot occur at the same time.*

Proof: : Since

$$\phi(v_i) = v_i[\lambda + (1-2\lambda)(1-\beta-\frac{c_0}{P(v_i)})] + \lambda(\overline{B} - \beta v_i) - \lambda\frac{c_0}{P(v_i)}q(v_i),$$

where $q(v_i)$ is the hazard function of distribution $f(v_i)$, we have

$$\frac{\partial\phi}{\partial P}P(v_i)f(v_i) + \phi(v_i)f(v_i) = [-v_i(1-2\lambda) - \lambda q(v_i)](-\frac{c_0}{P(v_i)})f(v_i) + [\lambda + (1-2\lambda)(1-\beta-\frac{c_0}{P(v_i)})]v_i f(v_i)$$

$$+\lambda(\overline{B} - \beta v_i)f(v_i) - \lambda\frac{c_0}{P(v_i)}q(v_i)f(v_i).$$

Thus, the first order condition $\frac{\partial H}{\partial P} = 0$ becomes

$$-\lambda_3\frac{c_0}{P^2(v_i)} + \lambda_1 f(v_i) + [v_i(1-\lambda)(1-\beta) + \lambda\overline{B}]f(v_i) = 0. \tag{056}$$

Suppose that in a neighborhood of v_i, $\alpha(v_i) > \underline{\alpha}$ and $P(v_i) \neq F^{n-1}(v_i)$. These imply that both λ_2 and λ_3 are 0 (the constraints are not binding). According to equation (056), one can see that $\lambda_1 = -(1-\lambda)(1-\beta)v_i - \lambda\overline{B}$. Thus, by equation (053), $\lambda_2 = -\lambda_1' = (1-\lambda)(1-\beta) > 0$. This is a contradiction. Therefore, at any point v_i, either $\alpha(v_i) = \underline{\alpha}$ or $P(v_i) = F^{n-1}(v_i)$.

Suppose the opposite is true, i.e. , $\alpha(v_i) = \underline{\alpha}$ and $P(v_i) = F^{n-1}(v_i)$. This is clearly impossible, since

$$\alpha' + (\alpha - \beta)P' = (\underline{\alpha} - \beta)(n-1)F^{n-2}(v_i)f(v_i) < 0,$$

Q.E.D.

i.e., constraint (15) is violated.

Let us check that the global incentive constraints are satisfied by our solution, i.e., $U(v_i, x_i)$ is globally concave in x_i. From lemma 4, we know that $P(v_i)[\alpha(v_i) - \beta] = c_0$ everywhere. Plugging in this condition into equation (038),

$$U(v_i, x_i) = c_0(v_i - x_i) + c_0 x_i + U_i(0,0) = c_0 v_i + U_i(0,0),$$

which is concave in x_i.

Step 4

An implication of the optimal scheme for privatization is the following:

Corollary 1 *If $\alpha^*(v_i)$ and $P^*(v_i)$ are those given by Proposition 5, then it is never incentive compatible to set reservation prices.*

Proof: Since $\alpha(v_i) < \beta$ always, $\phi(v_i, \alpha(v_i)) > 0$ always. Therefore, no reservation prices are needed.

Q.E.D.

Step 5

We will now show that if any "bunching" happens, it *only* happens at the lower end of the spectrum of v.

Proposition 8 *Assume that $\phi(v_i, \alpha)$ is monotonically increasing in v_i for any α constant. If there exists (a, b), such that $\alpha(v_i) = \underline{\alpha}$ for all $v \in (a, b)$, then there does not exist an interval (c, a) such that $P(v_i) = F^{n-1}(v_i)$ for all $v_i \in (c, a)$.*

Proof: It is illustrative to prove the case that $b = \bar{v}$. All other cases can be proved in the same spirit. Supposing the opposite, let x an arbitrary point in (c, a). In other words, $x < a$. From the last proposition, we know that $\alpha(a) = \underline{\alpha}$ and $\alpha(x) > \underline{\alpha}$. Since for all v_i, $\alpha(v_i) = \beta + \frac{c_0}{P(v_i)}$ ($c_0 < 0$) , this implies that $P(x) > P(a)$. In the following, a contradiction to this is found.

In a neighbourhood of x, by Proposition 5,

$$P(x) = F^{n-1}(x).$$

Moreover, from the proof of Proposition 5 (equations (056) and (053)), in the neighborhood of x, $\lambda_3 = 0$. This implies that in that neighborhood $s(x) = 0$, which means that, from equation (045),

$$\int_x^{\bar{v}} P(s)f(s)ds = \int_x^{\bar{v}} F^{n-1}(s)f(s)ds.$$

However,

$$\int_x^b P(s)f(s)ds = \int_x^a P(s)f(s)ds + \int_a^b P(s)f(s)ds = \int_x^a F^{n-1}(s)f(s)ds + \int_a^b P(s)f(s)ds.$$

Therefore,

$$\int_a^b P(s)dF(s) = \int_a^b F^{n-1}(s)dF(s).$$

That is,

$$P(a)[F(b) - F(a)] = \int_a^b F^{n-1}(s)dF(s) > F^{n-1}(a)[F(b) - F(a)],$$

since $P(v_i)$ is constant in (a, b) and $F(\cdot)$ is increasing in (a, b). Therefore

$$P(a) > F^{n-1}(a) > F^{n-1}(x) = P(x).$$

Thus, a contradiction is found.

Q.E.D.

Step 6

Finally, we can show that some bunching is always optimal. Suppose that there is no bunching at all in $[0, \bar{v}]$, then $P(v_i) = F^{n-1}(v_i)$ holds for all v_i. Thus, $P(0) = 0$, which is impossible.

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