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Some Aspects of Risk Sharing in Nonclassical Environments

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Abstract. Under the usual convexity assumptions concerning technology the tendency to diversify to reduce risk and the tendency to diversify to achieve technological efficiency reinforce each other. In nonclassical environments, where technology may be nonconvex, these two goals may be in conflict. We present several examples of this phenomenon and show how risk sharing arrangements can eliminate this conflict. We also consider the effects of introducing nonconcavities into the objective function of certain resource allocation problems.

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One of the most fundamental lessons economists have learned in their study of markets for sharing risk has concerned the benefits of diversification. Indeed, diversification is the major route through which risk sharing arrangements work to improve the overall allocation of risk bearing. In the absence of an insurance market each member of society may bear the risk of his house burning down; when an insurance market is available, each member of society can exchange this "concentrated" portfolio for one that offers a share of everyone's risk of a fire. This "diversification" can lower each individual's risk and thereby bring about a Pareto improvement in social welfare.

Any phenomenon this general and far reaching deserves to have some exceptions. In this paper we shall describe two interesting classes of exceptions to the diversification principle. Each involves, of course, a variation on the classical postulates. But the variations seem to us to be quite plausible; and the circumstances where diversification is harmful may be more common than hitherto assumed.

1. Economies of Scale

In the usual analysis of the benefits of markets for sharing risk, the primary benefit of the risk sharing arrangements is a utility benefit. The chief benefit of an insurance market, for

example, is usually taken to be the peace of mind it affords. Similarly with a stock market: an entrepreneur is viewed as desiring diversification of ownership due to the increment of expected utility resulting from the reduction of the risk he has to bear.

Little attention has been paid to the productive benefits of risk sharing arrangements. It is our thesis that in many cases there can be considerable increases in technological efficiency, resulting from markets for risk sharing; these concrete benefits may be the essential motivation behind many existing arrangements.

For example consider the rise of the stock market and the corporation as economic entities. According to most accounts these economic arrangements arose not as a way of diversifying existing risk, but as a way of reducing the risk of new investments to tolerable levels. The profitability of the East India trade was apparent to all; but the risks were equally apparent. Only by spreading these risks via stock markets and insurance markets could the massive investments needed to launch such trade expeditions be financed.

The case of ocean voyages brings out an essential feature of the productive benefits of risk sharing arrangements we wish to discuss; namely, the importance of economies of scale. The technology of ocean trade involved a very definite element of increasing returns to scale: a voyage halfway across the Atlantic was worth nothing; a voyage across and back could be worth an immense amount. But in order to exploit these economies of scale some method of reducing risk was needed.

Let us try to formalize this argument by means of a simple example.

Let us use x to denote the amount of some input and w its price. In the example given above we might think of x as the size of the ship constructed as outfitted by some entrepreneur. The value of the output of this enterprise is given by

$$f(x)\varepsilon + \delta \tag{1}$$

where ε and δ are random variables with $E\varepsilon = 1$ and $E\delta = 0$. Hence $f(x)$ measures the expected value realized by a ship of size x.

Let us suppose that $f(x)$ exhibits increasing returns to scale as illustrated in Figure 1. What will be the efficient pattern of production for this technology? What size voyages should be undertaken?

If we ignore the uncertainty aspect of the problem, it is clear that the optimal size voyage is at x^* as illustrated in Figure 2, that is the point where average product equals marginal product. But will this level of output be achieved by the functioning of a private market?

In the absence of markets for spreading or diversifying risk the optimal scale may well not be attained. Suppose for example that a single entrepreneur is determines output so as to maximize expected utility of profit. Let $u(W)$ be the entrepreneur's expected utility function of wealth; we assume it has the conventional properties of monotonicity and risk aversion ($u'(W) > 0$, $u''(W) < 0$ for all W). Then the maximization problem becomes:

$$\max \quad Eu\,[f(x)\varepsilon + \delta - wx] \tag{2}$$

The first and second order conditions for this problem are:

$$Eu'(W)\,[f'(x)\varepsilon \quad - w] = 0 \tag{3}$$

$$Eu'(W)\,f''(x)\varepsilon + Eu''(W)\,[f'(x)\varepsilon \quad - w^2] \leqq 0 \tag{4}$$

Note that the second order condition is composed of two parts: the first term is the technological effect - its sign depends on the sign of $f''(x)$

at the optimal scale of production. The second term is the utility effect; under the assumption of risk aversion it is unabiguously negative.

It can easily happen that the second term dominates the sign of the expression so that the preferred operating position occurs where $f''(x)$ is positive - that is, in the inefficientregion of the production function. This is illustrated in Figure 3.

Here the riskiness of the investment has induced the entrepreneur to choose an inefficient scale of operation. He deliberately keeps expected output low so as to reduce the level of risk he bears.

This type of inefficiency can be eliminated by various sorts of risk sharing and risk pooling arrangements; indeed, it is this opportunity for profit that stimulates the development of such institutions for minimizing risk.

Let us distinguish two logically separate roles for risk minimizing institutions. The first is that of risk sharing. This is simply the fact that coalitions of entrepreneurs can form to share the risk of some single investment activity. The second is that of risk pooling. This refers to the fact that when several investment activities are available that are not perfectly correlated one can reduce overall risk by investing in each of the projects; that is, part of the risk involved is eliminated by portfolio diversification.

In actual economies both effects are present of course, and the literature on risk bearing often treats the two phenomena interchangeably. However they are quite distinct concepts.

Let us first consider risk sharing. We will first discuss the special case where $\delta \equiv 0$, and later examine the more general case. We suppose that entrepreneurs can pool their resources and each can purchase a share of the investment being considered. That is, each entrepreneur i can provide $s_i wx$ of the costs involved and then reap $s_i f(x)\varepsilon$ of the output produced.

Hence the maximisation problem facing such a shareholder is simply to determine how much of the risky investment to purchase.

$$\max_{s_i}[Eu\ s_i f(x)\ \varepsilon - s_i wx] \tag{5}$$

Note that we are not considering any diversification behaviour on the part of the shareholder; we are only concerned with the risk sharing aspect of the problem of this stage.

Of course the stockholder cannot determine the optimal level of share investment in the activity until he knows at what level the activity itself is going to operate. This decision has to be made jointly by all of the shareholders involved, but we can at least ask what level any given shareholder would prefer if he had the dictatorial power to arrange this decision.

Accordingly we differentiate shareholder i's expected utility function with respect to s_i - his share of the profits - and x - the scale of operation. We find the first order conditions:

$$Eu'(W)\ [f(x^*)\,\varepsilon - wx^*] = 0 \tag{6}$$

$$Eu'(W)\ [f'(x^*)\,\varepsilon - w]\ s_i^* = 0 \tag{7}$$

Multiply the second equality by x^*/s_i^* and subtract it from the first to get:

$$Eu'(W)\varepsilon[f(x^*) - f'(x^*)/x^*] = 0 \tag{8}$$

Or,

$$f'(x^*) = f(x^*)/x^* \tag{9}$$

Note that this result is <u>independent</u> of the utility function involved; hence all shareholders will unanimously agree that the investment should be operated at the technologically efficient level where marginal product equals average product.[1]

Note further that this result has nothing to do with diversification; it is purely concerned with risk <u>sharing</u> arrangements. Unfortunately once we leave the world of multiplicative uncertainty this simple feature vanishes. Suppose for example that we return to the case where output can be written as $f(x)\varepsilon + \delta$. In this case the first order conditions take the form:

$$Eu'(W)\ [f(\hat{x})\varepsilon + \delta - w\hat{x}] = 0 \tag{10}$$

$$Eu'(W)\ [f'(\hat{x})\varepsilon - w]\ \hat{s}_i = 0 \tag{11}$$

These conditions can be combined to give:

$$Eu'(W)\ [f(\hat{x}) - \hat{x}f'(\hat{x}) + \delta] = 0 \tag{12}$$

or

$$Eu'(W)\ [f(\hat{x}) - \hat{x}f'(\hat{x})] = -Eu'(W)\delta \tag{13}$$

Since $E\delta = 0$, the term on the right is simply the (negative of the) covariance between marginal utility and δ. The concavity of $u(W)$ implies that this term is therefore positive, which in turn implies:

$$f(\hat{x})/\hat{x} < f'(\hat{x}) \tag{14}$$

Or marginal product <u>exceeds</u> average product at the optimal level of operation for individual i. Thus each investor will prefer that the investment be undertaken at a scale that is too small from a technological viewpoint. Of course investors will typically disagree about what exactly the "optimal" scale should be, since this will generally depend on their expected utility functions. However they all agree that the technologically efficient scale is too large!

Let us turn now to a description of the risk spreading effect.

Now we suppose that there are several ex ante identical investment activities described by production functions of the form $f(x)\ \varepsilon_i + \delta_i\ i = 1, \ldots, m$. We further assume that all of the random variables ε_i and δ_i are independently distributed across projects. In this case it will pay each individual investor to diversity his portfolio across the m firms. A typical individual's portfolio would satisfy the maximization problem:

$$\max\ EU\ \left(\sum_{i=1}^{m} s_i\ [f(x_i)\ \varepsilon_i + \delta_i - wx_i]\right) \tag{15}$$

$$\sum_{i=1}^{m} s_i = 1 \tag{16}$$

The first order conditions for this problem are just (10) and (11) with $\hat{x}_i = \hat{x}$ and $\hat{s}_i = \hat{s}$. The analog of (13) is:

$$Eu'\ (f(\hat{x}_i) - \hat{x}_i f'\ (\hat{x}_i)\ ) = -Eu'(W)\delta_i \tag{17}$$

Now if the individual's optimal portfolio is highly diversified, and all of the risks $(\varepsilon_i, \delta_i)$ are independent, then wealth will be nearly certain. Hence

$$Eu'\ (W)\delta_i \simeq u'(W)E\delta_i = 0 \tag{18}$$

Thus we find that the optimal level of investment is again technologically efficient: average product equals marginal product. This time the fact that diversification is possible plays an essential role in the argument.

Note also that this result is <u>independent</u> of the form of the utility functions. Each shareholder agrees about the scale operation of each of the investments.

The argument is depicted in Figure 3. Since the risks are all independent by assumption, the per capita production set society faces is simply $f(x)$. To operate most efficiently given this technology it pays to operate each investment at the optimal level $\hat{x}$ and simply vary the number of investments to achieve any point along the indicated straight line. If the returns to scale are sufficiently great the increase in technological efficiency from the improved institutions for risk bearing could be quite large.

2. Economies of Specialization

Our next example of the productive benefits of risk sharing can be cast in an agricultural framework. Suppose that we have a large number of identical farmers who can devote their resources to farming corn or wheat. Let us suppose that there are some sort of economies of scale (or other economies of specialization) so that the individual production possibilities sets are concave, as in Figure 4, rather than convex as is usually assumed. In the case illustrated, the farmer can specialize in wheat and produce 100 bushels of wheat, or specialize in corn and produce 100 bushels of corn. But if he produces both wheat and corn he ends up with only 40 bushels of each. Thus the individual production possibility sets exhibit diseconomies of scope or economies of specialization.

Society's (per capita) production possibilities set is quite different: it is the convex hull of the individual's production set. If we want to produce an average of 50 bushels of wheat and 50 bushels of corn per farm we simply have half the farms produce wheat and half produce corn. If we want 75 bushels of wheat and 25 bushels of corn per farm, we have three fourths of the farms produce wheat and one fourth produce corn, and so on. For society as a whole any combination along the indicated straight line is feasible. Where society chooses to operate is of course determined by the tastes for wheat and corn; one example is given in Figure 4.

But will the private market induce producers to operate in this efficient manner? In the absence of uncertainty - or when markets exist which can eliminate uncertainty - the answer is yes. If the relative prices of wheat and corn are 1:1 specializing in wheat or in corn is equally profitable and either option is more profitable than diversification. The market induces the optimal technological choices even in the presence of nonconvexities.

When uncertainty is present the situation is considerably different. Suppose for example that the price of corn, p_c, and the price of wheat, p_w, are random variables. Let c be the amount of corn produced by an individual farmer and $w(c)$ the corresponding maximal amount of wheat; i.e. $w(c)$ is the boundary of the production possibilities set.

If a farmer behaves as an expected utility maximizer his problem is:

$$\max Eu[p_c c + p_w w(c)]$$

This has first and second order conditions given by:

$$Eu'(W)[p_c + p_w w'(c)] = 0$$

$$Eu'(W)p_w w''(c) + Eu''(w)[p_c + p_w w'(c)]^2 \leqq 0$$

Just as before the second order condition has two terms. Even if economies of specialization are present (so that $w''(c) > 0$) the risk averse behavior indicated by the second term may lead to an interior optimum. The farmer is led to diversify in order to hedge against fluctuations in income even though this leads to technological inefficiency. (Figure 5.)

This tendency to diversify is quite strong, even in the face of the adverse technological repercussions. In the Appendix we extend Samuelson's (1967) argument that shows diversification generally pays for an individual expected utility maximizer.

At the social level of course things may be rather different. If the risk to the farmers can be eliminated or shifted to other agents, an improved pattern of production can be brought about. In the case described above, futures markets can be used to eliminate the uncertainty about price fluctuations and thereby induce the technologically advantageous specialization.

3. The Desirability of Diversification

The previous example has indicated that specialization may be desirable at the social level even though private interests indicate diversification. This phenomena arises because of nonconvexities in production.

Presumably a convex production set and a nonconcave objective function would give the same result. Consider for example the case in which production possibilities set is linear as in Figure 6 - i.e. constant returns to scale prevails. Suppose however that the social objective function is not of the proper shape: For example suppose it is max $[x_1,x_2]$. In such a case the optimal policy involves specialization in x_1 or x_2.

When might such an objective function arise? One common circumstance is that of a race: where coming in first is all that matters. Suppose for example that a firm is allocating resources to several research projects in an attempt to develop a new product before its competitors. Then of course only the project that develops the product first is relevant; only the winner matters.

Let us formalize this statement in the following way. Suppose a firm is allocating funds to various projects; let x_i be the funds allocated to project i and write the budget constraint of the firm as $\Sigma x_i = B$. Each project produces output $f_i(x_i)$ and the objective function of the firm is given by $W(f_1(x_1),\ldots,f_n(n))$. The resource allocation problem is then:

$$\max W(f_1(x_1),\ldots,f_n(x_n))$$

$$\text{s.t. } \Sigma x_i = B$$

When will the optimal solution involve specialization?

THEOREM 1. *Suppose that* $W(y_1,\ldots,y_n)$ *is increasing and convex as a function of* $(y_1,\ldots,y_n)$ *and that each* $f_i(x_i)$ *is a convex function of* x_i. *Then there is an optimal solution that involves specialization: for some* i, $x_i^* = B$, $x_j^* = 0$ for $j \neq i$.

Proof. Clearly all that we need show is that

$$V(x_1,\ldots,x_n) = W(f_1(x_1),\ldots,f_n(x_n))$$

is a convex function of $(x_1,\ldots,x_n)$. But this follows directly from the hypotheses:

$$\begin{aligned} V(tx + (1-t)x') &= W[f_1(tx_1 + (1-t)x_1'),\ldots,f_n(tx_n + (1-t)x_n')] \\ &\leqq W[tf_1(x_1) + (1-t)f_1(x_1'),\ldots,tf_n(x_n) \\ &\qquad + (1-t)f_n(x_n')] \\ &\leqq tW[f_1(x_1),\ldots,f_n(x_n)] \\ &\qquad + (1-t)W[f_1(x_1'),\ldots,f_n(x_n')] \\ &= tV(x_1) + (1-t)V(x') \quad \square \end{aligned}$$

The above (trivial) argument establishes the desired result: when the objective function is convex and the production functions are convex, specialization is optimal. Note that only weak convexity is needed: a linear welfare function and linear production functions are perfectly compatible with the above result.

Now let us ask how this result might be modified in the presence of uncertainty. There is an intuition that suggests that if uncertainty is present, it might pay to diversify: to hedge one's bets so that complete specialization is not desirable.

Let us model this in a rather general way by writing production functions as dependent on n random variables $(\varepsilon_1,\ldots,\varepsilon_n)$ with joint probability density $g(\varepsilon_1,\ldots,\varepsilon_n)$. The social objective function now becomes:

$$EW[f_1(x_1,\varepsilon_1),\ldots,f_n(x_n,\varepsilon_n)] = \int W[f_1(x_1,\varepsilon_1),\ldots,f_n(x_n,\varepsilon_n)]g(\varepsilon_1,\ldots,\varepsilon_n)d\varepsilon_1,\ldots,d\varepsilon_1$$

We now have the main result of this section:

THEOREM 2. <u>Suppose that the hypotheses of Theorem 1 hold for each realization of</u> $(\varepsilon_1,\ldots,\varepsilon_n)$. <u>Then there is an optimal solution</u> x^* <u>that involves specialization</u>.

<u>Proof</u>. Simply note that for each realization of $(\varepsilon_1,\ldots,\varepsilon_n)$, $V(x_1,\ldots,x_n)$ is a convex function by Theorem 1. But a weighted average of convex functions is still convex. Hence the result. $\square$

The proof of Theorems 1 and 2 are mathematically trivial but surprisingly nonintuitive. Let us illustrate this result in one simple case. (This case was originally examined in Nalebuff (1980) who proved that in this situation specialization was desirable. Theorem 2 resulted from our attempt to generalize this result.)

Suppose that output of project i is given by $y^i = a^i x^i + b^i$

where a^i and b^i are random variables. Suppose that only the winner matters so that the social objective function is $W(y_1,\ldots,y_n) = \max\,[y_1,\ldots,y_n]$. Then the hypotheses of Theorem 2 are satisfied and the optimal policy involves specialization. In fact in this case the policy is especially simple: we just compute the overall unconditional expected values of output resulting from specialization:

$$\bar{y}_i = \bar{a}_i B + \bar{b}_i \qquad i = 1,\ldots,n$$

and then choose to specialize in that activity with the highest expected output. In general we will <u>not</u> want to "hedge our bets" and diversify.

It seems nonintuitive that we will never want to diversify but on reflection it becomes clear: diversification is implied by the convexity of the constraints or the concavity of the objective function. If these conditions are not met, diversification is not necessarily optimal.

This simple point can have some interesting consequences for resource allocation. In the absence of a social planner who omnisciently chooses optimal behavior, decisions must be made by individuals - who may or may not have the same objective function as society. It seems that there can be errors of two sorts: society has a convex objective while the individual's objective function is concave, or vice versa.

As an example of the first case, consider research. Society does not really care whether there is a <u>second</u> firm to discover a new piece of technology: only the winner matters. Yet individual rewards to research directors may be such as to encourage diversification. The opposite kind of distortion may occur with political decisions: since winning is much more highly rewarded than coming in second, overly extreme social policies may be promulgated. Even though social welfare may be concave in the relevant variables the convex nature of the individual rewards could lead to inappropriate decisions.

Appendix

The farmer's maximization problem is given by:

$$\max Eu[p_c c + p_w w]$$

$$\text{s.t.} \quad w = f(c)$$

We want to find conditions under which the optimal solution has $c^* > 0$, $w^* > 0$, that is under which some diversification is deemed desirable. Let us suppose then that we are currently operating at the boundary $c = 1$, $w = f(c) = 0$ and that we contemplate a feasible change $\Delta c < 0$, $\Delta w > 0$. The change in utility will be:

$$\Delta u = Eu'(p_c c)p_c \Delta c + Eu'(p_c c)p_w \Delta w$$

Define:

$$\bar{m} = Eu'(p_c c)$$

$$\bar{p}_c = Ep_c$$

$$\bar{p}_w = Ep_w$$

$$\sigma_{mc} = \text{cov}\ (u'(p_c c), p_c)$$

$$\sigma_{mw} = \text{cov}\ (u'(p_c c), p_w)$$

Then using the standard covariance identity that $EXY = \text{cov}\ (X,Y) + (EX)(EY)$ we can rewrite the expression for Δu as:

$$U = (\sigma_{mc} + \bar{m}\bar{p}_c)\Delta c + (\sigma_{mw} + \bar{m}\bar{p}_w)\Delta w$$

$$= \sigma_{mc}\Delta c + \sigma_{mw}\Delta w + \bar{m}(\bar{p}_c \Delta c + \bar{p}_w \Delta w)$$

Consider the sign of each of these three terms. First σ_{mc} is certainly negative since p_c and the marginal utility of income move in opposite directions; since $\Delta c < 0$, $\sigma_{mc}\Delta c$ is positive. The sign of the second term depends on the covariance between p_w and p_c. If they are nonpositively correlated, we will have $\sigma_{mw} \geqq 0$. Thus σ_{mw} w will also be positive.

The sign of the last term depends on the expected profitability of the change. As long as $\bar{p}_c/\bar{p}_w$ exceeds the marginal rate of substitution - $\Delta w/\Delta c$ the last term will also be positive.

Hence the sign of ΔU will be sure to be positive when the expected profitability is positive, but even if the expected profits are negative the gain from the reduction in risk may induce some amount of diversification.

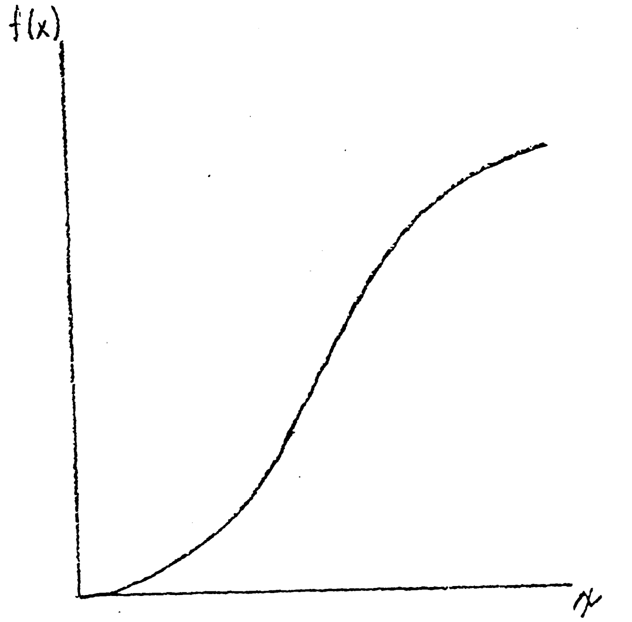

Fig 1. Individual Production Function

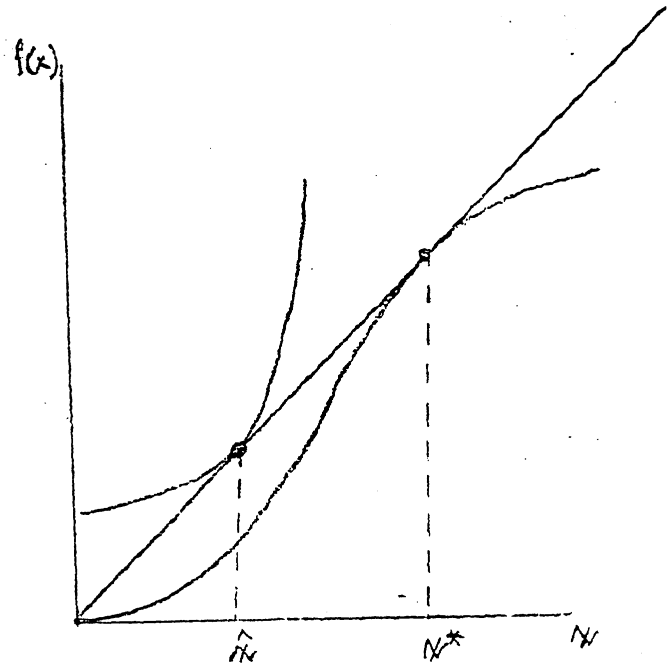

Fig 2. Social Production Function

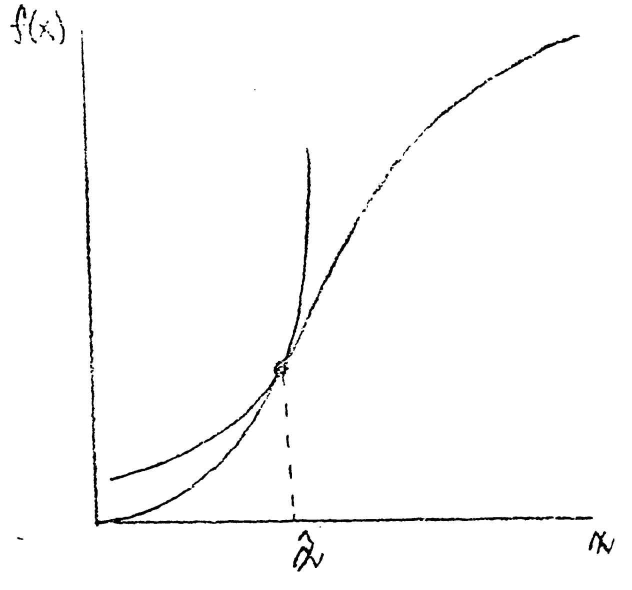

Fig 3. Inefficient Scale of Production

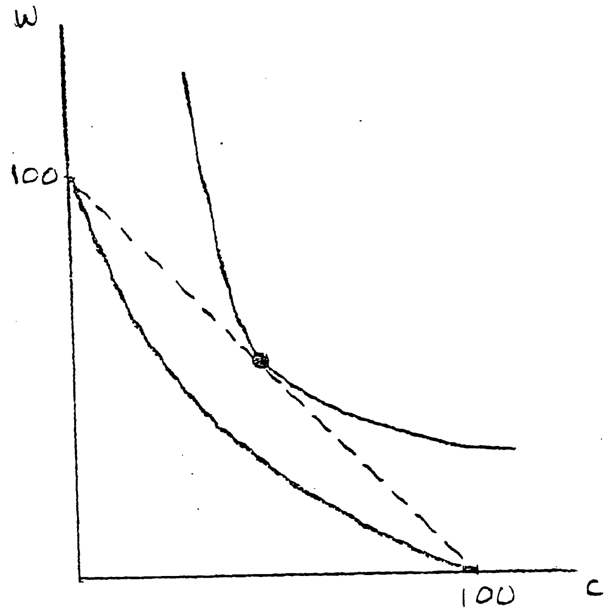

Fig 4. Individual and Social Production Possibilities

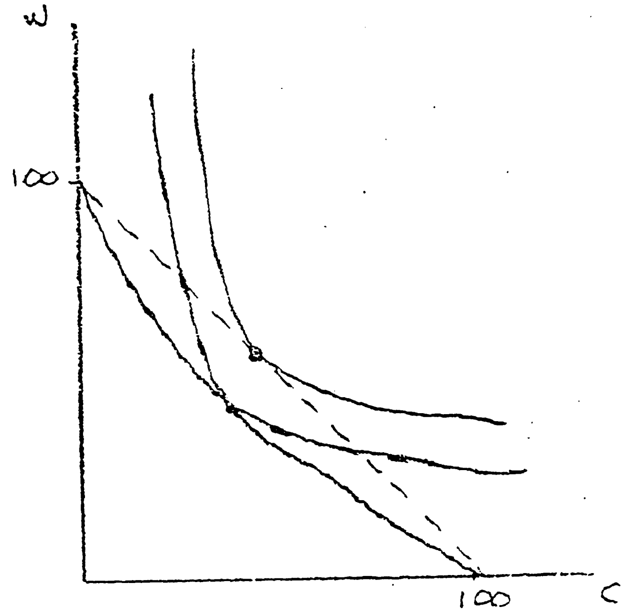

Fig 5. Inefficient Solution and Efficient Solution

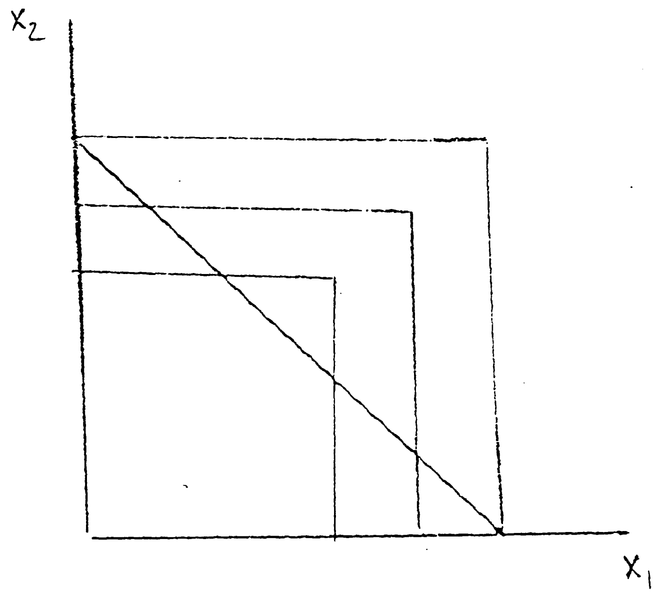

Fig 6. Nonconcave Objective Function

FOOTNOTES

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1. This result is of course closely related to Diamond's result that a competitive stock market is pareto efficient under multiplicative uncertainty.

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