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An Application of Control Theory To Rural-Urban Migration and Urban Unemployment

by
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AN APPLICATION OF CONTROL THEORY
TO RURAL-URBAN MIGRATION
AND URBAN UNEMPLOYMENT

Izevbuwa Osayimwese*

I

This paper is designed to be both expository and provocative. It is an attempt to pose the problem of rural-urban migration and urban unemployment as one in optimal control. Since this application is rather unusual, it can be expected to produce more cynicism than positive interest, but the novelty of the approach may offer its own fascination.

The paper is divided into four sections. In Section II we introduce, pose, and solve the control problem. Section III is devoted to a discussion of an alternative solution procedure, and the last section to some concluding remarks.

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II

The large influx of people from low-income areas to relatively high income urban and semi-urban areas is a particularly noticeable feature in transitional economies. This fact is well recognized by many development economists (3, 5). More recently, Todaro has produced some useful insights into the subject. The essence of his approach lies in the positing of a formal behavioral model in which migration depended not only on the urban-rural wage differentials, but also on the existence of job opportunities. Through the latter, urban-rural wage differentials are translated into expected wage differentials. Todaro then used the notion of stationary market equilibrium to solve for that urban unemployment rate at which the proportional change in excess demand for labor equals zero (10).

A "control theory" formulation of the problem of rural-urban migration presupposes that labor is responsive to discretionary variations in some variable(s). In our model we will assume that labor always seeks to behave rationally in the sense of preferring more money to less money, and hence must seek more money where it is expected to be available.

THE MODEL:

Let us suppose that there are two geographic sectors in a region — a rural sector and an urban sector. The urban sector may be thought of as comprising two sub-sectors: a traditional urban sector and a modern commercial sector.

Further, let u represent the rate of urban unemployment, and M_{ru} the volume of migration from rural to urban. We will assume that the time rate of change of urban unemployment rate is entirely due to migration. That is

$$\overset{\circ}{u} \text{ (u-dot)} = M_{ru} .$$

This assumption may be rationalized as follows: Following Kalacheck, unemployment is composed of three main parts — lay-offs, quits, and labor force entry. It seems reasonable to expect the relative importance of each part to depend on the level of aggregate employment. That part of unemployment which is due to labor force entry will tend to have a greater relative importance in a less developed country than in materially advanced countries. This is largely the result of a high and rising growth rate of population in the less developed countries and the reinforcing effects of rapidly expanding educational systems. Although this is a testable hypothesis, we shall simply accept it as true.

Next we postulate the following behavioral equation

$$\mathring{u} = \alpha u + \beta d + \gamma v \qquad (1)$$

where u-dot is as mentioned before,

d is the ratio of urban-rural wage rate differential to rural wage rate; that is, $d = (Y_u - Y_r)/Y_r$

v = d-dot is the time rate of change of d.

We shall assume that the parameters in (1) have the following signs:

$\alpha < 0;\quad \beta > 0;\quad \gamma > 0.$

α is assumed to be negative because we expect a high urban unemployment rate to deter potential migrants, in principle at least, from migrating into the urban sector. But the assumption that $\alpha < 0$ seems to have been persistently contradicted by experience. This is perhaps not surprising, since $\alpha < 0$ only ceteris paribus. Yet the world (including the migration phenomenon) is mutatis mutandis![1]

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[1] J. R. Harris and Michael Todaro described this phenomenon as a "curious economic phenomenon...." Later in a footnote they referred to Gugler's explanation in terms of the great disparity between urban and rural wages which makes the "mathematical expectation of urban wage rate higher than the certain prospect of rural wage rate." Gugler also stressed that the rural sector is not without its risks and uncertainties.

It is important to note that the choice of relation (1) was not arbitrary. The equation was chosen because of the desire to "catch" the expectations phenomenon which underlies the migration process. Previous empirical studies which aimed at predicting internal migration in terms of either unemployment rate or wage differentials seemed to have produced unsatisfactory results, viz., the unemployment variable was found statistically insignificant. Equation (1) thus attempts to combine the unemployment approach with the wage-differential approach into a single behavioral relation.[2]

THE FORMAL MODEL:

$$\text{Minimize } T = \int_o^T dt \text{ , with respect to } v, \tag{2}$$

Subject to,

$$\begin{aligned} &\overset{\circ}{u} = \alpha u + \beta d + \gamma v \\ &\quad u(o) = u_o,\ u(T) = u_T \\ &\quad u(t) \geqq 0 \text{ for all } t \\ &\quad \alpha < 0;\ \beta > 0;\ \gamma > 0 \ . \end{aligned} \tag{3}$$

$$\begin{aligned} &\overset{\circ}{d} = v; \quad d(o) = d_o, \\ &\quad |d| \leqq 1 \\ &\quad |v| \leqq 0.1 \end{aligned} \tag{4}$$

The constraint on d may be rationalized as a political constraint.

Verbally, the problem is to minimize the time it takes to move the urban

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[2]See Bruce H. Herrick, "Urban Migration and Economic Development in Chile," M·I·T, 1965, for an example of the unemployment approach that produced unsatisfactory results.

traditional sector from an initial unemployment rate u_o to a final arbitrary but known value u_T. Such that conditions in relations (3) and (4) are not violated.

Some remarks on the objective function will be in order. Although time minimization problems have been posed by some economists,[3] the approach is more familiar to engineers and physicists. Therefore our usage in this paper demands an explanation. First, the minimum time optimand is simple in structure, and simplicity is certainly desirable in an expository paper. Moreover, to reach full employment at the fastest time may be an important social goal, and under certain circumstances it could be a dominant one. The criticism may be levelled also that a time minimizing problem does not take account of probable "adjustment costs," which may arise because attainment of full employment may necessitate drastic changes in certain "slow" variables. Against such criticism one may argue that if the net costs of being out of equilibrium are proportionally related to time, minimizing the transition time will imply minimizing total (including adjustment) costs. Be it as it may, the objective of minimizing the time to attain "full employment" is no more extreme than the more familiar goal of maximizing the sum of discounted utilities.[4]

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[3]Avinash Dixit, "Marketable Surplus and Dual Development," Journal of Economic Theory I, 1969, pp. 203-219.

[4]Mr. Gary Fields and Professor J. Cross of The University of Michigan have pointed out to me that there is another cost which might be included in the "straight" time optimal problem. This cost arises from the arbitrariness of the constraint we imposed on the "control" variable, v. The initial choice of the constraint on v may have been wrong; this can be corrected only after the first problem, by solving a new problem, a process which lengthens the optimal transition time. This kind of cost can be incorporated by replacing equation (2) by

$$\int_o^T (1 + v^2)dt.$$

SOLUTION BY PONTRYAGIN'S MAXIMUM PRINCIPLE[5]

The Hamiltonian which corresponds to the system of equations (2) - (4) is

$$H(u, d, v, \Psi_1, \Psi_2) = -1 + \Psi_1(\alpha u + \beta d + \gamma v) + \Psi_2 v \ , \tag{5}$$

where Ψ_1, Ψ_2 are continuous time functions (or dynamic multipliers) associated with u-dot and d-dot.

$\Psi_1 \geqq 0$; $\Psi_2 \geqq 0$ (by virtue of the "shadow price" interpretation of dynamic multipliers).

From the maximum principle, we obtain the following necessary conditions for an optimum

$$\mathring{\Psi}_1 = -\alpha\Psi_1 = -\partial H/\partial u \tag{6}$$

$$\mathring{\Psi}_2 = -\beta\Psi_1 = -\partial H/\partial d \tag{7}$$

The solution to (6) is

$$\Psi_1 = C_1 e^{-\alpha t} \tag{8}$$

and that to (7) is

$$\Psi_2 = (\beta/\alpha) C_1 e^{-\alpha t} + C_2^* \tag{9}$$

The boundary conditions on Ψ_1 and Ψ_2 at the optimal time T^* are shown in Appendix A.

It should be noted that we are seeking to minimize time, and hence the terminal time is unspecified and should be solved for in the system. The fact that T is variable necessitates the introduction of a stopping condition for the system. According to Rosonoer,[6] this condition requires that at the optimizing time, the Hamiltonian function should be zero. That is

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[5] See L. Pontryagin, et alia, "The Mathematical Theory of Optimal Processes," Interscience, John Wiley, 1962.

[6] L. L. Rosonoer, "Pontryagin's Maximum Principle in Optimal Systems Theory," II, Translated in Automation and Remote Control, vol. 20, 1959, pp. 1405-1421.

$$H(u,\ d,\ v^*,\ \Psi_1,\ \Psi_2)\Big|_{t=T^*} = 0\ . \tag{10}$$

But Pontryagin et alia have shown that (10) is true at any t, provided an admissible v^* exists, and (6) and (7) are satisfied.[7] In the sequel we will indicate how one may solve for T^*.

By substituting (8) and (9) into (5) we obtain

$$H = -1 + C_1 e^{-\alpha t}(\alpha u + \beta d + \gamma v) + ((\beta/\alpha)C_1 e^{-\alpha t} + C_2^*)v \tag{5'}$$

From (5) and (5') it follows that the optimal policy is[8]

$$v^* = \begin{cases} +0\cdot 1 \text{ if } \Psi_1\gamma > \Psi_2 \\ \text{undefined in the interval } [-0\cdot 1,\ +0\cdot 1] \text{ if } \Psi_1\gamma = \Psi_2 \\ -0\cdot 1 \text{ if } \Psi_1\gamma < \Psi_2 \end{cases} \tag{11}$$

or $\qquad v^* = \text{Sign}\ (C_1 e^{-\alpha t}(\gamma - \beta/\alpha) - C_2^*)\ .$ (11')

If $\qquad v^* = +0\cdot 1$, then

$\qquad d = +0\cdot 1t + k$, since $v^* = \mathring{d}^*$.

Then we may write equation (1) as

$$\mathring{u} = \alpha u + \beta(0\cdot 1t + k_1) + 0\cdot 1\gamma\ . \tag{1'}$$

The solution to this first order differential equation is given by

$$u(t) = C_2 e^{\alpha t} + C_3 t + C_4 \tag{12}$$

where

$$C_2 = k_2 + (\beta/\alpha)k_1 + (0\cdot 1\gamma)/\alpha + (0\cdot 1\beta)/\alpha^2\ ;$$

$$C_3 = (=0\cdot 1\beta)/\alpha\ ;$$

$$C_4 = -[(\beta k_1 + 0\cdot 1\gamma)/\alpha + (0\cdot 1\beta)/\alpha^2]$$

By virtue of (11) and the constraint on d, the following cases will not be permissible:

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[7]Pontryagin et alia, "The Mathematical...," pp. 18-19.

[8]Expression (11) follows from the maximum principle which requires that the Hamiltonian be maximized with respect to the control. Hence Ψ_1, Ψ_2 must have the same signs as v in any terms involving Ψ_1 and v, and Ψ_2 and v.

$d = +1 \quad \text{if} \quad \gamma\Psi_1 > \Psi_2$

$d = -1 \quad \text{if} \quad \gamma\Psi_1 < \Psi_2$

$d \ \varepsilon \ (-1, +1) \quad \text{if} \quad \gamma\Psi_1 = \Psi_2$ because at T^*, $\Psi_2(T^*)$ equals zero and $\Psi_1(T^*)$ is identically zero since $\gamma > 0$, by assumption. The first condition says that because $\gamma\Psi_1 > \Psi_2$ calls for a policy $v = +0{\cdot}1$, d must have been less than unity, otherwise the right hand side constraint on d would be violated. The meanings of the other two conditions are straightforward.

Three permissible cases may be distinguished:

Case I: $d < 1, \ \gamma\Psi_1 > \Psi_2$

Case II: $d > -1, \ \gamma\Psi_1 < \Psi_2$

Case III: $d = \pm 1, \ \gamma\Psi_1 = \Psi_2$

We shall consider Case I, but for no special reason. Let us suppose that at final T^*, $d < 1$. Therefore prior to final time, the policy is $v = +0{\cdot}1$ because of the above end conditions. In this case,

$$\mathring{d} = +0{\cdot}1, \text{ and}$$

$$d = 0{\cdot}1t + k_1,$$

$$u(t) = C_2 e^{\alpha t} + C_3 t + C_4 \ .$$

The graphs of d and u for Case I are drawn below, and have been drawn "backwards" in time. The advantage in drawing the curves backwards in time is that it then becomes necessary to solve for the optimal T. This could have been done by using equation (5'), conditions (10) and (11). Also T^* could be obtained by solving the equation

$$u_T = C_2 e^{\alpha T} + C_3 T + C_4 \ .$$

But this is a difficult equation to solve for T. However, a lower bound can be obtained on T.[9]

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[9] If we assume that $C_2 = C_3 = C_4 = 1$. Then $u_T = e^{\alpha T} + T + 1 > e^{\alpha T}$; hence $T^* > (\log u_T)/\alpha$ and T^* approaches zero as α approaches minus infinity. From this we may infer that the more responsive each potential migrant is to the prevailing rate of urban unemployment, the quicker it takes to attain full employment.

For different d_T we shall have different curves for the u function, and a map is thus generated in the u-t plane. Eventually, the curves from the top and those from the bottom (above the t-axis, because a negative u is meaningless in our problem) coincide. At the resulting "boundary" trajectory (see Figure 1), the policy-makers must decide whether or not to follow this path to u_T. Formally, the question is: Is the "boundary" trajectory better (in the sense of the maximum principle) than "neighboring" trajectories? In Appendix B, we suggest how the proposition that the "boundary" trajectory will always be preferred to neighboring ones may be proved by construction.

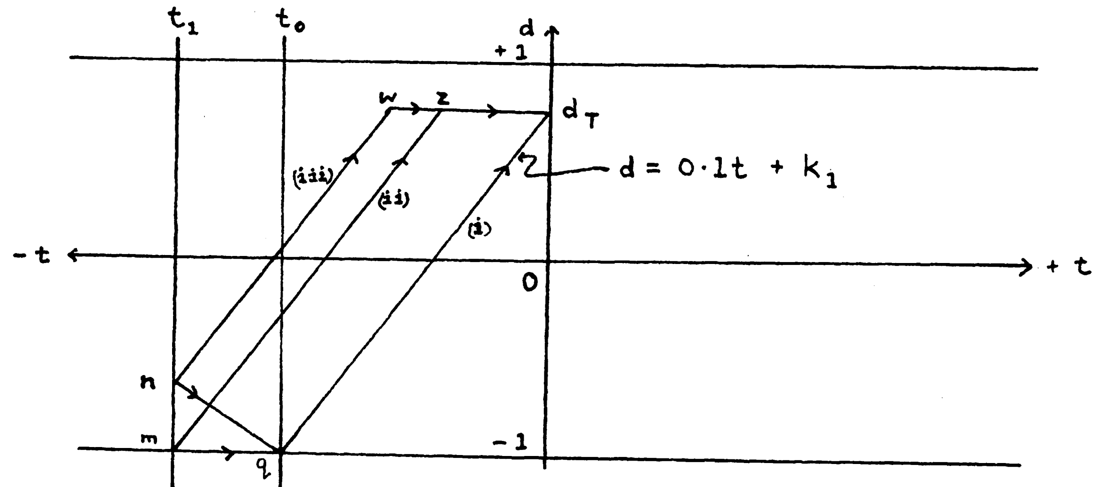

Figure 1

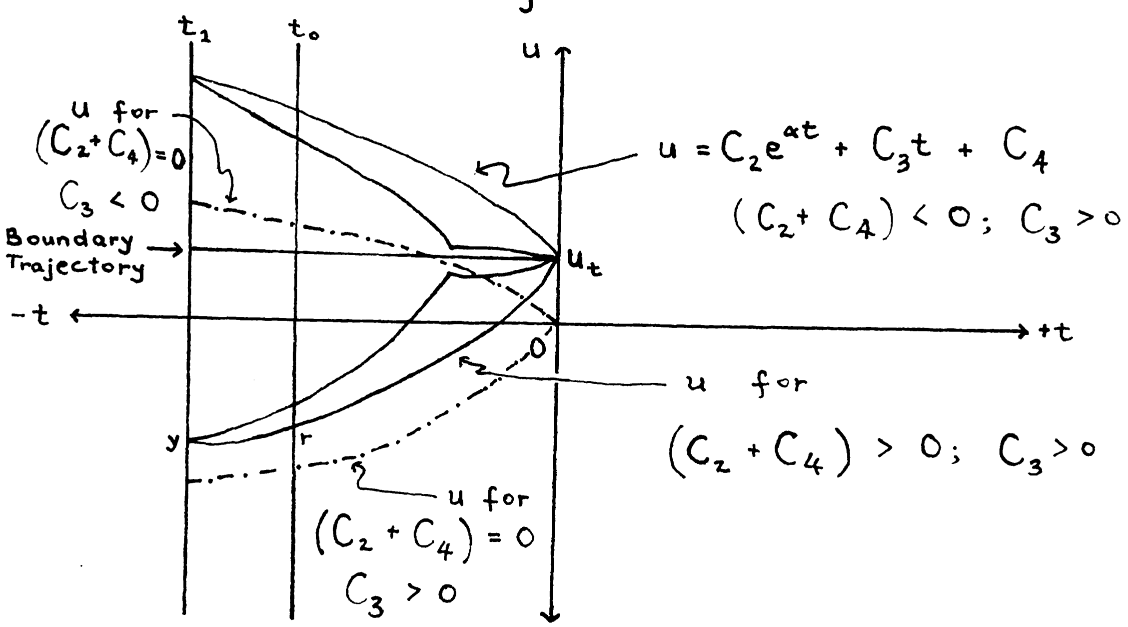

Figure 2

On the basis of the foregoing we shall state the following tentative propositions.

Proposition 1

Let us suppose that in Figure 1 the initial ratio of rural-urban wage rate differential to rural wage rate is at n. Then the optimal path for reaching the equilibrium d_T is path (iii).

Proof[10]

Now suppose that the initial d is at m; then there is no unique optimal trajectory; d_T can be reached either by first moving from m to q for some time and then following path (i), or by moving along path (ii) to Z and thence to d_T. If the initial d were at n, only the path via w to d_T would be used; the path through n and q to d_T would never be used since v = +0·1 indicates that the desired d_T is higher than the initial value. The path through n and q to d_T indicates that for part of the "journey" to d_T, the control variable switches signs more than once: first, just before n, and second, after q. Thus a basic postulate of "bang-bang" systems is violated. Hence proposition 1 must hold in order to rule out the above contradiction.

Proposition 2

The optimal control, v*, cannot assume values in the interior of the allowable interval, [-0·1, +0·1] for any finite time interval.

Proof

Assume that the optimal control can take on values in the interior of [-0·1, +0·1]. Then it follows that

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[10]This proof (or the proposition for that matter) is only intuitive and has no claim to mathematical rigor. The cardinal theorems of bang-bang systems are rigorously stated and proved in Chapter 3 of Pontryagin, et alia.

$H/\alpha v = 0$ exists in the interior of the interval.

$$H/\alpha v = 0 = \gamma C_1 e^{-\alpha t} + (\beta/\alpha) C_1 e^{-\alpha t} + C_2^* \quad (13)$$

Equation (13) cannot be satisfied unless $C_1 = C_2^* = 0$. This implies that equation (5') equals minus unity. But the optimal value of the Hamiltonian must be zero as was noted above. Hence v* cannot be in the interior of [=0·1, +0·1]. This is the meaning of "bang-bang" in proposition 1 and the basis of its proof.

Proposition 2a

The proportional change in the "social cost"[11] of altering the unemployment rate and the urban-rural wage differential ratio is always positive in the interior of [-0·1, +0·1]; this change in "social cost" may be very high the lower is the parameter α, especially in the neighborhood of the target u_T, where "overshooting" (of the target) could easily occur as α approaches minus infinity.

The importance of this proposition lies in the fact that it makes proposition 2 directly relevant to the economic problem in hand.

Proof

Recall expression (11).

$$v \varepsilon [-0·1, +0·1] \text{ when } \Psi_1 \gamma = \Psi_2 .$$

That is, when

$$C_1 e^{-\alpha t} = (\beta/\alpha) C_1 e^{-\alpha t} + C_2^*$$

or

$$C_1 / C_2^* = e^{\alpha t} / (1-(\beta/\alpha)) \quad (14)$$

Equation (14) is finite and positive if $(\beta/\alpha) < 1$ for finite time t. But $(1 - (\beta/\alpha))$ is positive because $\beta > 0$ and $\alpha < 0$, by assumption.

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[11] Social cost in this problem should be understood only as the "shadow prices," Ψ_1, Ψ_2, corresponding to the behavioral relations, (3) and (4). Social cost here may be interpreted as the lengthening of the minimum time.

Hence C_1 and C_2^* must each be positive and finite for finite time t, and consequently, Ψ_1, Ψ_2, and $\mathring{\Psi}_1$, $\mathring{\Psi}_1$ are positive and finite for finite time in view of (6) and (7). The proof of the second part of the proposition now follows.

For $v^* \varepsilon [-0{\cdot}1, +0{\cdot}1]$, $\Psi_1\gamma = \Psi_2$, and for all t $\mathring{\Psi}_1\gamma = \mathring{\Psi}_2$.

Let $\mathring{\Psi}_1\gamma = \mathring{\Psi}_2 = \mathring{\Psi}$.

Then $\mathring{\Psi}/\Psi = \gamma\mathring{\Psi}_1/\Psi_2 = -\alpha\gamma\Psi_1/\gamma\Psi_1$

by virtue of (6) and (7).

Hence $\mathring{\Psi}/\Psi = -\alpha > 0$ since $\alpha < 0$

$$\lim_{\alpha\to-\infty} \mathring{\Psi}/\Psi = +\infty \tag{15}$$

It should be noted that the possibility in equation (15) is very strong in the neighborhood of the target u_T; farther away from this target, a very negative α may be all that is required to drive the urban sector exactly (without "overshooting") to u_T.

III

This section is devoted to a consideration of an alternative approach to solving the optimal control problem we have posed in Section II. We desire another solution procedure because there are various problems inherent in the maximum principle, which after all gives us only qualitative impressions. Any other approach which gives qualitative results, but is less demanding in its mathematical specifications, is certainly to be desired.

Given the problem (2)-(4), we sketch the feasible values for u and d as follows.

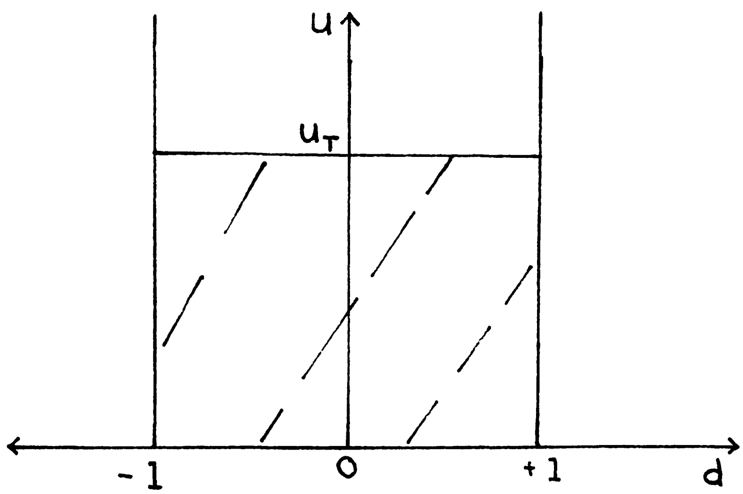

Figure 3

From equation (3) we may write the target unemployment rate as

$$u_T = e^{\alpha T}\int_0^T e^{-\alpha T}(\beta d+\gamma \dot{d})dt + u_o$$

$$= e^{\alpha T}\int_0^T e^{-\alpha T}(\beta+\alpha\gamma)d\ dt + \gamma d\Big|_o^T + u_o \tag{16}$$

after integrating by parts.

If the initial urban unemployment rate, u_o is less than the target one, u_T, we want to find a d* such that for any feasible d,

$$u_T = e^{\alpha T}\int_0^T e^{-\alpha T}(\beta+\alpha\gamma)d\ dt + \gamma d\Big|_o^T + u_o$$

$$\geqq e^{\alpha T}\int_0^T e^{-\alpha T}(\beta+\alpha\gamma)d^*\ dt + \gamma d^*\Big|_o^T + u_o \tag{17}$$

If $(\beta+\alpha\gamma) > 0$, it is evident from (16) or (17) that we want d* to be as large as possible at each instant; hence the optimal policy is

$$v^* = +0\cdot 1 \ ,$$

and this policy will be pursued until either $u = u_T$ or $d = 1$, whichever comes first. If u_T comes first, we are done; if $d = 1$ comes first, we then (that is after $d = 1$) put v at zero unit $u = u_T$. After $u = u_T$, we put u-dot equals zero (for stationary equilibrium) by setting v so that

$$0 = \alpha u_T + \beta d + \gamma v \ .$$

This may not be possible at all times; if it is not possible, u will overshoot u_T.

Similarly, if $u(o) > u_T$, we want d^* to be as small as possible at each instant, assuming that $(\beta+\alpha\gamma) > 0$. The policy in this case is,

$$v^* = -0\cdot 1 \ .$$

This will continue to be so until $d = -1$ or $u = u_T$. If $d = -1$ first, we will set $v = 0$ until $u = u_T$, after which u-dot is set at zero by choosing v to satisfy

$$0 = \alpha u_T + \beta d + \gamma v, \text{ if possible.}$$

If not, u will undershoot u_T.

Our policy diagram in the case $(\beta+\alpha\gamma) > 0$, assuming u-dot can be set at zero at time T is:

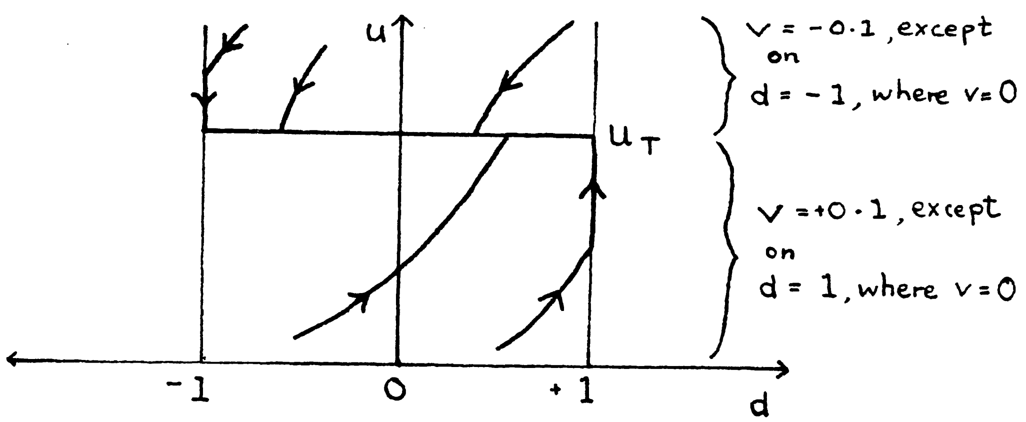

Figure 4

Finally one assumption on α, β, γ and u_T that guarantees that u-dot can be set to zero when $u = u_T$ is

$$\alpha u_T - \beta + 0{\cdot}1\gamma \gtreqless 0 \ , \quad \text{or} \quad u_T \geqq (\beta/\alpha) - (0{\cdot}1\gamma)/\alpha \geqq 0$$

$$\alpha u_T + \beta - 0{\cdot}1\gamma \lesseqgtr 0 \ , \quad \text{or} \quad u_T \geqq -(\beta/\alpha) - (0{\cdot}1\gamma))/\alpha \leqq 0$$

That is, $(\beta/\alpha) \geqq (0{\cdot}1\gamma)/\alpha$ since $u_T \gtreqless 0$, by assumption.

IV

In conclusion we should note that implicit in equation (1) is a theory of unemployment which says that the change in the rate of urban unemployment depends not only on the urban-rural wage rate differential ratio, but also on the existing rate of urban unemployment and the rate of change in the wage differential ratio. What is perhaps of interest in this formulation is that we have gone at least one step further in characterizing the behavioral dynamics of rural-urban migration. Economic studies of rural-urban migration, like Todaro's, have stressed the role of expectations only in relation to the level of the wage rate. To the extent that expected wage differentials have not been statistically significant in many empirical studies, our introduction (among other variables) of the rate of change of the wage differential introduces a new dimension to the expectations phenomenon. Indeed, the assumed responsiveness of u-dot to v makes v a good candidate for policy variable.

From an analysis of the basic model we see that there are in general two types of costs incurrable from adjusting to an equilibrium unemployment rate. The first and more obvious cost is that of changing the dependent variable, u from one value to another. The second and perhaps less obvious cost is that arising from changing policy instruments (or independent variables). This distinction is conceptually of interest and may guide analysis of economic policy problems.

In the foregoing analysis, α, β and γ were assumed to be constraints in the problem. There seems to be little reason to expect these parameters to remain constant from one problem to another. Indeed, the parameters may be (stable or unstable) functions of time.[12] The government can conceivably modify these response coefficients through specific fiscal and other measures. For example, these coefficients may depend on other parameters like the aggregate rate of job acceptance (which in a dynamic "general" theory need not be identically equal to unity), the rate at which unemployed migrants in the urban sector contact prospective employers, etc. In the medium-run, the government will have a trade-off between each response coefficient and the type, volume and spatial allocation of fiscal and other programs.

There are several possible objections to the basic model of this paper and its elaboration. One such objection is that the demand side of the picture is apparently ignored. Certainly, no labor demand relations were specified, and this seems unsatisfactory.

Finally the omission of demand considerations in our basic model may be a less serious matter in economies where the government is the predominant employer, and hence demand for labor may be assumed exogenous, than in advanced economies where such assumption will be patently unacceptable.

Acknowledgements: Many thanks are due to Professor David Peterson of Northwestern University, Evanston, Illinois, for his great interest and invaluable assistance in the conception and writing of this paper. While he rightfully has a claim to whatever merit this paper may have, I am solely responsible for all the shortcomings.

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[12] See Pontryagin, et alia, "Mathematical...," p. 181 for a discussion of application of the maximum principle to non-autonomous problems.

Appendix A

The boundary conditions on the dynamic multipliers of a control problem depend on the final values specified for the state variables, and on whether the terminal time is specified or not.

In our problem, the boundary conditions on Ψ_1 and Ψ_2 at the optimal time, T*, are given by

$\Psi_1(T^*)u(T^*) = 0$,

$\Psi_1(T^*)$ is unspecified because u(T) is fixed.

$\Psi_2(T^*)d(T^*) = 0$,

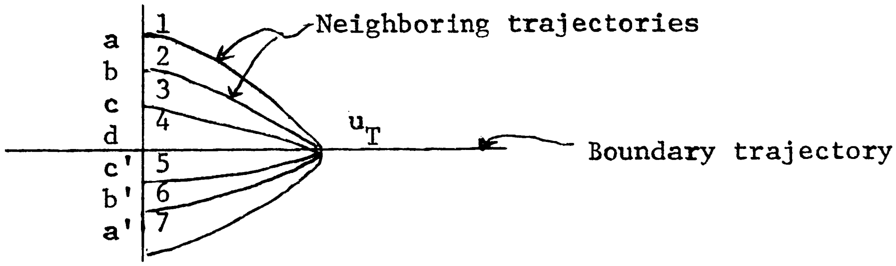
$\Psi_2(T^*) = 0$ since d(T) is free when d(T*) lies in the open interval (-1, +1).

$\Psi_2(T^*)$ may be unspecified when d(T) is in the closed interval [-1, +1], because in that case, d(T) is effectively fixed at the boundary of the interval.

Mathematically accessible treatments of the derivation of boundary conditions are rare. A clear presentation may be found in Chapter 4 of Morton M. Denn's <u>Optimization by Variational Methods</u>, McGraw Hill.

Appendix B

We now conjecture a method of proof of the proposition that the "boundary" trajectory will be preferred to "neighboring" trajectories. Such a proposition might be proved by construction.



Let curves a 1 and 7 determine the "neighborhood" of curve 4.

Consider triangles adu_T, bdu_T, Cdu_T ;

clearly (by construction) $au_T > bu_T > Cu_T > du_T$.

Hence to the extent that minimum time implies minimum distance, curve 4 will be preferred to either 1, 2 or 3. By a similar reasoning, curve 4 will be preferred to curves 5, 6 and 7. Hence we conclude that the boundary trajectory, 4, will always be preferred to those that be in its neighborhood.

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