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Ellet's Transportation Model of an Economy with Differentiated Commodities and Consumers, I: Generic Cumulative Demand Functions

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1. Introduction.

Theoretical microeconomics has relied heavily on the neo-classical Arrow-Debreu model of general equilibrium theory, as outlined in Debreu (1959). In this model, a commodity is a carefully specified good or service delivered at a specific location and at a specific time. Two related, but not identical, varieties of the same good must be treated as separate, independent commodities. Similarly, there are a finite number of consumers in such an economy, each specified by his preference relation and initial wealth. One cannot easily compare characteristics of similar but different commodities or tastes of similar but different consumers. Nor can one easily handle the addition or deletion of commodities from the market.

There are many situations, such as housing markets (see Sweeney (1974)) and labor markets (see Becker (1965) or Lewis (1969)), where the distribution of the qualities and characteristics of the studied goods and the spectrum of preferences of the consumers are the important objects of study. As a result, economic models with differentiated products and consumers have been widely developed and discussed over the past twenty-five years. Among the important references in this area are Houthakker (1952), Lancaster (1966), (1975), and Rosen (1974). Commonly in such models, an open subset of Euclidean space is used to parametrize an n-dimensional set of product characteristics or qualities. "The consumer is assumed to derive his actual utility or satisfaction from these characteristics which cannot in general be purchased directly, but are incorporated into goods. The consumer obtains his optimum bundle of characteristics by purchasing a collection of goods so chosen as to possess in toto the desired characteristics... . Furthermore, one generally assumes that the population consists of a very large number of consumers with difference preference patterns, so that there is a continuous spectrum of preferences." (Lancaster (1975)) Thus, the consumers' preferences are assumed to be parametrized by another open set of Euclidean space. Furthermore, both of these parametrizations are assumed to be continuous, or even differentiable, so that nearby parameters correspond to similar characteristics or similar preferences relations.

More concretely, if X in $\mathbb{R}^N$ is the space of characteristics and C in $\mathbb{R}^P$ represents the space of consumers, then for each $c \in C$ there is a utility function $u(x; c)$ which is smooth in both variables, and probably concave in x. If, for some given price system $p(x)$ and distribution of initial incomes, $x = \xi(c)$ maximizes $u(\cdot,c)$ on its budget set, then ξ might be considered an individual demand function. The function ξ by matching up points in C with points in X casts the problem into a spacial setting reminiscent of the spacial approach to economics of Harold Hotelling. In fact, such models with differential commodities and consumers have historically been closely associated with the theory of monopolistic competition.

There is no reason that the individual demand function ξ be surjective; in other words, some characteristics and commodities may not be purchased. As the price function varies, the image of ξ may change also; goods demanded at one price may be ignored at another. One can perform a similar analysis from the supply side with the cost and production functions of a continuous spectrum of firms leading to a supply function η. One can then interpret changes in the image of η as entrance and exit of firms from the market--a phenomenon which is rather difficult to model in the Arrow-Debreu model.

Besides providing an effective method of comparing qualities of similar commodities and tastes of similar consumers, these models often have other advantages over the traditional model in some applications. For example, one can easily include uncertainties about consumer behavior, as illustrated in Quandt (1956). One can also bring some extra mathematical tools into the analysis. While the mathematics used in the Arrow-Debreu model consists mainly of linear algebra, the implicit function theorem, nonlinear programming, and fixed-point theorems, the study of models with a continuum of commodities and traders often includes, in addition to these, techniques from capital theory, such as calculus of variations, optimal control theory, and partial differential equations.

However, the mathematics of the Arrow-Debreu model is much more straightforward, at least as it is used in most applications. One assumes that each of the finite number of consumers has a preference relation which is representable by a smooth, monotone non-singular, strictly concave utility function defined on a convex commodity space. In concrete applications, the utility function will usually be quadratic, logarithmic, Cobb-Douglas, or constant elasticity. Price functions are always linear. The production function defined on the convex production set usually has properties similar to those of the utility function on its commodity space. As a result of these specifications, mathematical analysis in the Arrow-Debreu model follows a carefully-marked path where the effects of various economic assumptions in the model are fairly well understood.

The picture is much more complicated and uncertain in models with differentiated products and consumers. Since prices on goods can often be reinterpreted as prices on the corresponding characteristics or attributes (the so-called "hedonic prices"), there is no reason to assume that such prices depend linearly on the qualities being studied. Neither is it clear whether price functions should be concave or convex or a little of both.

While it is reasonable that the above-mentioned utility function $u(x;c)$ be concave in x, there is no reason for u to be convex or concave in c. Since $p(x)$ may be neither concave nor convex, we will be maximizing $u(\cdot;c)$ on a (possibly) non-convex budget set. As a result, the maximizer $\xi(c)$ can behave rather wildly. Among the important questions one might ask are: i) what does ξ look like in a practical economic model with differentiated commodities, and ii) how should we aggregate

ξ over consumers to obtain a meaningful concept of aggregate demand or aggregate excess demand?

To quote Lancaster (1975) again, "It is obvious that opportunities for ill behavior abound in a general model." Carefully worked-out examples of reasonable economies with differentiated commodities and consumers are badly needed to understand what forms $u(x;c)$, $p(x)$, and $\xi(c)$ may take. These forms should be clearly tied to the application in question and not some ad-hoc formulae chosen as a mathematical convenience or curiosity. The resulting study of such examples should lead to deeper insights and a stronger intuition for exactly what kind of behavior one can expect in concrete economies with differentiated products and consumers.

The goal of this paper is to study in detail just such an example. The concrete example we will study is a generalization of one of the first economic treatises written in the United States--Charles Ellet's treatise on the theory underlying the determination of canal and railroad tariffs. Ellet (1810-1862) was a civil engineer who studied in Paris after some initial work on the Erie Canal. He returned to the United States with many exciting engineering ideas and began to work on the James River and Kanawha Canal Project. He built the first permanent American suspension bridge in Philadelphia over the Skuylkill River; and he also built a bridge in Wheeling, Ohio, which was at the time the world's longest suspension bridge. Not only is he responsible for Virginia's canal network but he also proposed a system of reservoirs for western rivers to aid flood control and shipping--proposals which were finally approved a century after his death. He advocated and designed ram boats for the North in the Civil War, and he died fighting in one. For further details on Ellet's life, see Calsoyas (1950), G. Lewis (1968), or the archives at the University of Michigan Transportation Library.

In 1839, Ellet wrote An Essay on the Laws of Trade, in Reference to the Works of Internal Improvement in the United States, a work which led Viner (1928) to rank Ellet "with Cournot...as a pioneer formulator of the pure theory of monopoly price in precise terms." The goal of this paper was to describe the principles by which the charges for the use of a new canal" should be regulated so that it may be rendered most profitable to the stockholders, and most beneficial to the community." (Ellet (1839)). He was especially interested in maximizing the profits obtained from transporting heavy, cheap materials, which will not be transported if the freight charge is above some fixed lower limit, and in discovering the optimal locations of roads which would feed into the main canal. For more complete details, see Ellet (1839) and Calsoyas (1950).

To model Ellet's problem mathematically, consider a canal or railroad lying along the positive x-axis with a large number of small farmers in the plane around the axis. Further, approximate this problem by assuming that the points in the right half plane parametrize the farmers and that each farmer wants to ship a bundle of farm goods to a warehouse or cannery located at the origin. There is a linear cost for shipping along land to the railroad and a non-linear cost function $p(x)$ which gives the cost of transporting the freight along the railroad from $(x,0)$ to $(0,0)$. Each farmer maximizes his utility by minimizing his shipping costs, but no farmer will pay more than v dollars to ship his bundle.

This is clearly a problem with differentiated commodities and consumers which is best handled as a spatial problem--matching farmers in the right half plane with their optimal connection point with the canal or railroad. Since the railroad may have steep grades in some places or the width and the depth of the canal may vary widely, the price function $p(x)$ may be concave in some places and convex in others.

In this paper, we will focus our attention on the individual demand function ξ and the corresponding cumulative demand function μ. We will illustrate that, while a broad class of demand functions arise, the typical individual demand function is smooth except on a set of measure zero while the generic cumulative demand function is smooth except on a finite set S of points. We will then give conditions for S to be empty, i.e., for μ to be globally smooth.

In one respect, this paper can also be considered as an extension of the work of Sondermann (1975), Araujo and Mas-Collel (1978), Mas-Collel and Neuefeind (1977), Neuefiend (1977), Dierker-Dierker-Trockel (1978), and others--all of whom have shown that for a generic economy with differentiated products the mean excess demand correspondence is a smooth function. This paper deals with a concept more closely related to the usual aggregate demand function of the Arrow-Debreu model (at least for the Ellet problem), the cumulative demand function. Using no assumptions or techniques of convexity, we show that the cumulative demand function in our model is generically smooth, except perhaps on a finite set of points. We also indicate that for an open set of economies in our model, this set of singularities is non-empty.

The fact that these demand functions are generically well-behaved is an important step in the analysis of economic equilibria and of the corresponding comparative statics. For example, if one can show that a closed property like the existence of a competitive equilibrium holds for an open dense subset of some class of economies, then it is straightforward matter to show that all the economies in the class have that property. See, for example, Mas-Collel and Neuefiend (1977).

In a sequel to this paper, we will study the supply side of this model--both in a competitive and monopolistic framework--and will describe the equilibria which exist and how they are affected by changes in the price systems.

The author acknowledges his deep gratitude to Hugo Sonnenschein, who not only introduced him to the work of Charles Ellet but also enkindled his interest in models of economics with differentiated products and consumers.

§2. PRELIMINARY ANALYSIS

In this section, the mathematical model will be described and the formulae and equations which are central to this investigation will be calculated. Assume that there are fruit growers evenly distributed throughout the right-half-plane. Each grower would like to ship a bushel of fruit to a cannery at the origin (0, 0). Assume that there is a canal running along the positive x-axis with shipping firms evenly distributed along the canal. In addition, suppose that the cost of shipping the bushel <u>along land</u> is one unit of currency (say, one dollar) and that there is a function $p: \mathbb{R}_+ \to \mathbb{R}_+$ such that the shipper at $(x, 0)$ charges $p(x)$ dollars to ship a bushel of fruit to $(0, 0)$. It is reasonable to make the following assumptions on $p(x)$:

i) $p(0) = 0$,

ii) p is a smooth (C^∞) function, i.e., every derivative of p exists and is continuous,

iii) $0 < p'(x) < 1$; for, if p' were greater than one there would be no advantage to using the canal. Finally, assume that no farmer is willing to spend more than v dollars to ship his fruit to $(0, 0)$.

Consequently, if the farmer at (a, b) ships his goods by land to $(x, 0)$ and then by canal to $(0, 0)$, his shipping cost will be

$$(2.1) \qquad C(a, b, x) \equiv C^{a,b}(x) = p(x) + \sqrt{(a - x)^2 + b^2} .$$

He wants to choose x to minimize $C^{a,b}(x)$. Define $\xi(a, b)$ and $\mu(x)$ by

$$\xi(a, b) = \{x | x \text{ is a global minimizer of } C^{a,b}\} \quad \text{and}$$

$$\mu(x) = \text{measure}\{(a, b) | 0 < \xi(a, b) \leq x\} .$$

The goal of this paper is to describe how the choice function ξ and the cumulative demand function μ behave for a large class of price functions p.

Of course, the first and second derivatives of $C^{a,b}$ will play a central role in our analysis:

$$\text{(2.2)} \quad \begin{aligned} C_x^{a,b}(x) &\equiv \frac{\partial C^{a,b}}{\partial x}(x) = p'(x) - \frac{(a - x)}{\sqrt{(a - x)^2 + b^2}} \\ C_{xx}^{a,b}(x) &\equiv \frac{\partial^2 C^{a,b}}{\partial x^2}(x) = p''(x) + \frac{b^2}{[(a - x)^2 + b^2]^{3/2}} \end{aligned}$$

Note that a subscript x will often be used to denote the partial derivative with respect to x. If $x^* > 0$ is a minimizer of $C^{a,b}$, then $C_x^{a,b}(x^*) = 0$, i.e.,

$$p'(x^*) = \frac{a - x^*}{\sqrt{(a - x^*)^2 + b^2}}$$

Since $p'(x^*) > 0$, $x^* < a$. Note also that $C_x^{a,b}(a) > 0$. Consequently, one need only consider $C^{a,b}$ as a function on the interval $[0, a)$.

Clearly, the x-axis is an axis of symmetry for the problem under study. To simplify matters, we will only consider (a, b) in the positive quadrant, i.e. $a > 0$ and $b > 0$.

Let $R(x)$ denote the ray from $(x, 0)$ into the first quadrant which makes an angle $\theta(x)$ with the positive x-axis, where

$$(2.3) \qquad \tan\theta(x) = \frac{\sqrt{1 - p'(x)^2}}{p'(x)} > 0 .$$

It follows from (2.2) that $C_x^{a,b}(x^*) = 0$ if and only if

$$(2.4) \qquad \frac{b}{a - x^*} = \frac{\sqrt{1 - p'(x^*)^2}}{p'(x^*)} ,$$

i.e., if and only if $(a, b) \in R(x^*)$. Thus, a necessary condition that $\xi(a, b) = x > 0$ is that $(a, b) \in R(x)$. Note that by equation (2.2), $(a, b) \in R(x)$ if and only if

$$\cos\theta(x) = p'(x) .$$

An intuitive way to study $C^{a,b}(x)$ for a given $p(x)$ is to note that x^* is a critical point of $C^{a,b}$ if and only if the graphs of $p'(x)$ and

$$B^{a,b}(x) \equiv \frac{a - x}{\sqrt{(a - x)^2 + b^2}}$$

intersect above the point x^*. Figures 2.1 a, b, and c illustrate three of the possible configurations for the graphs of p'

and $B^{a,b}$. Figures 2.2 a, b, and c present the corresponding graphs of $C^{a,b}$.

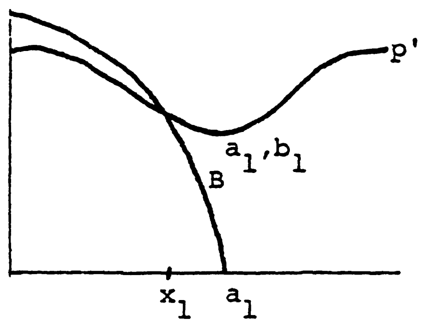


Figure 2.1 a

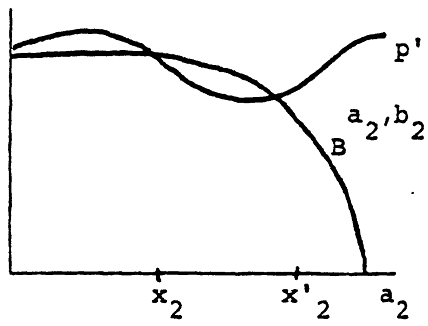


Figure 2.1 b

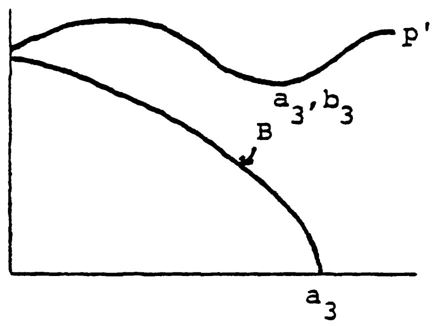


Figure 2.1 c

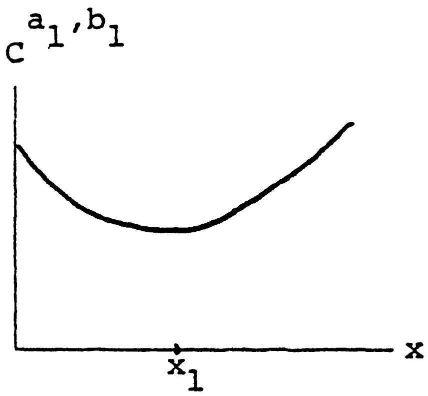


Figure 2.2 a

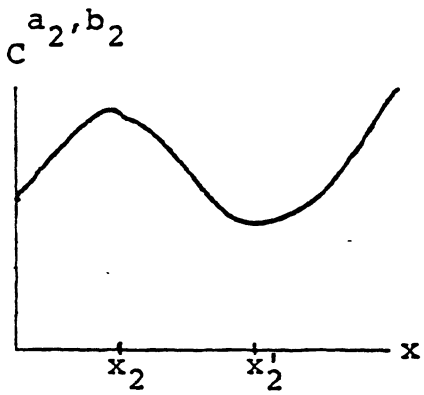


Figure 2.2 b

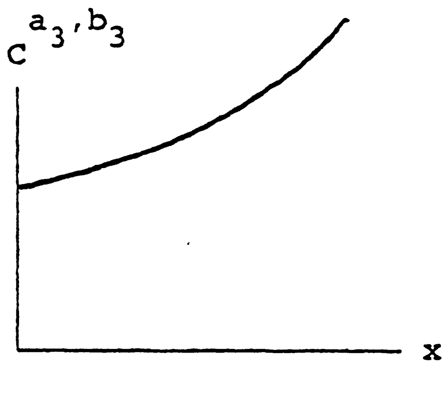


Figure 2.2 c

In Figure 2.1 a, $C_x^{a_1 b_1} < 0$ for $x < x_1$, and $C_x^{a_1 b_1} > 0$ for $x > x_1$; thus the graph is as in Figure 2.2 a and $\xi(a_1, b_1) = x_1$. By Figure 2.1 b, C^{a_2,b_2} has two critical points: a local max at x_2 and a local min at x_2' . However, one cannot tell from Figure 2.1 b whether 0 or x_2' is the <u>global</u> minimizer of C^{a_2,b_2} . Finally, $C_x^{a_3,b_3} > 0$ for all x in Figure 2.1 c; thus, the C^{a_3,b_3} is an increasing function with $\xi(a_3, b_3) = 0$.

Although one cannot tell from Figure 2.1 b whether or not C^{a_2, b_2} has a global minimum at 0 or at some interior x^*, one can calculate a formula for the curve Z of (a, b) such that $C^{(a,b)}$ has a minimum both at zero and at an interior point z . In many cases, on one side of Z , $\xi(a, b) = 0$; while for (a, b) on the other side of Z , $C^{(a,b)}$ has an interior global minimum. The equations for Z are given by:

$$(2.5)\begin{cases} C_x^{a,b}(x) = 0 \quad \text{or} \quad b = (a - x)\dfrac{\sqrt{1 - p'(x)^2}}{p'(x)} , \\ C^{a,b}(x) = C^{a,b}(0) \quad \text{or} \quad p(x) + \sqrt{(a-x)^2 + b^2} = \sqrt{a^2 + b^2} \end{cases}$$

We can solve these explicitly for a and b as functions of x by substituting the first equation into the second to obtain:

$$(2.6) \qquad \begin{cases} a(x) = x + \dfrac{p'(x)}{2} \cdot \dfrac{x^2 - p^2(x)}{p(x) - p'(x)x} \\ b(x) = \dfrac{\sqrt{1 - p'(x)^2}}{2} \cdot \dfrac{x^2 - p^2(x)}{p(x) - p'(x)x} \end{cases}$$

Now, let us bring the constraint $C^{a,b} \leq v$ into the picture. For those (a, b) with $\xi(a, b) > 0$, the curve Φ of (a, b) such that $(a, b) \in R(x)$ and $C^{a,b}(x) = v$ is an important curve. Using (2.4) instead of (2.2), one notes that the equations of Φ are:

$$(2.7) \qquad \begin{cases} b = (a - x) \dfrac{\sqrt{1 - p'(x)^2}}{p'(x)} , \\ p(x) + \sqrt{(a - x)^2 + b^2} = v . \end{cases}$$

Substituting from the first equation into the second, one finds the curve Φ parameterized by

$$(2.8) \qquad \begin{cases} a(x) = x + (v - p(x))p'(x) \ , \\ b(x) = (v - p(x))\sqrt{1 - p'(x)^2} \end{cases}$$

The curve Φ meets the x-axis at $(x^*, 0)$ where $p(x^*) = v$.

Finally, since $\mu(x)$ is defined as the area swept out by the ray $R(x)$ from $(x, 0)$ to the curve $(a(x), b(x))$, the following lemma will be an important tool for studying μ .

<u>Lemma 2.1</u> Let $(a(x), b(x))$ parameterize a continuous curve in the positive quadrant of the plane. Assume that $a(x)$ is non-decreasing. Let $R(x)$ be the ray from $(x, 0)$ to $(a(x), b(x))$, forming an angle $\theta(x)$ with the positive x-axis. Suppose that $x_1 \leq x_2$ and

$$0 < \theta(x) < \pi \quad \text{for all} \quad x \in (x_1, x_2) \ .$$

Then, the area A of the region bounded by $R(x_0)$, $\{(a(x), b(x)) | x \in [x_0, x_1]\}$, $R(x_1)$, and the x-axis is given by

$$A = \int_{x_0}^{x_1} [b(s) + \frac{b^2(s)}{2} \frac{d}{ds}(\cot \theta(s))]ds \ .$$

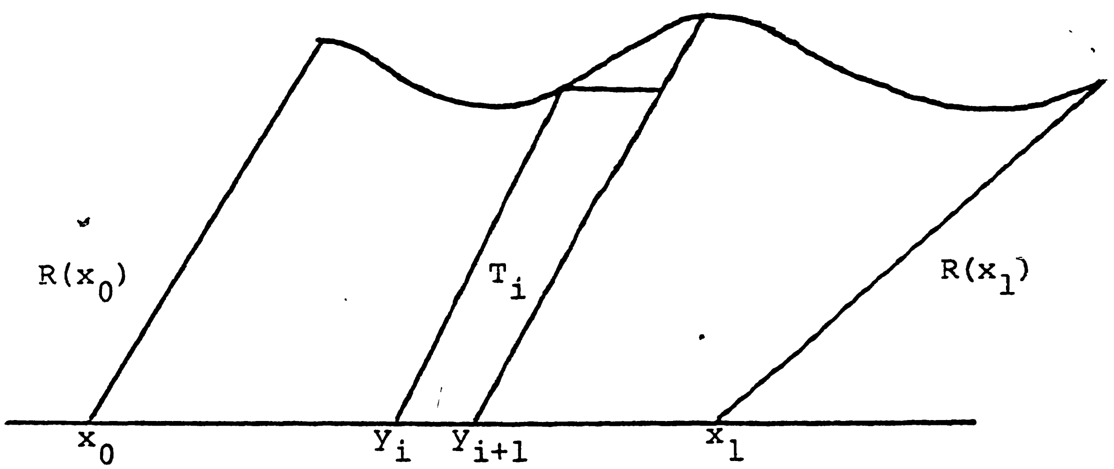
<u>Proof</u>: Partition $[x_0, x_1]$ into n equal intervals, each of length Δx, with endpoints

$$x_0 = y_0 < y_1 < y_2 < \cdots < y_n = x_1 .$$

Approximate the area A by the sum of the areas of the trapezoids T_i, where T_i has vertices:

$$(y_i, 0),\ (y_{i+1}, 0),\ (a(y_i), b(y_i)),\ ((y_{i+1} + b(y_i) \cot \theta(y_{i+1}), b(y_i)) .$$

See Figure 2.3. We are using the formula: $a(y) - y = b(y) \cot \theta(y)$.



<u>Figure 2.3</u>

The area of T_i is

$$\frac{b(y_i)}{2}\left[(y_{i+1}-y_i)+(y_{i+1}+b(y_i)\cot\theta(y_{i+1}))-(y_i+b(y_i)\cot\theta(y_i))\right]$$

$$=\left[b(y_i)+\frac{b^2(y_i)}{2}\frac{d}{dy}(\cot\theta)(\xi_i)\right]\Delta x\ ,\quad \text{for some}\quad \xi_i\in(y_i,y_{i+1})$$

Summing over i and taking the limit as $\Delta x \to 0$ yields the theorem. ∎

<u>Proposition 2.2</u>. If, in Lemma 2.1, $\cot\theta(x) = \dfrac{p'(x)}{\sqrt{1-p'(x)^2}}$ for some smooth function $p(x)$, then the corresponding area equals $\displaystyle\int_{x_0}^{x_1}\left[b(y)+\frac{b^2(y)\,p''(y)}{2[1-p'(y)^2]^{3/2}}\right]dy$. If p is C^∞ and $0 < p' < 1$ on $[x_o, x_1]$ and if b is k-times continuously differentiable (i.e., C^k), then A as a function of x_1 is (k+1)-times differentiable. If b is continuous, then $A(x_1)$ is C^1 .

§3. Some Concrete Examples.

In this section, we present three concrete examples in order to assist the reader's understanding of and intuition for the concepts invovled and to preview some of the difficulties which arise.

A. Linear price functions. The simplest price function to investigate is the linear one

$$p(x) = Px .$$

In such functions, one can actually compute the corresponding ξ and μ to see how well behaved they are. The assumption that $0 < p' < 1$ means that the constant P is in $(0,1)$. Formulae (2.1), and (2.2) become in this case:

$$C^{a,b}(x) = Px + \sqrt{(a-x)^2 + b^2} , \tag{3.1}$$

$$\frac{\partial C^{a,b}}{\partial x}(x) = P - \frac{a-x}{\sqrt{(a-x)^2 + b^2}} , \tag{3.2}$$

$$\frac{\partial^2 C^{a,b}}{\partial x^2}(x) = \frac{b^2}{[(a-x)^2 + b^2]^{3/2}} . \tag{3.3}$$

Since the graph of p' is a horizontal straight line, the situation in figure 2.1.b cannot occur and the graph of $C^{a,b}(x)$ is either as in figure 2.2.a or as in figure 2.2.c. The former occurs if

$P < \frac{a}{\sqrt{a^2 + b^2}}$; the latter occurs if $P \geq \frac{a}{\sqrt{a^2 + b^2}}$ (and therefore $C^{a,b}_{x} > 0$) . Consequently, $\xi(a,b)$ is a well-defined (single-valued) function.

Since p' is constant, the angles $\theta(x)$ described in section two are also constant and the rays $R(x)$ of constant demand are all parallel. In fact, it is easy to solve for $\xi(a,b)$ explicitly. If $(a,b) \in R(x)$, then by (2.3) ,

$$\frac{\sqrt{1 - P^2}}{P} = \tan\theta = \frac{b}{a - x} \quad .$$

Consequently, $x = a - \frac{bP}{\sqrt{1 - P^2}}$ and

$$\xi(a,b) = \max\{0, a - \frac{bP}{\sqrt{1 - P^2}}\} \quad . \tag{3.4}$$

Clearly, ξ is a smooth (in fact, linear) function of (a,b) for $\frac{a}{b} > \frac{P}{\sqrt{1 - P^2}}$.

Next, the value constraint needs to be considered,i.e., the assumption that no farmer wants to spend more than v dollars to transport his goods to $(0,0)$. Using equations (2.8), one finds that the set of (a,b) such that $(a,b) \in R(x)$ and $C^{(a,b)}(x) = v$ is the parameterized line:

$$\begin{cases} a = Pv + (1 - P^2)x \ , \\ b = \sqrt{1 - P^2}(v - Px) \ . \end{cases} \tag{3.5}$$

Eliminating x from equations (3.5) yields the equation of the line

$$Pa + b\sqrt{1 - P^2} = v \quad (3.6)$$

which runs from $(0, \frac{v}{\sqrt{1-P^2}})$ to $(\frac{v}{P}, 0)$.

Putting the above calculations together yields the information described in figure 2.1. Line segment α in this figure in part of the ray $R(0)$. Line segment β is part of the line described by equation (3.6). Curve γ is an arc of the circle of radius v about the origin. Farmers who are shipping from region A will ship to the x-axis along the appropriate line $R(x)$ parallel to α . Farmers starting in region B will ship their fruit along a straight line which ends directly at the origin. Farmers in region C wil decide not to ship since their shipping costs will exceed v_{ω}.

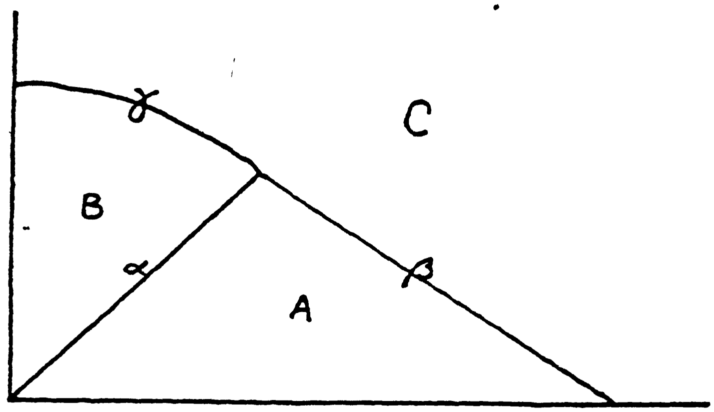


Figure 3.1

The cumulative demand function

$$\mu(x) \equiv \text{measure } \{(a,b) \,|\, 0 < \xi(a,b) \leq x\}$$

can now be explicitly computed for points in region A in figure 3.1. Since line β meets line α at the point $(Pv, \sqrt{1 - P^2}v)$ and it meets the x-axis at the point $(\frac{v}{P}, 0)$, the area of triangle A is

$$\mu(\tfrac{v}{P}) = \tfrac{1}{2}(\tfrac{v}{P})(\sqrt{1 - P^2}v) = \tfrac{1}{2}v^2\frac{\sqrt{1 - P^2}}{P} .$$

If $x \in (0, \frac{v}{P})$, the ray $R(x)$ meets β at the point $(Pv + (1 - P^2)x, \sqrt{1 - P^2}(v - Px))$. Consequently, the area of the triangle bordered by β, $R(x)$, and the x-axis is

$$\tfrac{1}{2}[\sqrt{1 - P^2}(v - Px)][\tfrac{v}{P} - x] = \frac{\sqrt{1 - P^2}}{2P}(v - Px)^2 .$$

The difference between these two areas is $\mu(x)$, i.e.,

$$\mu(x) = \frac{\sqrt{1 - P^2}}{2P}[v^2 - (v - Px)^2]$$

$$= \frac{\sqrt{1 - P^2}}{2}(2vx - Px^2) ,$$

a smooth (in fact, quadratic) function of x.

Alternatively, we could have calculated μ directly by applying Proposition 2.2 to the line (3.5). Thus, for linear price functions, both $\xi(a,b)$ and $\mu(x)$ are smooth functions whose explicit form is easy to calculate.

B) Quadratic price functions. As we will see later, price functions whose second derivative is everywhere non-negative are almost as easy to work with as linear price functions. Therefore, it makes sense to consider as a second example a quadratic price function with a negative second derivative, i.e., $p(\underline{x}) = Px - \varepsilon x^2$ where $0 < P < 1$ and $0 < \varepsilon << 1$.

Let $\theta(x)$ be the angle described in section two, (see (2.3)) and let $R(x)$ again be the ray from $(x,0)$ which makes an angle $\theta(x)$ with the positive x-axis. Since

$$\cos\theta(x) = p'(x) \quad \text{and} \quad p''(x) = -\varepsilon < 0 ,$$

$\theta(x)$ increases as x increases. It follows that for $x_1 \neq x_2$, $R(x_1)$ meets $R(x_2)$ in a point. Should the farmer located at this special point ship to x_1 or to x_2 or to the origin?

Another way of looking at this problem is to note that the phenomenon pictured in figures 2.1.b and 2.2.b can occur for these quadratic price functions. As a result, the necessary condition (2.2) for optimization is no longer sufficient. Note, however, that because the graph of p' is a negatively-sloped straight line, the graph of p' can meet the graph of some $B^{a,b}(x)$ in at most two points as in figure 2.1.b and figure 3.2. If the graphs do meet in two points, the first intersection point must be a local maximum of $C^{a,b}$ and the second must be a local mimimum (as a quick examination of the derivatives of $C^{a,b}$ indicates); and the graph of $C^{a,b}$ is similar to that in figure 2.2.b. In this case, the major question is whether $C^{a,b}$ has its <u>global</u> minimum at $x = 0$ or at the interior local minimizer. The curve Z of (a,b) such that $C^{a,b}$ has two global minimizers (one at 0 and one at an interior min) now becomes important. The choice function ξ is single-valued at (a,b) if and only if $(a,b) \notin Z$. For (a,b) above Z, $\xi(a,b) = 0$; for (a,b) below Z, $\xi(a,b) > 0$. The equations for Z are given by (2.6) along with the condition that $C^{a,b}_{xx} \geq 0$. Substituting the quadratic p into (2.6) yields:

$$(3.7)\quad \begin{cases} a = x + \frac{1}{2\varepsilon}[1 - P^2 + 2P\varepsilon x - \varepsilon^2 x^2][P - 2\varepsilon x] , \\ b = \frac{1}{2\varepsilon}[1 - P^2 + 2P\varepsilon x - \varepsilon^2 x^2][1 - (P - 2\varepsilon x)^2]^{1/2} . \end{cases}$$

Finally, the curve of Φ of (a,b) such that $\xi(a,b) = x$ and $C^{a,b}(x) = v$ must be determined. This time, after substituting the formulae for the quadratic p into equations (2.8), one finds that Φ can be parametrized by:

$$(3.8)\quad \begin{cases} a = Pv + (1 - P^2 - 2\varepsilon v)x + 3\varepsilon P x^2 - 2\varepsilon^2 x^3 , \\ b = (v - Px + \varepsilon x^2)[1 - (P - 2\varepsilon x)^2]^{1/2} . \end{cases}$$

Let us fix some values for ε , P , and v:

$$(3.9)\qquad \varepsilon = .1 , \; P = .9 , \text{ and } v = 1.4 .$$

Using a HP-25 hand calculator, the author has computed the curves Z and Φ for this special quadratic price function. The curves are plotted in figure 3.2.

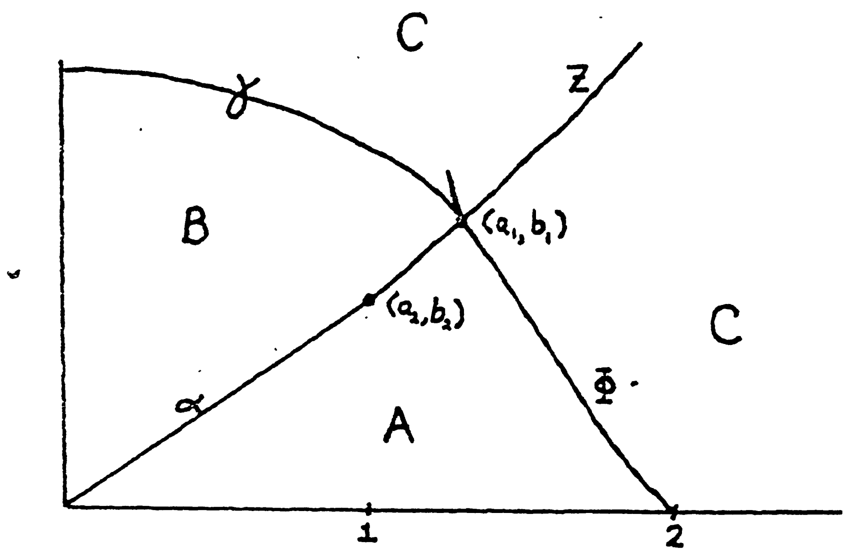


Figure 3.2

The regions A, B, and C in figure 3.2 have the same properties as the corresponding regions in figure 3.1: $\xi(a,b) = 0$ for $(a,b) = B$, $\xi(a,b) > 0$ for $(a,b) \in A$, and $C^{a,b}(\xi(a,b)) > v$ for $(a,b) \in C$. The curve γ is once again an arc of the circle of radius v. The line α is an interval on the ray $R(0)$.

The curves Z and C meet at the point $(a_1,b_1) \cong (1.2452, .6219)$. The curves α and Z meet at the point $(a_2,b_2) \cong (.855, .414)$. The interval between these points on Z is a locus of discontinuities for ξ with $\xi = 0$ above this curve in B and jumping to positive values as one crosses this curve. The interior minimizer for C^{a,b_1} occurs at $x_1 \cong .2554$. For (a,b) in the interior of the region A, $C^{(a,b)}$ has a unique global minimizer which is also a non-degenerate critical point of $C^{(a,b)}$. By the implicit function theorem, $\xi(a,b)$ is an analytic function for such (a,b).

Finally, it should be noted that the cumulative demand function $\mu(x)$ is a C^1 (but not C^2) function. Let $\beta(x)$ be the following continuous function, defined from equations (3.7) and (3.8) with the numerical values of (3.9) substituted for ε, P, and v:

$$\beta(x) = \begin{cases} (.95 + .9 - .05x^2)(.19 + .36x - .04x^2)^{1/2}, & 0 < x < x_1 \cong .2554 \\ (1.4 - .9x + .1x^2)(.19 + .36x - .04x^2)^{1/2}, & x_1 \le x \le 2 \end{cases}$$

Then, by Proposition 2.2,

$$\mu(x) = \int_0^x [\beta(y) - (.05)\beta^2(y)(.19 + .36y - .04y^2)^{-3/2}]dy .$$

Since β is continuous, μ is C^1 for $0 < x < 2$.

As a final example, consider

$$p(x) = 1 - e^{-x} \quad \text{with} \quad v = 1.5 .$$

Once again, $p''(x) = -e^{-x}$ is always negative. The only significant difference between this example and the previous one is that $p(x)$ never reaches v . The regions and curves in figure 3.3 below correspond to the regions and curves in figure 3.2 above.

In this case, the curve Φ tends asymptotically to the the line $b = .5$. The point (a_1,b_1) is approximately (1.3,.84) with $x_1 \cong .9$. Once again, $\mu(x)$ is a C^1 function which is analytic except at the point x_1 .

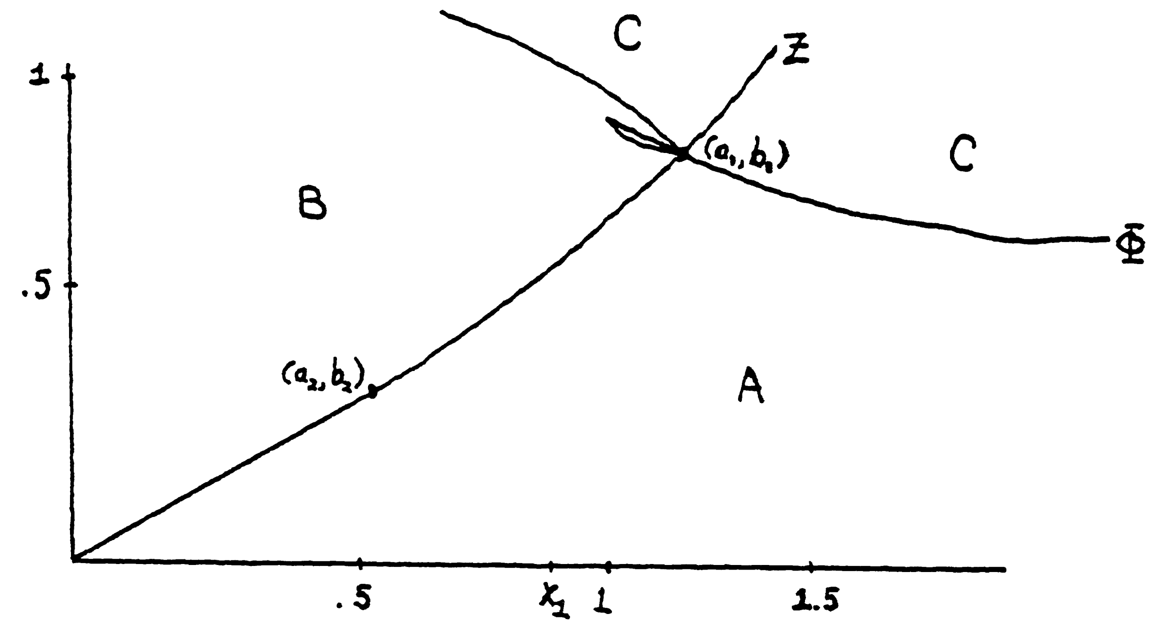


Figure 3.3

§4. GENERIC CURVES

In section two, we defined the curves $Z(p)$ and $\Phi(p)$, which bound the range of ξ for any price function p. In section three, we constructed some examples of such curves. In this section, we will characterize $Z(p)$ and $\Phi(p)$ for a generic set of price functions p; and we will introduce two other important curves, $\Sigma(p)$ and $\Gamma(p)$.

In the Appendix, we describe a natural topology for the vector space

$$\mathcal{C} = \{p:[0,\infty) \to \mathbb{R} \mid p(0) = 0 \text{ and } p \text{ is } C^{\infty}\} .$$

In this section, we will be working with the following open subset $\mathcal{C}_1$ of $\mathcal{C}$:

$$\mathcal{C}_1 = \{p \in \mathcal{C} \mid 0 < p'(x) < 1 \text{ for all } x\} .$$

Since $\mathcal{C}_1$ is open in $\mathcal{C}$, if $\mathcal{D}$ is open (or dense) in $\mathcal{C}$, then $\mathcal{D} \cap \mathcal{C}_1$ will be open (or dense) in $\mathcal{C}_1$.

Throughout this section, we will be working with a fixed bounded region B in $\mathbb{R}^2_+$, say the intersection of the closed ball of radius r about $(0,0)$, $B_r(0,0)$, with $\mathbb{R}^2_+$, $r >> 0$.

Let $(a(t),b(t))$ be a smooth parameterized curve γ in B. Call $(a(t_0),b(t_0))$ a <u>regular point</u> of γ if $a'(t_0) \neq 0$ or

$b'(t_0) \neq 0$, i.e., if γ has a well-defined tangent vector at $(a(t_0),b(t_0))$. If $a'(t_0) = b'(t_0) = 0$, then $(a(t_0),b(t_0))$ will be called a <u>singular point</u> of γ . If γ contains only regular points, we will call γ a <u>regular arc</u>.

We first work with the curve $Z(p)$, where

$$Z(p) = \{(a,b) \in B | C^{(a,b)}(0) = C^{(a,b)}(y) \text{ for some } y \text{ for which } C_x^{(a,b)}(y) = 0\} .$$

See equations (2.5). Equations (2.6) show that equations (2.5) can be solved for (a,b) in terms of y . In Proposition 4.1 below, we prove that for a generic price function $p \in C_1$, $Z(p)$ crosses itself only finitely often, each crossing is transverse; and except for these crossing points and another finite set of cusps, $Z(p)$ is a finite, disjoint union of smooth regular arcs.

<u>Proposition 4.1</u>. There is a residual set A_1 of p's in C_1 so that for any $p \in A_1$, there exists a finite subset $S_1(p)$ in B with the property that $S_1(p)$ contains all the points where $Z(p)$ crosses itself and all the singular points of $Z(p)$. So, $Z(p) \setminus S_1(p)$ is a finite, disjoint union of smooth, regular arcs. Furthermore, for $p \in A_1$, $Z(p)$ has no triple crossings.

<u>Proof</u>: We will use the transversaltiy theorems discussed in the appendix. Step one is to show that the set of (a,b,y) in $\mathbb{R}^3$ such that $C^{(a,b)}(0) = C^{(a,b)}(y)$ and $C_x^{(a,b)}(y) = 0$

is generically (in p) a one-manifold in $\mathbb{R}^3_+$, i.e., that (0,0) is generically a regular value of

$$F^1_{(p)}(a,b,y) = \left(p(y) + \sqrt{(a-y)^2 + b^2} - \sqrt{a^2 + b^2}\ ,\ p'(y) - \frac{a-y}{\sqrt{(a-y)^2 + b^2}}\right)$$

By the Thom Transversality Theorem, we need only show that (0,0) is always a regular value of

$$F^2 : C \times \mathbb{R}^3_+ \to \mathbb{R}^2$$

defined by $F^2(p,a,b,y) = F^1_{(p)}(a,b,y)$. Since C is a vector space, the partial derivative of F^2 with respect to p at $(p°,a°,b°,y°)$ maps C to $\mathbb{R}^2$ by

$$D_pF^2(p°,a°,b°,y°)q = (q(y°),q'(y°))\ ,\ \text{for}\ q \in C\ .$$

For any fixed $y° > 0$, there exists a $q \in C$ such that $(q(y°),q'(y°))$ takes on any prespecified value, i.e., $D_pF^2(p°,a°,b°,y°)$ is surjective. Thus, $DF^2(p°,a°,b°,y°)$ is surjective and (0,0) is a regular value of F^2. By the Thom Transversality Theorem, (0,0) is a regular value of $F^1_{(p)}$ for all p in some residual subset $A_{1,1}$ of C.

Secondly, we need to show that $(F^1_{(p)})^{-1}(0,0)$ in $\mathbb{R}^3$ projects nicely into the (a,b)-plane for a generic p, i.e., that the set of (a,b,y) which lie on $(F^1_{(p)})^{-1}(0,0)$ and for which the tangent vector is parallel to the y-axis (i.e., has no a- or b-component) is a finite set. The defining equations for this set are $F^1 = (0,0)$ and $\frac{\partial F^1}{\partial y} = (0,0)$. Consequently, we need to show that $(0,0,0)$ is generically a regular value of the map

$$F^3_{(p)}(a,b,y) = \left(p(y)+\sqrt{(a-y)^2+b^2}, p'(y) - \frac{a-y}{\sqrt{(a-y)^2+b^2}}, p''(y) + \frac{b^2}{[(a-y)^2+b^2]^{3/2}}\right)$$

By the Thom Transversality Theorem, we need only show that $(0,0,0)$ is always a regular value of

$$F^4(p,a,b,y) = F^3_{(p)}(a,b,y) \ .$$

However, the partial derivative of F^4 with respect to p is $D_pF^4(p^\circ,a^\circ,b^\circ,y^\circ)q = (q(y^\circ),q'(y^\circ),q''(y^\circ))$. For any $y^\circ > 0$ and any pre-assigned $(c_1,c_2,c_3) \in \mathbb{R}^3$, one can always find a $q \in C$ with $q(y^\circ) = c_1$, $q'(y^\circ) = c_2$, $q''(y_0) = c_3$. Since D_pF^4 and, a fortiori, DF^4 are surjective, $(0,0,0)$ is a regular value of $F^3_{(p)}$ for all p in some residual subset $A_{1,2} \subset A_{1,1} \subset C_1$. For such p , $(F^3_{(p)})^{-1}(0,0,0)$ is a zero-dimensional manifold, i.e., a discrete subset of $\mathbb{R}^3_+$ and its projection $S_1(p)$ on the (a,b)-plane is a discrete subset of B .

To prove that the points in $S_1(p)$ are generically non-degenerate cusps, we use Thom's Catastrophé Theorem, as described in the Appendix. We must show that where $F^1_{(p)} = (0,0)$ and $\frac{\partial F^1}{\partial y}(p) = (0,0)$, then $\frac{\partial^2 F^1}{\partial y^2}(p) \neq (0,0)$ for a generic set of p's in C . The proof is similar to the other parts of the proof of this proposition and will be omitted.

To complete the proof of Proposition 4.1, we show that double points of $Z(p)$, i.e., points where $Z(p)$ crosses itself once, form a discrete subset of B for generic p and that for such p , there are no triple points. Consider the map $F^5_{(p)} : \mathbb{R}^2_+ \times (\mathbb{R}^2 \setminus \Delta)$, where $\Delta \equiv \{(y_1,y_2) \in \mathbb{R}^2_+ | y_1 = y_2\}$, defined by $F^5_{(p)}(a,b,y_1,y_2) = (F_{(p)}{}^1(a,b,y_1), F_{(p)}^1(a,b,y_2))$. If $F^5_{(p)}(a,b,y_1,y_2) = (0,0,0,0)$ and $y_1 \neq y_2$, then $Z(p)$ crosses itself at (a,b) . If $(0,0,0,0)$ is a regular value of $F^5_{(p)}$, then $(F^5_{(p)})^{-1}(0,0,0,0)$ is a zero-dimensional manifold, i.e., a discrete subset of $\mathbb{R}^2_+ \times (\mathbb{R}^2_+ \setminus \Delta)$. Its projection on B in the (a,b)-plane will be a finite set.

To show that $(0,0,0,0)$ is a regular value of $F^5_{(p)}$ for generic p , we need only show that it is always a regular value of $F^6(p,a,b,y_1,y_2) = F^5_{(p)}(a,b,y_1,y_2)$. The partial derivative of F^6 with respect to p at $(p°,a°,b°,y_1°,y_2°)$ is

$$D_pF^6(p°,a°,b°,y_1°,y_2°)q = (q(h_1°),q'(y_1°),q(y_2°,),q'(y_2°)) .$$

Once again, for any given $y_1°,y_2° > 0$, $y_1° \neq y_2°$, and any assigned (c_1,c_2,c_3,c_4) , there is a $q \in C$ with

$$q(y_1^\circ) = c_1 \ , \ q'(y_1^\circ) = c_2 \ , \ q(y_2^\circ) = c_3 \ , \ q'(y_2^\circ) = c_4 \ .$$

Since D_pF^6 and DF^6 are surjective, $(0,0,0,0)$ is a regular value of F^6 and for a residual set of p's $A_{13} \subset A_{12} \subset A_{11} \subset C_1$ a regular value of F^5 .

To see that there are generically no triple self-intersections of $Z(p)$, one applies the method of the preceeding paragraphs to

$$F^7_{(p)} : \mathbb{R}^2_+ \times (\mathbb{R}^3 \setminus \Delta^3) \to \mathbb{R}^6 \ ,$$

defined by $F^7_{(p)}(a,b,y_1,y_2,y_3) = (F^1_{(p)}(a,b,y_1), F^1_{(p)}(a,b,y_2), F^1_{(p)}(a,b,y_3))$, where y_1, y_2, and y_3 are pairwise distinct. As a result, there will be a residual subset $A_{1,4} \subset A_{1,3} \subset C$ such that $(0,0,0,0,0,0)$ is a regular value of $F^7_{(p)}$ for $p \in A_{14}$. Since the domain of $F^7_{(p)}$ is 5-dimensional and its range is 6-dimensional, $(0,0,0,0,0,0)$ a regular value of $F^7_{(p)}$ means tha $(F^7_{(p)})^{-1}(0,0,0,0,0,0)$ is empty. Since this set corresponds to the triple crossings of $Z(p)$, it follows that for $p \in A_{1,4}$, $Z(p)$ has no triple points (and consequently, no quadruple points, etc.). This completes the proof of Proposition 4.1. ∎

We now want to demonstrate that the curves $\Phi(p)$ are generically as well-behaved as the curves $Z(p)$ are. Recall that $(a,b,y) \in \Phi(p)$ if and only if $(a,b) \in R(y)$ and $C^{(a,b)}(y) = v$ i.e., if and only if

$$p'(y) - \frac{a - y}{\sqrt{(a-y)^2 + b^2}} = 0 \quad \text{and}$$

(2.7)'

$$p(y) + \sqrt{(a-y)^2 + b^2} = v \ .$$

PROPOSITION 4.2. There is a residual subset A_2 of C_1 such that for any $p \in A_2$, there exists a finite subset S_2 of B in the (a,b)-plane which contains all the singular points of $\Phi(p)$ and all the self-crossings of $\Phi(p)$. Thus, $\Phi(p) \setminus S_2(p)$ is a finite union of disjoint, smooth, regular arcs in B. Furthermore, for $p \in A_2$, $\Phi(p)$ has no triple crossings.

The proof of Proposition 4.2 is virtually identical to that of Proposition 4.1 and so will be omitted. Actually, the set $\Phi(p)$ has more structure as Proposition 4.3 below indicates.

PROPOSITION 4.3. For any $p \in C$ and any $v \in \mathbb{R}_+$, the tangent line to $\Phi(p)$ at any regular point of this curve has negative slope. Furthermore, for any $p \in C_1$, there is a residual set of v's in $\mathbb{R}_+$ such that the corresponding $\Phi(p)$ is a regular curve except for a finite number of cusp points.

See figures 3.1, 3.2, and 3.3 for examples of well-behaved $\Phi(p)$-curves.

Proof: Let $(a,b) = (a(x^\circ), b(x^\circ))$ be a regular point of $\Phi(p)$, in the parameterization given by equations (2.8). We want to show that $\frac{b'(x_0)}{a'(x_0)} < 0$. We will use the implicit formula (2.7) for $\Phi(p)$ rather than the more explicit (2.8). Define

$$F^8_{(p)}(a,b,x) = (F^{8,1},F^{8,2}) = \left(p(x) + \sqrt{(a-x)^2 + b^2} - v,\ p'(x) - \frac{a-x}{\sqrt{(a-x)^2 + b^2}}\right)$$

Then, $\Phi(p)$ is defined implicitly by $F^8_{(p)}(a,b,x) = (0,0)$. By the Implicit Function Theorem,

$$a'(x^\circ) = \frac{\det \begin{vmatrix} -\frac{\partial F^{8,1}}{\partial y} & \frac{\partial F^{8,1}}{\partial b} \\ -\frac{\partial F^{8,2}}{\partial y} & \frac{\partial F^{8,2}}{\partial b} \end{vmatrix}}{D}$$

$$b'(x^\circ) = \frac{\det \begin{vmatrix} \frac{\partial F^{8,1}}{\partial a} & -\frac{\partial F^{8,1}}{\partial y} \\ \frac{\partial F^{8,1}}{\partial a} & -\frac{\partial F^{8,1}}{\partial y} \end{vmatrix}}{D},$$

where $$D = \det \begin{vmatrix} \frac{\partial F^{8,1}}{\partial a} & \frac{\partial F^{8,1}}{\partial b} \\ \frac{\partial F^{8,2}}{\partial a} & \frac{\partial F^{8,2}}{\partial b} \end{vmatrix}$$

and all partial derivatives are evalutated at (a,b,x°).

Since $\frac{\partial F^{8,1}}{\partial y}(a,b,x^\circ) = F^{8,2}(a,b,x^\circ) = 0$, one calculates quickly that

$$\frac{a'(x^\circ)}{b'(x^\circ)} = \frac{-b(x^\circ)}{a(x^\circ) - x^\circ} < 0 .$$

To prove the second statement in Proposition 4.3, let $F^9(a,b,x,v)$ be $F^8(a,b,x)$ with v allowed to vary also. Then, evaluating at $(a°,b°,x°,v°) \in (F^9)^{-1}(0,0)$,

$$DF^9(a°,b°,x°,v°) = \begin{pmatrix} \frac{a-y}{d^{1/2}} & \frac{b}{d^{1/2}} & 0 & -1 \\ -\frac{b^2}{d^{3/2}} & \frac{(a-y)b}{d^{3/2}} & \frac{\partial F^{9,2}}{\partial y} & 0 \end{pmatrix} .$$

where $d = [(a - y)^2 + b^2]$. Since $d > 0$, $DF^9(a°,b°,x°,v°)$ has maximal rank, i.e., rank two. Since $(0,0)$ is a regular value of F^9, $(F^8)^{-1}(0,0)$ is a smooth curve in (a,b,x)-sphere for generic v. To show that this curve is vertical at only finitely many points for a generic v, we need to show that $(0,0,0)$ is a regular value of $F^{10}(a,b,x,v) = (F^9(a,b,x,v), \frac{\partial F^{9,2}}{\partial x}(a,b,x,v))$, following the argument in Proposition 4.1. One checks readily that $DF^{10}(a,b,y,v) =$

$$\begin{pmatrix} \frac{a-y}{d^{1/2}} & bd^{-1/2} & 0 & -1 \\ -\frac{b^2}{d^{3/2}} & b(a-y)d^{-3/2} & 0 & 0 \\ -3b^2(a-y)d^{-5/2} & (2b(a-y)^2-b^3)d^{-5/2} & \frac{\partial^2 F^{9,2}}{\partial x^2} & 0 \end{pmatrix}$$

evaluated at $(a,b,y,v) \in (F^{10})^{-1}(0,0,0)$. Since the determinant of the lower left-hand 2×2 sub-matrix is $b^3 d^{-3} \neq 0$, DF^{10} is surjective and $(0,0,0)$ is a regular value of F^{10}. Consequently, for any $p \in C_1$, $\Phi(p)$ is a smooth regular curve, sloping downward in the (a,b)-plane, except possibly for a finite number of cusp points.

The $Z(p)$-curves have a similar structure, as Proposition 4.4 demonstrates.

Proposition 4.4. For any $p \in C$, the tangent line to $Z(p)$ at any regular point of this curve has a positive slope.

Proof: The proof of this proposition is similar to that of the previous proposition and will only be sketched. Let $(a(x^\circ),b(x^\circ))$ be a regular point of $Z(p)$, in the parameterization given by equations (2.5). We want to show that $\frac{b'(x_0)}{a'(x_0)} > 0$.

Use the $F^{1}_{(p)}$Ia,b,x) of Proposition 4.1,

$$\frac{b'(x^\circ)}{a'(x^\circ)} = \frac{\det\begin{vmatrix} -\dfrac{\partial F_1^1}{\partial x} & \dfrac{\partial F_1^1}{\partial b} \\ -\dfrac{\partial F_2^1}{\partial x} & \dfrac{\partial F_2^1}{\partial b} \end{vmatrix}}{\det\begin{vmatrix} \dfrac{\partial F_1^1}{\partial a} & -\dfrac{\partial F_1^1}{\partial x} \\ \dfrac{\partial F_2^1}{\partial a} & -\dfrac{\partial F_2^1}{\partial x} \end{vmatrix}}$$

$$= \frac{-\dfrac{\partial F_1^{\,1}}{\partial b}}{\dfrac{\partial F_1^{\,1}}{\partial a}} \quad , \text{ since } \frac{\partial F_1^{\,1}}{\partial x} = F_2^1 = 0$$

$$= \frac{\dfrac{b}{\sqrt{a^2 + b^2}} - \dfrac{b}{\sqrt{(a-x)^2 + b^2}}}{\dfrac{a-x}{\sqrt{(a-x)^2 + b^2}} - \dfrac{a}{\sqrt{a^2 + b^2}}}$$

$$= \frac{\sin\phi - \sin\theta(x)}{\cos\theta(x) - \cos\phi}$$

$$> \quad 0 \ , \text{ since } \ 0 < \phi < \theta(x) \ ,$$

where $\theta(x)$ is the angle defined in section 2 and ϕ is the angle that the vector $\overrightarrow{(a,b)}$ makes with the positive a-axis. See figure 4.1.

The curves $Z(p)$ and $\Phi(p)$ bound the domain of ξ, the choice function. We next study curves which affect the continuity of ξ within this domain, beginning with the curve $\Gamma(p)$, across which the number of critical points of $C^{(a,b)}$ changes. More precisely, for $p \in C_1$, let $\Gamma(p)$ be the set (a,b) in $B \subset \mathbb{R}^2_+$ for which $C^{(a,b)}$ has a degenerate critical point, i.e.,

$$\Gamma(p) \equiv \{(a,b) \in B \mid \text{for some } y > 0 , C_x^{(a,b)}(y) = 0 \text{ and } C_{xx}^{(a,b)}(y)=0\}.$$

If $(a^\circ,b^\circ) \notin \Gamma(p)$, then for all (a,b) in some neighborhood of (a°,b°), each $C^{(a,b)}$ will have the same number of critical points and the same number of local minima. Proposition 4.5 below is the analogue of Propositions 4.1 and 4.2 for the curve $\Gamma(p)$.

Proposition 4.5. There is a residual set A_3 of p's in C_1 so that for any $p \in A_3$, there is a finite subset $S_3(p)$ of (a,b)'s in $B \subset \mathbb{R}^2_+$ which contains all the self-crossings and all the singular points of $\Gamma(p)$. Furthermore, $\Gamma(p)$ is a cusp at these singular points and $\Gamma(p)$ has no triple crossings for $p \in A_3$.

Proof: Let $F^{11}_{(p)}(a,b,y) = \left(p'(y) - \frac{a-y}{\sqrt{(a-y)^2+b^2}}, p''(y) + \frac{b^2}{[(a-y)^2+b^2]^{3/2}}\right)$. Then $(a,b) \in \Gamma(p)$ if and only if $F^{11}_{(p)}(a,b,y) = (0,0)$ for some y. Furthermore, one easily checks that $\Gamma(p)$ has the parameterization

$$a(y) = y - \frac{p'(y)p''(y)}{[1-p'(y)^2]^2} ,$$

(4.1)

$$b(y) = - \frac{p''(y)}{[1-p'(y)^2]^{3/2}} .$$

The proof of Proposition 4.5 is similar to that of Proposition 4.1 and will only be sketched here. Let $F^{12}(p,a,b,y) = F^{11}_{(p)}(a,b,y)$. Since

$$D_p F^{12}(p^\circ,a^\circ,b^\circ,y^\circ)q = (q'(y^\circ),q''(y^\circ)) ,$$

DF^{12} is surjective and $(0,0)$ is a regular point of F^{12} . By the Thom Transversality Theorem, $(0,0)$ is a regular value of $F^{11}_{(p)}$ for a residual set $A_{3,1}$ of p's C . For these p's , $(F^{11}_{(p)})^{-1}(0,0)$ is a smooth regular curve in (a,b,y)-space. One next shows that for such p's , the set of points where $(F^{11}_{(p)})^{-1}(0,0)$ has a vertical tangent is a discrete set in (a,b,y) space (by showing that $(0,0,0)$ is a regular value of $(p,a,b,y) \mapsto (C^{(a,b)}_x(y),C^{(a,b)}_{xx}(y),C^{(a,b)}_{xxx}(y)))$. Such p's will be the singular points of $\Gamma(p)$, and one can show that generically they are non-degenerate cusps. Similarly, one shows by using the techniques of the proof of Proposition 4.1 that $\Gamma(p)$ has a discrete set of double points and no triple points for a residual subset $A_{3,2} \subset A_{3,1} \subset C$. ∎

In any component of $B \setminus \Gamma(p)$, each $C^{(a,b)}$ has the same number of local minima - all non-degenerate, of course. However, $\xi(a,b)$, the <u>global</u> minimizer of $C^{(a,b)}$, may be discontinuous within this component if one local minimizer replaces another as global minimizer. To keep track of this phenomenon, we need to look at the set of all (a,b) such that $C^{(a,b)}$ does not have a unique (interior) global minimizer. More generally, let $\Sigma(p)$ be the set of all (a,b) such that $C^{(a,b)}$ has at least two critical points which take on the same critical value. Formally,

$\Sigma(p) = \{(a,b) \in B \mid$ there exist $y_1 \neq y_2$ in $\mathbb{R}_+$ such that

$$C_x^{(a,b)}(y_1) = p'(y_1) - \frac{a-y_1}{\sqrt{(a-y)^2+b^2}} = 0 ,$$

$$C_x^{(a,b)}(y_2) = p'(y_2) - \frac{a-y_2}{\sqrt{(a-y_2)^2+b^2}} = 0 , \text{ and}$$

$$C^{(a,b)}(y_1) = C^{(a,b)}(y_2) , \text{ i.e.,}$$

$$p(y_1) + \sqrt{(a-y_1)^2+b^2} = p(y_2) + \sqrt{(a-y_2)^2+b^2} .$$

<u>PROPOSITION 4.6</u>. There is a residual subset A_4 of C_1 such that for $p \in A_4$, there is a finite set $S_4(p)$ in the (a,b)-plane which contains all the self-crossings and all the singular points of $\Sigma(p)$. At the singular points, $\Sigma(p)$ inter-

sects $\Gamma(p)$. Furthermore, for $p \in A_4$, $\Sigma(p)$ has no triple crossings.

<u>Proof</u>: The proof of Proposition 4.6, although analogous to that of previous propositions in this section, is slightly complicated by the fact that $\Sigma(p)$ is defined by a map from $\mathbb{R}^2 \times (\mathbb{R}^2 \setminus \Delta^2)$ into $\mathbb{R}^3$. Define $F^{13}: C \times \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \Delta^2) \to \mathbb{R}^3$ by

$$F^{13}(p,a,b,y_1,y_2) = (p'(y_1) - \frac{a-y_1}{\sqrt{(a-y)^2+b^2}}, p'(y_2) - \frac{a-y_2}{\sqrt{(a-y_2)^2+b^2}}, p(y_1) + \sqrt{(a-y_1)^2+}$$

$$- p(y_2) - \sqrt{(a-y_2)^2 - b^2}) .$$

To see that $(0,0,0)$ is a regular value of F^{13} , note that

$$D_p F^{13}(p^\circ, a^\circ, b^\circ, u_1^\circ, y_2^\circ)q = (q'(y_1^\circ), q'(y_2^\circ), q(y_1^\circ) - q(y_2^\circ)) ,$$

which is clearly a sujective map from C onto $\mathbb{R}^3$ since $y_1^\circ \neq y_2^\circ$. Letting $F^{14}_{(p)}(a,b,y_1,y_2) = F^{13}(p,a,b,y_1,y_2)$, it follows from the Thom Transversality Theorem that, for a residual subset of p's $\tilde{A}_{4,1}$ in C , $(F^{14}_p)^{-1}(0,0,0)$ is a regular curve in $\mathbb{R}^2 \times (\mathbb{R}^2 \setminus \Delta^2)$.

We next show that this curve generically projects to a nice curve in the (a,b)-plane, with only a finite number of singularities.

By the Implicit Function Theorem, at the points where this projection is singular, the vectors

$$\left(\frac{\partial F_1^{13}}{\partial y_1}, \frac{\partial F_2^{13}}{\partial y_1}, \frac{\partial F_3^{13}}{\partial y_1}\right) \text{ and } \left(\frac{\partial F_1^{13}}{\partial y_2}, \frac{\partial F_2^{13}}{\partial y_2}, \frac{\partial F_3^{13}}{\partial y_2}\right) \text{ are linearly}$$

dependent, where F_1^{13}, F_2^{13}, and F_3^{13} are the three components of F^{13}. However, $\frac{\partial F_1^{13}}{\partial y_2} = \frac{\partial F_2^{13}}{\partial y_1} = 0$; and on $(F^{13})^{-1}(0,0,0)$,

$\frac{\partial F_3^{13}}{\partial y_1} = F_1^{13} = 0$ and $\frac{\partial F_3^{13}}{\partial y_2} = F_2^{13} = 0$. Consequently, the projections of $(F^{13})^{-1}(0,0,0)$ is singular at points where

where the vectors $\left(\frac{\partial F_1^{13}}{\partial y_1},0,0\right)$ and $\left(0,\frac{\partial F_2^{13}}{\partial y_2},0\right)$ are linearly independent, i.e., where

$$\frac{\partial F_1^{13}}{\partial y_1} \cdot \frac{\partial F_2^{13}}{\partial y_2} = C_{yy}^{(a,b)}(y_1) \cdot C_{yy}^{(a,b)}(y_2) = 0 \ .$$

Now, the usual argument shows that $(0,0,0,0,0)$ is a regular value of

$$F_{(p)}^{14}(a,b,y_1,y_2) = (C_y^{(a,b)}(y_1), C_y^{(a,b)}(y_2), C_y^{(a,b)}(y_1) - C^{(a,b)}(y_2) ,$$

$$C_{yy}^{(a,b)}(y_1) , C_{yy}^{(a,b)}(y_2))$$

for a residual set $A_{4,2}$ of p's in C. Since the range of $F^{14}_{(p)}$ is 5-dimensional, while its domain is 4-dimensional, $(0,0,0,0,0)$ a regular value of $F^{14}_{(p)}$ means that $(F^{14}_{(p)})^{-1}(0,0,0,0,0)$ is empty.

To finish the proof that $(F^{13})^{-1}(0,0,0)$ generically projects nicely into (a,b)-space, we must show that $(0,0,0,0)$ is a regular value of $F^{15}: C \times \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \Delta^2) \to \mathbb{R}^4$

$$F^{15}(p,a,b,y_1,y_2) = (F^{13}(p,a,b,y_1,y_2), C^{(a,b)}_{yy}(y_1) \cdot C^{(a,b)}_{yy}(y_2)) .$$

The derivative of F^{15} with respect to p at $(p°,a°,b°,y_1°,y_2°)$ is $D_pF^{15}(p°,a°,b°,y_1°,y_2°,)q = (q'(y_1°), q'(y_2°), q(y_1°) - q(y_2°)$,

$$C_{yy}^{(a,b)}(y_1°)q''(y_2°) + C_{yy}^{(a,b)}(y_2°)q''(y_1°))$$

If $p° \in A_{4,2}$ and $F^{13}(a°,b°,y_1°,y_2°,) = (0,0,0)$, exactly only of $C_{yy}^{(a,b)}(y_1°)$ and $C_{yy}^{(a,b)}(y_2°)$ is non-zero; and D_pF^{15} is easily seen to be surjective. So, $(0,0,0,0)$ is a regular value of F^{15} for $p° \in A_{4,2}$; and there is a residual set $A_{4,3} \subset A_{4,2} \subset C$ of p's for which $\Sigma(p)$ has only finitely many singular points. Furthermore, the above argument indicates that $\Sigma(p)$ and $\Gamma(p)$ intersect at these points.

To show that crossings points of $\Sigma(p)$ generically form a discrete subset of $\Sigma(p)$ and that triple crossings do not occur,

we must show that $(0,0,0,0,0)$ is a regular value of $F^{16}: C \times \mathbb{R}^2 \times (\mathbb{R}^3 \setminus \Delta^3) \to \mathbb{R}^5$ defined by
$F^{16}(p,a,b,y_1,y_2,y_3) =$

$$(C_y^{(a,b)}(y_1), C_y^{(a,b)}(y_2), C_y^{(a,b)}(y_3), C^{(a,b)}(y_1)-C^{(a,b)}(y_2), C^{(a,b)}(y_1)-C^{(a,b)}(y_3))$$

and that $(0,0,0,0,0,0,0)$ is a regular value of $F^{17}: C \times \mathbb{R}^2 \times (\mathbb{R}^4 \setminus \Delta^4) \to \mathbb{R}^7$, defined by

$$F^{17}(p,a,b,y_1,y_2,y_3,y_4) = (C_y^{(a,b)}(y_1), C_y^{(a,b)}(y_2), C_y^{(a,b)}(y_3), C^{(a,b)}{}_y(y_4), C^{(a,b)}(y_1)-C^{(a,b)}(y_2), C^{(a,b)}(y_1)-C^{(a,b)}(y_3), C^{(a,b)}(y_1)-C^{(a,b)}(y_4)) .$$

The proofs of both of these facts are analogous to previous computations and will be omitted. ∎

To conclude our argument that $Z(p), \Phi(p), \Gamma(p)$, and $\Sigma(p)$ divide B into a finite number of well-behaved regions, we now show that for a generic $p \in C_1$ these 4 curves cross each other at only finitely many points in B.

PROPOSITION 4.7. There is a residual subset A_5 of C_1 such that for $p \in A_5$ the curves $Z(p), \Phi(p), \Gamma(p)$ and $\Sigma(p)$ cross each other at only finitely many points in B in the (a,b)-plane. For $p \in A_5$, there are no triple crossings.

<u>Proof</u>: The proof is a straightforward application of transversality theory. To show that the intersection of $Z(p)$ and $\Gamma(p)$ is generically a finite set, one must show that $(0,0,0)$ is generically a regular value of

$$H^1_{(p)}(a,b,y) = (C_y^{(a,b)}(y), C^{(a,b)}(y) - C^{(a,b)}(0), C_{yy}^{(a,b)}(y)) .$$

To prove that Z and Σ intersect generically in a finite set, one must show that $(0,0,0,0)$ is generically a regular value of

$$H^2_{(p)}(a,b,y_1,y_2) = (C_y^{(a,b)}(y_1), C_y^{(a,b)}(y_2), C^{(a,b)}(y_1) - C^{(a,b)}(0), C^{(a,b)}(y_2) - C^{(a,b)}(0)) ,$$

where $y_1 \neq y_2$, and that $(0,0,0,0,0)$ is generically a regular value of

$$H^3_{(p)}(a,b,y_1,y_2,y_3) = C_y^{(a,b)}(y_1), C_y^{(a,b)}(y_2), C_y^{(a,b)}(y_3), C^{(a,b)}(y_1) - C^{(a,b)}(y_2) , C^{(a,b)}(y_3) - C^{(a,b)}(0)) ,$$

where $(y_1,y_2,y_3) \in \mathbb{R}^3_+ \setminus \Delta^3$. To show that Φ and Γ intersect in a finite set for generic p, one must show that $(0,0,0)$ is generically a regular value of

$$H^4(p)(a,b,y) = (C^{(a,b)}(y) - v, C_y^{(a,b)}(y), C_{yy}^{(a,b)}(y))$$

and that $(0,0,0,0)$ is generically a regular value of

$$H^5(p)(a,b,y_1,y_2) = (C^{(a,b)}(y_1) - v, C_y^{(a,b)}(y_2), C_{yy}^{(a,b)}(y_2))$$

on the set $\mathbb{R}_+^2 \times (\mathbb{R}_+^2 \setminus \Delta^2)$. The mappings to be considered for $\Phi \cap \Sigma$ are $H^6_{(p)}: \mathbb{R}_+^2 \times (\mathbb{R}_+^2 \setminus \Delta^2) \to \mathbb{R}^4$ and $H^7_{(p)}$: $\mathbb{R}_+^2 \times (\mathbb{R}_+^3 \setminus \Delta^3) \to \mathbb{R}^5$ defined by

$$H^6_{(p)}(a,b,y_1,y_2) = (C_y^{(a,b)}(y_1), C_y^{(a,b)}(y_2), C^{(a,b)}(y_1) - C^{(a,b)}(y_2), C^{(a,b)}(y_1) - v) \quad \text{and}$$

$$H^7_{(p)}(a,b,y_1,y_2,y_3) = (C_y^{(a,b)}(y_1), C_y^{(a,b)}(y_2), C_y^{(a,b)}(y_3), C^{(a,b)}(y_1) - C^{(a,b)}(y_2), C^{(a,b)}(y_3) - v).$$

The mappings to be considered for $\Sigma \cap \Gamma$ are $H^7_{(p)}: \mathbb{R}_+^2 \times (\mathbb{R}_+^2 \setminus \Delta^2) \to \mathbb{R}^4$ and $H^8_{(p)} \mathbb{R}_+^2 \times (\mathbb{R}_+^3 \setminus \Delta^3) \to \mathbb{R}^5$ defined by

$$H^{7}_{(p)}(a,b,y_1,y_2) = (C_y^{(a,b)}(y_1), C_y^{(a,b)}(y_2), C^{(a,b)}(y_1) - C^{(a,b)}(y_2), C_{yy}^{(a,b)}(y_1)) \quad \text{and}$$

$$H^{8}_{(p)}(a,b,y_1,y_2,y_3) =$$

$$(C_y^{(a,b)}(y_1), C_y^{(a,b)}(y_2), C_y^{(a,b)}(y_3), C^{(a,b)}(y_1) - C^{(a,b)}(y_2), C_{yy}^{(a,b)}(y_3)) .$$

The proofs for all these mappings is a straightforward application of transversality theory as illustrated in the earlier proofs in this section.

The only situation which offers any complication is the intersection of Z and Φ, which can be written as $(H^{9}_{(p)})^{-1}(0,0,0)$ and $(H^{10}_{(p)})^{-1}(0,0,0,0)$, where

$$H^{9}_{(p)}(a,b,y) = (C_y^{(a,b)}(y), C^{(a,b)}(y) - v, C^{(a,b)}(y) - C^{(a,b)}(0)) , \text{ and}$$

$$H^{10}_{(p)}(a,b,y_1,y_2) = (C_y^{(a,b)}(y_1), C_y^{(a,b)}(y_2), C^{(a,b)}(y_1) - v, C^{(a,b)}(y_2) - 0) ,$$

where $y_1 \neq y_2$. While $H^{10}_{(p)}$ can be treated in the usual way, it is simpler to replace $H^{9}_{(p)}$ by $\tilde{H}^{9}_{(p)} : T_v \times \mathbb{R}_+ \to \mathbb{R}^2$ defined by

$$\tilde{H}^{9}_{(p)}((a,b),y) = (C_y^{(a,b)}(y), C^{(a,b)}(y) - v) ,$$

where $T_v \equiv \{(a,b) \in \mathbb{R}^2_+ | C^{(a,b)}(0) - v = 0 \text{ , i.e., } \sqrt{a^2 + b^2} = v\}$.

One uses the usual technique to show that $(0,0)$ is a regular value of $\tilde{H}^9_{(p)}$ for generic p. Since the range and domain of $\tilde{H}^9_{(p)}$ are two-dimensional, it follows that $Z(p) \cap \Phi(p)$ is a finite set for generic p.

The demonstration that there are no triple crossings for a generic p follows along the usual lines and will be omitted. ∎

For a fixed price function $p \in C_1$, one can always guarantee that $\Phi(p)$ intersects the $Z(p)$ and $\Gamma(p)$ curves only finitely often by proper choice of the value number v.

PROPOSITION 4.8. For any fixed $p \in C_1$, there is an open and dense set values v in $\mathbb{R}_+$ such that the corresponding $\Phi_v(p)$ curve intersects $Z(p)$ and $\Gamma(p)$ only finitely often.

Proof: We will prove this proposition for $Z(p)$ since the proof for $\Gamma(p)$ is similar (and, in fact, can also be deduced from Proposition 4.3). Recall the parameterization $(a(x),b(x))$ of $Z(p)$ as given by equations (2.6). We need only show that v is a regular value of

$$C^{(a(x),b(x))}(x) = C^{(a(x),b(x))}(0) =$$

$$= \left[\left(x + \frac{p'(x)}{2} \cdot \frac{x^2-p^2(x)}{p(x)-p'(x)x}\right)^2 + \left(\frac{\sqrt{1-p'(x)^2}}{2} \cdot \frac{x^2-p^2(x)}{p(x)-p'(x)x}\right)^2\right]^{1/2} .$$

By Sard's Theorem, the set of finite regular values of this map is open and dense for any $p \in C_1$. ∎

Finally, we summarize the results of this section by letting A be the intersection of the residual subsets described in Proposition 4.1, 4.2, 4.5,4.6, and 4.7.

<u>Theorem 4.9</u>. There exists a residual subset A of price functions p in C_1 with the property that for any $p \in A$ there is a finite set of points $S(p)$ in B in the (a,b)-plane which contains:

i) all the singular points of $Z(p), \Phi(p), \Gamma(p)$, and $\Sigma(p)$,

ii) all the self-crossings of $Z(p), \Phi(p), \Gamma(p)$, and $\Sigma(p)$,

iii) all the crossings of Z, Φ, Γ, and Σ with each other.

If ii) or iii) holds at $(a,b) \in S(p)$, only two of the above curves pass through (a,b) . If (a,b) is a singular point of $Z(p)$, $\Phi(p)$, or $\Gamma(p)$, it is a non-degenerate cusp. Singular points of $\Sigma(p)$ lie on $\Gamma(p)$. In particular, the complement of $Z(p) \cup \Phi(p) \cup \Gamma(p) \cup \Sigma(p) \cup R(0)$ is a finite union of $\{B_i(p)\}_{i=1}^{k(p)}$ of open sets, each homeomorphic to a two-dimensional disk.

§5. GENERIC CHOICE AND DEMAND FUNCTIONS

The goal of this section is to prove our main results: i) for every $p \in C_1$, μ is continuous; ii) for a residual set of p in C_1, ξ is C^∞ except on a set of measure zero (in fact, except on a finite union of smooth curves); and iii) for this same residual set, μ is C^∞ except at a finite number of x-values. In Section 7, we will discuss hypotheses on p which guarantee that the corresponding μ is C^1 (or even C^∞) everywhere.

Theorem 5.1. For every $p \in C_1$, the corresponding cumulative demand function $\mu_p(x)$ is continuous.

Proof: Since continuity is a local property, we need only work with an interval about some x^* on $\mathbb{R}_+$. Recall that $R(x^*)$ is the ray from $(x^*, 0)$ into the positive quadrant which meets the positive x-axis in an acute angle $\theta(x^*)$ where

$$\tan \theta(x^*) = \frac{\sqrt{1 - p'(x^*)^2}}{p'(x^*)} ,$$

and that $\xi(a, b) = x^*$ implies that $(a, b) \in R(x^*)$. Since p' is smooth and lies between 0 and 1, $\theta(x)$ is a smooth function of x. Let $(a(x), b(x))$ be the continuous curve $\Phi(p)$ as described by equations (2.8). For $x \neq x^*$, $|\mu(x) - \mu(x^*)|$ is bounded above by the area between the rays $R(x)$ and $R(x^*)$ and under the curve $\Phi(p)$. Since $\theta(x)$ is continous, this area goes to zero as x approaches x^*. Consequently, $|\mu(x) - \mu(x^*)| \to 0$ as $|x - x^*| \to 0$, i.e., μ is continuous at x^*. ∎

What we would really like to know is how frequently can μ and ξ be C^1. We first work with ξ.

Theorem 5.2. Let A be the residual subset of price functions p in C_1 described in Theorem 4.9. For $p \in A$, let $\xi_p(a,b)$ be the usual choice function:

$$\xi_p(a, b) = \{x | x \text{ is a global minimizer of } C^{(a,b)}\} .$$

Then, there exists a set $\Delta(p)$ of measure zero in $\mathbb{R}_+^2$, such that ξ is a C^∞ function on $\mathbb{R}_+^2 \setminus \Delta(p)$. In fact, $\Delta(p)$ is a finite union of smooth curves.

Proof: Consider one of the cells $B_j(p)$ defined in the statement of Theorem 4.9. Let $\Delta(p) = Z(p) \cup \Phi(p) \cup \Gamma(p) \cup \Sigma(p) \cup R(0)$. So, $B_j(p)$ is a component of $B - \Delta(p)$. We claim that each $C^{(a,b)}$ has qualitatively the same graph for each (a, b) in $B_j(p)$. Since $\Sigma(p)$ does not cross $B_j(p)$, each $C^{(a,b)}$ has a unique global minimizer $\xi(a, b)$ for $(a, b) \in B_j(p)$, which varies continuously with (a, b). Since $Z(p)$ does not cross $B_j(p)$, either $\xi(a, b) = 0$ for all $(a, b) \in B_j(p)$ or $\xi(a, b) > 0$ for all (a, b) in $B_j(p)$. Since $\Delta(p)$ does not cross $B_j(p)$, each $\xi(a, b)$ is a non-degenerate critical point of each $C^{(a,b)}$. By the Implicit Function Theorem, $\xi(a, b)$ is a C^∞ function of (a, b) on $B_j(p)$. Since $\Delta(p)$ is a finite union of smooth curves, it has measure zero in $\mathbb{R}_+^2$. ∎

Before stating the corresponding result for μ , we prove a proposition which formalizes some (fairly obvious) structure on ξ and μ .

PROPOSITION 5.3. Let $p \in C_1$; let $(a, b) \in \mathbb{R}_+^2$ with $\xi(a, b) = x$. If (a', b') lies on $R(x)$ between $(x, 0)$ and (a, b) ,then $\xi(a', b') = x$.

Proof: One needs to show that $C^{(a',b')}(y) \geq C^{(a',b')}(x)$ for all $y \in \mathbb{R}_+$. Suppose first that $y > x$. Of course, we can assume that $y < a'$. Let α be the (acute) angle that the ray from $(y, 0)$ to (a, b) makes with the positive a-axis. Let α' be the angle that the ray from $(y, 0)$ to (a', b') makes with the a-axis. See Figure 5.1.

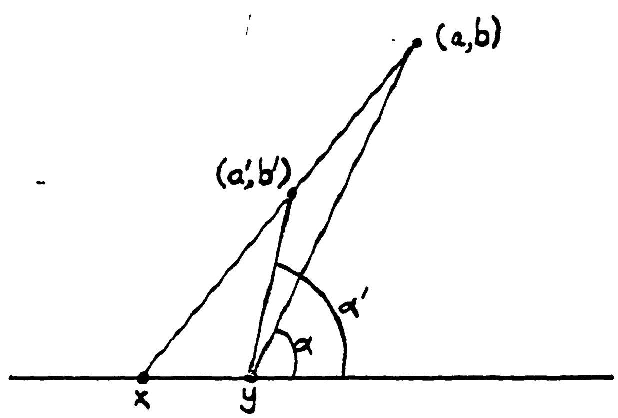


Figure 5.1

Since (a', b') lies between $(x, 0)$ and (a, b) on $R(x)$, $0 < \alpha < \alpha' < \pi$ and $\cos \alpha' < \cos \alpha$. In other words,

$$\frac{a' - y}{\sqrt{(a' - y)^2 + b'^2}} < \frac{a - y}{\sqrt{(a - y)^2 + b^2}} \quad \text{and}$$

$$p'(y) - \frac{a' - y}{\sqrt{(a' - y)^2 + b'^2}} > p'(y) - \frac{a - y}{\sqrt{(a - y)^2 + b^2}} \quad \text{, i.e.,}$$

$$C_x^{(a',b')}(y) > C_x^{(a,b)}(y) \quad \text{for all} \quad y \in (x, a') \ .$$

Consequently, $\int_x^y C_x^{(a',b')}(s)\,ds > \int_x^y C_x^{(a,b)}(s)\,ds$ and

$$(5.1) \quad C^{(a',b')}(y) - C^{(a',b')}(x) > C^{(a,b)}(y) - C^{(a,b)}(x) \ .$$

Since $\check{x}$ minimizes $C^{(a,b)}$, the right-hand term in non-negative in (5.1). One can therefore conclude that

$$C^{(a',b')}(y) \geq C^{(a',b')}(x) \quad \text{for all} \quad y \in (x, a') \ .$$

A similar argument shows that

$$C^{(a',b')}(y) \geq C^{(a',b')}(x) \quad \text{for all} \quad y \in (0, x) \ .$$

and the Proposition follows. ∎

The proof of Proposition 5.3 also demonstrates the following result.

PROPOSITION 5.4. For any $p \in C_1$, the set of $\{(a, b) \in \mathbb{R}^2_+ | \xi(a, b) = 0\}$ is star-like with respect to the origin. In other words, if $\xi(a, b) = 0$, then $\xi(a', b') = 0$ for all (a', b') on the line segment between (a, b) and $(0, 0)$.

(The set $\xi^{-1}(0)$ is probably even convex.)

Before proceeding to our main theorem, we need one more technical result.

PROPOSITION 5.5. There is a residual set $\tilde{A}$ in C_1 such that for any $p \in \tilde{A}$, there is no (a, b, x) such that $C_x^{(a,b)}(x) = 0$, $C_{xx}^{(a,b)}(x) = 0$, $C_{xxx}^{(a,b)}(x) = 0$, $C_{xxxx}^{(a,b)}(x) = 0$. In particular, for $p \in \tilde{A}$, each $C^{(a,b)}$ has only finitely many critical points.

Proof: To prove the first statement, we must show that $(0, 0, 0, 0)$ is a regular value of the map

$$(p, a, b, x) \longmapsto (C_x^{(a,b)}(x), C_{xx}^{(a,b)}(x), C_{xxx}^{(a,b)}(x), C_{xxxx}^{(a,b)}(x)).$$

However, this follows by the methods of Section 4. Now, if $p \in \tilde{A}$ and if x is a critical point of the corresponding $C^{(a,b)}$ for some (a, b), then the second, third, or fourth derivative of $C^{(a,b)}$ is non-zero at x . This implies that the critical points of $C^{(a,b)}$ are isolated. ∎

We now state and prove the main theorem.

<u>THEOREM 5.6</u>. Let A and $\tilde{A}$ be the residual subsets of C_1 defined in Theorem 4.9 and Proposition 5.5. Let p be a price function in the residual set $A \cap \tilde{A}$, and let $\mu(x)$ be the cumulative demand function for p . Then, not only is μ continuous (Theorem 5.1), but except for a <u>finite</u> (possibly empty) set of points in $\mathbb{R}_+$ μ is a C^∞ function.

<u>Proof</u>: Let $\hat{R}(x) = \{(a, b) \in R(x) | \xi(a, b) = x\}$. By Proposition 5.3, each $\hat{R}(x)$ is connected and the union of the $\hat{R}(x)$'s fill up the set on which ξ is positive. Off $\Sigma(p)$, $\hat{R}(x) \cap \hat{R}(y)$ is empty for $x \neq y$. Let $A(x)$, $B(x)$ be the upper endpoint of $\hat{R}(x)$. Then, $(A(x), B(x))$ must lie on $\Delta(p) \equiv Z(p) \cup \Phi(p) \cup \Gamma(p) \cup \Sigma(p)$, i.e., cannot lie in the interior of some $B_j(p)$. For, if it did lie in the interior of $B_j(p)$, the smoothness of ξ there (see Theorem 5.1) would enable us to extend the endpoint of $\hat{R}(x)$ beyond $(A(x), B(x))$. Let S denote the finite (by Theorem 4.9) set of points which are singular points or crossing points for $\Delta(p)$. If $(A(x), B(x)) \notin S$, then $(A(x), B(x))$ is a smooth, regular curve in a neighborhood, say (x_1, x_2) , of x . It follows from Proposition 2.2 that

$$\mu(x;\ x_1) = \text{area}\{(a,\ b) \mid x_1 \leq \xi(a,\ b) \leq x\}$$

is a C^∞ function on $(x_1,\ x_2)$.

Thus, the only possible values of x at which μ is not C^∞ are those x such that $(A(x),\ B(x)) \in S$. By Proposition 5.5, only finitely many $R(x)$'s meet at any point in the (a,b)-plane, since $p \in \tilde{A}$. Letting $x_1, \ldots, x_r$ be the points on the x-axis at which $(A(x_i),\ B(x_i)) \in S$, it follows that μ is C^∞ except possibly on $\{x_1,\ \ldots,\ x_r\}$. ∎

<u>Theorem 5.7</u>. The conclusion of Theorem 5.2 and 5.6 hold for any $p \in C_1$ which is <u>real analytic</u>.

<u>Proof</u>: If $p \in C_1$ is real analytic, then $Z(p)$, $\Phi(p)$, $\Gamma(p)$, and $\Sigma(p)$ will be real analytic curves in B . Since any two real analytic curves in the plane meet each other in a discrete set or coincide (see Simon-Titus (1978)), and since any real analytic curve has a discrete set of singular points, the set S of crossings, self-crossings, and singular points of the Z , Φ , Γ , and Σ curves will still be constant. Since each $C^{(a,b)}$ is real analytic, each will either be constant or have a discrete set of critical points. The case where some $C^{(a,b)}$ is constant is worked out in the example following this proof. So, we can assume that a finite number of $R(x)$'s meet at any $(a,\ b) \in \mathbb{R}^2$. The rest of the proof is identical to that of the previous theorem. ∎

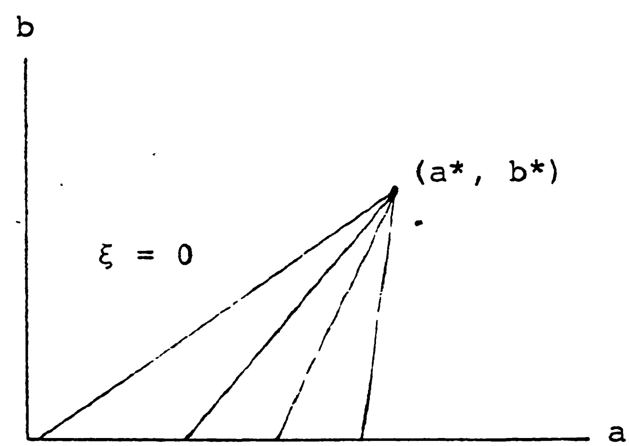
<u>Example</u>. What happens if some $C^{(a^*,b^*)}$ is constant? Then, there exists a constant $w > 0$ such that

$$p(x) = w - \sqrt{(a^* - x)^2 + b^{*2}} .$$

To keep $p' > 0$, we must restrict to (a, b) with $a \leq a^*$. To have $p(0) = 0$, w must equal $\sqrt{a^{*2} + b^{*2}}$. One now computes easily that the curves of constant choice ξ are line segments joining (a^*, b^*) to the a-axis, as in Figure 5.2. Furthermore,

$$\xi(a, b) = \begin{cases} 0 & , \text{ if } \dfrac{a}{\sqrt{a^2 + b^2}} \leq \dfrac{a^*}{\sqrt{a^{*2} + b^{*2}}} , \\ a^* - b^*\left(\dfrac{a - a^*}{b - b^*}\right) & , \text{ otherwise;} \end{cases}$$

and $\mu(x) = \frac{1}{2} b^* x$, for $0 \leq x \leq a^*$.



<u>Figure 5.2</u>

Example: If p is continuous but not C^1, then the corresponding μ may not be continuous, as the following example indicates. Let $p(x)$ be the following piecewise linear price function:

$$p(x) = \begin{cases} P_1 x & , 0 \le x \le x^* ; \\ P_2 x + (P_1 - P_2)x^* & , x^* \le x < \infty , \end{cases}$$

where $0 < P_1 < P_2 < 1$. So, $p'(x)$ has the graph indicated in figure 5.3.

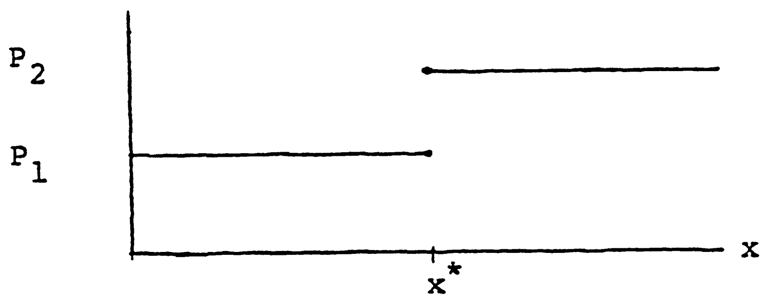


Figure 5.3

Use the analysis of figures 2.1 and 2.2 to describe the choice function ξ. No $B^{a,b}(x)$ will have a graph intersecting both segments in figure 5.3. If (a,b) is such that the graph of $B^{a,b}(x)$ crosses the first segment $(0 < x < x^*, y = P_1)$, say at $x°$, then $\xi(a,b) = x°$ and $\xi^{-1}(x°)$ is a ray $R(x°)$ with $\cos\theta(x_0) = P_1$. If (a,b) is such that the graph of $B^{a,b}(x)$ crosses the second segment $(x^* < x, y = P_2)$, say at x^1, then

$\xi(a,b) = x^1$ and $\xi^{-1}(x^1)$ is a ray $R(x^1)$ with $\cos\theta(x_1) = P_2$. If (a,b) is such that $P_1 \leq B^{a,b}(x^*) \leq P_2$, then $\xi(a,b) = x^*$ since

$$p'(x) - B^{a,b}(x) \quad \begin{cases} < 0 & \text{for } x < x^*, \\ > 0 & \text{for } x > x^*. \end{cases}$$

Accordingly, the level sets of ξ are as pictured in Figure 5.4. Since $\xi^{-1}(x^*)$ has positive area, μ is not continuous at x^*.

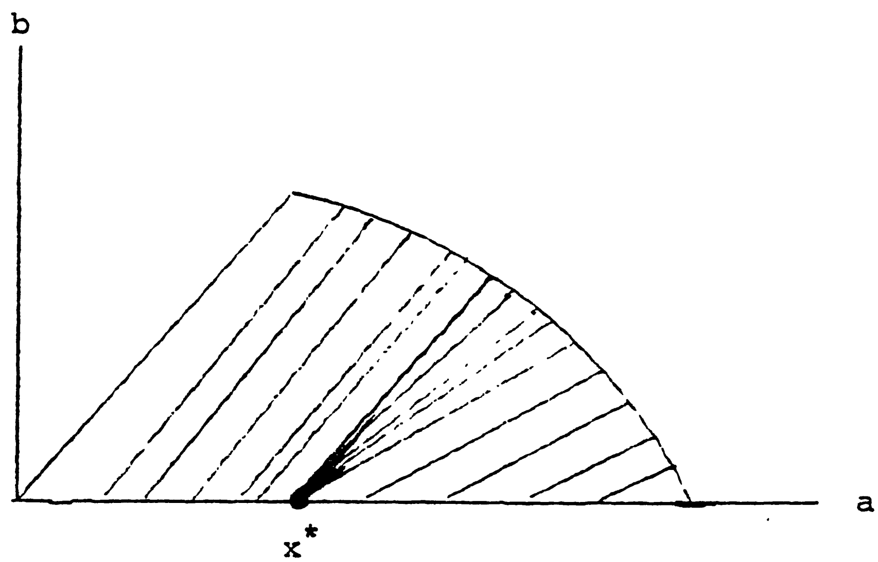


Figure 5.4: Level sets of ξ.

§6. A CLOSER LOOK AT THE SMOOTHNESS OF μ AND ξ.

From the analysis of the previous sections, it follows that there are four ways in which ξ can become discontinous and μ non-smooth; i) across Z, with $\xi(a, b) = 0$ on one side of Z and jumping to non-zero values on the other side; ii) across Φ, with $C^{(a,b)}(\xi(a, b)) < v$ on one side of Φ and $> v$ on the other side; iii) across Γ, where one (degenerate) global minimum can bifurcate into two local minima; and iv) across Σ, where the global minimizer jumps from one local minimizer to another.

In this section, we will analyze more carefully how each of these curves affects ξ and μ and will conclude that it is the self-crossings of $\Sigma(p)$ which leads to the stable continuities of ξ and non-smooth points of μ. We begin by showing that if $\Sigma(p) = \phi$, then ξ is continuous and μ is C^1 everywhere.

PROPOSITION 6.1. For $p \in C_1$, if $\Sigma(p)$ is empty, then ξ is continuous away from $\xi^{-1}(0)$ and μ is C^1 everywhere.

Proof Sketch: For each $x > 0$, let $(\alpha(x), \beta(x))$ be the point in $R(x)$ with the property that $\xi(a, b) = x$ if and only if $(a, b) \in R(x)$ and $a \leq \alpha(x)$, $b \leq \beta(x)$. By arguments of Section 5, each $(\alpha(x), \beta(x))$ must lie on $Z(p) \cup \Phi(p) \cup \Gamma(p) \cup \Sigma(p)$.

Since $\Sigma(p) = \emptyset$, we can disregard it. Suppose $(\alpha(x), \beta(x)) \in \Gamma(p) \setminus [Z(p) \cup \Phi(p)]$. Since $C^{(a,b)}$ is a continuous family of functions, its minimizer $\xi(a, b)$ is lower semicontinuous. (See p. 19 of Debreu (1959.)) Since $\Sigma(p) = \emptyset$, ξ is single-valued off $Z(p)$ and therefore continuous in a neighborhood of $(\alpha(x), \beta(x))$. Now $\xi^{-1}(x)$ lies on $R(x)$ by arguments of Section 2. If $\xi^{-1}(x)$ does not extend beyond $(\alpha(x), \beta(x))$ on $R(x)$, then the level curves of ξ are as sketched in Figure 6.1 and $(\alpha(x), \beta(x))$ must be a local extremum of ξ . However, this would contradict $\xi^{-1}(y) \subset R(y)$ for all y . Therefore, $\xi^{-1}(x)$ does extend beyond $(\alpha(x), \beta(x))$ on $R(x)$. This contradiction to the definition of $(\alpha(x), \beta(x))$ means that we can ignore points of $\Gamma \setminus [Z \cup \Phi]$ in computing ξ and μ .

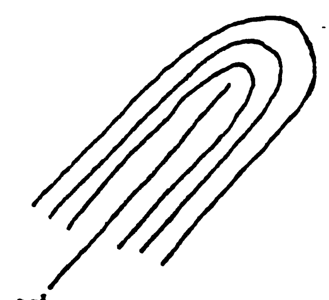


Figure 6.1: Level sets of ξ

Therefore, each $(\alpha(x), \beta(x))$ lies on $Z(p) \cup \Phi(p)$. By Proposition 2.2, we need only show that $(\alpha(x), \beta(x))$ is a continuous curve in x . If $(\alpha(x), \beta(x))$ is on $Z \setminus \Phi$ or $\Phi \setminus Z$, then α and β are continuous in a neighborhood of x by equations (2.6) and (2.8). So, we need only consider a neighborhood of $(\alpha(x^*), \beta(x^*)) \in \Phi(p) \cap Z^*(p)$, where Z^* is a branch of Z across which ξ actually changes from zero to non-zero values.

Recall from Propositions 4.3 and 4.4 that Φ slopes downwards while Z^* slopes upwards. If $(\alpha(x), \beta(x))$ moves from $\bar{\Phi}$ to Z^* at x^* as x increases, then, as in Figure 6.2, Z^* separates part of $\xi^{-1}(0)$ from $\xi^{-1}(x^*)$ and the origin. This contradicts the fact that $\xi^{-1}(0)$ is starlike with respect to the origin (Proposition 5.4). Thus, $(\alpha(x), \beta(x))$ moves from Z^* to Φ at $x = x^*$. It follows that Z^* and Φ can cross each other only once, i.e., $(\alpha(x), \beta(x)) \in Z$ for $x \leq x^*$ and $(\alpha(x), \beta(x))$ Φ for $x \geq x^*$. See Figures 3.2 and 3.3 for

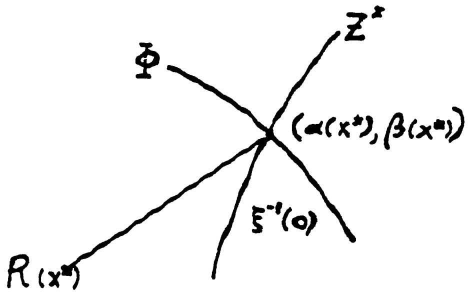


Figure 6.2

examples. Again, by equations (2.6) and (2.8) and Proposition 2.2, it follows that $(\alpha(x), \beta(x))$ is continuous, and that μ is C^1 everywhere and C^∞ except at one value of x.

Consequently, it is the existence of the Σ-curves which causes non-smoothness of μ. There are two phenomena associated with $\Sigma(p)$: i) the intersection of $\Gamma(p)$ and $\Sigma(p)$ curves where one minimum splits into two (and a $\Sigma(p)$-curve begins); and ii) self-crossing of $\Sigma(p)$ at a point (a, b) where $C^{(a,b)}$ has three distinct global minimizers. We now indicate that, for generic p, case i) does not hurt the smoothness of μ_p, while case ii) does.

PROPOSITION 6.2. There is a residual subset A' of C_1 such that if $p \in A'$ and if (a, b) is a point of $\Gamma(p) \cap \Sigma(p)$ with $\xi(a, b) = x$, then μ_p is C^∞ at x.

Proof: The proof is an application of Thom's Catastrophe Theorem. However, although most of Zeeman's applications of this Theorem (see Zeeman (1977)) use the "delay rule" to choose which of several local minima plays the major role, our model uses "Maxwell's Convention" which singles out the global minimizer. By the argument of Proposition 5.5 (or the argument of Section 4), one easily proves that there is a residual set A' of p's in C_1 such that for p in A' whenever $C_x^{(a,b)} = 0$, $C_{xx}^{(a,b)} = 0$, and $C_{xxx}^{(a,b)} = 0$, then $C_{xxxx}^{(a,b)}$, $C_{xxa}^{(a,b)}$, and $C_{xb}^{(a,b)}$ are non-zero.

Let $p \in A'$ and let $(a', b') \in \Gamma(p) \cap \Sigma(p)$. Unless $x' = \xi_p(a', b')$ is a degenerate minimum of $C^{(a',b')}$ which splits into two local minima, ξ and μ will not be affected. So, we will assume this situation. By Thom's Theorem (see Martinet (1974) or the Zeeman-Trotman notes in Zeeman (1977)), there are neighborhoods U_1 of $(x'; a', b')$ and U_2 of $(0; 0, 0)$ and a C^∞ change of coordinates (diffeomorphism) $h: U_1 \to U_2$ of the form:

$$\eta = h_1(x;\ a,\ b)\ ,$$

$$\alpha = h_2(x;\ a,\ b)\ ,$$

(6.1) $$\beta = h_3(x;\ a,\ b)\ ,$$

$$\frac{\partial h_2}{\partial x} = \frac{\partial h_3}{\partial x} = 0 \quad \text{on} \quad U_1\ , \text{ and}$$

$$h(x',\ a',\ b') = (0,\ 0,\ 0)\ .$$

In these new coordinates, C has the equation:

(6.2) $$\tilde{C}(\eta;\ \alpha,\ \beta) \equiv C \circ h^{-1}(\eta;\ \alpha,\ \beta) = \frac{\eta^4}{4} - \frac{\beta\eta^2}{2} + \alpha\eta\ .$$

Since h_2 and h_3 are independent of x, and since the smoothness of ξ and of μ is independent of choice of coordinates of the type described in (6.1), the smoothness of μ at 0 in the new coordinates will be equivalent to the smoothness of μ_p at x' in our original coordinates.

Examining (6.2), note that

$$\frac{\partial\tilde{C}}{\partial\eta} = \eta^3 - \beta\eta + \alpha \quad \text{and} \quad \frac{\partial^2\tilde{C}}{\partial\eta^2} = 3\eta^2 - \beta\ .$$

In these coordinates, the set Γ, defined by $\tilde{C}_\eta = \tilde{C}_{\eta\eta} = 0$, is the curve $4\beta^3 = 27\alpha^2$ as described in Figure 6.3; while Σ is the positive β-axis.

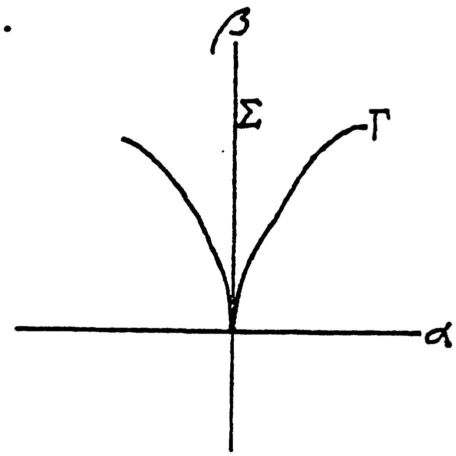


Figure 6.3

Now, $\frac{\partial \tilde{C}}{\partial \eta} = 0$ for each fixed η° along the line

$$\alpha - \beta\eta^\circ + {\eta^\circ}^3 = 0 \tag{6.3}$$

in the (α,β)-plane. Therefore, the curves of constant choice ξ lie on these lines and, in fact, are the segments as pictured in Figure 6.4.

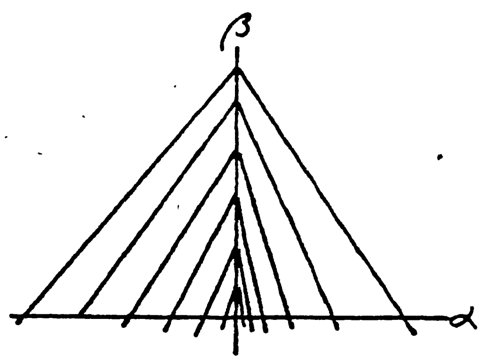


Figure 6.4: Level sets of ξ

Since we are only interested in the smoothness of μ at 0, we can let η vary along the α-axis between -1 and $+1$, without loss of generality. Let $\tilde{\mu}(\eta)$ be the line segment as described by equation (6.3) or as pictured in Figure 6.4 between $(\alpha, 0)$ and the positive β-axis. Let $\tilde{\mu}(\eta) = \text{measure}\{(\alpha, \beta) | \beta \geq 0$ and (α, β) lies between $\tilde{\xi}(-1)$ and $\tilde{\xi}(\eta)$. Thus, since $\tilde{\xi}(\eta)$ runs from $(\eta^3, 0)$ to $(0, \eta^2)$, $\tilde{\mu}(-1) = 0$, $\tilde{\mu}(0) = \frac{1}{2}$, and $\tilde{\mu}(1) = 1$. One checks easily that $\tilde{\mu}(\eta) = \frac{1}{2}(1 + \eta^5)$ for $\eta \in (-1, 1)$. Since $\tilde{\mu}$ is C^∞ at 0 , the original μ is C^∞ at x' .

Therefore, for generic price function p , lack of smoothness of the corresponding μ arises not at the points where $\Sigma(p)$ is created but at the crossings of $\Sigma(p)$. In particular, when two branches of Σ cross each other, ξ may jump abruptly and μ may lose smoothness. This phenomenon occurs for an open subset of C_1 , i.e., cannot be removed by perturbing p . To illustrate that such crossings do cause μ to lose smoothness, we describe the analogous phenomenon in one less dimension where it is simpler to understand.

Consider the problem of minimizing

$$J(a, x) = \frac{1}{4} x^4 - \frac{1}{2} x^2 + ax$$

for each a . Since $\frac{\partial J}{\partial x} = x^3 - x + a$, the graph of $a(x) = x - x^3$ is the set of all (x, a) such that x is a critical point of $J(a, \cdot)$. Further analysis yields that the subset $\{(x, a) | a = x - x^3, |x| \geq 1\}$ is the graph of $x = \xi(a)$, the global minimizers. See Figures 6.5 and 6.6.

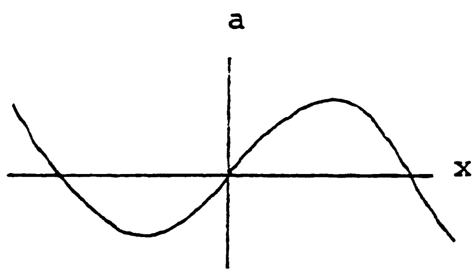


Figure 6.5: $a = x - x^3$

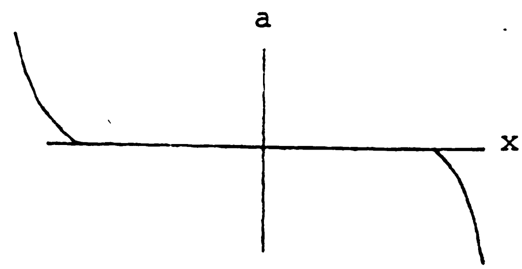


Figure 6.6: $x = \xi(a)$

Furthermore, for all $x \geq -2$,

$$\mu(x) = \text{measure}\{a \leq 6 \mid -2 \leq \xi(a) \leq x\}$$

$$= \begin{cases} 6 + x^3 - x \ , & \text{for } -2 \leq x \leq -1 \ , \\ 6 \ , & \text{for } -1 \leq x \leq 1 \ , \\ 6 + x^3 - x \ , & \text{for } 1 \leq x \ . \end{cases}$$

Since $\mu(-1) = \mu(+1) = 6$, μ is continuous. However, μ is not C^1 at $x = -1$ or $x = +1$.

7. CONDITIONS WHICH GUARANTEE SMOOTHNESS

Theorem 5.6 asserts that the cumulative demand function $\mu(x)$ is C^∞ except on a finite (possibly empty) set of x-values for a generic price function p. In this section, we give conditions on $p(x)$ which guarantee that the corresponding ξ and μ are everywhere smooth. Some conditions have already been discussed in Section 3, namely that p be linear or quadratic. The conditions described in this section are natural generalizations of the cases studied in Section 3.

THEOREM 7.1. If $p \in C_1$ and $p''(x) \geq 0$ for all x, then the corresponding ξ and μ are everywhere C^∞ and can easily be computed. (See equation (7.1) below.)

Proof: Since $p''(x) \geq 0$ for all x, $C_{xx}^{(a,b)}(x)$ is strictly positive for all a,b, and x. (See equation (2.2).) Therefore, $C_x^{(a,b)}(x)$ is strictly increasing for each (a,b). If $C_x^{(a,b)}(0) < 0$, then $C_x^{(a,b)}$ will have a unique interior zero which will be the global minimizer $\xi(a,b)$ of $C^{(a,b)}$. Since $C_{xx}^{(a,b)}$ is never zero, $\xi(a,b)$ will be a non-degenerate minimizer and (by the Implicit Function Theorem) will be a C^∞ function of (a,b).

If $C_x^{(a,b)}(0) \geq 0$, then $C^{(a,b)}$ will be a strictly increasing function of x and $\xi(a,b)$ must be zero. It follows that the curve $Z(p)$ coincides with the ray

$$R(0) = \{(a,b) | C_x^{(a,b)}(0) = 0\} .$$

Since p and p' are increasing functions of x, $b(x)$ in equation (2.8) is strictly decreasing and the curve $\Phi(p)$ has no cusps. Both $\Gamma(p)$ and $\Sigma(p)$ are empty. Therefore, $\mu(x)$ can be explicitly calculated by substituting equations (2.8) in the formula of Proposition 2.2. The result is

$$\text{(7.1)} \qquad \mu(x) = \int_0^x [(v - p(y))[1 - p'(y)^2]^{1/2} + \frac{1}{2}(v - p(y))^2 p''(y)[1 - p'(y)^2]^{-1/2}]dy ,$$

a C^∞ function of x for $0 \leq x \leq p^{-1}(v)$.

THEOREM 7.2. If $p \in C_1$ and $p''' \geq 0$ (e.g., $p'' < 0$ but increasing), then the corresponding ξ is continuous and μ is everywhere C^1.

Proof: One computes easily that

$$\frac{\partial^3 C^{(a,b)}(x)}{\partial x^3} = p'''(x) + \frac{b^2}{[(a-x)^2+b^2]^{3/2}} ;$$

so, $p''' \geq 0$ implies that $C_{xxx}^{(a,b)} > 0$ and that each $C_{xx}^{(a,b)}$ is strictly increasing. If $C_{xx}^{(a,b)}(x) > 0$ for all x or

$C_{xx}^{(a,b)}(x) < 0$ for all x, then $C^{(a,b)}$ can have only one interior critical point. If $C_{xx}^{(a,b)}(x) < 0$ for $x \in [0,x^*)$ and > 0 for $x > x^*$, then $C_x^{(a,b)}$ is decreasing on $[0,x^*)$ and increasing on (x^*,∞). Such a $C^{(a,b)}$ can have at most two critical points - one local max and one local min. It follows that $\Sigma(p)$ is empty; and therefore, by Proposition 6.1, ξ is continuous on the set $\xi > 0$, μ is everywhere C^1, and μ is C^∞ except possibly at one point. ∎

§8. Characterization of Cumulative Demand Functions

We have described how the cumulative demand function $\mu_p(x)$ behaves for a generic price function $p \varepsilon \mathcal{C}_1$. A natural question to ask is: what functions $\mu(x)$ can arise as cumulative demand functions from price functions in $\mathcal{C}_1$? Clearly, both $\mu(x)$ and $\mu'(x)$ must be positive. We will indicate in this section that these are the only constraints on possible cumulative demand functions.

Sonnenschein (1973) and Debreu (1974) have demonstrated the corresponding result for neo-classical exchange economies with strictly concave preferences. They proved that continuity, Walras' identity, and some natural boundary conditions are both necessary and sufficient for a function to be an aggregate excess demand for some economy.

First, we demonstrate a partial result in this direction, showing that there are no other necessary conditions on the signs of the derivatives of a given μ (except, of course, $\mu > 0$ and $\mu' > 0$) for μ to be a cumulative demand function μ_p in the Ellet transportation problem. For, let μ be an arbitrary C^2 function on $[0, \delta_1)$ with μ and μ' both positive, $\mu'(0) > 0$, and $\mu(0) = 0$. Let v be a positive constant. One can rewrite equation (7.1) as the second order differential equation

$$p'' = \frac{\mu'(1-p'^2)^{\frac{1}{2}}}{2(v-p)^2} - \frac{(1-p'^2)}{2(v-p)} \quad . \tag{8.1}$$

Choose $\rho \varepsilon (0,1)$ so that

$$\mu'(0) > v\sqrt{1-\rho^2}\ . \tag{8.2}$$

By the fundamental existence theorem of differential equations, there is a unique, smooth solution $p : [0,\delta_2) \to \mathbb{R}$ to equation (8.1) which satisfies the initial conditions $p(0) = 0$ and $p'(0) = \rho$. By (8.2), this solution will have $p''(0) > 0$. Since p is smooth, there is a δ with $0 < \delta < \min\{\delta_1, \delta_2\}$ such that for all $x \in (0,\delta)$,

$$0 < p(x) < v, \qquad 0 < p'(x) < 1,$$

$$\text{and} \quad 0 \leq p''(x).$$

By (7.1) and the uniqueness of the solution to (8.1) with the above initial values, the corresponding cumulative demand function μ_p for our solution p agrees with the original μ, at least on the interval $[0, \delta)$.

As H. Sonnenschein has pointed out, there is another way to show that any μ with $\mu > 0$ and $\mu' > 0$ can be a cumulative demand function [illegible] the Ellet problem. First, add an extra degree of freedom to this problem by allowing the density of the farmers filling up the right half plane to be non-uniform. Let $\eta : \mathbb{R}^2 \to \mathbb{R}_+$ be the corresponding density function. (Thus far, we have assumed $\eta \equiv 1$. With η non-constant, the formula analogous to (7.1) will be much more complicated than (7.1).)

To be more specific, consider example one of section 2 with linear $p(x) = Px$ and value v. As shown in section two, the rays $R(x)$ of constant individual demand all have slope $P^{-1}\sqrt{1-P^2}$. Choose a density function η which is also constant on any such ray $R(x)$, i.e., (8.3) $$\eta(a,b) = \tilde{\eta}(Pb - a\sqrt{1-P^2})$$

for some positive function $\tilde{\eta}$ on $\mathbb{R}$. An analysis similar to that of section two shows that the corresponding cumulative demand function is

$$\mu(x) = \int_0^x \sqrt{1-P^2}\,(v - Py)\tilde{\eta}(y)dy, \text{ i.e.,}$$

(8.4) $$\mu'(x) = \sqrt{1-P^2}\,(v - Px)\tilde{\eta}(x).$$

Conversely, let $\hat{\mu}(x)$ be a candidate for a cumulative demand function with $\hat{\mu} > 0$ and $\hat{\mu}' \geq 0$ on $(0,\alpha)$. Suppose further that $\hat{\mu}'(\alpha) = 0$, i.e., cumulative demand chokes off (stops accumulating) at $x = \alpha$. Choose v and P so that $0 < P < 1$ and $\frac{v}{P} = \alpha$. Using (8.4), define $\tilde{\eta} : [0,\alpha) \to (0,\infty)$ by

$$\tilde{\eta}(x) \equiv \frac{\hat{\mu}'(x)}{\sqrt{1-P^2}\,(v-Px)} .$$

(Since $\hat{\mu}'(\frac{v}{P}) = 0$, $\tilde{\eta}$ is differentiable at $x = \frac{v}{P} \equiv \alpha$.) If one now chooses price function $p(x) = Px$ and constructs density function $\eta(a,b)$ from $\tilde{\eta}$ using formula (8.3), the above analysis shows that the corresponding cumulative demand function for this model will be the original $\hat{\mu}$.

Appendix

In this appendix, the reader is reminded of some of the basic definitions and theorems of differential calculus and topology which are used throughout this paper. For further mathematical details, see Martinet (1974), Golubitsky-Guillemin (1973), or Zeeman (1977). For discussions of these concepts in economics settings, see Dierker (1974), Simon-Titus (1975), Saari-Simon (1977), and the five papers of Smale on "Global Analysis and Economics" in the Journal of Mathematical Economics.

To begin, recall that a function $f: \mathbb{R} \to \mathbb{R}$ is k-times continuously differentiable (writen C^k) if for each $j = 0, 1, \ldots, k$

$$\frac{d^j f}{dx^j}: \mathbb{R} \to \mathbb{R}$$

is defined and continuous. (By usual convention, $\frac{d^0 f}{dx^0}$ is just f itself.) If f is C^k for every k, $1 \leq k < \infty$, then we say f is C^∞. If for each $x \in \mathbb{R}$, the Taylor Series of f at x exists and has a positive radius of convergence, then f is real analytic (written C^ω). Note that $C^\omega \Rightarrow C^\infty \Rightarrow C^k \Rightarrow C^{k-1} \Rightarrow C^0 =$ continuous. If f is a function on $\mathbb{R}^n$, then f is C^k if every partial derivative of f of order $\leq k$ exists and is continuous on $\mathbb{R}^n$. If A is a closed subset of $\mathbb{R}^n$, then we'll say that f is C^k on A if f has a C^k extension to an open neighborhood of A.

Let C denote the following vector space of functions:

$$C = \{p\colon [0, \infty) \to \mathbb{R} \mid p(0) = 0 \text{ and } p \text{ is } C^\infty\} .$$

Topologize C as follows. Let $p_0 \in C$, let $\eta\colon [0, \infty) \to \mathbb{R}_+$ be a continuous, positive function, and let k be an integer. Define

$$N_{\eta,k}(p^\circ) = \{p \in C\colon \left| \frac{d^i p}{dx^i}(x) - \frac{d^i p_0}{dx^i}(x) \right| < \eta(x) \text{ for } i = 0, \ldots, k\} .$$

The sets $N_{\eta,k}(p^\circ)$ form a neighborhood basis for a topology on C, i.e., a subset M of C is <u>open</u> if for any $p^\circ \in M$, there is an $\eta(x)$ and a k as above such that $p^\circ \in N_{\eta,k}(p^\circ) \subset M$. For more details on this, the Whitney C^∞-topology, see Golubitsky-Guillemin (1973).

A subset D of C is <u>dense</u> if D has a non-empty intersection with <u>every</u> open subset of C. A subset R of C is <u>residual</u> if R is the intersection of a countable set of open and dense subsets of C. Residual subsets of C are dense in C (see Golubitsky-Guillemin (1973)) and are roughly analogous to sets of full measure. Furthermore, the countable intersection of residual subsets of C is itself residual. Let P be a property that holds for some functions in C. Then, P is called a <u>generic property</u> if a residual subset of maps in C have property P.

We now introduce some "transversality theory". Let f be a C^∞ mapping from $\mathbb{R}^n \to \mathbb{R}^m$, i.e., each component f^j of f is a C^∞ function from $\mathbb{R}^n$ to $\mathbb{R}$. A point $\underline{x}^\circ \in \mathbb{R}^n$ is called a <u>regular point</u> of f if the Jacobian derivative of f at $\underline{x}^\circ$:

$$Df(\underline{x}^\circ) \equiv \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(\underline{x}^\circ) & \cdots & \frac{\partial f^m}{\partial x_1}(\underline{x}^\circ) \\ \vdots & & \\ \frac{\partial f^1}{\partial x_n}(\underline{x}^\circ) & \cdots & \frac{\partial f^m}{\partial x_n}(\underline{x}^\circ) \end{pmatrix}$$

has rank m, i.e., is surjective. Otherwise $\underline{x}^\circ$ is called a singular point or critical point of f. A point $\underline{y}^\circ$ in the range $\mathbb{R}^m$ is called a regular value of f if either $f^{-1}(\underline{y}^\circ) = \emptyset$ or each $\underline{x}^\circ \in f^{-1}(\underline{y}^\circ)$ is a regular point of f. Otherwise, $\underline{y}^\circ$ is a critical value. The basic theorems surrounding these concepts are Sard's Theorem and the Implicit Function Theorem. See Golubitsky-Guillemin (1973) for discussion and proofs.

Sard's Theorem: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞ mapping. Then, the regular values of f form a residual subset of $\mathbb{R}^m$.

Implicit Function Theorem: Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞ mapping and let $\underline{y}^\circ$ be a regular value of f with $f^{-1}(\underline{y}_\circ)$ non-empty. Then, if $m < n$, $f^{-1}(\underline{y}^\circ)$ is a C^∞ submanifold of $\mathbb{R}^n$ of dimension $n - m$ (codimension m). (In other words, for each $\underline{x}^\circ \in f^{-1}(\underline{y}^\circ)$, there is a neighborhood U of $\underline{x}^\circ$ in $\mathbb{R}^n$, a decomposition of $\mathbb{R}^n$ into $\mathbb{R}^m \times \mathbb{R}^{n-m}$, and a C^∞ function

$$\lambda: U \cap [\{0\} \times \mathbb{R}^{n-m}] \to [\mathbb{R}^m \times \{0\}]$$

such that $f^{-1}(\underline{y}°) \cap U$ is the smooth graph of λ .) If $m = n$, $f^{-1}(\underline{y}°)$ is a discrete set of points. Note that $m > n$ is ruled out by hypothesis.

In sections 4, 5, and 6 of this paper, we show that, for certain p in C , a curve $\Delta(p)$ is well-behaved by characterizing it as the inverse image of the regular value $\underline{0}$ of a C^∞ map $F^{\Delta}_{(p)}: \mathbb{R}^n \to \mathbb{R}^{n-1}$. We also show that a set $S(p)$ is discrete by characterizing it as the inverse image of the regular value $\underline{0}$ of a C^∞ map $F^S_{(p)}: \mathbb{R}^n \to \mathbb{R}^n$. In fact, our goal is to prove that for a residual set of p's , $F^{\Delta}_{(p)}$ and $F^S_{(p)}$ have $\underline{0}$ as a regular value. The following theorem provides the machinery for just such a proof.

<u>Thom Transversality Theorem</u>. Let C be a normed vector space of mappings. Let $F: C \times \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞ mapping. If $\underline{0}$ is a regular value of F , then $\underline{0}$ is a regular value of the map $F_{(p)}: \mathbb{R}^n \to \mathbb{R}^m$, defined by $F_{(p)}(x) = F(p, x)$, for a residual subset of p's in C .

To show that $DF(p°, x°)$ is surjective, it often suffices to show that the partial derivative of F with respect to p , $D_pF(p°, x°)$, is surjective. The latter is the derivative of $p \longmapsto F(p, x°)$ at $p°$, with x fixed at $x°$ and is best viewed as the linear map from $C \to \mathbb{R}^m$ which best approximates the map $p \longmapsto F(p , x°)$ at $p°$. Thus, if $G: C \to \mathbb{R}^m$ is smooth, then its derivative at $p°$, $DG(p°)$, is the (unique) linear map from $C \to \mathbb{R}^m$ which satisfies

$$\lim_{h \to 0} \frac{G(p^\circ + h) - G(p^\circ) - DG(p^\circ)h}{\|h\|} = 0$$

for $h \in C$. For example, let $G(p) = p(x^\circ)$ for some fixed x°. Then, since G is linear in p, its derivative is itself, i.e.,

$$DG_{(p^\circ)}(h) = h(x^\circ) .$$

Finally, we will need a version of Thom's Catastrophé Theorem, which gives especially nice local forms of some mappings around degenerate critical points.

<u>Theorem</u>. Let $C(a, b; x)$ be a C^∞ map from $\mathbb{R}^2 \times \mathbb{R}^1 \to \mathbb{R}^1$. Suppose that $C(a^\circ, b^\circ; x^\circ)$, $\frac{\partial C}{\partial x}(a^\circ, b^\circ; x^\circ)$, $\frac{\partial^2 C}{\partial x^2}(a^\circ, b^\circ; x^\circ)$, $\frac{\partial^3 C}{\partial x^3}(a^\circ, b^\circ; x^\circ)$ are all zero and that the matrix

$$\begin{pmatrix} \frac{\partial^2 C}{\partial a \partial x} & \frac{\partial^2 C}{\partial b \partial x} & 0 \\ \frac{\partial^3 C}{\partial a \partial^2 x} & \frac{\partial^3 C}{\partial b \partial x^2} & 0 \\ \frac{\partial^4 C}{\partial a \partial x^3} & \frac{\partial^4 C}{\partial b \partial x^3} & \frac{\partial^4 C}{\partial x^4} \end{pmatrix}$$

is non-singular at $(a^\circ, b^\circ; x^\circ)$. Then, there exists a neighborhood U about $(a^\circ, b^\circ; x^\circ)$, a neighborhood V about $(0, 0; 0)$ in $\mathbb{R}^3$, and a C^∞ change of variables

$H = (H^1, H^2, H^3): U \to V$ (local diffeomorphism) which preserves vertical lines $(\frac{\partial H^1}{\partial x} \equiv \frac{\partial H^2}{\partial x} \equiv 0)$ and which sends $(0, 0; 0)$ to $(a°, b°; x°)$, so that in the new coordinates C has the form

$$C \circ H(\alpha, \beta; \eta) = \frac{\eta^4}{4} - \frac{\beta}{2}\eta^2 + \alpha\eta .$$

Under the above hypotheses, the curve in the (a,b)-place near $(a°, b°)$ defined by

$$\{(a, b) \left| \frac{\partial C}{\partial x}(a, b; x) = \frac{\partial^2 C}{\partial x^2}(a, b; x) = 0 \quad \text{for some} \quad x\right\}$$

is called a non-degenerate cusp. For the proof of this theorem, see Martinet (1974) or Trotman's notes in Zeeman (1977). This particular result was originally proved in Whitney (1955) and generalized by Morin (1965).

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