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A Theory of Partnership Dynamics: Learning, Specific Investment, and Dissolution

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A THEORY OF PARTNERSHIP DYNAMICS:

LEARNING, SPECIFIC INVESTMENT, AND DISSOLUTION [1]

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Abstract This paper explores the benefits and drawbacks of potential partnership dissolution through an infinite–period, dynamic game–theoretic model of learning and endogenous dissolution. As partners learn about the quality of their partnership relative to their outside opportunities, the rents associated with the partnership change, effecting a related change in the strength of incentives to provide effort.

The paper develops an incentive–constrained dynamic programming algorithm for the computation of optimal symmetric equilibria of dynamic games with known worst punishments (such as dissolution here). The scheme is much simpler than the more general set–valued approach pioneered by Abreu, Pearce, and Stacchetti in that it only requires the computation of one value function at each iteration. The algorithm is then used to show that rather mild supermodularity conditions lead to effort levels in the optimal equilibria which rise in the expected quality of the partnership.

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1. INTRODUCTION

The potential for business relationships to dissolve has both benefits and drawbacks. The ability to pull out of an existing relationship provides economic actors with the flexibility to reallocate their resources to uses which they deem more productive, as in job matching models of the labor market (e.g., Jovanovic (1979)). Furthermore, the threat of discontinuing future relations with a partner who greatly values one's business may discipline the behavior of the partner (Klein and Leffler (1981), Shapiro (1983)). Both of these are important benefits of the proper functioning of any competitive economic system. However, commitment to a relationship may also be of great value. If parties to a relationship may always threaten to take their business elsewhere at no great loss, then there may be no force disciplining the behavior of the parties within their present relationship.

This paper is concerned with the tradeoffs and interactions between these benefits and drawbacks of potential partnership dissolution. In order to study these interactions, I develop a dynamic game–theoretic model of partnerships. This is a natural framework within which to study the effects of potential partnership dissolution. Many rules of business behavior are implicit and are not written into enforceable contracts. MaCaulay (1963) finds that a large proportion of business relations are conducted without recourse to the safeguards provided by the legal system through contracting. Also, numerous contributions to business partnerships have public good qualities, and incentives must be provided for partners to contribute to the general good of the partnership. Finally, the expected productivity of assets within a given relationship may vary over time relative to that in their best alternative use.

These are important determinants of the dynamics of performance of many business as well as social relationships. Industrial firms entering into a research and development joint venture, for example, may at first be optimistic about the prospects of success in a particular R&D project. The benefits of the firms' personnel, financial, equipment, and planning contributions to the partnership will be at least partially shared, yet the firms may nevertheless be motivated to provide efficient levels of these inputs because of the threat of dissolution of the valuable partnership. Were the firms to become more pessimistic about the prospects of success of the project, however, or alternatively, were they to discover outside projects that may turn out to be more productive, they may no longer be sufficiently disciplined by the threat of loss of the small rent remaining in the partnership to contribute efficiently. A purchaser/supplier relationship of an intermediate product may display similar dynamics. As the value of continuing the relationship declines, the product quality provided by the supplier may decline while the purchaser's timeliness of payments and devotion of personnel to the relationship for planning and technical expertise may falter. Lawyers within a legal partnership may too decrease their effort in bringing business into the firm as the firm's expected time horizon shortens.

The economics of the family provides another fruitful area of application of the theory I develop in this paper. In an agricultural household model, for example, we may expect to see optimal levels of work effort prevail in families without recourse to attractive modern sector employment elsewhere. On the other hand, rising city wages and/or stochastic arrival of opportunities for migration may serve to decrease the efficiency of effort on the farm. This may be a source of the apparent disruption of rural life brought about through the development of labor markets during early stages of industrialization (e.g., Polanyi (1944)). In social relations more generally, we should expect to see low levels of inputs, broadly speaking, into relationships that face probable rapid dissolution.

In each of these cases potential dissolution displays the important benefits and drawbacks noted above. Which of the effects dominates has to do with their relative magnitudes and frequencies over time. Since the expected quality of a partnership must fall in order to reap the potential gains from dissolution, the partnership will generally pass through a phase in which its performance deteriorates as a result of its likely demise.

Therefore, the more quickly partners can ascertain a partnership's true quality and the easier it is to uphold efficient effort levels, the more likely that attractive outside opportunities will be on balance beneficial.

Since poor incentives for proper performance in partnerships are associated with low rents, we may expect to find a solution to the motivational problem in partnerships in the creation of excess rents (see, e.g., Klein and Leffler (1981), Shapiro (1983), Williamson (1983)). As long as today's effort is limited by the fact that the relationship is not valuable enough to the partners to support full efficiency, *productive* relationship–specific investment will be adjusted upwards at the margin. In Section 6, I present an incentive constraint–augmented Euler equation which displays this effect. This incentive–based enhancement of investment is often strongest for partnerships expected to be poor since supportable effort there is low and may have high marginal returns.

In Section 2, I present the model which will form the basis of the analysis. The model is an infinite–period symmetric game of input (effort) provision and perfect monitoring with two dynamic elements: the option of unilateral partnership dissolution in favor of a known constant outside opportunity, and passive learning through productivity signals about the quality of the partnership. Assume that there are only two types of partnerships, good and bad; then each period players will have a common prior/posterior probability assessment $\pi \in [0,1]$ that their partnership is good. For notational simplicity and conceptual clarity, I restrict attention to this simple case, deferring until Section 6 various extensions.

In Section 3, I compute the optimal symmetric sequential equilibria (SSEs) of the game. If an optimal SSE exists, then one will exist whose equilibrium actions are stationary in the payoff–relevant state variable π; the optimal SSE value function will also be stationary. I therefore proceed with stationarity as a working assumption, and analyze value functions on the state space of probabilities.

Theorem 1 shows that optimal SSEs exist and provides a method for their computation. This proceeds as follows. Note that partnership dissolution is the worst punishment to any player, as all players can choose this outcome but may also have it imposed upon them. The importance of using the severest possible punishment to a player in supporting equilibrium behavior has been emphasized by Abreu (1988). In analyzing symmetric solutions we need only analyze a single player's strategic decisions. In the spirit of dynamic programming, I decompose the decision problem into a first period and the entire remainder of the game. Transform an arbitrary value function v on [0,1] as follows. Treat v as a promised continuation value for periods 2 onward and the dissolution value (zero) as a threatened punishment. For the first period, take the most efficient action which is incentive compatible with respect to v, that is, from which no player would like to deviate and face dissolution rather than follow the action and receive the promised reward of the expected transition–probability weighted v. Choose a starting value v_0 which is everywhere greater than the maximal one–period payoff forever, and let, for all t, v_{t+1} be the utility of the most efficient efforts incentive compatible with respect to v_t, plus the discounted v_t. Since v_0 was chosen artificially high, $v_1 \leq v_0$. By induction, $\{v_t\}$ is a decreasing sequence, and thus $\bar{v} := \lim_{t\to\infty} v_t$ exists. By a continuity argument, $\bar{v}$ is a fixed point of this transformation, and is therefore an SSE value function. A simple comparison shows that it is the optimal SSE value function.

This approach is a maximum–valued analog of the set–valued iterative approach to the computation of equilibrium values in dynamic games (Theorem 5 in Abreu, Pearce, and Stacchetti (1990)). In symmetric games with known worst punishments, using a starting value v_0 which is artificially high and threatening the worst punishment ensures us that the actions taken in the optimal SSE will be supportable at every iteration of the algorithm of the preceding paragraph, exactly as beginning with a set containing the entire equilibrium value set ensures that the entire value set will be carried to the next iteration in their work. By continuity, then, the limits are the optimal SSE value function and the entire

equilibrium value set, respectively.

The present approach is much simpler than the set–valued approach, as it relies only upon the computation of a single value function at every iteration. Also, here, it is clear exactly how the optimal equilibrium is computed through the transformation, and thus we will be able to use properties of the transformation, such as preservation of weakly increasing functions in certain situations, to analyze optimal SSEs. However, this computational simplicity and fruitfulness of analysis comes at the cost of a strong restriction in the class of games to which the algorithm applies: It applies only to symmetric games with known worst punishments. Section 3 contains details.

The introductory discussion given above of how partnership performance deteriorates with the decrease in π implicitly assumed a monotonicity result: Effort levels rise with the expected quality of the partnership. This is confirmed in Theorem 2 assuming only rather weak supermodularity conditions. For this to be the case, players must both wish to and be able to uphold a higher level of effort for higher π. Increasing differences in effort and π and the public–good nature of effort ensure that partners wish to support higher levels of effort for higher π. A monotone likelihood ratio property on the signal distributions to ensure stochastic monotonicity in the transition probabilities, the public–good payoff structure, and increasing differences in efforts and π ensure that rents rise with π. Finally, increasing differences in own efforts and the efforts of others, and the public–good nature of effort ensure that in the region in which it is difficult to support effort — above the one–period Nash levels — partners wish to put in *less* effort, and the gain from cheating from a given effort level is less for higher π by increasing differences in efforts and π. Note that the approach taken in this paper is novel in the theory of supermodular games (e.g., Topkis (1979), Milgrom and Roberts (1990), Milgrom and Shannon (1991)), for I do not restrict attention to state–space strategies; rather, my results rest as much upon the supportability of an equilibrium action as upon the nature of one–shot game best–response curves.

Because many of the economic examples of interest and theoretical insights we hope to gain do not fall within the scope of the basic framework of Sections 2–4, I provide extensions of the model in Section 5. I consider evolving outside opportunities, active learning, and relationship–specific investment in turn. I defer discussion of the results until then.

The remainder of the paper is structured as follows. Section 2 introduces the basic model under consideration. Section 3 presents the algorithm for the computation of optimal SSE value functions. In Section 4, I analyze the monotonicity of optimal SSE action functions in the expected quality of the partnership. Section 5 provides an analysis of three important economic extensions of the basic framework. A numerical example is presented in Section 6. Section 7 concludes.

2. THE MODEL

The game to be studied is a dynamic, infinite–period, symmetric partnership game of perfect monitoring. In each period of the life of the partnership, partners learn about the quality of the partnership, and each partner may either contribute effort to the partnership, or dissolve it. Players discount the future at the common discount factor of $\delta \in (0,1)$, and discount to the *end* of the first period. (First period payoffs are not discounted.) Players are interested in maximizing the expected disounted present value of supergame payoffs.

The Stage Games

There are two types of partnerships, good and bad. This quality represents something about the partnership as a whole — such as whether or not the partners enjoy working together or whether or not they are productive as a team — and does not

represent how well the partners perform in equilibrium. Partners are unaware of their partnership's true quality, and at the beginning of any period have a common prior belief of the probability that their partnership is good. Call this variable π; the stage games will then be indexed by the state variable π. Each of N players simultaneously chooses an effort level $e \in [0,\infty)$ within the partnership, or dissolves it (plays D). Let $S_n := D \cup [0,\infty)$, $n=1,\dots,N$, be the set of actions possible for the n^{th} player, and $S := \times_{n=1}^{N} S_n$ be the set of action profiles for all players. In each period players choose an action and then receive a productivity signal
$\omega \in \Omega \subseteq \mathbb{R}$ of the quality of the partnership. The probability densities over Ω for good and bad partnerships are f_g and f_b, respectively, with associated cumulative density functions F_g and F_b.

Player n has the symmetric utility function $U_n : S \times \Omega \to \mathbb{R}$ over actions and productivity. The expected utility $u_n : S \times [0,1] \to \mathbb{R}$ accruing to player n when actions $(a_1,\dots,a_N) \in S$ are played and $\pi \in [0,1]$ is the probability of a good partnership is then

$$\text{(D1)} \quad u_n(a_1,\dots,a_N;\pi) := \int_\Omega U_n(a_1,\dots,a_N;\omega)\,[\pi\, f_g(\omega) + (1-\pi)\, f_b(\omega)]\, d\omega.$$

For example, ω may be a productivity parameter, and utility may be based upon a partner's receiving a share of the joint output and some disutility of effort. Make the following assumptions so that we may interpret the relationships as being good and bad:

(A1) U_1 is weakly increasing in ω,

(A2) $F_b(\omega) \geq F_g(\omega)$, $\forall\, \omega \in \Omega$.

These imply that u_1 rises weakly in π. Finally, if anyone dissolves the partnership, the utility received by all players is normalized to zero:

(A3) $\forall\, n,\ U_n(D;a_{-1};\omega) = 0,\ \forall\ a_{-1}\in S_{-1},\ \omega\in\Omega,$

where $S_{-1} := \overset{N}{\underset{n=2}{\times}} S_n$ is the action space for all players other than player 1.

The Dynamic Game

Realizations of productivity provide information with which partners update their beliefs about the likelihood of a good partnership. If π is the prior going into a period that the partnership is good, then, by Bayes' Rule,

$$(1)\qquad \pi'(\pi;\omega) = \frac{\pi\ f_g(\omega)}{\pi\ f_g(\omega)\ +\ (1-\pi)\ f_b(\omega)}$$

is the posterior probability of a good partnership given productivity ω. Associated with this learning process is a class of induced transition probability density functions $f(\pi'|\pi)$ and cumulative density functions $F(\pi'|\pi)$.[2]

For the remainder of the paper, I restrict attention to symmetric strategy profiles. I will use the notation σ for both strategies and strategy profiles. Let u: $S_1\times[0,1]\to\mathbb{R}$ be the symmetric action utility function: $u(a;\pi) := u_1(a,\dots,a;\pi)$. Let $H^t=\Omega^t$ be the set of t-period productivity realizations. Let $H^\infty = \bigcup_{t=0}^{\infty} H^t$ be the set of all finite-period productivity realizations, including the null history $H^0 := \{\phi\}$. Then a strategy σ tells a player what to do as a function of all finite histories of previous actions of the players, and realizations of productivity. Since monitoring is perfect and all players receive the same productivity signal in any given period, the strategy σ is a sequence $\{\sigma_t\}_{t=1}^{\infty}$ of functions

[2] Most commonly-used learning distributions have two state variables, for the mean and accuracy of a posterior. However, when there are only two possible distributions then there is only one state variable, π, the probability that one of the two distributions is correct, which summarizes both the expected value and the variance of the posterior distribution. I have chosen this case for ease of illustration.

telling the player what to do after any complete t-period history: For t=1,2,..., σ_t:$S^{t-1}\times H^{t-1}\to S_1$ determines period t action as a function of history, with $S^0 := \{\phi\}$.

The strategy profile σ induces a stochastic sequence of actions for each player which is dependent only upon finite histories of realizations of productivity. In the first period, the action taken is $\sigma_1(\phi,\phi)$. Let $e := (1,\dots,1)\in\mathbb{R}^N$. After any one-period realization $\omega\in\Omega$, $a_\sigma(\omega) = \sigma_2(e\cdot\sigma_1(\phi,\phi);\omega)$; after any two-period history $h=(\omega,\omega')\in H^2$, $a_\sigma(h) = \sigma_3(e\cdot\sigma_1(\phi,\phi),e\cdot\sigma_2(e\cdot\sigma_1(\phi,\phi);\omega);h)$. Actions prescribed after longer histories are similarly inductively defined. Thus, fixing σ, the actions taken are purely a function of the realizations of productivity. Associated with the strategy profile σ is then a value function v_σ: $H^\infty\to\mathbb{R}$, which is the expected present discounted value of future utilities accruing to each player after the productivity history $h\in H^\infty$, discounted to the *end* of the period following the given history. Let $\pi(h)$ be the posterior likelihood of a good partnership after the history h. We then have the following value recursion:

$$(2)\qquad v_\sigma(h) = u(a_\sigma(h);\pi(h)) + \delta\int_\Omega v_\sigma(h,\omega)\,[\pi(h)\,f_g(\omega) + (1-\pi(h))\,f_b(\omega)]\,d\omega.$$

The value accruing to each player under σ after history h is the one-period payoff prescribed by play of the equilibrium action in the present period, plus the expected discounted future value of all future play after all possible successor histories (h,ω).

For $\pi\in[0,1]$, let $H(\pi)\subset H^\infty$ be the set of productivity histories such that, after that history, π is the posterior likelihood of a good match. If, $\forall\ \pi\in[0,1]$, $h_1,h_2\in H(\pi) \Rightarrow a_\sigma(h_1) = a_\sigma(h_2)$, then we say that σ and a_σ are *stationary*. Notice that the transition probabilities in the learning process are in a related sense stationary as well, since they only depend on π, and not on details of how π was reached. If a_σ is stationary, we may define an action function a: $[0,1]\to S_1$ which is consistent with a_σ after all histories. That is, if $h\in H(\pi)$, let $a(\pi)=a_\sigma(h)$. We may then define the associated value function v_a: $[0,1]\to\mathbb{R}$ as follows:

(3) $v_a(\pi) = u(a(\pi);\pi) + \delta \int v_a(\pi')\, f(\pi'|\pi)\, d\pi'$,

if $a(\pi)\neq D$, and $v_a(\pi) = 0$ if $a(\pi)=D$.[3] Again, this provides players with the first–period payoff $u(a(\pi);\pi)$ provided by playing the stationary action $a(\pi)$ at probability π, and expected discounted continuation value of the rest of the game $\delta \int v_a(\pi')\, f(\pi'|\pi)\, d\pi'$ for all possible posteriors π'. This will not necessarily be defined for all action functions a since I will not assume bounded payoffs; however, it will be well–defined whenever necessary in the equilibrium analysis to come.

3. COMPUTATION OF OPTIMAL SYMMETRIC SEQUENTIAL EQUILIBRIA

In this section, I devise a method for the computation of the optimal symmetric sequential equilibrium (SSE) value function; this is presented in Theorem 1. The algorithm is an incentive–constrained analog of the value iteration technique of dynamic programming (Howard (1960)). The algorithm performs a transformation only upon a single value function at every iteration, rather than upon an entire value set as in Abreu, Pearce, and Stacchetti (1990) (APS). The algorithm also extends the APS algorithm for purely repeated games to dynamic games with a state variable, an extension that has been shown to be perfectly straightforward (Atkeson (1991)).

I begin with the intuition that an optimal SSE σ is likely to be stationary. I therefore proceed with stationarity as a working assumption, and analyze value and action

[3] We must make the exception for $a(\pi)=D$ for the stationary value function but not for the full productivity realization–dependent value function. When we keep track of full histories, whenever a partner dissolves at some point, she has therefore dissolved at all successor histories, and thus the future expected utility stream corresponds to being out of the partnership. However, when just posterior probabilities of a good relationship are followed, updates may occur through f which represent learning, even though the partnership has dissolved.

functions on the state space of probabilities [0,1]. From principles of optimality of dynamic programming, it seems natural that only payoff–relevant aspects of history will affect *optimal* equilibrium play. This intuition is formally confirmed in Theorem 1, which shows that any arbitrary (nonstationary) SSE value function is bounded above by the stationary SSE value function computed by the algorithm.

Let $A := \{a\colon [0,1] \to S_1\}$ be the set of all action functions. For $a\in A$ to be played on the equilibrium path of a symmetric sequential equilibrium, it must induce a value function v_a which satisfies incentive compatibility requirements. For analysis of incentive compatibility, let us make the following assumptions and definitions:

(A4) $\forall\, \pi,\ \lim_{e\to\infty} u(e;\pi) = -\infty$,

(A5) U_1 is nondecreasing in e_{-1},

(A6) U_1 is continuous in efforts.

(D2) $u^*(e;\pi) := \max_{e'} \{u_1(e',e,\ldots,e;\pi)\}$,

(D3) $d(e;\pi) := u^*(e;\pi) - u(e;\pi)$.

Assumptions (A4)–(A6) ensure that u^*, d are well–defined and continuous. A number of different sets of assumptions would suffice as well; this set is chosen for its interpretation within the model at hand, where we want to interpret effort as a public good (implied by (A5)) and for there to be efficient levels of effort for each π ((A4), (A6)). I will hereafter denote an arbitrary symmetric action profile, including the possibility of dissolution, as a, and an effort level specifically as e. Here $u^*(e;\pi)$ is the total utility received by deviating from effort level e when the state variable is π; $d(e;\pi)$ is the gain from cheating from the symmetric action profile in which effort level e is chosen by all partners at prior π. Let $G(\pi)$ be the stage game at probability π with domain restricted to effort levels. Let $e^*(e;\pi)$

be the best response correspondence in $G(\pi)$ for any player from the symmetric action profile in which all players play e; let $\bar{e}^*(e;\pi)$ be its maximal element. Again, (A4)–(A6) ensure that $\bar{e}^*(e;\pi)$ is compact and nonempty, $\forall\ e,\ \pi$

Let us make the following further definitions. Let $C=\{v:[0,1]\to\mathbb{R}\}$ be the set of all value functions. Then:

(D4) For $v\in C$, an effort level e is *incentive compatible with respect to v at π* if

$$d(e;\pi) \leq \delta \int v(\pi')\, f(\pi'|\pi)\, d\pi' .$$

(D5) $IC(v)(\pi) := \{e\in[0,\infty) \mid e \text{ is incentive compatible with respect to } v \text{ at } \pi\}$

Then an effort level is incentive compatible with respect to a value function at π if, promised its discounted expected value tomorrow for following the action and threatened with partnership dissolution, any player would rather put in that level of effort than shirk. (A4)–(A6) imply that $IC(v)(\pi)$ is compact, $\forall\ \pi$, as the gain from cheating is continuous and rises without bound as e increases. Notice that (A4)–(A6) ensure that there will exist some effort level $\bar{e}$ such that no effort level above $\bar{e}$ will ever be played in an SSE, for short–term losses could never be recouped (see, e.g., Abreu (1986)). Since SSE action functions will always lie on $S_1\backslash(\bar{e},\infty)$, the corresponding one–period payoffs possible in (1) will be bounded, and thus there will be a unique solution v_a for any possible stationary SSE action function a.

We can now define the set of stationary SSE value functions as follows:

(D6) $V := \{v\in C \mid \exists\ a\in A \text{ s.t. } v=v_a, \text{ and, } \forall\ \pi,\ a(\pi)\in[0,\infty) \Rightarrow (v(\pi) \geq 0 \text{ and } a(\pi)\in IC(v)(\pi)\ \}$.

For a stationary value function to be an SSE value function, it must come from some stationary action function and must satisfy incentive compatibility requirements. Players can deviate in two ways, leaving the partnership or shirking, from a prescribed equilibrium effort level. The value of following the prescribed equilibrium $v(\pi)$ must therefore be positive; it also must be incentive compatible with respect to the value function itself. Implicit is that dissolution is the severest SSE punishment to any partner, and that therefore any equilibrium actions can be upheld by the threat of dissolution (Abreu (1988)); this will be true as long as anybody can dissolve the partnership unilaterally at any time. We therefore also do not need an incentive compatibility condition for dissolving the partnership in the definition of V.

Now let us shift attention to optimal SSEs. The following is an expository outline of the issues addressed in Theorem 1, whose formal proof appears in the Appendix. Suppose $v^*: [0,1]\to\mathbb{R}$ is the optimal SSE value function; anticipating Theorem 1, we can say without loss of generality that there exists a stationary SSE value function which is optimal in the class of all SSEs after all histories. v^* must satisfy both the recursive and incentive compatibility requirements which make it a member of V. But it must also be derived from an action function which at all $\pi\in[0,1]$ is the *best* supportable action: It will never pay to play an action in a given period when better incentive compatible actions are available. This motivates the following definition of the transformation B: $C\to C$:

(D7) $$Bv(\pi) := \max\ \{\ 0,\ \max_{e\in IC(v)(\pi)} u(e;\pi) + \delta \int v(\pi')\, f(\pi'|\pi)\, d\pi'\ \}.$$

This mapping chooses the best of the values obtainable by playing the actions in the present period, with continuation values v, subject to the incentive compatibility constraints applied to the continuation values. Since $IC(v)(\pi)$ is compact and u continuous, B is well–defined.

The optimal SSE value function v^* must be a fixed point of B: At each value of π, v^* must be equal to the value of the best action which is incentive compatible with respect to v^* itself, plus the discounted, transition–weighted continuation value of v^*. To compute the optimal SSE value function, we would then like to have a method of finding fixed points of B. But since B is not a contraction mapping, due to the incentive constraints, it may have multiple fixed points. If an iterative procedure based upon the mapping converges, the point to which it converges may be sensitive to the starting point.[4]

Nonetheless, we can ensure that an iterative procedure *will* converge to the *optimal* SSE value function if the actions supported in the optimal SSE are supported at every step of the iteration; they will then be supported in the limit. If the promised continuation value function, therefore, is at least as great everywhere as the optimal SSE value function, it will provide *both* at least the value of the optimal equilbrium actions today, since optimal SSE actions themselves are supported, and a continuation value higher than the optimal SSE value itself. Thus, the property of being uniformly greater than or equal to the optimal SSE is preserved by the mapping. Note that (A4) and the definition of u imply that there exists $M\in\mathbb{R}$ such that $u(e;\pi) \leq M$, for all e, π. If we choose the starting point of our iterative procedure to be equal to $M/(1-\delta)$ everywhere, then $v_{t+1} := Bv_t \leq v_t$, $\forall\, t$, by induction, and thus the point–by point limit $\bar{v} := \lim_{t\to\infty} v_t$ exists.

This limit is a fixed point of B by continuity. And since the optimal SSE actions are incentive compatible at each iteration, they will be in the limit as well. Any fixed point of B is an SSE value function, as we can simply read off supportable actions which give the value function.

Finally, there is no nonstationary equilibrium value function on the space of all

[4] The following is an intuitive example of multiple fixed points of B. Consider the infinitely repeated prisoners' dilemma game. Finking forever gives the worst symmetric equilibrium payoff. If we were to use the value of shirking forever as a promised continuation and threatened punishment, then only finking will be incentive compatible. Thus the transformed value function still equals the value of finking forever; this is a fixed point of B. However, were we to start with the value of playing mum forever as the promised continuation, then, if it is in fact supportable in the best equilibrium, it will be incentive compatible, and the transformed value will again equal the value of mum forever, and this will be a fixed point as well.

histories of productivity which is anywhere better than the stationary value function $\bar{v}$. It is in this strong sense that we say that $\bar{v}$ is the optimal SSE value function. This follows since any SSE value function v' on the extended state space of full productivity histories will be less than or equal to the payoff from the *best* supportable action plus v' as continuation, since it must come from *some* supportable action and itself as continuation. So on the extended state space, $B^h v' \geq v'$; therefore $\bar{v}' := \lim_{t\to\infty} (B^h)^t v'$ exists. (B^h is the natural extension of B to functions on the space of all productivity histories.) But by choice of v_0, $v' \leq v_0$; the same weak inequality will hold in the limit of the procedure. Therefore $\bar{v} \geq \bar{v}' \geq v'$. Theorem 1 summarizes these results:

__Theorem 1__: *Assume* (A3)–(A6), *set* $v_0 := M/(1-\delta)$ *everywhere, and let* $v_{t+1} := Bv_t$, $t\geq 0$. *Then:*

(a) $\bar{v} := \lim_{t\to\infty} v_t$ *exists,*

(b) $\bar{v} = B\bar{v}$,

(c) $\bar{v} \in V$, *and*

(d) *suppose σ is an SSE. Then,* $\forall\, \pi$, $\forall\, h\in H(\pi)$, $v_\sigma(h) \leq \bar{v}(\pi)$.

The proof follows the lines drawn above and appears in the Appendix.

To reiterate, this is a maximum–valued analog of the set–valued approach to computation of equilibrium values pioneered by Abreu, Pearce, and Stacchetti (1986, 1990). Their method generalizes to all finite or compact–continuous games of perfect monitoring, and finite–move constant–support games of imperfect monitoring.

I have focussed on symmetric dynamic games with known worst punishments not only due to the nature of the economic problem under study: No significant generalization is possible. Such simplified computation is clearly ruled out for the conputation of optimal asymmetric equilibria, for the entire Pareto frontier of the equilibrium value set must be computed generally. Games with unknown worst punishments are similarly troublesome, as worst punishments are generally asymmetric and have continuation values which are not extremal points of the equilibrium set. Finally, and most interestingly in terms of its potential for related research, imperfect monitoring is ruled out. In the present setup under imperfect monitoring, deviations from equilibrium behavior provide the deviator with a posterior assessment different from that held by all others. The value of deviating must take this into account. I know of no research yet along these lines.

4. MONOTONICITY

We might expect any optimal equilibrium action function $\bar{a}$ which induces the value function $\bar{v}$ to have certain monotonicity properties. With the expected quality of the partnership should rise rents and therefore effort levels. Partnerships very likely to be bad are those that should dissolve. Monotonicity is confirmed in Theorem 2: Effort levels rise with π, and the criterion for dissolution is a simple trigger rule in π.

In order to derive the most general monotonicity result possible, I utilize the theory of supermodular games. For an extremely thorough and clear introduction to supermodular optimization and supermodular games and their implications for monotonicity, see Milgrom and Roberts (1990) and Milgrom and Shannon (1991). For monotonicity, I make assumptions which ensure both that players would like to be able to support higher effort levels at higher values of π, and that they are in fact able to. Let X be a lattice and T a partially ordered set. Then we say a function $f\colon X \times T \to \mathbb{R}$ has *increasing differences* in (x,t) if for $x' \geq x$, $f(x',t) - f(x,t)$ is nondecreasing in t. I will henceforth use the descriptive term "increasing" to mean nondecreasing. Make the following assumptions:

(A7) $f_g(\omega)/f_b(\omega)$ is increasing in ω,

(A8) U_1 has increasing differences in (e_1,e_{-1}) for fixed ω,

(A9) U_1 has increasing differences in $((e_1,\dots,e_N),\omega)$.

These are natural assumptions for the partnership model. Assumption (A7) ensures that high productivity not only brings high payoffs in the given period (along with (A1)), but also that it is "good news" concerning the likely quality of the partnership, in the sense of Milgrom (1981). For (A8), (A9), consider the following interpretation of single–period utility: Each partner shares equally in the joint output of the partnership, but dislikes providing effort. Then (A8) can be interpreted as each partner's marginal productivity rising in the efforts of others, and/or as her marginal disutility of effort falling with the effort put in by others. (A9) will hold for many functional forms for production where ω is simply a parameter. For example, if the production function were Cobb–Douglas with ω as a multiplicative parameter, (A8) and (A9) would hold. However, these assumptions are at odds with the interesting case where, once partners find themselves to be very successful, they would like to rest.

Notice that (A7)–(A9) imply the following, whose proof is straightforward and appears in the Appendix:

<u>Proposition 2.1</u>: *Assume* (A7)–(A9). *Then:*

(a) u_1 *has increasing differences in* (e_1,e_{-1}) *for fixed* π,

(b) u_1 *has increasing differences in* $((e_1,\dots,e_N),\pi)$,

(c) u *has increasing differences in* (e,π).

For monotonicity of actions with respect to a state variable to hold in best Nash equilibria of a class of one–period supermodular games with increasing differences, we need that the direction of desired action correspond with the direction of increasing differences.

Recall the stage game in efforts $G(\pi) := (\{S_n \backslash D\}_{n=1}^{N};\{u_n(\ ;\pi)\}_{n=1}^{N})$. By Proposition 2.1 (a) and continuity of u (from (A6)), each of these games is supermodular on any complete lattice subset of its domain, for example, any n–fold Cartesian product of a compact subset of $\mathbb{R}$. Since it is a game of public–good provision (by (A5)) and utilities fall off with high enough effort (A4), we can bound from above the possible symmetric one–shot Nash effort levels. Therefore, each $G(\pi)$ has a greatest Nash equilibrium (Milgrom and Roberts (1990, Theorem 5)). Since each partner's utility rises in the efforts of others, the greatest Nash equilibrium is also the best; see the welfare results presented as Theorem 7 by Milgrom and Roberts (1990) and Theorem 17 by Milgrom and Shannon (1991). Finally, Proposition 2.1 (b) ensures that the highest Nash effort levels are increasing in π (Milgrom and Roberts (1990, Corollary to Theorem 6)). Therefore, the best Nash equilibrium effort levels of the games $G(\pi)$ rise with π.

For monotonicity of actions in the state variable in optimal SSEs of the full supergame, much more is required. The analysis will no longer be based solely upon Nash levels of the one–shot games, but also upon how much better the partners can do due to the incentive power of the partnership. The proof of Theorem 2 presented in the Appendix proceeds as follows: Consider the solution $a_t : [0,1] \to S_1$ to the incentive–constrained maximization problem in the mapping B given value function v_t. If we can show that an increasing v_t gives an increasing policy function a_t and an increasing value $v_{t+1} := Bv_t$, then the limit policy function will be increasing, and we will be done.

Proposition 2.1 (c) and the public good nature of effort (A5) ensure that the players would like to uphold more effort at higher values of π. Are they able to? They will be if (1) the rents are higher and (2) the gain from deviation is less for a particular desirable action when the partnership is more likely to be good. Assumption (A7) gives increasing rents in π for an increasing v_t: It ensures that the transition denstities $f(\pi' | \pi)$ are stochastically increasing in π, and this integrated over the increasing v_t must be increasing in π (Ross (1983, p. 154)). The supermodularity of the stage games $G(\pi)$ when restricted to compact domains, the negative eventual returns to effort (A4), and the public good assumption (A5) ensure that above the highest one–shot Nash effort level, the best response functions lie strictly below the diagonal; that is, partners wish to cheat by providing less effort rather than more. This follows since best response functions are increasing in π by Topkis' Monotonicity Theorem (Topkis (1978)), they eventually must fall below the diagonal ((A4), (A5)), and thus, by Tarski's Fixed Point Theorem would somewhere fall on the diagonal. But this would be another Nash equilbrium, a contradiction. Thus, in the region of primary interest for choosing good effort levels — above the one–period Nash equilibria — the best responses are always to put in less effort. But then any supportable action at a low π will also be supportable at a higher π, by Proposition 2.1 (b).

We therefore get that the players would both like to uphold higher effort levels and are able to at higher π. The one–period utility will thus rise with π (by (A1), (A7)) as does the continuation. Therefore, if v_t is increasing, then a_t and v_{t+1} are as well, since not only are supportable effort levels rising, but dissolution will only be chosen in the lowest range of expected qualities where the best effort level would give a negative value function v_{t+1}. Choosing v_0 increasing, then, ensures that the limit value function $\bar{v}$ and the limit policy $\bar{a}$ are increasing, and we have:

<u>Theorem 2</u>: *Assume* (A1), (A3)–(A9). *Then $\bar{v}$ is increasing, and there exists an increasing SSE action function $\bar{a}:[0,1] \to S_1$ such that $v_{\bar{a}} = \bar{v}$, that is, the associated SSE is optimal.*

The proof follows the argument laid out above, and is included in the Appendix.

5. EXTENSIONS AND APPLICATIONS

In this section, I alter and extend the basic framework of Sections 2–4 in order to address a number of important related economic issues. I analyze evolving outside opportunities, active learning, patience, and relationship–specific investment in turn.

(1) *Evolving Outside Opportunities.* The rents in a relationship often change not through learning about its productivity, but rather through the changing value of the alternative use of assets. Suppose that the value of activity within a partnership is not changing over time, but rather that the value of dissolution is. Let $u: \mathbb{R}_+ \to \mathbb{R}$ be the single period symmetric utility function in effort, and let z, rather than 0, be the value of dissolution, which evolves according to a first–order Markov process with transition functions $f(z'|z)$ which are stochastically increasing in z. Then the computational approach taken in Theorem 1 applies directly as long as dissolution is an available option and is the worst punishment. Here it is no longer quite as natural to assume that no further learning will occur outside of the present partnership and therefore that once the partnership dissolves it will never start back up. Permanent dissolution will lie on some equilibrium paths, but generally not on the optimal path. Still, in certain situations, transactions costs may preclude the reinitiation of a partnership.

An analog of Theorem 2 will also hold. Here, rents fall with z even though the value function, including the exit option, may rise. The public–good quality of effort will still be important for the result, but the interactions between the state variable and marginal utility of effort will no longer play a role; there will be no corresponding increasing differences assumption such as (A9).

(2) *Active Learning.* Accurate information about the true productivity of assets in their various uses may be extremely valuable. It is generally when partners are most unsure about the quality of their partnership that it will be most worthwhile to expend resources on learning. Here, when π is close to 1/2 its variance is greatest. If the speed of learning is positively related to the amount of effort provided, then effort will be increased vis–a–vis the passive learning model where the most can be learned. Thus effort will

generally not be monotonic in expected partnership quality, for, although rents will still generally rise with π, the desired level of effort will not.

In the passive learning model, the transition probabilities representing learning are independent of the present period's action. The analysis extends straightforwardly to the more general case of dependence: If the density f is continuous in e, then $IC(v)(\pi) := \{e|\ d(e;\pi) \leq \delta \int v(\pi')\, f(\pi'|\pi;e)\, d\pi'\ \}$ will still be compact and thus B well–defined.

(3) *Patience.* Consider the behavior of optimal equilibria as the discount factor approahes 1. The option value of remaining in a potentially valuable partnership becomes positive for lower and lower values of π. Therefore, the partners must become arbitrarily pessimistic about the partnership before dissolving in the case of complete patience, and they will generally be able to support fully efficient effort. A previous version of this paper contains an appropriate "folk–theorem" result (Roth (1992)).

(4) *Relationship–Specific Investment.* Specific investment may help ameliorate incentive problems in partnerships by providing quasi–rents which the partners would have to forfeit were they to leave the partnership. Let specific capital k be a state variable which provides higher productivity of effort, and let investment i be a control. Let the single–period symmetric utility and deviation value now be a function of $(e;i;k;\pi)$. In a similar model of a firm's accumulation problem with a shut–down option, Pakes (1991) has shown that if the transition probabilities are stochastically increasing and certain restricted supermodularity conditions hold, then investment will rise with the stochastic state.

However, when the lack of sufficient rents constrain effort, it may be that at lower values of π the incentive value of investment is greatest. This is apparent in the example provided in Section 7: At the highest probabilities of a good partnership, where there are no incentive problems, no excess investment is necessary. In some intermediate range, commitment to the relationship is valuable for incentive reasons, and investment is undertaken even though its own direct productivity will not cover its costs and the project is in this respect of negative value. Finally, the investment is not undertaken in a partnership most likely to be of low quality, for the expected productive horizon of the capital is short enough to offset any positive incentive gains.

The derivation of an incentive constraint–augmented Euler equation is straightforward. Pakes (1991) shows that traditional variational arguments are still valid in the case of mixed continuous and discrete controls. Here, for investment to have an incentive effect the inequality in the definition of incentive compatibility must be holding with equality. Optimal effort is, through this equality, defined implicitly in terms of capital tomorrow, which is controlled by investment today.

Let $(\bar{e};\bar{i})$ be optimal effort and investment levels given today's capital stock and probability $(k;\pi)$, and $\bar{e}$, $\bar{i}$ be the optimal effort and investment functions more generally. Let tomorrow's rents be $r(k+\bar{i};\pi) := \delta \int \bar{v}(k+\bar{i};\pi')\, f(\pi'|\pi)\, d\pi'$. Let $\chi(k;\pi)$ be the indicator function which takes on the value 1 if the partnership continues at $(k;\pi)$ and 0 if it dissolves. Then the following must hold, if all terms are well–defined:

$$-\,\partial u(\bar{e};\bar{i};k;\pi)/\partial i = \partial u(\bar{e};\bar{i};k;\pi)/\partial e \,\frac{\partial r(k+\bar{i};\pi)/\partial k}{\partial d(\bar{e};\bar{i};k;\pi)/\partial e} + \delta \int \chi(k+\bar{i};\pi')$$
$$[\partial u(e(k+\bar{i};\pi');i(k+\bar{i};\pi');k+\bar{i};\pi')/\partial k - \partial u(e(k+\bar{i};\pi');i(k+\bar{i};\pi');k+\bar{i};\pi')/\partial i\,]\; f(\pi'|\pi)\; d\pi' .$$

This is the usual Euler equation for investment when a firm may shut down, augmented by an incentive term; it equates the marginal cost of investment to its marginal benefit. Notice that if $\partial u(\bar{e};\bar{i};k;\pi)/\partial e = 0$, the incentive term drops out, as we must then be at the efficient levels of effort today and do not need to increase rents for incentive effects. If $\partial u(\bar{e};\bar{i};k;\pi)/\partial e \neq 0$, it must be that $\partial u(\bar{e};\bar{i};k;\pi)/\partial e > 0$, for by Lemma 4 to Theorem 2 (in the Appendix), effort will never be on a decreasing portion of the symmetric utility function $u(\ ;\pi)$. The incentive compatibility constraint is thus binding, and any increase in rents will increase optimal effort accordingly. In this way, in a neighborhood of the optimum, the effort level that can be upheld is defined implicitly by the incentive constraint in i.

Therefore, investment has a marginal benefit through incentives of the marginal utility of effort $\partial u/\partial e$ times the increase in rents $\partial r/\partial i$ divided by the amount of extra rents needed for a unit of extra effort $\partial d/\partial e$.

6. AN EXAMPLE

In this section, I provide an example to illustrate a number of issues brought up in the preceding analysis. Consider the following 2–person, 2–move, 2–productivity realization (discrete) version of the game. There are two possible effort levels, so $S_1=\{e_1, e_2, D\}$. Productivity may be either high (h) or low (l), providing the players each with k>0 or 0 units of utility, respectively. In a good partnership, high productivity occurs with probability P_g , in a bad partnership, P_b , with $P_g>P_b$. Effort level e_1 is higher than level e_2, and provides higher utility; however, e_2 is Nash in $G(\pi)$, $\forall\ \pi$. I have computed the optimal SSE value function for this model with the following parameters:

δ=.9	(discount factor),
$U_1(e_1,e_1;l)$=–4.5	(payoff of high effort and low producitivity),
$U_1(e_2,e_2;l)$=–7.5	(payoff of low effort and low productivity),
k=10	(extra payoff to high productivity),
P_g=.7	(prob. of high productivity for a good partnership),
P_b=.3	(prob. of high productivity for a bad partnership),
$d(e_1)$=10	(gain from cheating from action e_1; independent of π),
$d(e_2)$=0	(gain from cheating from action e_2; independent of π).

I have followed the computational procedure described in Theorem 1 and have plotted the optimal SSE value function $\bar{v}$ in Figure 1. Players play e_1 in optimal SSEs inducing $\bar{v}$ as value function whenever $\pi \geq \bar{\pi}$, play e_2 when $\bar{\pi} > \pi \geq \underline{\pi}$, and leave the relationship when $\pi < \underline{\pi}$. (Find $\underline{\pi}$, $\bar{\pi}$ on the horizontal axis in Figure 1.) The jump of 3 units at $\pi = \bar{\pi}$ represents the increase in supergame payoff based upon today's gain in utility from putting in high effort rather than low. The diagonal dashed line is the optimal SSE value function v' for the same game without the option of dissolution; there, cooperation is upheld perfectly everywhere.

Starting from any initial probability of a good match π, players would be better off with the potential of dissolution than without if $\bar{v}(\pi) > v'(\pi)$. For the example chosen, there is a substantial region for which this is not true: For all priors $\pi > \pi^*$, partners are better off without outside opportunities than with. Notice that it is not only in regions in which low effort is provided that $\bar{v}(\pi) < v'(\pi)$. Even where high effort is provided, it is significantly more immanent and likely that the partnership will fall into noncooperation and lose utility than that it will fall so low as to gain from dissolution, that outside opportunities are detrimental on balance.

In Figure 2, I plot a family of best SSE value functions using all but one of the same parameter values as for Figure 1, varying only $d(e_1)$, the value of cheating from the cooperative action e_1. Notice that when $d(e_1)=0$, cooperation is upheld perfectly at all π for which players stay in the relationship. For sufficiently small values of cheating, good outside opportunities are always preferable.

In Figure 1, it is clear that partners would be willing to pay up to the difference $v'(\pi) - \bar{v}(\pi)$ in order to perfectly commit to the partnership, and receive the perfect cooperation value without the potential of dissolution thereafter. We may interpret this in terms of specific capital investment as well; see Section 6, subsection 3. I have computed the optimal SSE value function for the game of Figure 1 with the possibility of investment in a unit of assets that are so valuable to the partnership that with the assets in place perfect cooperation will be upheld everywhere and the partnership will never dissolve. However, the direct discounted value of the stream of production provided by the assets

(*without* incentive effects) is below their cost. Partners only invest in the assets, therefore, if their incentive effects more than offset their extra cost.

Figure 3 plots the value function for this game, and notes where the investment will and will not take place. Investment does not take place at highest likelihoods of a good partnership, as no extra incentives are there needed. Nor will partners invest if their partnership will very likely be of poor quality. Only where the relationship is both rather likely to be good and extra incentives are needed will the partners invest. Although this is a discrete model and thus the Euler equation of Section 6, subsection 3 does not apply, the primary determinants of investment are here the same.

7. CONCLUSION

This paper has been concerned with the analysis of the effects of learning about the quality of a partnership in relation to outside opportunities on the level of work effort within the partnership, relationship-specific investment, and dissolution. It developed an iterative procedure suggested by dynamic programming and by previous recursive formulations of equilibrium in dynamic games (Abreu, Pearce, and Stacchetti (1986, 1990)), for the computation and characterization of optimal SSEs. Optimal SSEs exist, can be readily constructed, and can be characterized.

Efficient effort levels are best upheld in partnerships most likely to be of high quality. Partnerships that dissolve will pass through a phase of poor performance prior to dissolution. How protracted is this period of poor performance depends primarily on the speed of the learning process.

The potential for partnership dissolution may or may not be beneficial on balance. Although I have not mentioned potential policy implications of the analysis, a word may here be in order. In situations where incentives for proper performance are drastically reduced by the potential of partnership dissolution, policymakers may consider imposition of commitment devices. For example, if we could tax dissolution and return the proceeds lump–sum to the population, the present analysis would generally recommend a nonzero tax. However, the remarkable diversity of business partnerships and difficulty of measuring the effects across types of arrangement, as well as the efficiacy of reputation–based sanctions within the business community itself, may be important reasons that such taxes and related policies are rare. The policymaker's lack of knowledge of the true parameters of the model seem potentially quite acute here. However, divorce laws may be interpreted as commitment devices to partnerships for the purpose of improving performance. Perhaps the parameters more consistently call for the imposition of commitment in marriage than in business relationships.

Finally, although the class of dynamic games to which the simplified computational scheme suggested here applies is quite limited, a number of important economic problems may be analyzed. Examples which have appeared in the literature include Bertrand oligopoly (Rotemberg and Saloner (1986), Haltiwanger and Harrington (1991), Kandori (1991)), and extraction of common–property resources (Levhari and Mirman (1980), Sundaram (1989), and Benhabib and Radner (1991)).

APPENDIX

Proof of Theorem 1

(a) I will show that $\{v_t\}$ is a decreasing sequence; (a) then follows immediately. The proof proceeds by induction on t. Suppose $v_{t+1} \leq v_t$; we wish to show that $v_{t+2} \leq v_{t+1}$. But:

$$v_{t+2}(\pi) = Bv_{t+1}(\pi)$$

$$\begin{aligned}
&= \max\{0, \max_{e \in IC(v_{t+1})(\pi)} u(e;\pi) + \delta \int v_{t+1}(\pi')\, f(\pi' \mid \pi)\, d\pi' \} \\
&\leq \max\{0, \max_{e \in IC(v_t)(\pi)} u(e;\pi) + \delta \int v_{t+1}(\pi')\, f(\pi' \mid \pi)\, d\pi' \} \\
&\leq \max\{0, \max_{e \in IC(v_t)(\pi)} u(e;\pi) + \delta \int v_t(\pi')\, f(\pi' \mid \pi)\, d\pi' \} \\
&= Bv_t = v_{t+1},
\end{aligned}$$

where the first weak inequality holds since $v_{t+1} \leq v_t$ implies $IC(v_{t+1})(\pi) \subseteq IC(v_t)(\pi)$, $\forall\, \pi$. Since v_0 was chosen artificially high, $v_1 \leq v_0$. Thus, $v_{t+1} \leq v_t$, $\forall\, t$.

(b) We must check the following two cases separately: (1) $IC(\bar{v})(\pi) = \emptyset$, and(2) $IC(\bar{v})(\pi) \neq \emptyset$. If $IC(\bar{v})(\pi) = \emptyset$, then $\exists\, T$ s.t. $\forall\, t \geq T$, $IC(v_t)(\pi) = \emptyset$, since $\{IC(v_t)(\pi)\}$ is a decreasing sequence of compact sets. Thus, for $t > T$, $v_t(\pi)=0$, and therefore $\bar{v}(\pi)=0=B(\bar{v})(\pi)$. If $IC(\bar{v})(\pi) \neq \emptyset$, then

$$\begin{aligned}
B(\bar{v}) &= \max\{0, \max_{e \in IC(\bar{v})(\pi)} u(e;\pi) + \delta \int \bar{v}(\pi')\, f(\pi' \mid \pi)\, d\pi' \} \\
&= \max\{0, \max_{e \in IC(\lim_{t\to\infty} v_t)(\pi)} u(e;\pi) + \delta \int \lim_{t\to\infty} v_t(\pi')\, f(\pi' \mid \pi)\, d\pi' \}
\end{aligned}$$

by definition of $\bar{v}$,

$$= \max\{0, \lim_{t\to\infty} \max_{e \in IC(v_t)(\pi)} u(e;\pi) + \delta \lim_{t\to\infty} \int v_t(\pi')\, f(\pi' \mid \pi)\, d\pi' \}$$

by interchanging limits and integrals, by the continuity of u, and since $\{IC(v_t)(\pi)\}$ is a decreasing sequence of compact, nonempty sets,

$$\begin{aligned}
&= \lim_{t\to\infty} \max\{0, \max_{e \in IC(v_t)(\pi)} u(e;\pi) + \delta \int v_t(\pi')\, f(\pi' \mid \pi)\, d\pi' \} \\
&= \lim_{t\to\infty} B(v_t)(\pi) \\
&= \bar{v}(\pi).
\end{aligned}$$

(c) Since $\bar{v} = B\bar{v}$, we can define the optimal SSE action function $\bar{a}\in A$ as follows: If $\bar{v}(\pi) = 0$, set $\bar{a}(\pi) = D$; if not, set $\bar{a}(\pi) = e$, where $e\in IC(v)(\pi)$ and and $\bar{v}(\pi) = u(e;\pi) + \delta \int \bar{v}(\pi')\, f(\pi' | \pi)\, d\pi'$. Such an e exists. Then $\bar{v} = v_{\bar{a}}$ as in the definition of V, and the incentive compatibility requirements are satisfied, since $\bar{a}(\pi) = e$ only when $\bar{v}(\pi) \geq 0$ and $\bar{v}(\pi) \geq u^{*}(e;\pi)$.

(d) Define the following extension B^h: $C^h \to C^h$ of the transformation function B, where $C^h := \{v\colon H^{\infty} \to \mathbb{R}\}$: For $h\in H(\pi)$, let

$$B^h v(h) := \max\{\, 0, \max_{e\in IC(v)(h)} u(e;\pi(h)) + \delta \int_{\Omega} v(h,\omega)[\pi(h)\, f_g(\omega) + (1-\pi(h))\, f_b(\omega)\,]\, d\omega \,\},$$

where $IC(v)(h)$ is the natural extension to the space of full histories of productivity realizations. The SSE σ induces an extended value function $v_{\sigma}\in C^h$ which is the expected utility accruing to each player after each history $h\in H^{\infty}$. Now, B^h can be used to compare any SSE value function on the extended domain H^{∞}, to $\bar{v}$. The main point of this proof is to show that, since v_{σ} is an SSE value function, $B^h v_{\sigma} \geq v_{\sigma}$. The comparison is straightforward: For all $h\in H^{\infty}$, $B^h v_{\sigma}(h) \geq v_{\sigma}(h)$, since σ is an SSE and therefore the actions taken after each history are incentive compatible with respect to v_{σ}, and therefore are available to be chosen from in the maximand for $B^h v_{\sigma}$.

Now, since $B^h v_{\sigma} \geq v_{\sigma}$, by the same argument as found in the proof of part (a), $(B^h)^{t+1} v_{\sigma} \geq (B^h)^t v_{\sigma}$, $\forall\, t\geq 0$. Thus, $\lim_{t\to\infty} (B^h)^t v_{\sigma} = \bar{v}_{\sigma}$ exists, and $\bar{v}_{\sigma} \geq v_{\sigma}$. Extend v_0 to the state space H^{∞} as follows: Set $v_0^h(h) = M/(1-\delta)$, $\forall\, h\in H^{\infty}$. Then by construction, $v_0^h \geq v_{\sigma}$ everywhere on H^{∞}. Thus, since B^h is weakly increasing, $B^h v_0^h \geq B^h v_{\sigma}$. Continuing in this fashion, $(B^h)^t v_0^h \geq (B^h)^t v_{\sigma}$, $\forall\, t\geq 0$. Thus, since the limits exist, the same weak inequality holds in the limit. Thus $\bar{v}^h := \lim_{t\to\infty} (B^h)^t v_0^h \geq \bar{v}_{\sigma}$. But $\bar{v}_{\sigma} \geq v_{\sigma}$. Thus $\bar{v}^h \geq v_{\sigma}$. But, $\forall\, \pi\in\Pi$, if $h\in H(\pi)$, then $\bar{v}^h(h) = \bar{v}(\pi)$, because at every step of the iteration, the same computation was made for each, since only π is payoff-relevant. Thus, $\forall\, \pi\in[0,1]$, if $h\in H(\pi)$, then $v_{\sigma}(h) \leq \bar{v}(\pi)$. *Q.E.D.*

Proof of Proposition 2.1

(a) This follows directly from (A8) and the definition of u.

(b) Let $(e_1,\dots,e_N) \leq (e_1',\dots,e_N')$. Since $u_1(e_1',\dots,e_N';\pi) - u_1(e_1,\dots,e_N;\pi) = \int [U_1(e_1',\dots,e_N';\omega) - U_1(e_1,\dots,e_N;\omega)]\, [\pi\, f_g(\omega) + (1-\pi)\, f_b(\omega)]\, d\omega$, this difference is an integral of an increasing function (by (A9)) weighted by densities which are first-order stochastically increasing in π (by (A7)), and must therefore be increasing in π.

(c) This follows directly from part (b).

Proof of Theorem 2

The proof shows that if the value function v is increasing:

(1) there is a best action function a supportable by v which is increasing, and

(2) Bv is increasing.

It will suffice to show that the best *effort level* function is increasing; see the end of the proof. For this reason, all analysis will be conducted only on optimal effort levels, disregarding the possibility of dissolution until the final choice between the optimal effort level and dissolution is taken. Given (1) and (2) above, set $v_0 := M/(1-\delta)$ everywhere; this is increasing. Then $\bar{v} := \lim_{t\to\infty} B^t v_0$, which by Theorem 1 is the best SSE value function, will also be increasing, and there will then exist an optimal SSE action function which is increasing. Thus, we need only show (1) and (2) above. I divide the analysis into a series of lemmas. Lemma 2.1 is self-explanatory:

__Lemma 2.1__: *Assume* (A4)–(A9). *Then:*

(a) $\forall\, \pi$, *there exists a greatest symmetric Nash equilibrium* $\bar{e}_N(\pi)$ *of* $G(\pi)$, *and*

$\bar{e}_N$ *is increasing in* π,

(b) $\forall\ \pi$, *there exists a greatest maximizer* $\bar{e}_{eff}(\pi)$ *of* $u(\ ;\pi)$, $\bar{e}_{eff}$ *is increasing in* π, *and* $\bar{e}_{eff} \geq \bar{e}_N$.

Proof of Lemma 2.1: (a) By (A4), for all π, $\underline{u} \in \mathbb{R}$, $\exists\ \bar{e}$ s.t. for $e \geq \bar{e}$, $u(e;\pi) < \underline{u}$. In particular, let $\underline{u} := u(0;\pi)$. Then, for $e \geq \bar{e}$, $u(e;\pi) < \underline{u} = u(0;\pi) \leq u_1(0,e,...,e;\pi)$, by (A5). So e cannot be a symmetric Nash effort level for $G(\pi)$. Therefore, we may restrict attention to the game on the restricted domain of $[0,\bar{e}]$ in considering symmetric Nash equilibria of $G(\pi)$. On this strategy space the game is supermodular by Proposition 2.1 (a), and thus has a greatest symmetric Nash effort level (Milgrom and Roberts (1990, definition of supermodular game and Theorem 5)). Proposition 2.1 (b) implies that the greatest Nash effort levels are increasing in π, by Milgrom and Roberts (1990, Corollary to Theorem 6).

(b) By (A6), u is also continuous, and by (A4) it therefore has a compact set of maximizers in e with π fixed, which must have a maximum element. By Proposition 2.1 (c) and Topkis' Monotonicity Theorem, this is increasing in π. Finally suppose that for some π, $\bar{e}_N(\pi) > \bar{e}_{eff}(\pi)$. Then $u(\bar{e}_{eff}(\pi);\pi) = u_1(\bar{e}_{eff}(\pi),...,\bar{e}_{eff}(\pi);\pi) \leq u_1(\bar{e}_{eff}(\pi),\bar{e}_N(\pi),...,\bar{e}_N(\pi);\pi) \leq u_1(\bar{e}_N(\pi),...,\bar{e}_N(\pi);\pi) = u(\bar{e}_N(\pi);\pi)$, contradicting the definition of $\bar{e}_{eff}(\pi)$, and where the first inequality comes from (A5), and the second from Nash equilibrium. *Q.E.D.*

Lemma 2.2 shows that rents are higher for higher π when v is increasing:

Lemma 2.2: *Assume* (A7), *and* $v \in C$ *increasing. Then* $\int v(\pi')\, f(\pi'|\pi)\, d\pi'$ *is increasing in* π.

Proof of Lemma 2.2: (A7) implies that the densities $f(\pi'|\pi)$ are first–order stochastically increasing in π, and thus their integrals over an increasing function are increasing (Ross (1983, p. 154)). *Q.E.D.*

Lemma 2.3 shows that above the highest one–period Nash level of effort, the best response correspondence must lie completely below the diagonal; therefore, any effort level above the highest Nash level at a high value of π which is supportable at a lower value of π, must also be supportable at the higher π:

Lemma 2.3: *Assume* (A4)–(A9). *Then:*

(a) *Fix an arbitrary* π, *and let* $e > \bar{e}_N(\pi)$. *Then* $\bar{e}^*(e;\pi) < e$.

(b) *Let* $\pi_1 \leq \pi_2$ *and* $e > \bar{e}_N(\pi_2)$. *Then,* $e \in IC(v)(\pi_1) \Rightarrow e \in IC(v)(\pi_2)$.

Proof of Lemma 2.3: (a) Suppose not. Let $\underline{u} < u(\bar{e}_{eff}(\pi);\pi)$ and let $\underline{e}$ be such that $u(\underline{e};\pi) = \underline{u}$ but for $e > \underline{e}$, $u(e;\pi) < \underline{u}$. By (A4), (A6) such an $\underline{e}$ exists for all such $\underline{u}$. Note that $\underline{e} > \bar{e}_{eff}(\pi)$. Suppose $\bar{e}^*(\underline{e};\pi) = \underline{e}$; then $\underline{e}$ is a symmetric Nash effort level for $G(\pi)$, a contradiction of Lemma 2.1 (a,b). Suppose $\bar{e}^*(\underline{e};\pi) > \underline{e}$. Then $u(\underline{e};\pi) \leq u_1(\bar{e}^*(\underline{e};\pi),\underline{e},...,\underline{e};\pi) \leq u_1(\bar{e}^*(\underline{e};\pi),...,\bar{e}^*(\underline{e};\pi);\pi) = u(\bar{e}^*(\underline{e};\pi);\pi)$, contradiction of the definition of $\underline{e}$, where the first inequality holds since $\bar{e}^*(\underline{e};\pi)$ is a best response to $\underline{e}$, and the second by (A5). So $\bar{e}^*(\underline{e};\pi) < \underline{e}$. Since u_1 has increasing differences in (e_1,e_{-1}) by Proposition 2.1 (a), the maximal best response function is increasing in e by Topkis's Monotonicity Theorem, it lies above the diagonal at e and below at $\underline{e}$, and thus by Tarski's Fixed Point Theorem has a fixed point in $[e,\underline{e}]$. There is therefore a symmetric Nash of $G(\pi)$ at least equal to e, a contradiction.

(b) Let $e > \bar{e}_N(\pi_2)$ be in $IC(v)(\pi_1)$. By Lemma 2.2, if we can show that $d(e;\pi_2) \leq d(e;\pi_1)$, then we are done. But $d(e;\pi_2) = u_1(\bar{e}^*(e;\pi_2),e,...,e;\pi_2) - u(e;\pi_2) \leq u_1(\bar{e}^*(e;\pi_2),e,...,e;\pi_1) - u(e;\pi_1) \leq u_1(\bar{e}^*(e;\pi_1),e,...,e;\pi_1) - u(e;\pi_1) = d(e;\pi_1)$, where the first inequality follows from Proposition 2.1 (b) and part (a), the second from the fact that $\bar{e}^*(e;\pi_1)$ is a best response. *Q.E.D.*

Lemma 2.4 shows that we may find an optimal effort level at the maximum of a compact set which is larger for higher π:

<u>Lemma 2.4</u>: *Assume* (A4)–(A9), *that* $v \geq 0$ *is increasing, and let* $E(\pi) := \{e \mid \forall\, e' \leq e,\ u(e';\pi) \leq u(e;\pi)\}$. *Then:*

(a) $u^*(e;\pi)$ *is increasing in e.*

(b) $e_{max}(\pi) := \max \{ [\bar{e}_N(\pi),\infty) \cap E(\pi) \cap IC(v)(\pi) \}$ *is an optimal effort level.*

(c) *Let* $\pi_1 \leq \pi_2$. *Then* $e_{max}(\pi_1) \leq e_{max}(\pi_2)$.

<u>Proof of Lemma 2.4</u>: (a) Fix π, and suppose that $e_1 \leq e_2$. Then $u^*(e_1;\pi) = u_1(\bar{e}^*(e_1;\pi),e_1,\ldots,e_1;\pi) \leq u_1(\bar{e}^*(e_1;\pi),e_2,\ldots,e_2;\pi) \leq u_1(\bar{e}^*(e_2;\pi),e_2,\ldots,e_2;\pi) = u^*(e_2;\pi)$, where the first inequality holds by (A5) and the second by the fact that $\bar{e}^*(e_2;\pi)$ is a best response.

(b) By part (a), we know that $\forall\, \pi$, $\bar{e}(\pi)$ may be chosen at least equal to $\bar{e}_N(\pi)$, since any Nash level is supportable since $v \geq z/(1-\delta)$, and for $e < \bar{e}_N(\pi)$, $u(e;\pi) \leq u^*(e;\pi) \leq u^*(\bar{e}_N(\pi);\pi) = u(\bar{e}_N(\pi);\pi)$. By part (a) we may also restrict attention to the set $E(\pi)$, for in an optimal SSE players would not choose any effort level below which there lies an effort level of higher symmetric utility, for that lower effort level would then also be supportable. Finally, choices are restricted to being in $IC(v)(\pi)$. Since utility is increasing in $E(\pi)$, we may choose the maximal available effort in the intersection.

(c) Suppose not. Then $e_{max}(\pi_1) \geq \bar{e}_N(\pi_2)$. By Proposition 2.1 (c), we have $E(\pi_1) \subseteq E(\pi_2)$. So, by this, part (b), and Lemma 2.3 (b), $e_{max}(\pi_1) = \max \{ [\bar{e}_N(\pi_2),\infty) \cap E(\pi_1) \cap IC(v)(\pi_1) \} \leq \max \{ [\bar{e}_N(\pi_2),\infty) \cap E(\pi_2) \cap IC(v)(\pi_2) \} = e_{max}(\pi_1)$. *Q.E.D.*

Finally note that one–period payoffs thus rise in π as well by (A1), (A7), and thus the value function Bv rises in π, since dissolution will only be chosen if at all where $u(e;\pi) + \delta \int v(\pi')\, f(\pi'|\pi)\, d\pi' \leq 0$, by (A3), which could only hold, if at all, in some

interval $[0,\underline{\pi}]$. The Theorem follows from this and Lemma 2.4 (c). *Q.E.D.*

FIGURE 1

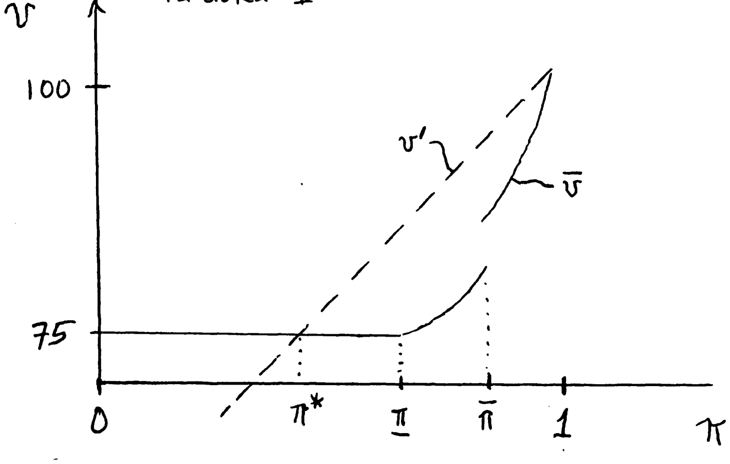


FIGURE 2

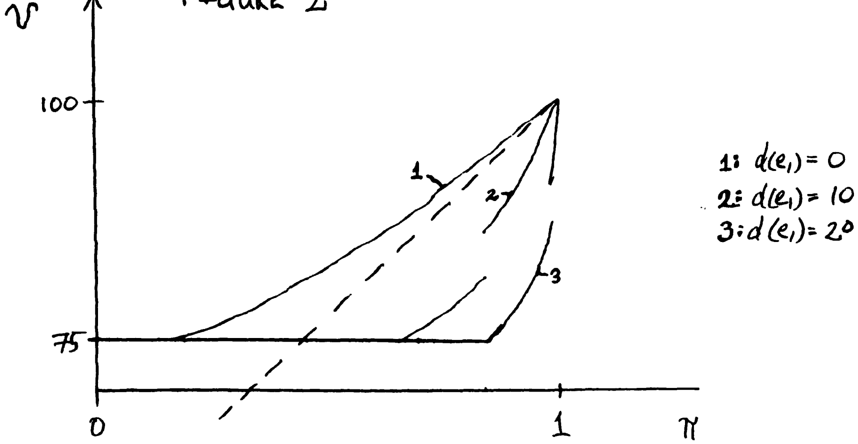
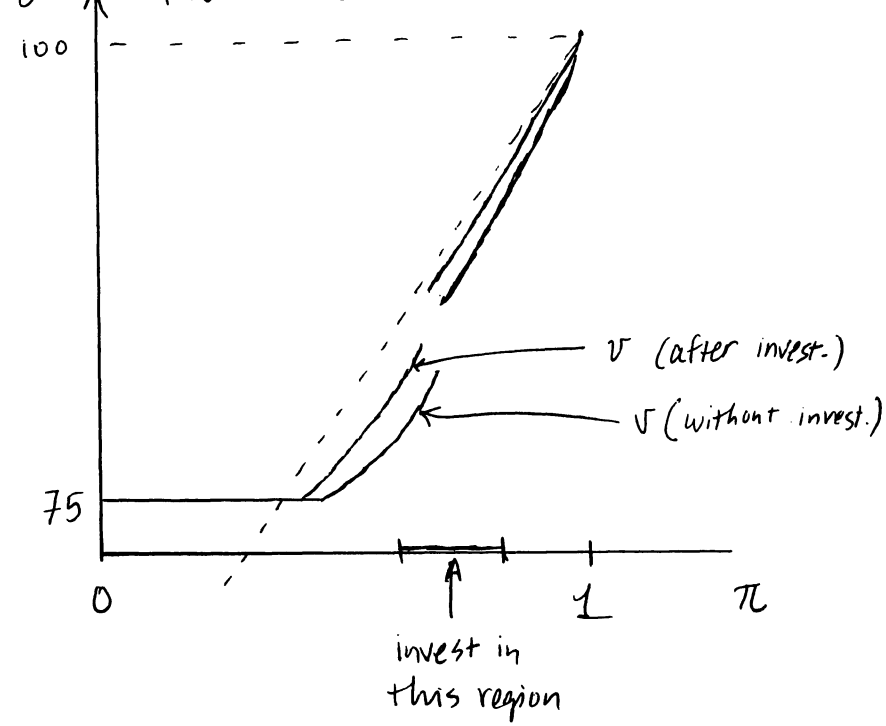


FIGURE 3



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