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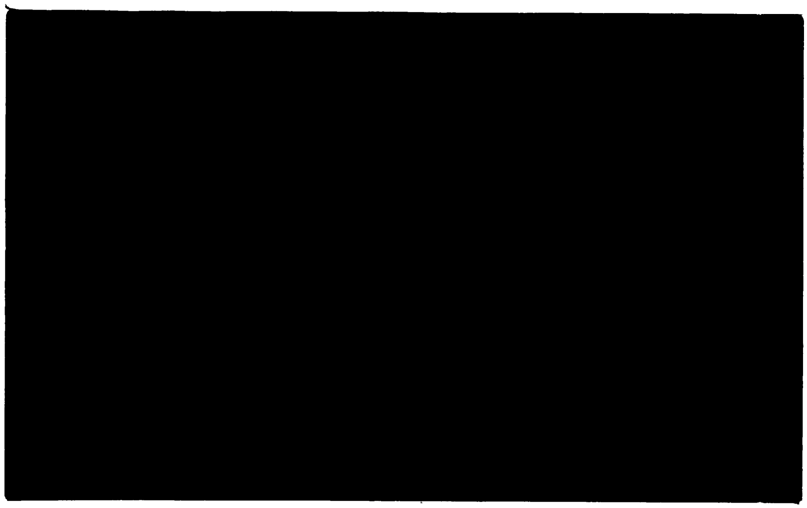

Scalar and Vector Maximization: Calculus Techniques with Economics Applications[1]

by

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The main purpose of this paper is to present a thorough and systematic study of the necessary and the sufficient conditions for a smooth non-linear mapping $u: \mathbb{R}^n \to \mathbb{R}^a$ to have a vector maximum (or Pareto optimum) on some (constraint) subset of $\mathbb{R}^a$ and to apply this study to some of the basic problems in microeconomics. The principal technique in this study will be to equate any given constrained vector maximization problem with a system of constrained scalar maximization problems, that is, problems of mathematical programming. This approach seems much simpler and more rewarding than the usual ad-hoc methods used in vector maximization problems. (See Theorem 7.1 and the remarks in secton 7.A.)

Consequently, we also need to present a thorough introduction to the theory of mathematical programming. We begin this presentation in chapter two by recalling the first and second order conditions involved in unconstrained maximization problems. In chapter three, we use these results to study constrained maximization problems where the constraint set is a smooth manifold, i.e., the derivative of the mapping which defines the constraints has maximal rank near the proposed solution. We also derive very general second order sufficient conditions for a constrained maximum in this chapter.

In chapter four, the strong non-degeneracy assumptions on the constraint set are replaced by the more general " constraint qualifications" of Kuhn-Tucker, Arrow-Hurwicz-Uzawa, and Slater. An attempt is made to keep the presentation of these different cases as unified, yet as simple as possible. The first order necessary conditions of chapter four are the basic ingredients of the general theorems on vector maximization presented in chapter seven.

In chapter five, we examine the situation where the first order necessary conditions are also sufficient - the economically important case of concave and almost concave objective and constraint functions. This chapter also includes a brief introduction to saddle point theorems and to duality.

Chapter six brings together the theory of the previous four chapters by using programming theorems to introduce the basic concepts and norms of the economic theories of the consumer and of the firm. We first derive the classical necessary (and often sufficient) conditions that describe a consumer's choice of a most preferred commodity vector from a set of feasible and affordable commodity vectors. We then turn to a similar study of a firm trying to choose the level of production that will maximize profits or revenues. This study includes an introduction to the activity analysis of production.

In chapter seven, all the theory developed for scalar maximization probelms is applied to vector maximization problems. This includes both necessary conditions and sufficient conditions, first order and rather strong second order conditions. We discuss both the "proper" solutions of Kuhn-Tucker and Geoffrion and the saddle point approach to vector maximization problems. We end this chapter by reviewing some of the insights into vector maxima that Smale and others have achieved by using techniques of differential topology.

Finally, the eighth chapter extends the applications of chapter six to the case where a number of consumers interact in an economy. Special properties of the utility mappings that arise in these situations are related to the hypotheses of theorems in chapter seven. Then, results of chapters five and seven are used to prove the Fundamental Theorems of Welfare Economics, which relate the concepts of Pareto optimum and competitive equilibrium. The chapter closes with an application of vector maximization to the choice of an efficient portfolio of securities.

The author hopes that after reading this paper the general reader will develop an understanding of and an intuition for some of the more basic concepts and techniques of mathematical economics and, as a result, will be adequately prepared to examine the more advanced topics in mathematical economics that are presented in this book.

With the exception of a few references to elementary facts about matrices, the only mathematical tools used are the basic theorems of multi-dimensional calculus, e.g., the Chain Rule, Taylor's Theorem, the Mean Value Theorem, and the Implicit Function Theorem. Consequently, this paper should be accessible to any reader who has taken the basic two-year sequence of differential calculus. To refresh the reader's familiarity with these theorems and to introduce the convenient coordinate-free notation which will be used throughout this paper, the author presents a mini-course in advanced calculus - without proofs - in the first section of chapter one. This chapter also contains an introduction - with proofs - to the properties of concave functions and their generalizations which are important in programming problems.

This paper contains no really new results in scalar and vector maximization, although a number of theorems in the last three chapters are presented with weaker hypotheses or stronger conclusions than the author has found in the literature. The emphas has been on presenting a very thorough description of the theory of non-linear vector maximization and as unified and as simple an approach as possible to the problems of scalar and vector maximization.

I am grateful for comments and suggestions by L. Blume, J. Tolle, and H. Varian on an earlier draft of this paper.

§1. MATHEMATICAL BACKGROUND

1.A Derivatives

In this section, we will summarize some of the important results from differential calculus which will be needed in later chapters. No proofs will be given. To facilitate later expositions, we will try to stay with a coordinate-free notation. See Courant (1947), Fleming (1965), and Edwards (1973), for example, for complete proofs and further discussions.

Let $\mathbb{R}^n$ denote the usual linear space of n-vectors $\{\underline{x} = (x_1,\ldots,x_n) \mid x_i \text{ is a real number}\}$. Let $\mathbb{R}^n_+$ denote the positive orthant of $\mathbb{R}^n$, i.e., $\{\underline{x} \in \mathbb{R}^n \mid x_i \geq 0 \text{ for } i = 1,\ldots,n\}$. If $\underline{x}$ and $\underline{y}$ are in $\mathbb{R}^n$, we will write $\underline{x} \leq \underline{y}$ if $x_i \leq y_i$ for $i = 1,\ldots,n$, and $\underline{x} < \underline{y}$ if $x_i < y_i$ for all i. We will denote the standard inner product between $\underline{x}$ and $\underline{y}$ as

$$\underline{x}\cdot\underline{y} = \sum_{i=1}^{n} x_i y_i$$

and the norm or length of $\underline{x}$ as

$$|\underline{x}| = \sqrt{\underline{x}\cdot\underline{x}}$$

On $\mathbb{R}^1$, write $[a,b]$ for $\{t \in \mathbb{R} \mid a \leq t \leq b\}$ and (a,b) for $\{t \in \mathbb{R} \mid a < t < b\}$, where a and $b \in \mathbb{R}^1$.

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a continuous mapping. Then, f has a derivative at $\underline{x}^\circ \in \mathbb{R}^n$ (or f is differentiable at $\underline{x}^\circ$) if there is a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\underline{h}\to 0} \frac{f(\underline{x}^\circ+\underline{h}) - f(\underline{x}^\circ) - L(\underline{h})}{|\underline{h}|} \text{ exists and is } \underline{0}\,.$$

In this case, L is called the first derivative of f at $\underline{x}^\circ$ and written as $Df(\underline{x}^\circ)$. Since $Df(\underline{x}^\circ)$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, it has an $(m \times n)$ matrix representation in the standard basis of $\mathbb{R}^n$ - the usual Jacobian matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\underline{x}^\circ) \cdots\cdots \frac{\partial f_1}{\partial x_n}(\underline{x}^\circ) \\ \vdots \qquad\qquad \vdots \\ \frac{\partial f_m}{\partial x_1}(\underline{x}^\circ) \cdots\cdots \frac{\partial f_m}{\partial x_n}(\underline{x}^\circ) \end{pmatrix}$$

where $f_1, \ldots, f_m$ are the components of f. More precisely, if the linear map $Df(\underline{x}^\circ)$ exists, then all the first order partial derivatives $\partial f_i / \partial x_j$ of f exist at $\underline{x}^\circ$ and the above Jacobian matrix represents $Df(\underline{x}^\circ)$. Conversely, if all the partial derivatives $\partial f_i / \partial x_j$ exist and are continuous on a neighborhood U of $\underline{x}^\circ$, then f is differentiable with derivative as above. In this case, we say f is continuously differentiable or C^1 on U since its derivative changes continously as $\underline{x}$ varies in U, i.e., the mapping

$$Df: \ U \to L(\mathbb{R}^n, \mathbb{R}^m)$$

is continuous where $L(\mathbb{R}^n, \mathbb{R}^m)$ is the vector space of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or equivalently, the $m \cdot n$-dimensional vector space of $m \times n$ matrices).

One can now go on to define the higher order derivatives of f. If f is C^1, one can ask whether the continuous map $Df: U \to L(\mathbb{R}^n, \mathbb{R}^m)$ has a derivative at $\underline{x}°$. If it does, one writes $D(Df)(\underline{x}°)$ or $D^2f(\underline{x}°)$ for its derivative, a linear map from $\mathbb{R}^n$ to $L(\mathbb{R}^n, \mathbb{R}^m)$, or equivalently a bilinear map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^m$. One usually takes the latter point of view and writes $D^2f(\underline{x}°)(\underline{v},\underline{w})$ instead of $(D(Df)(\underline{x}°)(\underline{v}))(\underline{w})$.

What is the bilinear map $D^2f(\underline{x}°)$ in coordinates? If $f: \mathbb{R}^n \to \mathbb{R}^1$ is C^1, then $Df: \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^1)$ is the map

$$\underline{x} \longmapsto \left(\frac{\partial f}{\partial x_1}(\underline{x}), \frac{\partial f}{\partial x_2}(\underline{x}), \ldots, \frac{\partial f}{\partial x_n}(\underline{x})\right). \tag{1}$$

Then, the matrix representation of $D^2f(\underline{x}°)$, the derivative of Df, is the Jacobian matrix of (1), i.e., the matrix $\left[\left[\frac{\partial^2 f}{\partial x_j \partial x_i}(\underline{x}°)\right]\right]_{ij}$. This matrix is usually called the Hessian matrix of f at $\underline{x}°$. If $\underline{v}$ and $\underline{w}$ are in $\mathbb{R}^n$, one checks easily that

$$D^2f(\underline{x}°)(\underline{v},\underline{w}) = \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x}°)\, v_i w_j$$

$$= (v_1 \ldots v_n) \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\underline{x}°) & \cdots\cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(\underline{x}°) \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(\underline{x}°) & \cdots\cdots & \frac{\partial^2 f}{\partial x_n^2}(\underline{x}°) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}.$$

Some authors write $\underline{v}^t D^2 f_{\underline{x}^\circ} \underline{w}$ for $D^2f(\underline{x}^\circ)(\underline{v},\underline{w})$. If $f = (f_1, \ldots, f_m): \mathbb{R}^n \to \mathbb{R}^m$, then

$$D^2f(\underline{x})(\underline{v},\underline{w}) = \left(D^2f_1(\underline{x})(\underline{v},\underline{w}), \ldots, D^2f_m(\underline{x})(\underline{v},\underline{w})\right).$$

Again, if the bilinear map $D^2f(\underline{x}^\circ)$, or equivalently all the second order partial derivatives $\left(\partial^2 f/\partial x_i \partial x_j\right)(\underline{x}^\circ)$, depend continuously on $\underline{x}^\circ$ in U, then f is called C^2 on U.

One can continue this process and define the third derivative of f at $\underline{x}^\circ$ as the derivative of the map $\underline{x} \longmapsto D^2f(\underline{x})$ from $\mathbb{R}^n$ to the space of bilinear maps on $\mathbb{R}^n$. The third derivative is a tri-linear mapping from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^m$ and is written $D^3f(\underline{x}^\circ)(\underline{u},\underline{v},\underline{w})$. If it is continuous in $\underline{x}^\circ$, i.e., if all the partial derivatives of f of order 3 exist and are continuous, we say f is C^3; and so on to define C^k. A central fact about these derivatives is that they are all symmetric multilinear maps. In particular for the second derivative, this symmetry means that $D^2f(\underline{x}^\circ)(\underline{v},\underline{w}) = D^2f(\underline{x}^\circ)(\underline{w},\underline{v})$ for all $\underline{v}$ and $\underline{w}$. In coordinates, this symmetry means that the Hessian is a symmetric matrix and the the appropriate mixed partial derivatives are equal $(\partial^2 f/\partial x_i \partial x_j = \partial^2 f/\partial x_j \partial x_i)$.

In our coordinate-free notation, the Chain Rule and Taylor's Theorem have particularly elegant formulations. The reader is encouraged to write these formulae in coordinates. Theorem 1.2.b is a form of the Mean Value Theorem.

<u>Theorem 1.1 (Chain Rule for First and Second Derivatives.)</u> If $f:\mathbb{R}^n \to \mathbb{R}^m$ and $g:\mathbb{R}^m \to \mathbb{R}^p$ are C^r maps, then $g \circ f:\mathbb{R}^n \to \mathbb{R}^p$ is C^r. If $r \geq 1$,

$$D(g\circ f)(\underline{x}^\circ)\underline{h} = Dg(f(\underline{x}^\circ))\circ Df(\underline{x}^\circ)\underline{h} .$$

So, the Jacobian matrix of the composition $g\circ f$ is the matrix product of the Jacobian matrix of g at $f(\underline{x}^\circ)$ and the Jacobian matrix of f at $\underline{x}^\circ$. If $r \geq 2$ and $y^\circ = f(x^\circ)$ in $\mathbb{R}^m$,

$$D^2(g\circ f)(\underline{x}^\circ)(\underline{h},\underline{k}) = D^2 g(\underline{y}^\circ)(Df(\underline{x}^\circ)\underline{h}, Df(\underline{x}^\circ)\underline{k})$$

$$+ Dg(\underline{y}^\circ)(D^2 f(\underline{x}^\circ)(\underline{h},\underline{k})) .$$

<u>Theorem 1.2 (Taylor's Theorem of order two)</u>. Suppose that $f: \mathbb{R}^n \to \mathbb{R}^m$ is a C^2 mapping on a convex neighborhood U of $\underline{x}^\circ \in \mathbb{R}^n$.

a) Then, there is a C^2 map $S: \mathbb{R}^n \to \mathbb{R}^m$ (depending continuously on $\underline{x}^\circ$) such that for all $\underline{h} \in \mathbb{R}^n$ with $\underline{x}^\circ + \underline{h} \in U$

$$f(\underline{x}^\circ+\underline{h}) = f(\underline{x}^\circ) + Df(\underline{x}^\circ)\underline{h} + \frac{1}{2!} D^2 f(\underline{x}^\circ)(\underline{h},\underline{h}) + S(\underline{h})$$

where $\dfrac{S(\underline{h})}{|h|^2} \to 0$ as $|h| \to 0$.

b) Let $m = 1$ and let $f, \underline{x}^\circ$, and $\underline{h}$ be as in a). There are $\underline{x}'$ $\underline{x}''$ on the line segment between x° and $\underline{x}^\circ + \underline{h}$ such that

$$f(\underline{x}^\circ+\underline{h}) = f(\underline{x}^\circ) + Df(\underline{x}')\underline{h} \quad \text{and}$$
$$f(\underline{x}^\circ+\underline{h}) = f(\underline{x}^\circ) + Df(\underline{x}^\circ)\underline{h} + \frac{1}{2} D^2 f(\underline{x}'')(\underline{h},\underline{h}) .$$

As an illustration of the chain rule, let $f: \mathbb{R}^n \to \mathbb{R}^1$ and $\alpha: \mathbb{R}^1 \to \mathbb{R}^n$ be C^2 maps with $\alpha(0) = \underline{x}^\circ$ and $\alpha'(0) \equiv D\alpha(0)1 = \underline{v} \in \mathbb{R}^n$. One says that α is a C^2 curve at $\underline{x}^\circ$ with tangent (or velocity) vector $\underline{v}$. The rate of change of f at $\underline{x}^\circ$ along α is

$$\begin{aligned}\frac{d}{dt}(f\circ\alpha)(0) &= D(f\circ\alpha)(0)(1) \\ &= Df(\alpha(0))\cdot D\alpha(0)1 \\ &= Df(\underline{x}^\circ)\underline{v}\,,\end{aligned}$$

which is called the <u>directional derivative</u> of f at $\underline{x}^\circ$ in the direction $\underline{v}$. Similarly, one computes that

$$\frac{d^2}{dt^2}(f\circ\alpha)(0) = D^2f(\underline{x}^\circ)(\underline{v},\underline{v}) + Df(\underline{x}^\circ)(\alpha''(0)),$$

where $\alpha''(0) = D^2\alpha(0)(1,1)$.

Putting together the definition of $Df(\underline{x}^\circ)$ and the above paragraph, one notes that

$$Df(\underline{x}^\circ)\underline{v} = \lim_{t \to_+ 0} \frac{f(\underline{x}^\circ + t\underline{v}) - f(\underline{x}^\circ)}{t} .$$

Let U be a subset of $\mathbb{R}^n$ with $\underline{x}^\circ \in U$. The set of all tangent vectors at $\underline{x}^\circ$ to C^1 curves which remain in U is called the <u>tangent space</u> to U at $\underline{x}^\circ$ and denoted by $T_{\underline{x}^\circ}U$. In other words,

$$T_{\underline{x}^\circ}U = \{v = \alpha'(0) \in \mathbb{R}^n \mid \alpha : [0,\varepsilon) \to \mathbb{R}^n \text{ is a } C^1 \text{ curve with } \alpha(0) = \underline{x}^\circ \text{ and } \alpha(t) \in U \text{ for all } t\}$$

Thus, if U is an open subset of $\underline{x}^\circ$ in $\mathbb{R}^n$, $T_{\underline{x}^\circ}U$ is $T_{\underline{x}^\circ}\mathbb{R}^n$, which is just $\mathbb{R}^n$ with the origin pictured at $\underline{x}^\circ$.

There is one more interpretation of the derivative of a C^1 $f : \mathbb{R}^n \to \mathbb{R}^1$. Instead of working with the $1 \times n$ matrix $\left(\frac{\partial f}{\partial x_1}(\underline{x}^\circ) \ldots\ldots \frac{\partial f}{\partial x_n}(\underline{x}^\circ)\right)$ which represents $Df(\underline{x}^\circ)$ as a linear map, one often thinks of the column vector

$$\nabla f(\underline{x}^\circ) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(\underline{x}^\circ) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\underline{x}^\circ) \end{pmatrix} \in \mathbb{R}^n$$

as a vector in $T_{\underline{x}^\circ}\mathbb{R}^n$, i.e., a geometric vector with its tail at $\underline{x}_0$. One notices easily that

$$Df(\underline{x}^\circ)\underline{v} = \nabla f(\underline{x}^\circ)\cdot\underline{v}$$

and that $\nabla f(\underline{x}^\circ)$, if non-zero, points into the direction in which f increases most rapidly at $\underline{x}^\circ$ and is perpendicular to the level set of f at $\underline{x}^\circ$, $\{\underline{x}|f(\underline{x}) = f(\underline{x}^\circ)\} = f^{-1}(f(\underline{x}^\circ))$.

As the linear approximation to a C^1 function $f:\mathbb{R}^n\to\mathbb{R}^m$ at $\underline{x}^\circ$, the derivative not only tells us much about f at $\underline{x}^\circ$ but can also yield important information about the behavior of f in a whole neighborhood of $\underline{x}^\circ$. The outstanding example of this phenomenon is the implicit function theorem - a result which will play an important role in later chapters of this paper.

Recall that an $m \times n$ matrix has maximal rank if either all its rows or all its columns are linearly independent or equivalently it contains a $p \times p$ non-singular square matrix where $p = \min\{m,n\}$.

Theorem 1.3 (Implicit Function Theorem). Suppose that $f: \mathbb{R}^n \to \mathbb{R}^m$ is a C^r mapping with $r \geq 1$ and that $\underline{x}^\circ \in \mathbb{R}^n$. Suppose that $Df(x^\circ)$ has maximal rank $p = \min\{m,n\}$.

a) If $n \leq m$, $p = n$ and $Df(\underline{x}^\circ)$ is a 1-1 linear map. Then, f itself is 1-1 on a neighborhood of $\underline{x}^\circ$, i.e., there is an open neighborhood U of $\underline{x}^\circ$ such that for each $\underline{y}$ in $f(U)$, the equation $f(x) = \underline{y}$ has at most one solution $\underline{x}$ in $\underline{U}$.

b) If $m \leq n$, $p = m$ and $Df(\underline{x}^\circ)$ is surjective. Then, there are neighborhoods U of $\underline{x}^\circ$ in $\mathbb{R}^n$ and V of $f(\underline{x}^\circ)$ in $\mathbb{R}^m$

such that f maps U onto V, i.e., for each $\underline{y}$ in V, there is at least one $\underline{x}$ in U such that $f(\underline{x}) = \underline{y}$. In addition, the (local) level set of f through $\underline{x}°$

$$f^{-1}(f(\underline{x}°)) \cap U = \{\underline{x} \in U | f(\underline{x}) = f(\underline{x}°)\}$$

is a C^r (n-m)-dimensional submanifold of $\mathbb{R}^n$. This means that it sits in U like a smooth (non-linear) (n-m)-dimensional slice or graph; one can find coordinates $y_1,\ldots,y_n$ on U in which $\underline{x}°$ corresponds to the origin and the level set through $\underline{x}°$ is the $y_{m+1},\ldots,y_n$ coordinate plane $(y_1=\ldots=y_m=0)$ in the new coordinate system. Furthermore, the set of tangent vectors to the level set at $\underline{x}°$, $T_{\underline{x}°}[f^{-1}(f(\underline{x}°)) \cap U]$, is the nullspace of $Df(\underline{x}°)$, $\{\underline{v} | Df(x°)v = 0\}$.

c) In particular, if $m = 1$ in b) and if $Df(x°)$ (or $\nabla f(x°)$) is non-zero, say $\frac{\partial f}{\partial x_1}(\underline{x}°) \neq 0$, then the level set of f through $\underline{x}°$ is the graph of a smooth function $x_1 = g(x_2,\ldots,x_n)$ around $\underline{x}°$ and the set of tangent vectors to the level set is precisely $\{\underline{v} | \nabla f(x°) \cdot \underline{v} = 0\}$.

d) Furthermore, if $m < n$, if we write $\mathbb{R}^n$ as $\mathbb{R}^m \times \mathbb{R}^{n-m} = \{(x_1,x_2) | x_1 \in \mathbb{R}^m, x_2 \in \mathbb{R}^{n-m}\}$, if $f(x_1^°,x_2^°) = 0$ and the square matrix $D_{x_1} f(x_1^°,x_2^°)$ has maximal rank m, then there is a neighborhood U of $x_2^°$ in $\mathbb{R}^{n-m}$ and a unique C^r map $g: U \to \mathbb{R}^m$ such that $g(x_2^°) = x_1^°$ and $f(g(x_2),x_2) = 0$ for all x_2 in U. (For each fixed x_2 near $x_2^°$, $x_1 = g(x_2)$ is the solution of $f(x_1,x_2) = 0$. In other words, $f(x_1,x_2) = 0$ defines x_1 as an (implicit) function of x_2.)

1.B Definite Symmetric Bilinear Maps

If $f:\mathbb{R}^n \to \mathbb{R}^1$ is a C^2 function, the second derivative of f at $\underline{x}°$, $D^2f(\underline{x}°)$, is a symmetric bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1$, as indicated above, and can be represented by the $n \times n$ (symmetric) Hessian matrix $\left(\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\underline{x}°)\right)\right)$. In studying second order tests for optimality, we will need to work with symmetric maps which are definite.

Let $L:\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric bilinear map. Then, L is <u>negative definite</u> if $L(\underline{v},\underline{v}) < 0$ for all $\underline{v} \neq \underline{0}$; L is <u>negative semi-definite</u> if $L(\underline{v},\underline{v}) \leq 0$ for all $\underline{v} \in \mathbb{R}^n$; L is <u>positive definite</u> if $L(\underline{v},\underline{v}) > 0$ for all $\underline{v} \neq \underline{0}$; L is <u>positive semi-definite</u> if $L(\underline{v},\underline{v}) \geq 0$ for all $\underline{v} \in \mathbb{R}^n$.

If L is symmetric and bilinear and if $\underline{e}^1,\ldots,\underline{e}^n$ is a basis of $\mathbb{R}^n$, then the matrix of L with respect to this basis is $((L(\underline{e}^i,\underline{e}^j))_{i,j}$ since

$$L(\sum_i a_i\underline{e}^i, \sum_j b_j\underline{e}^j) = \sum_{i,j} L(\underline{e}^i,\underline{e}^j)a_ib_j .$$

Conversely, if A is a symmetric matrix, then $L(\underline{v},\underline{w}) = \underline{v}^t A \underline{w}$ is its corresponding symmetric bilinear map. The most straightforward test for the positive or negative definitness of L uses the leading principal minors of A.

The $k \times k$ square submatrix of A obtained by deleting $(n-k)$ rows and <u>the same</u> $(n-k)$ columns from A is called a kth order <u>principal submatrix</u> of A. If this $k \times k$ submatrix is formed by deleting the <u>last</u> $(n-k)$ rows and columns from A, it is called the kth <u>leading principal submatrix of A</u>. The determinant of a (leading) principal submatrix is called a (leading) principal minor. The following important result relates the definiteness of L to the eigenvalues and principal minors of A.

Theorem 1.4. Let $L: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear, symmetric map with matrix $A = ((L(e^i,e^j)))_{i,j}$. Then, all the eigenvalues of A are real and A has a complete set of eigenvectors, i.e., A is diagonalizable. Furthermore, the following three statements are equivalent:

a) L is positive definite;

b) all the eigenvalues of A are positive;

c) the n leading principal minors of A are positive.

If one is testing for negative-definiteness, then the corresponding three equivalent statements are:

a') L is negative definite;

b') all the eigenvalues of A are negative;

c') the kth leading principal minor of A has the same sign as $(-1)^k$ for $k = 1,2,\ldots,n$.

There are corresponding results for semi-definiteness. For example, L is negative semi-definite if and only if all the eigenvalues of A are non-positive if and only if each of the non-zero kth order principal minors of A has the same sign as $(-1)^k$ for $k = 1,\ldots,n$. Note that to check for definiteness, one only checks the sign of n leading principal minors; but to check for semi-definiteness, one must check the sign of all 2^n-1

principal minors. The proofs of these results can be found in most linear algebra books and in Bellman (1960) and Debreu (1952).

In our necessary and sufficient conditions for constrained optimization problems, we will need to check whether the <u>restriction</u> of a symmetric bilinear map to some linear subspace of $\mathbb{R}^n$ is definite or not. The following theorem provides a sufficient condition for this phenomenon.

<u>Theorem 1.5.</u> Let $L(\underline{v},\underline{w}) = v^t A \underline{w}$ be a symmetric bilinear map on $\mathbb{R}^n$. Let B be an $m \times n$ submatrix with m linearly independent rows, $m < n$. Let S be the $(n-m)$-dimensional nullspace of B, $\{\underline{x} \in \mathbb{R}^n \mid B\underline{x} = 0\}$. Form the bordered $(n+m) \times (n+m)$ matrix $C = \begin{pmatrix} 0 & B \\ B^t & A \end{pmatrix}$. If each of the <u>last</u> $(n-m)$ leading principal minors of C (i.e., the ones of order $2m+1,\ldots,m+n$) has the same sign as $(-1)^m$, then $L(\underline{x},\underline{x}) > 0$ for all non-zero $\underline{x}$ such that $B\underline{x} = \underline{0}$. On the other hand, if the <u>last</u> $(n-m)$ leading principal minors of C alternate in sign with determinant of C having the same sign as $(-1)^n$, then $L(\underline{x},\underline{x}) < 0$ for all non-zero $\underline{x}$ such that $B\underline{x} = \underline{0}$.

The proof of this theorem is fairly intricate. See Debreu (1952) or Bellman (1960).

<u>1.C Concave and Convex Functions</u>

As we will see in chapters six and eight, concave functions arise naturally in problems of economics and concavity is a common and useful hypothesis in many theorems of maximization. In this section, we will survey some of the important properties of concave and almost concave functions. For further reading and more complete proofs, see Fenchel (1953), Karlin (1959), Gale (1960), and

Mangasarian (1969). Many of the ideas and proofs of this section are adopted from the excellent presentation of Mangasarian (1969).

Definitions. Let $\underline{x}$ and $\underline{y}$ lie in $\mathbb{R}^n$. We will denote the line segment from $\underline{x}$ to $\underline{y}$ by $\ell(\underline{x},\underline{y})$, i.e.,

$$\ell(\underline{x},\underline{y}) = \{t\underline{y}+(1-t)\underline{x} \mid 0 \leq t \leq 1\}.$$

A subset U of $\mathbb{R}^n$ is convex if whenever $\underline{x},\underline{y} \in U$, then $\ell(\underline{x},\underline{y}) \subset U$. Let $f:U \to \mathbb{R}^1$ be a function on the convex subset U of $\mathbb{R}^n$. Then, f is concave (convex) on U if for all $\underline{x},\underline{y} \in U$ and $t \in [0,1]$,

$$f(t\underline{y}+(1-t)\underline{x}) \geq tf(\underline{y}) + (1-t)f(\underline{x})$$

$$\left(f(t\underline{y}+(1-t)\underline{x}) \leq tf(\underline{y}) + (1-t)f(\underline{x})\right).$$

If, for all $\underline{x},\underline{y} \in U$ and for all $t \in (0,1)$, the above inequalities can be written as strict inequalities, then we say that f is strictly concave (or strictly convex) on U. Note that linear maps are concave and convex. Fleming (1965) gives a proof that any function which is convex or concave on an open subset of $\mathbb{R}^n$ is continuous.

IMPORTANT REMARK. Note that f is convex if and only if $-f$ is concave. Since minimizing f is equivalent to maximizing $-f$, all the results of this paper on maximization can be written as results on minimization. In this case, one naturally changes hypotheses about concave functions to hypotheses about convex functions.

If f is C^1 or C^2, there are some powerful criteria from calculus for concavity, as summarized in the following theorem.

Theorem 1.6. Let $f: U \to \mathbb{R}^1$ be a C^1 function on a convex open subset U of $\mathbb{R}^n$. Then, the following are equivalent:

a) f is concave on U,

b) $f(\underline{y}) - f(\underline{x}) \leq Df(\underline{x})(\underline{y}-\underline{x})$ for all $\underline{x}, \underline{y}, \in U$,

c) $[Df(\underline{y}) - Df(\underline{x})](\underline{y}-\underline{x}) \leq 0$ for all $\underline{x}, \underline{y} \in U$.

If f is C^2, a), b) and c) are equivalent to

d) $D^2f(\underline{x})(\underline{v},\underline{v}) \leq 0$ for all $x \in U$ and all non-zero $\underline{v} \in \mathbb{R}^n$ (i.e., $D^2f(\underline{x})$ is negative semi-definite on U).

Remark: Theorem 1.6 is true if one changes "concave" to "convex" in a) and reverses the inequalities in b), c) and d). If one changes "concave" to "strictly concave" in a) and make the inequalities strict in b), c) and d), then

$$d) \Rightarrow a) \Leftrightarrow b) \Leftrightarrow c),$$

with $f(x) = -x^4$ a counterexample to a) $\Leftrightarrow$ d) in the strict case.

Proof: Throughout this proof, $\underline{x}$ and $\underline{y}$ will denote arbitrary elements in U with $t \in [0,1]$.

a) $\Rightarrow$ b): Since f is concave,

$$t\, f(\underline{y}) + (1-t)\, f(\underline{x}) \leq f(t\underline{y}+(1-t)\underline{x}); \quad \text{or}$$

$$f(\underline{y}) - f(\underline{x}) \leq \frac{f(\underline{x}+t(\underline{y}-\underline{x}))-f(\underline{x})}{t}$$

Taking the limit as $t \to 0$ and using the remarks under Theorem 1.2, we see that $f(\underline{y}) - f(\underline{x}) \leq Df(\underline{x})(\underline{y}-\underline{x})$.

b) $\Rightarrow$ c): Add the two inequalities:

$$f(\underline{y}) - f(\underline{x}) - Df(\underline{x})(\underline{y}-\underline{x}) \leq 0 \quad \text{and}$$

$$f(\underline{x}) - f(\underline{y}) - Df(\underline{y})(\underline{x}-\underline{y}) \leq 0.$$

c) $\Rightarrow$ b): By Theorem 1.2b (Mean Value Theorem),

$$f(\underline{y}) - f(\underline{x}) = Df(\underline{y}+t_\circ(\underline{x}-\underline{y}))(\underline{y}-\underline{x}) \quad \text{for some} \quad t_\circ \in (0,1) .$$

By c), $[Df(\underline{y}+t_\circ(\underline{x}-\underline{y})) - Df(\underline{x})](1-t_\circ)(\underline{y}-\underline{x}) \leq 0$, or

$$Df(\underline{y}+t_\circ(\underline{x}-\underline{y}))(\underline{y}-\underline{x}) \leq Df(\underline{x})(\underline{y}-\underline{x}) .$$

Thus, $$f(\underline{y}) - f(\underline{x}) \leq Df(\underline{x})(\underline{y}-\underline{x}) .$$

b) $\Rightarrow$ a): By b),

$$f(\underline{x}) - f((1-t)\underline{x}+t\underline{y}) \leq -t\, Df((1-t)\underline{x}+t\underline{y})(y-\underline{x})$$

and $$f(\underline{y}) - f((1-t)\underline{x}+t\underline{y}) \leq (1-t)Df((1-t)\underline{x}+t\underline{y})(\underline{y}-\underline{x}) .$$

Now, a) follows immediately after one multiplies the first inequality by (1-t) and the second by t and then adds the two inequalities.

d) $\Rightarrow$ b): By Theorem 1.2b,

$$f(\underline{y}) - f(\underline{x}) = Df(\underline{x})(\underline{y}-\underline{x}) + \frac{1}{2}D^2f(\underline{z})(\underline{y}-\underline{x},\underline{y}-\underline{x})$$

for some $\underline{z} \in \ell(\underline{x},\underline{y})$. Since the last term is non-positive, b) follows.

b) $\Rightarrow$ d): Suppose there are $\underline{x}^\circ \in U$ and $\underline{v} \in \mathbb{R}^n$ such that $D^2f(\underline{x}^\circ)(v,v) > 0$. Since f is C^2, there is a convex neighborhood W of $\underline{x}^\circ$ in U such that $D^2f(\underline{x})(\underline{v},\underline{v}) > 0$ for all $\underline{x} \in W$. Also, $D^2f(\underline{x})(t\underline{v},t\underline{v}) = t^2D^2f(\underline{x})(\underline{v},\underline{v}) > 0$ for all $\underline{x} \in W$ and for all t.

Choose $t_\circ > 0$ and small enough so that $\underline{x}^\circ + t_\circ\underline{v} \in W$.

By Theorem 1.2b,

$$\begin{aligned} f(\underline{x}^\circ+t_\circ\underline{v}) - f(\underline{x}^\circ) &= Df(\underline{x}^\circ)(t_\circ\underline{v}) + \tfrac{1}{2}D^2f(\underline{x}^\circ+t_1\underline{v})(t_\circ\underline{v},t_\circ\underline{v}) \\ &> Df(\underline{x}^\circ)(t_\circ\underline{v}) \end{aligned}$$

for some $t_1 \in [0,t_\circ]$ - a contradiction to b). Note that the last two paragraphs, show that d) $\Rightarrow$ b) in the strict convexity case. ∎

In some problems that arise in economics, concavity is a little too strong as an hypothesis. Since many important maximization theorems hold with weaker forms of concavity, we will discuss some of these modifications now. One important property of a concave function is that its level sets bound a convex set, i.e., if f is concave, $\{\underline{x} | f(\underline{x}) \geq a\}$ is convex. Since any monotone function from $\mathbb{R}^1$ to $\mathbb{R}^1$ (e.g., $f(\underline{x}) = \underline{x}^3$) also has this property, it does not characterize concave functions. So, if $f: U \to \mathbb{R}$ is a function on a convex U of $\mathbb{R}^n$, it is natural to call f quasi-concave on U if $\{\underline{x} \in U | f(\underline{x}) \geq a\}$ is convex for all $a \in \mathbb{R}$. Similarly, f is quasi-convex on U if $\{\underline{x} \in U | f(\underline{x}) \leq a\}$ is convex for all $a \in \mathbb{R}$. Fortunately, there is a useful calculus criterion for quasi-concavity.

Theorem 1.7. Suppose that $f: U \to \mathbb{R}$ is a C^1 function on an open convex subset U of $\mathbb{R}^n$. Then, f is quasi-concave on U if and only if $f(\underline{y}) \geq f(\underline{x})$ implies that $Df(\underline{x})(\underline{y}-\underline{x}) \geq 0$.

Proof: Suppose f is quasi-concave on U and that $f(y) \geq f(x)$ for some $x, y, \in U$. Then, for all $\mu \in [0,1]$,

$$f(x+\mu(y-x)) \geq f(x) .$$

Since $\dfrac{f(\underline{x}+\mu(\underline{y}-\underline{x})) - f(\underline{x})}{\mu} \geq 0$ for all $\mu \in (0,1)$,

$$Df(\underline{x})(\underline{y}-\underline{x}) \geq 0 \text{, letting } \mu \to 0 .$$

To prove the converse, choose $\underline{x}^0$ and $\underline{x}^1 \in U$ with $\underline{x}^0 \neq \underline{x}^1$ and $f(\underline{x}^1) \geq f(\underline{x}^0)$. Let $\underline{x}^\mu \equiv \underline{x}^0 + \mu(\underline{x}^1-\underline{x}^0)$. We will prove that $f(\underline{x}^\mu) \geq f(\underline{x}^0)$ for all $\mu \in [0,1]$.

To reach a contradiction, suppose there is a $\mu^* \in (0,1)$ with $f(\underline{x}^{\mu^*}) < f(\underline{x}^0) \leq f(\underline{x}^1)$. Let $J = [\mu^1,\mu^2]$ be a (connected) interval in $(0,1)$ with $\mu^* \in J$, $f(\underline{x}^\mu) \leq f(\underline{x}^0)$ for all $\mu \in J$, and $f(\underline{x}^{\mu^1}) = f(\underline{x}^{\mu^2}) = f(\underline{x}^0)$. We first claim that $Df(\underline{x}^\mu)(\underline{x}^1-\underline{x}^0) = 0$ for all $\mu \in J$. If $\mu \in J$, $f(\underline{x}^\mu) \leq f(\underline{x}^0) \leq f(\underline{x}^1)$. By hypothesis,

$$Df(\underline{x}^\mu)(\underline{x}^0-\underline{x}^\mu) \geq 0 \quad \text{and} \quad Df(\underline{x}^\mu)(\underline{x}^1-\underline{x}^\mu) \geq 0 .$$

Since $\underline{x}^0 - \underline{x}^\mu = -\mu(\underline{x}^1-\underline{x}^0)$ and $\underline{x}^1-\underline{x}^\mu = (1-\mu)(\underline{x}^1-\underline{x}^0)$, we have

$$-\mu Df(\underline{x}^\mu)(\underline{x}^1-\underline{x}^0) \geq 0 \quad \text{and} \quad (1-\mu)Df(\underline{x}^\mu)(\underline{x}^1-\underline{x}^0) \geq 0 .$$

Since μ and $1-\mu$ are positive, $Df(\underline{x}^\mu)(\underline{x}^1-\underline{x}^0) = 0$.

On the other hand,

$$\begin{aligned} 0 < f(\underline{x}^0) - f(\underline{x}^{\mu^*}) &= f(\underline{x}^{\mu^1}) - f(\underline{x}^{\mu^*}) \\ &= Df(\underline{x}^{\mu^3})(\underline{x}^{\mu^1}-\underline{x}^{\mu^*}), \text{ by Theorem 1.2b, } \mu^3 \in J . \\ &= (\mu^*-\mu^1)Df(\underline{x}^{\mu^3})(\underline{x}^1-\underline{x}^0), \end{aligned}$$

since $x^{\mu^1}-x^{\mu^*}=(\mu^*-\mu^1)(x^1-x^0)$ – a contradiction to the last paragraph. ▯

<u>Remark</u>. Of course, there is an analogous result for quasi-convexity. One can also define and work with strict quasi-concave functions and strict quasi-convex functions.

Quasi-concave functions share with concave functions the property that local maxima are global maxima. However, there is an important difference. Because a) $\leftrightarrow$ b) in Theorem 1.6, a critical point of a concave function is a local (and therefore) global maximum. But $f(x) = x^3$ shows that quasi-concave functions do not have this property. To fill this gap, Mangasarian (1965) introduced the concept of a pseudo-concave function.

Definition. A C^1 function $f: \mathbb{R}^n \to \mathbb{R}$ is pseudoconcave at $\underline{x}^\circ \in \mathbb{R}^n$ if whenever $Df(\underline{x}^\circ)(\underline{y}-\underline{x}^\circ) \leq 0$, $f(\underline{y}) \leq f(\underline{x}^\circ)$. One defines a pseudoconvex function similarly.

Theorem 1.8. Let $f: U \to \mathbb{R}$ be a C^1 pseudoconcave function at all $\underline{x}$ in the convex subset U of $\mathbb{R}^n$. Then,

a) $\underline{x}^\circ$ maximizes f on U if and only if $Df(\underline{x}^\circ)(\underline{x}-\underline{x}^\circ) \leq 0$ for all $x \in U$;

b) if U is open, $\underline{x}^\circ$ maximizes f on U if and only if $Df(\underline{x}^\circ) = \underline{0}$.

The, "only if" parts of a) and b) hold for all C^1 functions f.

Proof: a) If $\underline{x}^\circ$ maximizes f on U, $f(\underline{x}^\circ+t(\underline{x}-\underline{x}^\circ)) - f(\underline{x}^\circ) \leq 0$ for all $\underline{x} \in U$ and all $t \in [0,1]$. Dividing by t and letting t tend to 0 yields $Df(\underline{x}^\circ)(\underline{x}-\underline{x}^\circ) \leq 0$. The converse is immediate from the definition of a pseudoconcave function.

b): That a maximizer on an open set is a critical point is a classical result (see Theorem 2.1) and also follows from a), since the openness of U implies that $Df(\underline{x}^\circ)\underline{v} \leq 0$ for all $\underline{v} \in T_{\underline{x}^\circ}\mathbb{R}^n$. The converse also follows from a).

Pseudoconcavity is a less geometric concept than quasi-concavity, although it is an important analytical one. But notice that the definition of pseudoconcavity is a slight strengthening of the contrapositive of the analytical characterization of quasi-concavity in Theorem 1.7. Part b) of Theorem 1.9 describes the mild conditions under which the two concepts are equivalent, namely f is C^2 and f is non-zero on a "solid convex" set. Part a) of Theorem 1.9 summarizes the hierarchy of concavity for C^1 functions, while part c) summarizes the principal-minor conditions which one can easily use to test a C^2 function for concavity or quasi-concavity.

<u>Theorem 1.9</u>. Let U be a convex of $\mathbf{R}^n$. Let $f : U \to \mathbf{R}$ be a C^1 function. Then,

a) f is strictly concave on $U \Rightarrow f$ is concave on U
$\Rightarrow f$ is pseudoconcave on U
$\Rightarrow f$ is strictly quasi-concave on U
$\Rightarrow f$ is quasi-concave on U.

Furthermore, none of the implications can be reversed. (The C^1 hypothesis is made only to include pseudoconcave functions.)

b) If U has a non-empty interior, if f is C^2 on U, and if ∇f is never $\underline{0}$ on U, then f is pseudoconcave if and only if it is strictly quasi-concave if and only if it is quasi-concave.

c) Let $H(\underline{x})$ be the $n \times n$ Hessian matrix for $D^2 f(\underline{x})$. Let $B(x)$ be the $(n+1) \times (n+1)$ bordered matrix

$$B(\underline{x}) = \begin{pmatrix} 0 & \nabla f(\underline{x})^t \\ \nabla f(\underline{x}) & H(\underline{x}) \end{pmatrix}.$$

If the kth leading principal minor of $H(\underline{x})$ has the same sign as $(-1)^k$ for $k = 1,\ldots,n$ and for all $\underline{x}$ in U, then f is strictly concave on U. If every non-zero $k \times k$ princpal minor of $H(\underline{x})$ has the same sign as $(-1)^k$ for $k = 1,\ldots,n$ and for all x in U, then f is concave on U. If the kth <u>leading</u> principal minor of $B(\underline{x})$ has the same sign as $(-1)^{k-1}$ for $k = 3,4,\ldots,n+1$, then f is pseudoconcave (and hence quasiconcave) on U.

<u>Proof</u>: a) Most of these implications follow from the definition or from Theorem 1.6. See Mangasarian (1969) for complete details. We will sketch a proof of the third implication here.

Suppose f is pseudoconcave but not strictly quasi-concave. Then, there are $\underline{x}^0, \underline{x}^1 \in U$ such that $f(\underline{x}^0) > f(\underline{x}^1)$ but for some $\mu \in (0,1)$, $f(\underline{x}^\mu) \leq f(\underline{x}^1)$, where $\underline{x}^\mu = \underline{x}^0 + \mu(\underline{x}^1 - \underline{x}^0)$. Choose $\bar{\mu}$ so that $f(\underline{x}^{\bar{\mu}}) \leq f(\underline{x}^\mu)$ for all $\mu \in [0,1]$. Since $\underline{x}^{\bar{\mu}}$ minimizes f on the line segment $\ell(\underline{x}^0, \underline{x}^1)$, $Df(\underline{x}^{\bar{\mu}})(\underline{x}^\mu - \underline{x}^{\bar{\mu}}) \geq 0$ for all $\mu \in [0,1]$ by Theorem 1.8a. By the method of proof of the claim in Theorem 1.7, $Df(\underline{x}^{\bar{\mu}})(\underline{x}^1 - \underline{x}^0) = 0$, since $\underline{x} - \underline{x}^{\bar{\mu}} = -\bar{\mu}(\underline{x}^1 - \underline{x}^0)$ and $\underline{x}^1 - \underline{x}^{\bar{\mu}} = (1-\bar{\mu})(\underline{x}^1 - \underline{x}^0)$. Similarly, $Df(\underline{x}^{\bar{\mu}})(\underline{x}^1 - \underline{x}^{\bar{\mu}}) = 0$. But now, since f is pseudoconcave, $f(\underline{x}^1) \leq f(\underline{x}^{\bar{\mu}})$. Since $f(\underline{x}^0) < f(\underline{x}^1)$, we have a contradiction to the minimizing property of $\underline{x}^{\bar{\mu}}$.

b): See Ferland (1972).

c): The first two sentences follow from Theorem 1.4, 1.5, and 1.6. Under the hypotheses of the last statement, Theorem 1.5 tells us that $D^2f(\underline{x})$ is negative definite on the nullspace of $Df(x)$. By Theorem 3.4 below, each $\underline{x}$ in U maximizes f on the constraint set $\{y|Df(\underline{x})\underline{y} - Df(\underline{x})\underline{x} \geq 0\} = \{y|Df(\underline{x})(\underline{x}-\underline{y}) \leq 0\}$. In other words, if $x,y \in U$ and $Df(x)(x-y) \leq 0$, then $f(x) \geq f(y)$. But this is the definition of pseudoconcavity on U. []

There is one important result, as described in the following theorem, which holds for concave functions but not for pseudoconcave or quasi-concave functions. This theorem is one reason why one cannot weaken the concavity hypotheses in some of the theorems of chapter seven.

Theorem 1.10. Let $f_1,\ldots,f_a : U \to \mathbb{R}$ be concave functions on a convex subset U of $\mathbb{R}^n$. Let $\lambda_1,\ldots,\lambda_a$ be non-negative numbers. Then, $\sum_1^a \lambda_i f_i : U \to \mathbb{R}$ is a concave function. This result is not true for pseudoconcave or quasi-concave functions.

Proof: Since each $\lambda_i \geq 0$ and each f_i is concave.

$$\lambda_i f_i(x^\circ+\mu(x'-x^\circ)) \geq \lambda_i f_i(\underline{x}^\circ) + \mu[\lambda_i f(\underline{x}') - \lambda_i f(x^\circ)] .$$

The theorem follows by adding these inequalities. To see the last sentence, note that $f_1(x) = -2x$ and $f_2(x) = x^3 + x$ are both pseudoconcave, but $(f_1 + f_2)(x) = x^3 - x$ is not even quasi-concave. []

2. UNCONSTRAINED MAXIMA

2.A. Necessary Conditions

Let C be some subset of $\mathbb{R}^n$. For example, C may be $\{\underline{x} \in \mathbb{R}^n | g_1(\underline{x}) \geq 0, \ldots, g_M(\underline{x}) \geq 0, h_1(\underline{x}) = 0, \ldots, h_N(\underline{x}) = 0\}$, where $g_1, \ldots, g_M, h_1, \ldots, h_N$ are functions from $\mathbb{R}^n$ to $\mathbb{R}^1$. Then, if $f\colon \mathbb{R}^n \to \mathbb{R}$ and if $\underline{x}^\circ \in C$, f has a <u>local maximum</u> on C at $\underline{x}^\circ$ (or $\underline{x}^\circ$ locally maximizes f on C) provided $\underline{x}^\circ$ has a neighborhood U in $\mathbb{R}^n$ with $f(\underline{x}) \leq f(\underline{x}^\circ)$ for all $\underline{x} \in U \cap C$.

If one can take U to be $\mathbb{R}^n$, then f has a <u>global</u> maximum on C at $\underline{x}^\circ$. In this paper, "maximum" will always mean "local maximum" unless stated otherwise.

If one uses calculus techniques, then there are usually two steps in a maximization problem. First, look at effective necessary conditions for a point to be a maximum. This step should quickly narrow down the number of candidates for a maximum point - possibly to a finite set of points. Secondly, apply some effective method for checking out each of these points. Such methods will usually involve the local convexity of f or the negative-definiteness of some second derivative.

Let us first examine the simplest such problem, i.e., find the maxima of a C^1 $f\colon \mathbb{R}^n \to \mathbb{R}^1$ with no other constraints. (Equivalently, C is some <u>open</u> subset of $\mathbb{R}^n$.) We first list the classical necessary conditions.

Theorem 2.1. If $f: \mathbb{R}^n \to \mathbb{R}$ is C^2 and if $\underline{x}^\circ$ is a (local) maximum point of f, then a) $Df(\underline{x}^\circ) = 0$, and b) $D^2f(x_0)(v,v) \leq 0$ for all $\underline{v}$.

Proof: We will assume that the reader is familiar with this theorem for the case $n = 1$. Let $g(t) = f(\underline{x}^\circ + t\underline{v})$ for some arbitrary $\underline{v} \in T_{\underline{x}^\circ}\mathbb{R}^n$. By hypothesis, $t = 0$ is a local maximum point of g. Therefore, by theorems of Calculus I, $g'(0) = 0$ and $g''(0) \leq 0$. By the Chain Rule,

$$g'(0) = \frac{d}{dt} f(\underline{x}^\circ + t\underline{v})\Big|_{t=0} = Df(\underline{x}^\circ)\underline{v} \qquad \text{and}$$

$$g''(0) = \frac{d^2}{dt^2} f(\underline{x}^\circ + t\underline{v})\Big|_{t=0} = D^2f(\underline{x}^\circ)(\underline{v},\underline{v}). \qquad \blacksquare$$

2.B. Sufficient Conditions

By Theorem 2.1, in searching for maxima one need only check out the critical points of f, i.e., $\{\underline{x} | Df(\underline{x}) = \underline{0}\}$. For most smooth functions, the critical points are isolated in $\mathbb{R}^n$. (See Section II.6 of Golubitsky-Guillemin (1973).) The next theorem gives the classical sufficient condition for a critical point to be a local maximum. Its proof uses the basic fact that any sequence on a compact set has a convergent subsequence.

Theorem 2.2. Suppose that $\underline{x}^\circ$ is a critical point of a C^2 $f: \mathbb{R}^n \to \mathbb{R}^1$. If $D^2f(\underline{x}^\circ)$ is negative definite, then $\underline{x}^\circ$ is a strict local maximum point of f.

Proof: Suppose $\underline{x}°$ is not a strict local maximum point of f. Then there is a sequence of distinct $\underline{x}^n$ approaching $\underline{x}°$ with $f(\underline{x}^n) \geq f(\underline{x}°)$. By the compactness of $\{\underline{v} \,\|\, |\underline{v}| = 1\}$ in R^n, we can choose $\{\underline{x}^n\}$ so that $\underline{v}^n \equiv \dfrac{\underline{x}^n - x°}{|\underline{x}^n - x°|}$ converges to some $\underline{v}°$ with $|\underline{v}°| = 1$. Using Taylor's Theorem,

$$\text{(A)} \quad 0 \leq f(\underline{x}^n) - f(\underline{x}°) = Df(\underline{x}°)(\underline{x}^n-\underline{x}°) + \tfrac{1}{2}D^2f(\underline{x}°)(\underline{x}^n-\underline{x}°,\underline{x}^n-\underline{x}°)+R(\underline{x}^n),$$

where $Df(x°)(x^n-x°) = 0$ for all n and $\dfrac{R(\underline{y})}{|\underline{y}-\underline{x}°|^2} \to 0$ as $\underline{y} \to \underline{x}°$.

Divide (A) by $|\underline{x}^n-\underline{x}°|^2$:

$$\text{(B)} \quad 0 \leq \frac{f(\underline{x}^n) - f(\underline{x}°)}{|x^n-x°|^2} = D^2f(x°)(v^n,v^n) + \frac{R(\underline{x}^n)}{|x^n-x°|^2}, \text{ for all } n.$$

Now, if $n \to \infty$ in (B), one finds $0 \leq D^2f(x°)(v°,v°)$, contradicting the negative definiteness of $D^2f(x°)$.

For another proof, note that by Theorem 1.4.C, negative definiteness is an "open" property in R^n. In other words, if $D^2f(\underline{x}°)$ is negative definite and f is C^2, $D^2f(\underline{x})$ will also be negative definite for all $\underline{x}$ in an open ball V around $\underline{x}°$. If $y \in V$ and $\underline{y} \neq x°$,

$$\text{(C)} \qquad f(y)-f(\underline{x}°) = Df(\underline{x}°)(\underline{y}-\underline{x}°) + \frac{1}{2} D^2f(\underline{y}')(\underline{y}-\underline{x}°,\underline{y}-\underline{x}°)$$

for some $\underline{y}'$ in V. Since $Df(\underline{x}°) = 0$ and $y-\underline{x}° \neq 0$ and $D^2f(\underline{y}')$ is negative definite, the right hand side of (C) is negative. So, $f(\underline{y}) < f(\underline{x}°)$ for all $\underline{y}$ in V.

This proof even works in infinite-dimensional spaces provided one replaces "$D^2f(\underline{x}°)$ negative-definite" by "$D^2f(\underline{x}°)$ strictly negative definite" to make sure the condition is valid for an open set around $\underline{x}°$. (One says that $D^2f(\underline{x}°)$ is strictly negative definite if there is a positive number c that $D^2f(\underline{x}°)(\underline{v},\underline{v}) \leq -c|\underline{v}|^2$ for all $\underline{v}$, or equivalently such that the eigenvalues of $D^2f(\underline{x}°)$ are strictly less than $-c$. This concept is the same as negative-definiteness in the finite dimensional case.)

For concave functions, the first order necessary conditions are also sufficient and yield <u>global</u> maxima. Theorem 2.3 is a restatement of Theorem 1.8 and is included here for completeness.

<u>Theorem 2.3</u>. Suppose $f : \mathbb{R}^n \to \mathbb{R}^1$ is C^1 and concave (or even pseudoconcave). Then f has a global maximum at $\underline{x}°$ if and only if $Df(\underline{x}°) = \underline{0}$

3. PROBLEMS WITH NON-DEGENERATE CONSTRAINTS

3.A. The Non-Linear Programming Problem

In most maximization problems arising in economics and engineering, there are constraints on the set of feasible states, i.e., the set C of Section 2.A is not an open subset of $\mathbb{R}^n$. In the next sections, we will discuss the following problem, often called "the classical problem of non-linear programming":

Maximize $f : \mathbb{R}^n \to \mathbb{R}$ on the set C, where

(D) $C = \{\underline{x} \in \mathbb{R}^n \mid g_i(\underline{x}) \geq 0,\ i = 1,\ldots,M;\ h_j(\underline{x}) = 0,\ j = 1,\ldots,N\}$

and g_i's and h_j's are smooth functions $\mathbb{R}^n \to \mathbb{R}$.

If $f(\underline{x}) = \underline{c} \cdot \underline{x}$, $g_i(\underline{x}) = \underline{A}_i \cdot \underline{x} + a_i$, and $h_j(\underline{x}) = \underline{B}_j\underline{x} + b_j$ for some vectors $\underline{A}_1,\ldots,\underline{A}_M,\underline{B}_1,\ldots,\underline{B}_N$, and $\underline{c}$ in $\mathbb{R}^n$, and scalars $a_k,\ldots,a_M,b_1,\ldots,b_N$, the problem (D) is the usual linear programming problem. Since the constraint set C for the linear problem is a polyhedral set and f is linear, the solution of this problem, if it exists, lies at a vertex of C (or sometimes at a complete bounding face of C). There are simple, but beautiful algorithms for solving the linear problem, which we will not discuss here. See (for example) Karlin (1959), Hadley (1962), Dantzig (1963), Intrilligator (1971), or Varaiya (1972) for further details and examples.

Returning to the non-linear problem, in this section we will first discuss conditions for maximization where the constraint set C is a "manifold" or the smooth boundary of a manifold. Analytically, this means that the Jacobian matrix of the constraint functions has maximal rank at the proposed solution. We'll call such constraints "non-degenerate". In this situation, it is convenient to consider first the case of equality constraints, i.e., $M = 0$ in (D) .

3.B. Non-degenerate Equality Constraints

The following theorem gives the classical necessary conditions of Lagrange for $\underline{x}°$ to maximize a function on a submanifold of $\mathbb{R}^n$. We write $h_1 = 0,\ldots,h_N = 0$ as $h \equiv (h_1,\ldots,h_N) = 0$.

Theorem 3.1. Suppose that $\underline{x}°$ maximizes $f: \mathbb{R}^n \to \mathbb{R}$ on the set $M_h = \{\underline{x} \in \mathbb{R}^n | h(\underline{x}) = 0$ where $h: \mathbb{R}^n \to \mathbb{R}^N, N < n\}$. Suppose further that f and h are C^1 and that $Dh(\underline{x}°)$ has maximal rank. a) Then, there exists a unique non-zero $(\mu_1^\circ,\ldots,\mu_N^\circ) \in \mathbb{R}^N$ such that $Df(\underline{x}°) + \sum_1^N \mu_i^\circ \, Dh_i(\underline{x}°) = \underline{0}$.
b) If f and h are C^2 , then

$$D^2[f + \sum_1^N \mu_i^\circ h_i](\underline{x}°)(\underline{v},\underline{v}) \le 0 \text{ , for all } \underline{v}$$

for which $Dh(\underline{x}°)\underline{v} = 0$, i.e., all $\underline{v} \in T_{\underline{x}°}M_h$.

Proof: We will work with the function $(f,h): \mathbb{R}^n \to \mathbb{R}^1 \times \mathbb{R}^N \cong \mathbb{R}^{N+1}$. We first claim that $D(f,h)(\underline{x}°): \mathbb{R}^n \to \mathbb{R}^{N+1}$ does not have maximal rank. For, if it does, then by the implicit function theorem (Theorem 1.3.b) (f,h) is locally onto, i.e., there is a neighborhood U of $\underline{x}^\circ$ in $\mathbb{R}^n$ and a neighborhood V of $\left(f(\underline{x}^\circ), h(\underline{x}^\circ)\right)$ in $\mathbb{R}^{N+1}$ so that f maps U onto V. So, we can choose $\underline{x}^1 \in U$ with $\left(f(\underline{x}^1), h(\underline{x}^1)\right) = \left(f(\underline{x}^\circ)+\varepsilon, h(\underline{x}^\circ)\right)$ in V for some $\varepsilon > 0$. Then $f(\underline{x}^1) > f(\underline{x}^\circ)$ and $h(\underline{x}^1) = h(\underline{x}^\circ) = \underline{0}$, contradicting the fact that $\underline{x}^\circ$ maximizes f on M_h.

Since $D(f,h)(\underline{x}^\circ)$ is not of maximal rank, its rows are linearly dependent, i.e., there exists non-zero $(\lambda_0,\ldots,\lambda_N) \in \mathbb{R}^{N+1}$ such that $\lambda_0 Df(\underline{x}^\circ) + \sum_1^N \lambda_i Dh_i(\underline{x}^\circ) = \underline{0}$. If $\lambda_0 = 0$, then $\sum_1^N \lambda_i Dh_i(\underline{x}^\circ) = 0$ for some non-zero $(\lambda_1,\ldots,\lambda_N)$, contradicting the maximal rank of $Dh(\underline{x}^\circ)$. So, let $\mu_i^\circ = \lambda_i/\lambda_0$. If there is another non-zero $(\mu_1^1,\ldots,\mu_N^1)$ with $Df(\underline{x}^\circ) + \sum_1^N \mu_i^1 Dh_i(\underline{x}^\circ) = 0$, we can subtract one equation from the other to obtain $\sum_1^N (\mu_i^1 - \mu_i^\circ) Dh_i(\underline{x}$ $\underline{0}$. Again, the non-degeneracy of $Dh(\underline{x}^\circ)$ implies that $\mu_i^1 - \mu_i^\circ = 0$ for all i.

To see part b), let $\underline{v} \in \ker Dh(\underline{x}^\circ) = T_{\underline{x}^\circ}M_h$. Again, by the implicit function Theorem (1.3.b), there is a C^2 curve $a\colon [0,\varepsilon) \to \mathbb{R}^n$ with $a(0) = \underline{x}^\circ$, $a'(0) = \underline{v}$, and $h\left(a(t)\right) = 0$ for all t. By hypothesis, $f \circ a\colon [0,\varepsilon) \to \mathbb{R}^1$ has a maximum at 0. By results of Calculus I,

$$0 \geq \frac{d^2}{dt^2}(f\circ a)\Big|_{t=0} = \frac{d^2}{dt^2}\left[f\circ a + \sum_i \mu_i^\circ (h_i \circ a)\right]\Big|_{t=0} \quad (h_i \circ a = 0)$$

$$= \frac{d}{dt}\left[Df\big(a(t)\big)a'(t) + \sum_i \mu_i^\circ\, Dh_i\big(a(t)\big)a'(t)\right]\Big|_{t=0} \quad \text{(chain rule)}$$

$$= D^2 f\big(a(0)\big)\big(a'(0), a'(0)\big) + Df\big(a(0)\big)\big(a''(0)\big)$$

$$+ \sum_i \mu_i^\circ\, D^2 h_i\big(a(0)\big)\big(a'(0), a'(0)\big) + \sum_i \mu_i^\circ\, Dh_i\big(a(0)\big)a''(0)$$

$$= D^2 f(\underline{x}^\circ)(\underline{v}, \underline{v}) + \sum_i \mu_i^\circ\, D^2 h_i(\underline{x}^\circ)(\underline{v}, \underline{v}) ,$$

since $Df(\underline{x}^\circ) + \sum_i \mu_i^\circ\, Dh(\underline{x}^\circ) = \underline{0}$. ∎

The geometric interpretation of Theorem 3.1 is simple since the maximal rank of $Dh(\underline{x}^\circ)$ implies that $M_h \equiv h^{-1}(\underline{0})$ is a submanifold around $\underline{x}^\circ$. Recall that $Dg(\underline{x}^\circ)v = \underline{v} \cdot \nabla g(\underline{x}^\circ)$ where $\nabla g(\underline{x}^\circ)$ is the gradient (column) vector of g at $\underline{x}^\circ$. Then, 3.1.a says that $\nabla f(\underline{x}^\circ)$ is a linear combination of $\nabla h_1(\underline{x}^\circ), \ldots, \nabla h_N(\underline{x}^\circ)$. Since each $\nabla h_i(\underline{x}^\circ)$ is perpendicular to $T_{\underline{x}^\circ}M_h$, so is $\nabla f(\underline{x}^\circ)$. This means that the projection of $\nabla f(\underline{x}^\circ)$ on $T_{\underline{x}^\circ}M_h$ is zero, i.e., that $f|M_h$ has a critical point. If one now uses coordinates that give M_h as a hyperplane of $\mathbb{R}^n$ around $\underline{x}^\circ$, then Theorem 2.1.b becomes Theorem 3.1.b in these coordinates.

However, one does not need non-degenerate constraint equations to derive second-order sufficient conditions, i.e., the analogue of Theorem 2.2. We will even use the more general first order condition of Section 4.

Theorem 3.2. (Second Order Sufficient Condition) Suppose that $f, h_1, \ldots, h_N$ are C^2 functions on $\mathbb{R}^n$. Suppose $h(\underline{x}^\circ) \equiv (h_1(\underline{x}^\circ), \ldots, h_N(\underline{x}^\circ)) = 0$. Suppose there is a non-zero $(\mu_0, \ldots, \mu_N)$ such that $\mu_0 \geq 0$, $D(\mu_0 f + \Sigma\mu_i h_i)(\underline{x}^\circ) = 0$, and $D^2(\mu_0 f + \Sigma\mu_i h_i)(\underline{x}^\circ)(\underline{v}, \underline{v}) < 0$ for all $\underline{v}$ with $Dh(\underline{x}^\circ)\underline{v} = 0$, $\underline{v} \neq \underline{0}$. Then, $\underline{x}^\circ$ is a strict local maximum point of f on $h^{-1}(\underline{0})$.

Proof: Let $M_h = h^{-1}(\underline{0})$ and let $F \equiv \mu_0 f + \Sigma\mu_i h_i : \mathbb{R}^n \to \mathbb{R}^1$. Now, just imitate the proof of Theorem 2.2, using F. That is, suppose there is $x^n \to x^\circ$ such that $x^n \neq x^\circ$ for all n, $h(\underline{x}^n) = \underline{0}$, $f(x^n) \geq f(\underline{x}^\circ)$, and $\underline{v}^n \equiv \dfrac{\underline{x}^n - \underline{x}^\circ}{|x^n - x^\circ|} \to v^\circ$. Since $\mu_0 \geq 0$, $0 \leq F(\underline{x}^n) - F(\underline{x}^\circ)$ for all n. As in the proof of Theorem 2.2., one finds that $D^2F(\underline{x}^\circ)(\underline{v}^\circ, \underline{v}^\circ) \geq 0$ using Taylor's series. Since $|\underline{v}^\circ| = 1 \neq 0$, we need only show that $Dh(\underline{x}^\circ)\underline{v}^\circ = \underline{0}$ to find a contradiction. But, for each $i = 1, \ldots, N$ and for each n,

$$0 = \frac{h_i(\underline{x}^n) - h_i(\underline{x}^\circ)}{|\underline{x}^n - \underline{x}^\circ|} = Dh_i(\underline{x}^{n,i})\underline{v}^n$$

for some $\underline{x}^{n,i}$ on the line between $\underline{x}^n$ and $\underline{x}^\circ$. As $n \to \infty$, $\underline{x}^n \to \underline{x}^\circ$, $\underline{x}^{n,i} \to \underline{x}^\circ$, and $\underline{v}^n \to \underline{v}^\circ$. Since each h_i is C^1, $Dh_i(\underline{x}^\circ)v^\circ = 0$. ∎

3.C. Non-Degenerate Inequality Constraints

The next step is to generalize the problem by allowing inequality constraints, $g_i(\underline{x}) \geq 0$, $i = 1, \ldots, M$, as in statement (D) . We will still focus on the situation where the "effective" constraints are non-degenerate

Let $x^{\circ} \in \mathbb{R}^n$ with $h(\underline{x}^{\circ}) = \underline{0}$ and $g(\underline{x}^{\circ}) = (g_1,\ldots,g_m)(x^{\circ}) \geq 0$. Let $E \equiv \{i \mid g_i(\underline{x}^{\circ}) = 0\}$ and $I \equiv \{j \mid g_j(\underline{x}^{\circ}) > 0\}$. Reparameterize so that $E = \{1,\ldots,\kappa\}$ and $I = \{\kappa+1,\ldots,M\}$ for some κ and $g = (g_E, g_I): \mathbb{R}^n \to \mathbb{R}^E \times \mathbb{R}^I \cong \mathbb{R}^M$. The mapping $(g_E, h): \mathbb{R}^n \to \mathbb{R}^E \times \mathbb{R}^N$ represents the <u>effective constraints</u> at $\underline{x}^{\circ}$. The next theorem states the necessary first and second order conditions for a maximum under non-degenerate constraints. The first such theorems were proved by Karush (1939) and Pennisi (1953).

<u>Theorem 3.3</u>. Suppose that $\underline{x}^{\circ}$ is a local maximum of f on $C_{g,h} \equiv \{\underline{x} \in \mathbb{R}^n \mid g(x) \geq 0,\ h(\underline{x}) = \underline{0}\}$. Suppose that f, g, and h are C^2 and that $D(g_E,h)(\underline{x}^{\circ})$ has maximal rank. Then, there is a unique non-zero $(\lambda_1,\ldots,\lambda_M,\mu_1,\ldots,\mu_N)$ such that

i) $D_x[f + \sum_1^M \lambda_i g_i + \sum_1^N \mu_j h_j](\underline{x}^{\circ}) \equiv DL(\underline{x}^{\circ}) = \underline{0}$,

ii) $\lambda_j \geq 0$ for $j = 1,\ldots,M$, and

iii) $\lambda_j g_j(\underline{x}^{\circ}) = 0$ for $j = 1,\ldots,M$, (i.e., $\underline{\lambda} \cdot g(\underline{x}^{\circ}) = 0$) .

Furthermore, iv) $D^2L(\underline{x}^{\circ})(\underline{v},\underline{v}) \leq 0$ for all $\underline{v}$ such that

$$Dg_E(\underline{x}^{\circ})\underline{v} = \underline{0} \text{ and } Dh(\underline{x}^{\circ})\underline{v} = \underline{0} .$$

Proof: If I is non-empty, let U denote the open set $\{\underline{x} \in \mathbb{R}^n | g_I(\underline{x}) > \underline{0}\}$. If I is empty, let U denote $\mathbb{R}^n$. Then, i), iii), and iv) follow immediately if one notices that $\underline{x}^\circ$ maximizes f on the set $\{\underline{x} \in U | g_E(\underline{x}) = \underline{0}, h(\underline{x}) = \underline{0}\}$ and then applies Theorem 3.2 setting $\lambda_j = 0$ for $j \notin E$, i.e., when $g_j(\underline{x}^\circ) \neq 0$. To prove the important statement ii), let $j \in E$. Without loss of generality, we will take j to be 1. Since $(Dg_E(\underline{x}^\circ), Dh(\underline{x}^\circ))$ has maximal rank, there is a vector $\underline{v}$ with

(E) $$Dg_1(\underline{x}^\circ)\underline{v} > 0,\ Dg_2(\underline{x}^\circ)\underline{v} = \ldots = Dg_K(\underline{x}^\circ)\underline{v} = 0, \text{ and } Dh(\underline{x}^\circ)\underline{v} = \underline{0} .$$

By the implicit function theorem (Theorem 1.3b) applied to $(g_2, \ldots, g_K, h) =$ $\underline{0}$, there is a smooth curve $c: [0,\varepsilon) \to \mathbb{R}^n$ such that $c(0) = \underline{x}^\circ$, $c'(0) = \underline{v}$, $g_2(c(t)) = \ldots = g_K(c(t)) = 0$ and $h(c(t)) = \underline{0}$ for all t. Since $Dg_1(\underline{x}^\circ)\underline{v} > 0$, $g_1(c(t)) > 0$ for $t \in (0,\varepsilon_1)$ for some $\varepsilon_1 > 0$, i.e., $c(t) \in C_{g,h}$ for $t \in [0,\varepsilon_1)$. Since $c(0) = \underline{x}^\circ$ maximizes f on $C_{g,h}$, $f(c(t)) \leq f(x^\circ)$ for t small and

$$Df(\underline{x}^\circ)\underline{v} = (f \circ c)'(0) \leq 0 .$$

By i) and (E), $Df(\underline{x}^\circ)\underline{v} + \lambda_1 Dg_1(\underline{x}^\circ)\underline{v} = 0$. Since $Df(\underline{x}^\circ) \leq 0$ and $Dg_1(\underline{x}^\circ)\underline{v} > 0$, $\lambda_1 \geq 0$. One argues similarly for $\lambda_2, \ldots, \lambda_K$. ☐

Finally, we consider second order sufficient conditions for a constrained maximum. Hestenes (1966), McCormick (1967), and Fiacco-McCormick (1968) seem to be the first ones to prove a strong second order sufficiency result for inequality-equality constraints without any non-degeneracy assumptions on the constraint set. Their proofs are basically similar to the one described below.

Theorem 3.4. Suppose $f, g_1, \ldots, g_M, h_1, \ldots, h_N$ are C^2 functions on $\mathbb{R}^n$. Suppose $g(\underline{x}^\circ) \geq \underline{0}$ and $h(\underline{x}^\circ) = \underline{0}$. Suppose there exist $\lambda_0, \lambda_1, \ldots, \lambda_M, \mu_1, \ldots, \mu_N$ so that

i) $\lambda_i \geq 0$ for $i = 0, \ldots, M$,

ii) $\lambda_i g_i(\underline{x}^\circ) = 0$ for $i = 1, \ldots, M$,

iii) $D[\lambda_0 f + \sum_1^M \lambda_i g_i + \sum_1^N \mu_j h_j](\underline{x}^\circ) = \underline{0}$, and

iv) $D^2[\lambda_0 f + \sum_1^M \lambda_i g_i + \sum_1^N \mu_j h_j](\underline{x}^\circ)(\underline{v}, \underline{v}) < 0$

for all non-zero $\underline{v}$ satisfying $\lambda_0 Df(\underline{x}^\circ)\underline{v} \equiv 0$, $\lambda_i Dg_i(\underline{x}^\circ)\underline{v} = 0$ $i = 1, \ldots, M$, and $Dh(\underline{x}^\circ)\underline{v} = \underline{0}$.

Then, there is a neighborhood U of $\underline{x}^{\circ}$ such that $f(\underline{x}) < f(\underline{x}^{\circ})$ for all $\underline{x} \neq \underline{x}^{\circ}$ in $U \cap C_{g,h}$.

Remark. One can further restrict $\underline{v}$ in iv) to those for which $Dg_i(\underline{x}^{\circ})\underline{v} \geq 0$ for i such that $\lambda_i > 0$ and $Df(\underline{x}^{\circ})\underline{v} \geq 0$.

Proof: First, choose a neighborhood V of $\underline{x}^\circ$ so that $g_i(\underline{x}) > 0$ for $i \in I$ and $\underline{x} \in V$. Working within V, our constraints are now $g_E \geq \underline{0}$ and $h = \underline{0}$ and our Lagrangian is

$$L' \equiv \lambda_0 f + \sum_{i \in E} \lambda_i g_i + \sum_1^N \mu_j h_j = \lambda_0 f + \sum_1^M \lambda_i g_i + \sum_1^N \mu_j h_j \equiv L .$$

Arguing by contradiction as in Theorem 3.2, suppose there exist $\underline{x}^n \to \underline{x}^\circ$ such that $\underline{x}^n \in C_{g,h}$, $f(\underline{x}^n) \geq f(\underline{x}^\circ)$, and $\underline{x}^n \neq \underline{x}^\circ$ for each n. As before, choose $\underline{x}^n$ so that $\underline{v}^n = \dfrac{\underline{x}^n - \underline{x}^\circ}{|\underline{x}^n - \underline{x}^\circ|}$ converges to some unit vector $\underline{v}^\circ$.

Next, we show that $\underline{v}^\circ$ satisfies the conditions of hypothesis iv). Arguing as in Theorem 3.2, one proves easily via the Mean Value Theorem that

(F) $Df(\underline{x}^\circ)\underline{v}^\circ \geq 0$, $Dh(\underline{x}^\circ)\underline{v}^\circ = \underline{0}$, and $Dg_i(\underline{x}^\circ)\underline{v}^\circ \geq 0$ for each $i \in E$.

Furthermore, $\lambda_i Dg_i(\underline{x}^\circ)\underline{v}^\circ = 0$ for each i. Otherwise, there exists a j such that $\lambda_j Dg_j(\underline{x}^\circ)\underline{v}^\circ > 0$ and then by (F) and i)

$$DL(\underline{x}^\circ)\underline{v}^\circ = \lambda_0 Df(\underline{x}^\circ)\underline{v}^\circ + \sum_{i \in E} \lambda_i Dg_i(\underline{x}^\circ)\underline{v}^\circ > 0 ,$$

which contradicts hypothesis iii). Similarly, $\lambda_0 \, Df(\underline{x}^\circ)\underline{v}^\circ = 0$.

Finally, by Taylor's Theorem (Theorem 1.2a), there exist C^2 functions R, S, and T such that for each n

$$0 \leq f(\underline{x}^n) - f(\underline{x}^\circ) = Df(\underline{x}^\circ)(\underline{x}^n - \underline{x}^\circ) + \frac{1}{2!}D^2 f(\underline{x}^\circ)(\underline{x}^n - \underline{x}^\circ, \underline{x}^n - \underline{x}^\circ) + R(\underline{x}^n) \ ;$$

(F') $$0 \leq g_i(\underline{x}^n) - g_i(\underline{x}^\circ) = Dg_i(\underline{x}^\circ)(\underline{x}^n - \underline{x}^\circ) + \frac{1}{2!}D^2 g_i(\underline{x}^\circ)(\underline{x}^n - \underline{x}^\circ, \underline{x}^n - \underline{x}^\circ) + S_i(\underline{x}^n), \ i \in E \ ;$$

$$0 = h_j(\underline{x}^n) - h_j(\underline{x}^\circ) = Dh_j(\underline{x}^\circ)(\underline{x}^n - \underline{x}^\circ) + \frac{1}{2!}D^2 h_j(\underline{x}^\circ)(\underline{x}^n - \underline{x}^\circ, \underline{x}^n - \underline{x}^\circ) + T_j(\underline{x}^n) \ ,$$

where $\frac{R(\underline{x}^n)}{|\underline{x}^n - \underline{x}^\circ|^2}$, $\frac{S_i(\underline{x}^n)}{|\underline{x}^n - \underline{x}^\circ|^2}$, and $\frac{T_j(\underline{x}^n)}{|\underline{x}^n - \underline{x}^\circ|^2}$ all tend to zero as $\underline{x}^n \to \underline{x}^\circ$. Divide each expression in (F') by $|\underline{x}^n - \underline{x}^\circ|^2$, multiply through by the corresponding Lagrange multiplier, and add the expressions to obtain

$$0 \leq \frac{DL(\underline{x}^\circ)(\underline{x}^n - \underline{x}^\circ)}{|\underline{x}^n - \underline{x}^\circ|^2} + \frac{1}{2!}D^2 L(\underline{x}^\circ)(\underline{v}^n, \underline{v}^n) + \frac{Q(\underline{x}^n)}{|\underline{x}^n - \underline{x}^\circ|^2} \ ,$$

where the last term tends to zero as $\underline{x}^n \to \underline{x}^\circ$. Using $DL(\underline{x}^\circ) = \underline{0}$ and then letting $\underline{x}^n \to \underline{x}$, one finds that $0 \leq D^2 L(\underline{x}^\circ)(\underline{v}^\circ, \underline{v}^\circ)$. Since $\underline{v}^\circ$ satisfies the conditions of iv), we have a contradiction to iv) and $\underline{x}^\circ$ must be a strict local maximum of f on $C_{g,h}$. ∎

Remark. Condition iv), the second order condition, in Theorem 3.4 is difficult to check as it is written. However, by Theorem 1.5, one can replace iv) by the following much more easily checked condition:

iv)': Let H be the bordered matrix

$$\begin{pmatrix} 0 & 0 & A \\ 0 & 0 & B \\ A^t & B^t & C \end{pmatrix}$$

where $A = ((\lambda_i \frac{\partial g_i(\underline{x}^\circ)}{\partial x_k}))$ for $k = 1,\ldots,n$ and for i such that $\lambda_i \neq 0$,

$$B = ((\frac{\partial h_j(\underline{x}^\circ)}{\partial x_k})) \quad \text{for} \quad k = 1,\ldots,n \quad \text{and} \quad j = 1,\ldots,N ,$$

$$C = ((\frac{\partial^2 L}{\partial x_i \partial x_j}(\underline{x}^\circ))) \quad \text{for} \quad i,j = 1,\ldots,n .$$

We require H to satisfy the conditions of Theorem 1.5, namely determinant H has the same sign as $(-1)^n$ and the last $(n-m)$ leading principal minors of H alternate in sign, where m = number of rows of A + number of rows of B = $\#\{i \mid \lambda_i > 0,\ i = 1,\ldots,M\} + N$.

Some authors, e.g., McShane (1942), Weinberger (1974) and Ben-tal (1980), have noticed that one can find an even stronger sufficiency test than that of Theorems 3.2 and 3.4. For, in the proofs of these results, one can easily allow the Lagrange multipliers to depend on the vector $\underline{v}$ being tested and thus prove the following result.

Theorem 3.5. Suppose that $f, g_1, \ldots, g_M, h_1, \ldots, h_N$ are C^2 functions on $\mathbb{R}^n$. Suppose that $g(\underline{x}^\circ) \geq \underline{0}$ and $h(\underline{x}^\circ) = \underline{0}$. Suppose that for each non-zero $\underline{v}$ such that $Df(\underline{x}^\circ)\underline{v} \geq 0$, $Dg_E(\underline{x}^\circ)\underline{v} \geq \underline{0}$, and $Dh(\underline{x}^\circ)\underline{v} = \underline{0}$, there exists $\lambda_0, \lambda_1, \ldots, \lambda_M, \mu_1, \ldots, \mu_N$ so that i), ii), and iii) of Theorem 3.4 are satisfied and

$$D^2[\lambda_0 f + \sum_1^M \lambda_i g_i + \sum_1^N \mu_j h_j](\underline{x}^\circ)(\underline{v},\underline{v}) < 0 .$$

Then, there exists a neighborhood U of $\underline{x}^\circ$ such that $f(\underline{x}) < f(\underline{x}^\circ)$ for all $\underline{x} \neq \underline{x}^\circ$ in U satisfying $g(\underline{x}) \geq 0$, $h(\underline{x}) = \underline{0}$.

As we will see later, an important variant of problem (D) is the following:

(G) Maximize $f: \mathbb{R}^n \to \mathbb{R}$ on the set $\{x \mid G_i(x) \geq 0,\ i = 1,\ldots,M;\ x_j \geq 0,\ j = 1,\ldots,n\}$.

We state without proof the application of Theorems 3.3 and 3.4 to this special problem.

Corollary 3.6. Suppose $f, G_1, \ldots, G_M$ are C^2 functions on $\mathbb{R}^n$.

a) If $\underline{x}^\circ$ is a solution of (G) and if $\frac{\partial G_E}{\partial x_B}(\underline{x}^\circ)$ has maximal rank where $E = \{i \mid G_i(\underline{x}^\circ) = 0\}$ and $B = \{j \mid x_j^\circ > 0\}$, then there exists unique non-zero $\underline{\lambda}^\circ = (\lambda_1^\circ, \ldots, \lambda_N^\circ)$ such that

(i) $\lambda_i^\circ \geq 0$ for all i,

(ii) if $L(x,\lambda) \equiv f(x) + \sum_1^M \lambda_i G_i(x)$, then

$$\frac{\partial L}{\partial \underline{x}}(\underline{x}^\circ,\underline{\lambda}^\circ) \le 0 \quad \text{and} \quad \underline{x}^\circ \cdot \frac{\partial L}{\partial \underline{x}}(\underline{x}^\circ,\underline{\lambda}^\circ) = 0 ,$$

$$\frac{\partial L}{\partial \underline{\lambda}}(\underline{x}^\circ,\underline{\lambda}^\circ) = G(\underline{x}^\circ) \ge 0 \quad \text{and} \quad \underline{\lambda}^\circ \cdot \frac{\partial L}{\partial \underline{\lambda}}(\underline{x}^\circ,\underline{\lambda}^\circ) = 0 .$$

(iii) $D_x^2 L(\underline{x}^\circ,\underline{\mu}^\circ)(\underline{v},\underline{v}) \le 0$ for all non-zero $\underline{v}$ with $DG_E(\underline{x}^\circ)\underline{v} = 0$ and $v_i = 0$ for $i \notin B$.

b) Conversely, suppose that $G(\underline{x}^\circ) \ge 0$ and $\underline{x}^\circ > 0$. Suppose further that there is $\underline{\lambda}^\circ = (\lambda_1^\circ,\ldots,\lambda_N^\circ) \ge 0$ such that L satisfies (ii) and (iii) at $(\underline{x}^\circ,\underline{\lambda}^\circ)$ with "$<$" replacing "$\le$" in (iii), then $\underline{x}^\circ$ is a strict local maximum of f on $\{\underline{x} \mid G(\underline{x}) \ge 0, \underline{x} \ge 0\}$.

Conditions i), ii), and iii) of Theorem 3.3 are usually called the Kuhn-Tucker conditions for problem (D). Conditions (i) and (ii) of Corollary 3.6 are called the Kuhn-Tucker conditions for problem (G).

El-Hodiri (1971) and Milleron (1972) both have complete, yet concise, discussions of the non-linear programming problem with non-degenerate constraints. El-Hodiri also adds some interesting historical comments.

3.D Lagrange Multipliers as Sensitivity Indicators

Consider the problem of maximizing $f : \mathbb{R}^n \to \mathbb{R}$ subject to the equality constraint $h(\underline{x}) = \underline{b}$, where $h : \mathbb{R}^n \to \mathbb{R}^M$ and $\underline{b}$ is viewed as a parameter. A natural and important question is: how does the optimal value of f change as $\underline{b}$ is allowed to vary. The following theorem shows that the Lagrange multipliers themselves measure the sensitivity of the optimal value of f to changes in the constraint $\underline{b}$. We will see a number of economic applications of this fact in Chapter six.

Theorem 3.7 Let $f, h_1, \ldots, h_N: \mathbb{R}^n \to \mathbb{R}$ be C^2 functions with $\underline{x}^\circ \in \mathbb{R}^n$ and $h(\underline{x}^\circ) = \underline{b}^\circ$. Suppose that the following sufficient conditions for a maximum of f on $h^{-1}(\underline{b}^\circ)$ are satisfied at $\underline{x}^\circ$:

i) There exist $\lambda_1^\circ, \ldots, \lambda_N^\circ$ such that $DL(\underline{x}^\circ) = \underline{0}$, where $L(\underline{x}) \equiv f(x) + \sum_1^N \lambda_i^\circ (b_i^\circ - h_i(\underline{x}))$.

ii) $D^2L(\underline{x}^\circ)(\underline{v},\underline{v}) < 0$ for all non-zero $\underline{v}$ in the kernel of $Dh(\underline{x}^\circ)$;

iii) $Dh(\underline{x}^\circ)$ has maximal rank.

Then, there is a neighborhood W of $\underline{b}^\circ$ in $\mathbb{R}^N$ and C^1 functions

$$\xi: W \to \mathbb{R}^n, \qquad \lambda: W \to \mathbb{R}^N$$

such that $\xi(\underline{b}^\circ) = \underline{x}^\circ$, $\lambda(\underline{b}^\circ) = \underline{\lambda}^\circ$, and for all $\underline{b} \in W$ $\xi(b)$ maximizes f on $h^{-1}(\underline{b})$ with Lagrange multipliers $\lambda_1(b), \ldots, \lambda_N(b)$.

Furthermore, $\lambda_i(\underline{b}) = \frac{\partial}{\partial b_i} (f \circ \xi)(\underline{b})$.

<u>Proof</u>: Define $M = (M_1, M_2): \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^n \times \mathbb{R}^N$ by $M(\underline{x},\underline{\lambda},\underline{b}) = (\nabla f(\underline{x}) - \lambda \cdot \nabla h(\underline{x}),\ \underline{b} - h(\underline{x}))$. Then $M(\underline{x}^\circ,\underline{\lambda}^\circ,\underline{b}^\circ) = (\underline{0},\underline{0})$ and

$$D_{(x,\lambda)} M(\underline{x}^\circ,\underline{\lambda}^\circ,\underline{b}^\circ) = \begin{pmatrix} D^2L(\underline{x}^\circ) & -Dh(\underline{x}^\circ)^T \\ -Dh(\underline{x}^\circ) & \underline{0} \end{pmatrix}$$

Here, $D^2L(\underline{x}^\circ)$ denotes the Hessian matrix of L at $\underline{x}^\circ$. To solve $M = 0$ for $\underline{x}$ and $\underline{\lambda}$ as functions of $\underline{b}$, we will use the implicit function theorem, of course. We need only show that the above $(n+N) \times (n + N)$ matrix is one-to-one and therefore non-singular by Theorem 1.3.d.

Suppose $D_{(x,\lambda)} M(\underline{x}^\circ,\underline{\lambda}^\circ,\underline{b}^\circ)(\underline{v},\underline{w}) = (\underline{0},\underline{0})$. Then,

$$(+) \qquad D^2L(\underline{x}^\circ)\underline{v} - Dh(\underline{x}^\circ)^T\underline{w} = \underline{0} \quad \text{and}$$

$$(++) \qquad -Dh(\underline{x}^\circ)\underline{v} = 0 \ .$$

Take the inner product of equation (+) with $\underline{v}$. Since $\underline{v} \cdot Dh(\underline{x}^\circ)^T = \underline{0}$ by (++), (+) becomes $v \cdot D^2L(\underline{x}^\circ)\underline{v} = D^2L(\underline{x}^\circ)(\underline{v},\underline{v}) = 0$. By hypothesis ii), $\underline{v}$ must be zero and (+) becomes $-Dh(\underline{x}^\circ)^T\underline{w} = \underline{0}$. By hypothesis iii), $Dh(\underline{x}^\circ)^T$ is injective and $\underline{w}$ is zero also.

Since our partial derivative is non-singular, the Implicit Function Theorem (Theorem 1.3.d) tells us that there is a neighborhood W of $\underline{b}^\circ$ in $\mathbb{R}^N$ and C^1 functions $\xi: W \to \mathbb{R}^N$, $\lambda: W \to \mathbb{R}^N$

such that $M(\xi(\underline{b}),\lambda(\underline{b}),\underline{b}) = (\underline{0},\underline{0})$ for all $\underline{b} \in W$ with $\xi(\underline{b}^\circ) = \underline{x}^\circ$ and $\lambda(\underline{b}^\circ) = \underline{\lambda}^\circ$. Choose W small enough so that hypotheses ii) and iii) hold for all $\underline{b} \in W$ and $x \in \xi(W)$. (Conditions ii) and iii) define open sets since they can be expressed by the non-vanishing of certain determinants.) By Theorem 3.4, each $\xi(b)$ maximizes f on $h^{-1}(\underline{b})$.

To see that $\lambda_i(\underline{b}) = \frac{\partial}{\partial b_i}(f\circ\xi)(\underline{b})$, note that

$$f(\xi(\underline{b})) = f(\xi(\underline{b})) + \sum_1^N \lambda_i(\underline{b})(b_i - h_i(\xi(\underline{b}))) \quad \text{since } h(\xi(\underline{b})) = \underline{b}$$

$$\equiv L(\xi(\underline{b}),\lambda(\underline{b}),\underline{b}) \quad .$$

Thus, $D_b(f\circ\xi)(b)\underline{v} = D_x f(\xi(b)) \circ D\xi(\underline{b})\underline{v} + \sum_i (D\lambda_i(\underline{b})\underline{v})(b_i - h_i(\xi(\underline{b}))$

$$+ \lambda(b) \cdot (I - Dh(\xi(b))\cdot D\xi(b))\underline{v}, \text{ by Theorem 1.1,}$$

$$= \lambda(b) + [D_x f(\xi(b)) - \lambda(\underline{b}) \cdot Dh(\xi(b))] D\xi(b) \cdot v \text{ , since } b_i - h_i(\xi(b)) = 0 \text{ ,}$$

$$= \lambda(\underline{b}) \text{ , since } M_1(\xi(\underline{b}),\lambda(\underline{b}),\underline{b}) = \underline{0} \text{ . } \blacksquare$$

<u>Remark 1</u>. Note that hypotheses ii) and iii) hold for most functions f and for most constraint values $\underline{b} \in \mathbb{R}^N$. The latter part follows from Sard's Lemma and the former from the fact that most functions on a manifold are "Morse functions", functions with only non-degenerate critical points. See Golubitsky-Guillemin (1974) for proofs of these results and Dierker (1974)

for further applications of these results to economics. The word "most" is used in the sense of an open-dense subset or second-category subset of the set of all constraint-values and of the set of all objective functions.

Remark 2. Lagrange multipliers yield the same sensitivity analysis when the constraints involve inequalities such as $g(\underline{x}) \geq \underline{a}$, $h(\underline{x}) = \underline{b}$. Let $I = \{i \mid g_i(\underline{x}^\circ) > a_i^\circ\}$. For $i \in I$, the Lagrange multiplier λ_i° must be zero. On the other hand, since these g_i give ineffective constraints, the optimal value of f does not change as one varies a_i for $i \in I$. Thus, for $i \in I$,

$$0 = \lambda_i^\circ = \frac{\partial (f \circ \xi)}{\partial a_i} (\underline{a}, \underline{b})$$

One is then led to the problem of maximizing f subject to $g_E(\underline{x}) = a_E$, $h(\underline{x}) = \underline{b}$; and one can argue as in Theorem 3.7.

Remark 3. Condition (iii) can be relaxed, though at some cost. See Gauvin and Tolle (1977) for results in this direction.

Remark 4. Theorem 3.7 is a special case of the "envelope theorem", a theorem which has begun to play a large role in comparative statics. In Theorem 3.8, we state the unconstrained and constrained versions of the envelope theorem. Their proofs are essentially the same as that of Theorem 3.7. Note that 3.8.a. states that the change in the objective function adjusting x optimally is equal to the change in the objective function when one does not adjust x.

Theorem 3.8. a) Let $f : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}$ be a C^1 function $f(x_1,\ldots,x_n;a)$ with parameter a. Let $x = \xi(a)$ be the maximizing value of x and let $M(a) = f(\xi(a))$. (M is called the indirect objective function.) If $\xi(a)$ is a C^1 function, e.g., if $D_x^2 f(\xi(a),a)$ is non-degenerate, then

$$\frac{dM}{da}(a) = \frac{\partial f}{\partial a}(\xi(a),a) \quad .$$

b) Let $f,h_1,\ldots,h_N$ be C^1 functions on $\mathbb{R}^n \times \mathbb{R}^1$ where the last variable is a parameter b. Let $\underline{x} = \xi(b)$ be the maximizer of $f(x;b)$ on the constraint set

$$C_b = \{\underline{x} \mid h_i(\underline{x};b) = 0,\ i=1,\ldots,N\} \quad .$$

Let $M(b) = f(\xi(b),b)$. Suppose $\xi(b)$ is a C^1 function of b, e.g., (f,h) satisfies hypotheses similar to those of Theorem 3.7. Then

$$\frac{dM}{db}(b) = \frac{\partial}{\partial b} L(\xi(b),b) \quad ,$$

where $L(x,b) = f(x;b) + \sum_{i=1}^{N} \lambda_i h_i(x;b)$.

4. CONSTRAINT QUALIFICATIONS

4.A. Fritz John's First Order Necessary Conditions

In the last section, we discussed necessary and sufficient conditions for constrained maxima under the condition that the Jacobian matrix of the effective constraint functions be of maximal rank. However, such a condition is too stringent for some applications and too difficult to check for others. In this section, we will examine much weaker and more geometric hypotheses on the constraint set. Since we will impose conditions only on the constraint set and not on the function to be maximized, such conditions are called "constraint qualifications". The most famous early paper on constraint qualifications is that of Kuhn and Tucker (1951). One can also find excellent surveys in Arrow-Hurwicz-Uzawa (1961) and Mangasarian (1969).

In contrast to the approach for nondegenerate constraints, one usually proves theorems about inequality constraints first, when working with constraint qualifications. Then, one can often handle equality constraints, like $h(\underline{x}) = 0$, by writing them as the set of inequality constraints: $h(\underline{x}) \geq 0$, $-h(\underline{x}) \geq 0$.

The following result of Fritz John (1948), is the broadest first order necessary conditions. Condition (H) is usually called the Fritz John Condition.

Theorem 4.1. Let $f, g_1, \ldots, g_M, h_1, \ldots, h_N$ be C^1 functions on $\mathbb{R}^n$ and let $\underline{x}^\circ$ be a local maximum of f on the set $C_{g,h} = \{x \in \mathbb{R}^n \mid g(\underline{x}) \geq 0,\ h(\underline{x}) = 0\}$. Then, there exists non-zero

$(\lambda_0, \lambda_1, \ldots, \lambda_M, \mu_1, \ldots, \mu_N)$ such that $\lambda_i \geq 0$, $\lambda_i g_i(\underline{x}^\circ) = 0$ for all $i \leq M$, and

$$\text{(H)} \qquad \lambda_0 Df(\underline{x}^\circ) + \sum_1^M \lambda_i \, Dg_i(\underline{x}^\circ) + \sum_1^N \mu_j \, Dh_j(\underline{x}^\circ) = 0.$$

Proof: We will first assume there are no equality constraints, i.e., that $N = 0$. We'll need the following important lemma, usually attributed to Gordan(1873).

Lemma. The following statements are equivalent for vectors $\underline{a}^1, \ldots, \underline{a}^m$ in $\mathbb{R}^n$: a) There exists no $\underline{v} \in \mathbb{R}^n$ such that $\underline{a}^i \cdot \underline{v} > 0$ for all i; b) There exists non-zero $(\lambda_1, \ldots, \lambda_m) \geq \underline{0}$ in $\mathbb{R}^m$ such that $\sum_1^m \lambda_i \underline{a}^i = \underline{0}$.

Proof of Lemma: b) $\Longrightarrow$ a): Suppose b) with $\lambda_k > 0$ and suppose there exists $\underline{v} > \underline{0}$ with $\underline{a}^i \cdot \underline{v} > 0$ for all i. Then,

$$\underline{a}^k \cdot \underline{v} = -\lambda_k^{-1} \Big(\sum_{i \neq k} \lambda_i \underline{a}_i \cdot \underline{v} \Big) \leq 0, \quad \text{a contradiction.}$$

a) $\Longrightarrow$ b): Let $X \subseteq \mathbb{R}^m$ be the linear subspace $\{(\underline{a}^1 \cdot \underline{b}, \ldots, \underline{a}^m \cdot \underline{b}) \in \mathbb{R}^m \mid \underline{b} \in \mathbb{R}^n\}$. By a), $X \cap P = \phi$, where $P \equiv \{\underline{x} \in \mathbb{R}^m | x_i > 0$ for all $i\}$. So, there is a non-zero $(\lambda_1, \ldots, \lambda_m) \in \overline{P}$ so that $(\lambda_1, \ldots, \lambda_m)$ is perpendicular to X. But then, $\sum_1^m \lambda_i \underline{a}^i \cdot \underline{b} = 0$ for all $\underline{b} \in \mathbb{R}^n$, which implies that $\sum_1^m \lambda_i \underline{a}^i = \underline{0}$.

Returning to the proof of Theorem 4.1, we claim that there is no $\underline{v} \in \mathbb{R}^n$ so that $Df(\underline{x}^\circ)\underline{v} > 0$ and $Dg_i(\underline{x}^\circ)\underline{v} > 0$ for all $i \in E \equiv \{j | g_j(\underline{x}^\circ) = 0\}$. For, if there were such a $\underline{v}$, f and each g_i for $i \in E$ would be increasing on the curve $t \to \underline{x}^\circ + t\underline{v}$ for small enough t. Since $g_i(\underline{x}^\circ + t\underline{v})$ would still be positive for $i \notin E$ and t small, $\underline{x}^\circ$ would not maximize f on $g \geq 0$.

We can now apply Gordan's Lemma with $\underline{a}^{\circ} = \nabla f(\underline{x}^{\circ})$ and $\underline{a}^{i} = g_i(\underline{x}^{\circ})$ for $i \in E$. So, there is a non-zero $(\lambda_0, \ldots, \lambda_M)$ with $\lambda_i = 0$ for $i \notin E$, $\lambda_j \geq 0$ for all j, and $\lambda_0 \nabla f(\underline{x}^{\circ}) + \sum_1^M \lambda_i \, g_i(\underline{x}^{\circ}) = \underline{0}$.

If one now includes the equality constraints: $h_1 = \ldots = h_N = 0$, the proof becomes a bit more complicated. We will outline the basic idea and leave the details to the reader. If $\nabla h_1(\underline{x}^{\circ}), \ldots, \nabla h_N(\underline{x}^{\circ})$ are linearly dependent, there is a non-zero $(\mu_1, \ldots, \mu_N)$ such that $\sum_1^N \mu_j \nabla h_j(\underline{x}^{\circ}) = 0$. In this case, take $\lambda_0 = \lambda_1 = \ldots = \lambda_M = 0$.

On the other hand, if $Dh(x^{\circ})$ has maximal rank N, $h^{-1}(\underline{0})$ is an $(n-N)$-dimensional submanifold around $\underline{x}^{\circ}$. In particular, by the implicit function theorem, there are coordinates $y_1, \ldots, y_N, z_1, \ldots, z_{n-N}$ on a neighborhood U of $\underline{x}^{\circ}$ such that in U:

i) $\underline{h} = \underline{0}$ if and only if $\underline{y} = \underline{0}$,

ii) $z_1, \ldots, z_{n-N}$ coordinatize $U \cap h^{-1}(\underline{0})$,

iii) $\frac{\partial h_i}{\partial z_j}(\underline{x}^{\circ}) = 0$ for all i and j, and

iv) $\frac{\partial h_i}{\partial y_j}(\underline{x}^{\circ})$ is 1 if $i = j$ and 0 if $i \neq j$.

Work first on $h^{-1}(\underline{0}) \cap U$ and apply the above arguments to find a non-zero $(\lambda_0, \ldots, \lambda_M)$ such that $\lambda_i \geq 0$ and $\lambda_i g_i(\underline{x}^{\circ}) = 0$ for each i and

$$\lambda_0 \frac{\partial f}{\partial z_k}(\underline{x}^{\circ}) + \sum_{i=1}^{M} \lambda_i \frac{\partial g_i}{\partial z_k}(\underline{x}^{\circ}) = 0, \text{ for } k = 1, \ldots, n-N.$$

Let $\mu_k = -\frac{\partial}{\partial y_k}[\lambda_0 f + \sum_1^M \lambda_i g_i](\underline{x}^\circ)$ for $k = 1,\ldots,N$. By iii) and iv),

$$\frac{\partial}{\partial z_k}[\lambda_0 f + \sum_1^M \lambda_i g_i + \sum_1^N \mu_j h_j](\underline{x}^\circ) = 0, \quad k = 1,\ldots,n-N$$

$$\frac{\partial}{\partial y_h}[\lambda_0 f + \sum_1^M \lambda_i g_i + \sum_1^N \mu_j h_j](\underline{x}^\circ) = 0, \quad h = 1,\ldots,N .$$

Therefore, the gradient of this Lagrangean is zero at x° in any smooth coordinate system (Theorem 1.3). ∎

John's statement of this result dealt only with inequality constraints. See Mangasarian and Fromowitz (1967) for the first proof involving both inequality and equality constraints.

4.B. Constraint Qualifications

The following simple example illustrates the difference between Theorem 3.1.a and Theorem 4.1. Let $f(x,y) = x$ and let $g(x,y) = y^2 + x^3$. Then, $g^{-1}(0)$ is the standard "cusp" in the left half-plane of $\mathbb{R}^2$; and $(0,0)$ is a global maximum of f on $g = 0$ and on $(-g) \geq 0$. Since $Df(0,0) = (1,0)$ and $Dg(0,0)$

$= (0,0)$, $\lambda_0 Df(\underline{0}) + \lambda_1 Dg(\underline{0}) = \underline{0}$ implies that $\lambda_0 = 0$ and λ_1 is arbitrary.

In situations like this where $\lambda_0 = 0$, the Fritz John Necessary Condition (H) says nothing about the maximization problem since it does not involve the function f at all. Thus, it is very important to introduce some conditions on g and h that will guarantee the existence of a non-zero λ_0 in (H). These are the above-mentioned "constraint qualifications". Roughly speaking, we need to eliminate the case where the constraint set C has a cusp at the point in question, i.e., we want C to satisfy some weak convexity assumption.

Let us write C_g for our constraint set $\{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1,\ldots,M\}$. As before, if $\underline{x}^\circ \in C_g$, $E(\underline{x}^\circ) \equiv E = \{i \mid g_i(\underline{x}^\circ) = 0\}$ and $I = \{j \mid g_j(\underline{x}^\circ) > 0\}$. A <u>constrained path from $\underline{x}^\circ$ in direction $\underline{v}$</u> is a smooth arc $a:[0,\varepsilon) \to \mathbb{R}^n$ so that $a(0) = \underline{x}^\circ$, $a'(0) = \underline{v}$, and $a(t) \in C_g$ for all t. For such $\underline{v}$, it follows immediately that $Dg_E(\underline{x}_0)\underline{v} \geq \underline{0}$.

<u>Definition</u>. The mapping g satisfies the <u>Karush-Kuhn-Tucker constraint qualification</u> (KKT) at $\underline{x}^\circ \in C_g$, if for each $\underline{v}$ with $Dg_E(\underline{x}^\circ)\underline{v} \geq 0$ ("<u>constrained direction</u>") there is a constrained path from x° in direction $\underline{v}$. See Karush (1939), Kuhn and Tucker (1951), and Kuhn (1976).

It is easy to see that the above example does not satisfy (KKT) at (0,0) and that (KKT) rules out such cusp-like constraint sets. A slightly weaker constraint qualification is due to Arrow, Hurwicz and Uzawa (1961).

Definition. The mapping g satisfies the Kuhn-Tucker weak constraint qualification (w - K - T) at $\underline{x}^\circ \in C_g$ if every constrained direction at $\underline{x}_0$ lies in the smallest closed convex cone containing $\{a'(0) \mid a$ is a constrained path from $\underline{x}^\circ\}$, i.e. if $Dg_E(\underline{x}^\circ)\underline{v} \geq 0$ implies that there are non-negative $\lambda_1,\ldots,\lambda_k$ and smooth $a_i: [0,\varepsilon) \to C_g$ for $i = 1,\ldots,k$ with $a_i(0) = \underline{x}^\circ$ and $\underline{v} = \sum_i \lambda_i a_i'(0)$.

It is easy to see that $g_1 = x_1, g_2 = x_2, g_3 = -x_1x_2$ satisfies (w-K-T) at (0,0) but not (K K T). See Arrow-Hurwicz-Uzawa (1961).

The following algebraic lemma is the key step in many optimization theorems where the constraints may be degenerate.

Farkas' Lemma: Let A be an $(n \times m)$ matrix and let $\underline{b}$ be a fixed vector in $\mathbb{R}^m$. If $\underline{b} \cdot \underline{v} \geq 0$ for all $\underline{v}$ in $\mathbb{R}^m$ such that $A\underline{v} \geq \underline{0}$, then there exist $\lambda_1,\ldots,\lambda_n$ all ≥ 0 such that

$$A^T\begin{pmatrix}\lambda_1\\ \vdots \\ \lambda_n\end{pmatrix} = b, \quad \text{i.e.,} \quad \sum_{i=1}^{n} \lambda_i \underline{a}_i = \underline{b}$$

where the $\underline{a}_i$ are the columns of A.

Proof : We first recall some simple properties of convex cones from Fenchel (1953) or Gale (1960). If B is a set of vectors, let $B' = \{\underline{u} \mid \underline{u} \cdot \underline{x} \geq 0$ for $\underline{x} \in B\}$. Then, B' is a closed, convex cone, called the polar cone of B. If $B_1 \subset B_2$, then $B_2' \subset B_1'$; and if B is a closed convex cone, $B = (B')'$.

Let $L = \{\underline{v} \mid A\underline{v} \geq 0\}$. Let $B = \{\Sigma\lambda_i \underline{a}_i \mid \lambda_i \geq 0\}$, a closed convex cone. To see that $B' \subset L$, let $\underline{v} \in B'$, i.e.,

$$\Sigma\lambda_i \underline{a}_i \underline{v} \geq 0 \qquad \text{for all} \quad \lambda_i \geq 0.$$

Thus, $A\underline{v} \geq 0$ (taking $\lambda_j = (0,\ldots,0,1,0,\ldots)$); and $\underline{v} \in L$. Finally, $B' \subset L$ implies that $L' \subset B'' = B$. []

The fundamental result on constraint qualifications is the following theorem.

<u>Theorem 4.2</u>. Suppose that $g: \mathbb{R}^n \to \mathbb{R}^M$ satisfies (KKT) or (w-K-T) at $\underline{x}°$ and that $\underline{x}°$ maximizes f on C_g. Then, there exist non-negative $\lambda_1,\ldots,\lambda_N$ such that

$$Df(\underline{x}°) + \sum_1^N \lambda_i \, Dg_i(\underline{x}°) = \underline{0} \quad \text{and} \quad \lambda_i g_i(\underline{x}°) = 0 \quad \text{for all } i.$$

<u>Proof</u> [Arrow-Hurwicz-Uzawa (1961)]: Since (KKT) implies (w-K-T), we will assume (w-K-T) at $\underline{x}°$ and apply Farkas' Lemma with $A = Dg_E(\underline{x}°)$ and $\underline{b} = -\nabla f(\underline{x}°)$. To see that $-\nabla f(\underline{x}°) \in L'$, choose $\underline{v} \in L$, i.e., $Dg_E(\underline{x}°)\underline{v} \geq 0$. By (w-K-T), there are constrained paths $a_1,\ldots,a_k$ from $\underline{x}°$ and non-negative $\mu_1,\ldots,\mu_k$ with

$$\underline{v} = \Sigma \mu_i \, a_i'(0).$$

Then,

$$\begin{aligned} \underline{v} \cdot (-\nabla f(\underline{x}°)) &= -Df(\underline{x}°)\underline{v} \\ &= -Df(\underline{x}°)(\Sigma \mu_i \, a_i'(0)) \\ &= -\Sigma \mu_i \frac{d}{dt} f(a_i(t)) \Big|_{t=0} \\ &\geq 0, \end{aligned}$$

since f is non-increasing along each a_i.

Applying Farkas' Lemma, there exist non-negative $\lambda_1,\ldots,\lambda_N$, with $\lambda_i = 0$ for $i \notin E$, such that

We are now in a position to describe some other successful constraint qualifications--all of which guarantee some sort of convexity or concavity for the constraint set. Condition d) below is the non-degeneracy condition of Section 3. Here, we see how it implies the weaker constraint qualifications of this section.

Theorem 4.3. Let $f, g_1, \ldots, g_M$ be C^1 functions on $\mathbb{R}^n$. Suppose that $\underline{x}°$ maximizes f on C_g. Suppose g satisfies one of the following constraint qualifications at $\underline{x}°$:

a) [Arrow-Hurwicz-Uzawa(1961)] There is a vector $\underline{v}$ with $Dg_{E_1}(\underline{x}°)\underline{v} \geq 0$ and $Dg_{E_2}(\underline{x}°)\underline{v} > \underline{0}$, with $E_1 = \{i \in E \mid g_i$ is pseudo-convex around $\underline{x}°\}$ and $E_2 = E - E_1$;

b) [Slater(1950)] There is a convex neighborhood U of $\underline{x}°$ such that g is concave on U and $g(x') > 0$ for some $x' \in U$;

c) g is convex (e.g., linear);

d) $Dg_E(\underline{x}°)$ has maximal rank.

Then, there exist $\lambda_1, \ldots, \lambda_N \geq 0$ with $\lambda_i g_i(\underline{x}°) = 0$ for all i and

$$D[f + \Sigma \lambda_i g_i](\underline{x}°) = \underline{0}.$$

Proof: Following Arrow-Hurwicz-Uzawa(1961), one shows that a) implies condition (w-K-T) and that b), c), and d) each imply a). To see that a) implies (w-K-T), let $\underline{w}$ be a constrained direction. For $\varepsilon > 0$, let $\phi^{\varepsilon}(t) = \underline{x}° + t(\underline{w} + \varepsilon\underline{v})$, where $\underline{v}$ is as in qualification a).

We first show that $\phi^{\varepsilon}(t)$ is a constrained path. For $i \in E$, $\frac{d}{dt} g_i \circ \phi^{\varepsilon}(t)\Big|_{t=0} = Dg_i(\underline{x}°)(\underline{w} + \varepsilon\underline{v}) \geq \underline{0} + \varepsilon Dg_i(\underline{x}°)\underline{v} \geq 0.$

If $i \in E_2$, $\frac{d}{dt}(g_i \circ \phi^{\varepsilon})(0) > 0$ and so $0 = g_i(x^\circ) = g_i(\phi^{\varepsilon}(0)) <$ $g_i(\phi^{\varepsilon}(t))$ for t small. If $i \in E_1$, $g_i \circ \phi^{\varepsilon}$ is pseudo-convex; and so $\frac{d}{dt}(g_i \circ \phi^{\varepsilon})(0) \geq 0$ implies that $0 = g_i(\phi^{\varepsilon}(0)) \leq g_i(\phi^{\varepsilon}(t))$ for t small. If $i \notin E$, then $g_i(\phi^{\varepsilon}(t))$ will be positive for t small. So ϕ^{ε} is a constrained path. Thus,

$$\underline{w} = (\phi^\circ)'(0) = \lim_{\varepsilon \to 0}(\phi^{\varepsilon})'(0)$$

lies in the closure of $\{a'(0) \mid a(t)$ is a constrained path from $\underline{x}^\circ\}$; and constraint qualification (w-K-T) is satisfied.

b) $\Longrightarrow$ a): Since g_j is concave, $Dg_j(\underline{x}^\circ)(\underline{x}'-\underline{x}^\circ) \geq g_j(\underline{x}')-g_j(\underline{x}^\circ)$ $= g_j(x') > 0$ for any $j \in E$. (See Theorem 1.6). Take $\underline{v} = \underline{x}'-\underline{x}^\circ$ in a).

c) $\Longrightarrow$ a): Here E_2 is empty. So, take $\underline{v} = \underline{0}$.

d) $\Longrightarrow$ a): Let $\underline{b}$ be a positive vector in $\mathbb{R}^E$. Since $Dg_E(\underline{x}^\circ)$: $T_{\underline{x}^\circ}\mathbb{R}^n \to \mathbb{R}^E$ has maximal rank, it is onto and there is a $\underline{v} \in T_{\underline{x}^\circ}\mathbb{R}^n$ with $Dg_E(\underline{x}^\circ)\underline{v} = \underline{b} > \underline{0}$. ▯

Once can now add equality constraints $h_1(\underline{x}) = \ldots = h_N(\underline{x}) = 0$ to the inequality constraints $g(\underline{x}) \geq \underline{0}$. In this case, the standard device to replace the equality $h_j(\underline{x}) = 0$ by the two inequalities $h_j(\underline{x}) \geq 0$, $-h_j(\underline{x}) \geq 0$. Parts i), ii), and iii) of the following proposition then follow immediately from Theorems 4.2 and 4.3. See Mangasarian-Fromowitz(1967) or Mangasarian(1969) for a proof of part iv), or use the techniques described in the last paragraph of our proof of Theorem 4.1.

<u>Theorem 4.4</u>. Suppose that $f, g_1, \ldots, g_M, h_1, \ldots, h_N$ are C^1 functions on $\mathbb{R}^n$. Suppose $\underline{x}^\circ$ maximizes f on $\{\underline{x} | g_i(\underline{x}) \geq 0,$ $i = 1, \ldots, M$; $h_j(\underline{x}) = 0$, $j = 1, \ldots, N\}$. Suppose any one of the following constraint qualifications hold:

i) If $Dg_E(\underline{x}^\circ)\underline{v} \geq \underline{0}$ and $Dh(\underline{x}^\circ)\underline{v} = \underline{0}$, then there is a C^1 path $a: [0,\varepsilon] \to \mathbb{R}^n$ with $a(0) = \underline{x}^\circ$, $a'(0) = \underline{v}$, $g(a(t)) \geq 0$, and $h(a(t)) = \underline{0}$;

ii) h is pseudo-concave and pseudo-convex (e.g., linear) and there is a $\underline{v} \in T_{\underline{x}^\circ}\mathbb{R}^n$ such that $Dg_{E_1}(\underline{x}^\circ)\underline{v} \geq 0$, $Dg_{E_2}(\underline{x}^\circ)\underline{v} > 0$, and $Dh(\underline{x}^\circ)\underline{v} = \underline{0}$, with E_1 and E_2 as in Theorem 4.3a;

iii) g is convex and h is linear;

iv) $Dh(\underline{x}^\circ)$ has maximal rank and $Dg_E(\underline{x}^\circ)\underline{v} > \underline{0}$, $Dh(\underline{x}^\circ) = \underline{0}$ for some $\underline{v} \in T_{\underline{x}^\circ}\mathbb{R}^n$.

For a more complete discussion of constraint qualifications and their intrinsic geometry, see Mangasarian(1969) and Gould-Tolle(1972).

4.C. Second Order Conditions

Since Theorems 3.2 and 3.5 do not make non-degeneracy assumptions on $\{x \mid g(x) \geq 0, h(\underline{x}) = 0\}$, they are just about the most effective second order sufficient conditions around. (However, stronger sufficient conditions using constraint qualifications are required for theoretical convergence of many algorithms for solving non-linear problems.) We now stop for a second to consider second order necessary conditions. Since one would think that some second derivative would have to be negative semi-definite at a maximum, it is surprising that the non-degeneracy of the constraint set is not an easy hypothesis to remove in looking for second order necessary conditions. Consider the following example of McCormick (1967):

Maximize $f(x,y) = -y$, subject to $g_1(x,y) \equiv -x^9 + y^3 \geq 0$, $g_2(x,y) \equiv x^9 + y^3 \geq 0$, and $g_3(x,y) \equiv x^2 + (y+1)^2 - 1 \geq 0$.

It is easy to see that (0,0) is such a maximum and that constraint qualification (KKT) is satisfied at (0,0). The Lagrangian is

$$L = f + \lambda_1 g_1 + \lambda_2 g_2 + \frac{1}{2} g_3 ,$$

where λ_1 and λ_2 are arbitrary. But $D^2L(0,0)$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

a positive definite matrix.

McCormick(1967) also proves the following second order necessary condition.

Theorem 4.5. Suppose $f, g_1, \ldots, g_M, h_1, \ldots, h_N$ are C^2 functions on $\mathbb{R}^n$ and $\underline{x}^\circ$ maximizes f subject to $g(\underline{x}) \geq 0$ and $h(\underline{x}) = \underline{0}$. Suppose further that (g,h) satisfies (KKT) and the following constraint qualification: for any $\underline{v} \in T_{\underline{x}^\circ} \mathbb{R}^n$ such that $Dg_E(\underline{x}^\circ)\underline{v} = 0$ and $Dh(\underline{x}^\circ)\underline{v} = 0$ there is a C^2 arc $a: [0,1] \to \mathbb{R}^n$ such that $a(0) = \underline{x}^\circ$, $a'(0) = \underline{v}$, $g(a(t)) \equiv \underline{0}$ and $h(a(t)) \equiv \underline{0}$. Then, there exist $\lambda_1, \ldots, \lambda_M$ non-negative and $\mu_1, \ldots, \mu_N$ such that

$$D[f + \sum_1^M \lambda_i g_i + \sum_1^N \mu_j h_j](\underline{x}^\circ) = \underline{0},$$

$$(\underline{\lambda}, \underline{\mu}) \neq 0, \quad \underline{\lambda} \geq \underline{0}, \quad \lambda_i g_i(\underline{x}^\circ) = 0, \quad \text{and}$$

$$D^2[f + \Sigma \lambda_i g_i + \Sigma \mu_j h_j](\underline{x}^\circ)(\underline{v},\underline{v}) \leq 0 \quad \text{for all } \underline{v}$$

with $Dg_E(x^\circ)v = 0$ and $Dh(x^\circ)v = 0$.

The proof of this theorem is very similar to that of Theorem 3.1.b and will be omitted. McCormick (1967) also shows that the above second order constraint qualification holds if $(Dg_E(\underline{x}°), Dh(\underline{x}°))$ has maximal rank.

See Kuhn (1976) for an interesting historical survey of the theorems of this chapter. He describes the various applied problems which motivated the papers of Karush (1939), John (1948) and Kuhn-Tucker (1951).

§5. CONCAVE PROGRAMMING

5A. First Order Necessary Conditions

In many optimization problems, one finds conditions that lead naturally to concave constraint and objective functions. Fortunately, for these situations one never has to use second order tests since, as in Theorem 2.3, the first order necessary conditions are also sufficient. While discussing these results, we will first assume that there are only inequality constraints.

Theorem 5.1. Suppose that $f, g_1, \ldots, g_m$ are differentiable concave functions on $\mathbb{R}^n$ and that $\underline{x}^\circ \in C_g \equiv \{\underline{x} \in \mathbb{R}^n \mid g(\underline{x}) \geq \underline{0}\}$. If there exist non-negative $\lambda_1, \ldots, \lambda_m$ such that

$$Df(\underline{x}^\circ) + \Sigma\, \lambda_i Dg_i(\underline{x}^\circ) = \underline{0} \quad \text{and} \quad \lambda_i g_i(\underline{x}^\circ) = 0 \quad \text{for all } i ,$$

then $\underline{x}^\circ$ maximizes f (globally) on C_g. Furthermore, the set of all such maximizers is convex.

Proof: Note that $L(\underline{x}) = f(\underline{x}) + \Sigma \lambda_i g_i(\underline{x})$ is a non-negative linear combination of concave functions and thus is concave. Since the g_i are concave, C_g is convex. By Theorem 2.3, $\underline{x}^\circ$ is a global maximizer of L since $DL(\underline{x}^\circ) = \underline{0}$. If $\underline{x}' \in C_g$ and $f(\underline{x}') > f(\underline{x}^\circ)$, then

$$\begin{aligned} L(\underline{x}') &= f(\underline{x}') + \Sigma\, \lambda_i g_i(\underline{x}') \geq f(\underline{x}') \\ &> f(\underline{x}^\circ) = f(\underline{x}^\circ) + \Sigma\, \lambda_i g(\underline{x}^\circ) = L(\underline{x}^\circ) , \end{aligned}$$

a contradiction. So, $\underline{x}^\circ$ maximizes u on C_g.

If $\underline{x}'$ and $\underline{x}^2$ maximize f in C_g, then $t\underline{x}' + (1-t)\underline{x}^2 \in C_g$ and $f(t\underline{x}' + (1-t)\underline{x}^2) \geq tf(\underline{x}') + (1-t)f(\underline{x}^2) = f(\underline{x}')$ (or $f(\underline{x}^2)$. So, $t\underline{x}' + (1-t)\underline{x}^2$ is also a maximizer.

The converse of Theorem 5.1 is true provided there is an $\underline{x}'$ with $g(\underline{x}') > 0$ by Theorem 4.3.b. One can add equality constraints $\{h_1 = \ldots = h_N = 0\}$ to the hypothesis of Theorem 5.1 provided the h_i are affine functions, i.e., $h_i(\underline{x}) = A_i\underline{x} + \underline{b}_i$. For then, $-h$ and h are concave and, as in section 4, one replaces the N equality constraints $h = 0$ by the $2N$ inequality constraints $h \geq \underline{0}$, $-h \geq \underline{0}$.

Theorem 5.1 appears in Kuhn-Tucker (1951). Arrow-Enthoven (1961) and Mangasarian (1969) prove the following generalizations of Theorem 5.1, relaxing the concavity hypothses.

<u>Theorem 5.2</u>. Suppose that $f, g_1, \ldots, g_M$ are C^1 functions on $\mathbb{R}^n$, that f is pseudoconcave, and that the g_i are quasi-concave. Suppose that $g(\underline{x}^\circ) \geq \underline{0}$ and that there are non-negative $\lambda_1, \ldots, \lambda_M$ with $\lambda_i g_i(\underline{x}^\circ) = 0$ for all i

(I) and $D[f + \Sigma\, \lambda_i g_i](\underline{x}^\circ)(\underline{x} - \underline{x}^\circ) \leq 0$ for all $\underline{x} \in C_g$.

(For example, $D[f + \Sigma\, \lambda_i g_i](\underline{x}^\circ) = \underline{0}$.) Then, $\underline{x}^\circ$ maximizes f (globally) on C_g.

<u>Remark</u>: One can now include more general equality constraints, i.e., h_i that are both pseudoconcave and pseudoconvex.

Proof: Let $\underline{x} \in C_g$. Then, $g_E(x) \geq 0 = g_E(\underline{x}^\circ)$. Since g_E is quasiconcave, $Dg_E(\underline{x}^\circ)(\underline{x} - \underline{x}^\circ) \geq \underline{0}$. Since $\lambda_I = \underline{0}$ and $\lambda_E \geq 0$, $\Sigma \lambda_i Dg_i(\underline{x}^\circ)(\underline{x} - \underline{x}^\circ) \geq 0$. By (I) , $Df(\underline{x}^\circ)(\underline{x} - \underline{x}^\circ) \leq 0$. Since f is pseudoconcave, $f(\underline{x}) \leq f(\underline{x}^o)$.

Note that we really only needed g_E to be quasiconcave.

Finally, we note that for concave problems, not only is the solution set convex but the optimal value function is a convex function of the parameters.

Theorem 5.3 a) Suppose that $f, g_1, \ldots, g_m$ are concave functions on $\mathbb{R}^n \times \mathbb{R}^p$, where the last p variables are treated as parameters. Let $C_b = \{\underline{x} \in \mathbb{R}^n \mid g(\underline{x}, \underline{b}) \geq 0\}$ and let $\xi(b)$ be the set of maximizers of $f(\cdot, b)$ on C_b . Finally, let $v(\underline{b}) = f(\xi(\underline{b}), \underline{b})$. Then, $v(\underline{b})$ is a concave function of $\underline{b}$.

b) Now drop the dependence of g on the parameter b and the concavity assumption on g . Suppose only that $f(\underline{x}, \underline{b})$ is convex as a function of $\underline{b}$. Then, $v(\underline{b})$, the maximum value of $f(x, \underline{b})$ subject to the constraint $g(x) \geq 0$, is also a convex function.

Proof: a) Let $\underline{b}_1$ and $\underline{b}_2$ be two parameter values. Let $\underline{x}_i = \xi(\underline{b}_i)$, $i = 1, 2$; so $g(\underline{x}_i, \underline{b}_i) \geq 0$ for $i=1,2$. Consider the convex combination $b_3 = \lambda \underline{b}_1 + (1-\lambda)\underline{b}_2$ for some λ in $[0,1]$. Since g is concave in $(\underline{x}, \underline{b})$,

$$g(\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2, \lambda \underline{b}_1 + (1-\lambda)\underline{b}_2) \geq \lambda g(\underline{x}_1, b_1) + (1-\lambda) g(\underline{x}_2, \underline{b}_2) \geq 0 .$$

Therefore, $\lambda x_1 + (1-\lambda) x_2$ is in C_{b_3} . Now by the definition of v and the concavity of f ,

$$v(\lambda \underline{b}_1+(1-\lambda)\underline{b}_2) \geq f(\lambda \underline{x}_1+(1-\lambda)\underline{x}_2,\ \lambda \underline{b}_1+(1-\lambda)\underline{b}_2)$$

$$\geq \lambda f(\underline{x}_1,\underline{b}_1) + (1-\lambda)f(\underline{x}_2,\underline{b}_2)$$

$$= \lambda v(\underline{b}_1) + (1-\lambda)v(\underline{b}_2) \ .$$

b): Using the same notation as in a), let $\underline{x}_3$ denote $\xi(\underline{b}_3)$, where $\underline{b}_3 = \lambda \underline{b}_1+(1-\lambda)\underline{b}_2$. Note that $g(\underline{x}_i) \geq 0$ for $i=1,2,3$; in particular $\underline{x}_3$ is in the constraint set when $\underline{x}_1$ and $\underline{x}_2$ were chosen. This implies

$$f(\underline{x}_3,\underline{b}_1) \leq f(\underline{x}_1,\underline{b}_1) \equiv v(\underline{b}_1) \quad \text{and}$$

$$f(\underline{x}_3,\underline{b}_2) \leq f(\underline{x}_2,\underline{b}_2) \equiv v(\underline{b}_2) \ .$$

Therefore, $v(\lambda \underline{b}_1+(1-\lambda)\underline{b}_2) = f(\underline{x}_3, \lambda \underline{b}_1+(1-\lambda)\underline{b}_2)$

$\leq \lambda f(\underline{x}_3,\underline{b}_1) + (1-\lambda)f(\underline{x}_3,\underline{b}_2)$ by the convexity of f in b

$\leq \lambda v(\underline{b}_1) + (1-\lambda)v(\underline{b}_2)$,

and v is convex in $\underline{b}$. []

Let $v(\underline{b})$ denote the maximum value function for the problem of maximizing $f(\underline{x})$ subject to the constraints $g(\underline{x}) \geq \underline{b}$, where $f,g_1,\ldots,g_m$ are concave functions. Let $\lambda(\underline{b})$ denote the corresponding multiplier. Theorem 3.7 showed that in the smooth, non-degenerate case $\lambda_i(\underline{b}) = \frac{\partial v}{\partial b_i}(b)$. One can use Theorem 5.3 and some basic theory about concave functions to show that when one replaces "smooth, non-degenerate" by "concave":

$$\lim_{h \to 0+} \frac{v(\underline{b}+h\underline{e}_i)-v(b)}{h} \leq \lambda_i(b) \leq \lim_{h \to 0-} \frac{v(\underline{b}+h\underline{e}_i)-v(\underline{b})}{h}$$

See Dixit (1976) for further discussion.

5.B. Saddle Point Conditions

In order to compute maxima of f under constraints, one often considers the corresponding "saddle point problem", especially when the functions involved are concave.

Definition. Let $f, g_1, \ldots, g_M$ be continuous functions on R^n . Consider the Lagrangian $L(\underline{x}, \lambda_1, \ldots, \lambda_M) = f(\underline{x}) + \Sigma \lambda_i g_i(\underline{x})$ as a function of $\underline{x}$ and $\underline{\lambda}$. Then, $(\underline{x}, \underline{\lambda})$ is a (non-negative) saddle point of L if

(J) $$L(\underline{x}, \underline{\lambda}^\circ) \leq L(\underline{x}, \underline{\lambda}^\circ) \leq L(\underline{x}^\circ, \underline{\lambda})$$

for all $\underline{\lambda} \geq \underline{0}$ in R^M and all $\underline{x} \in R^n$ (and all $\underline{x} \geq \underline{0}$ in R^n , where $\underline{x}^\circ \geq \underline{0}$.)

Theorem 5.4. If $(\underline{x}^\circ, \underline{\lambda}^\circ)$ is a (non-negative) saddle point for L as above, then $\underline{x}^\circ$ maximizes f subject to $g \geq \underline{0}$ (and $\underline{x} \geq 0$) .

Proof: First, show $g(x^\circ) \geq 0$. The right side of (J) means that $\sum_i (\lambda_i - \lambda_i^\circ) g_i(\underline{x}^\circ) \geq 0$ for all $\lambda_i \geq 0$. For any fixed K , plug in $\lambda_K = \lambda_K^\circ + 1 \geq 0$ and $\lambda_j = \lambda_j^\circ$ for $j \neq K$. Then, $g_K(\underline{x}^\circ) \geq 0$ and $\Sigma\, \lambda_i^\circ g_i(\underline{x}^\circ) \geq 0$.

Setting $\underline{\lambda} = 0$ in (J) yields, $\Sigma\, \lambda_i^\circ g_i(\underline{x}^\circ) \leq 0$. So, $\Sigma\, \lambda_i^\circ g_i(\underline{x}^\circ) = 0$ and thus each $\lambda_i^\circ g_i(\underline{x}^\circ) = 0$. If $g(\underline{x}) \geq 0$ (and $\underline{x} \geq \underline{0}$),

$$
\begin{aligned}
f(\underline{x}) &\leq f(\underline{x}) + \Sigma\, \lambda_i^\circ g_i(\underline{x}), \text{ since each } \lambda_i^\circ g_i(\underline{x}) \geq 0, \\
&\leq f(\underline{x}^\circ) + \Sigma\, \lambda_i^\circ g_i(\underline{x}^\circ), \text{ by (J)}, \\
&= f(\underline{x}^\circ). \ \square
\end{aligned}
$$

In concave programming, solutions to the saddle point problem are more or less equivalent to solutions of the programming problem, as Kuhn and Tucker (1951) pointed out:

<u>Theorem 5.5.</u> Suppose that $f, g_1, \ldots, g_M$ are C^1 concave functions and that $\underline{x}^\circ$ maximizes f subject to $g \geq 0$ (and $\underline{x} \geq 0$). Suppose further that $g(\underline{x}') > \underline{0}$ for some $\underline{x}'$ (constraint qualification 4.3.b) or that g is linear. Then, there exists $\underline{\lambda}^\circ \geq \underline{0}$ such that $(\underline{x}^\circ, \underline{\lambda}^\circ)$ is a (non-negative) saddle point of

$$L(\underline{x}, \underline{\lambda}) = f(\underline{x}) + \underline{\lambda} \cdot \underline{g}(\underline{x}).$$

<u>Proof:</u> By Theorem 4.3, the Kuhn-Tucker conditions are satisfied, i.e., there exists $\underline{\lambda}^\circ \geq \underline{0}$ with $\underline{\lambda}^\circ \cdot \underline{g}(\underline{x}^\circ) = 0$ and

$$\text{(K)} \qquad Df(\underline{x}^\circ) + \Sigma\, \lambda_i^\circ Dg_i(\underline{x}^\circ) = \underline{0}.$$

Since $L(\underline{x}, \underline{\lambda}^\circ)$ is a concave function of $\underline{x}$, for any $\underline{x} \in C_g$

$$L(\underline{x}, \underline{\lambda}^\circ) - L(\underline{x}^\circ, \underline{\lambda}^\circ) \leq \frac{\partial L}{\partial \underline{x}}(\underline{x}^\circ, \underline{\lambda}^\circ)(\underline{x} - \underline{x}^\circ) = \underline{0} \text{ by (K) and}$$

Theorem 1.6.

On the other hand, for any $\underline{\lambda} \geq \underline{0}$ in $\mathbb{R}^M$,

$$\begin{aligned} L(\underline{x}^\circ,\underline{\lambda}^\circ) &= f(\underline{x}^\circ) + \underline{\lambda}^\circ \cdot \underline{g}(\underline{x}^\circ) \quad \left(\text{since } \underline{\lambda}^\circ \cdot g(\underline{x}^\circ) = 0\right) \\ &= f(\underline{x}^\circ) \leq f(\underline{x}^\circ) + \underline{\lambda} \cdot \underline{g}(\underline{x}^\circ) \\ &= L(\underline{x}^\circ,\underline{\lambda}) \quad . \end{aligned}$$

We will see in chapter six that the saddle point approach has certain advantages in economic problems. Furthermore, as we mentioned earlier, one can use this approach to compute solutions of concave programming problems and their corresponding multipliers.

5.C Duality in Linear Programming

An important special case of concave programming is linear programming; and one of the most powerful tools in the theory of linear programming is the existence of a dual problem to every linear problem. If the original (or primal) problem arises from an economics question, the dual problem usually is filled with economic significance. An illustration of this fact will be discussed in section 6.C. Consider the linear problem of maximizing

$$\text{(L)} \qquad \begin{aligned} &f(\underline{x}) = \underline{c} \cdot \underline{x} \text{ , subject to the constraints} \\ &A\underline{x} \leq \underline{b} \text{ in } \mathbb{R}^M \text{ and } \underline{x} \geq \underline{0} \text{ in } \mathbb{R}^n . \end{aligned}$$

Then, the dual problem is that of minimizing

$$\text{(M)} \qquad F(\underline{y}) = \underline{y} \cdot \underline{b}\ , \text{ subject to the constraints}$$
$$\underline{y}A \geq \underline{c} \text{ in } \mathbb{R}^n \text{ and } \underline{y} \geq 0 \text{ in } \mathbb{R}^M .$$

If the j^{th} inequality in the constraint $A\underline{x} \leq \underline{b}$ in (L) becomes an equality constraint, then the constraint $\underline{y}_j \geq 0$ is dropped in (M).

We will use the above saddle point theorems to give simple proofs of the basic facts on duality.

Theorem 5.6. Let $\underline{c} \in \mathbb{R}^n$, $\underline{b} \in \mathbb{R}^M$, and let $A: \mathbb{R}^n \to \mathbb{R}^M$ be a linear map. Let (L) denote the above primal problem and let (M) denote its dual. Then,

i) $\underline{x} \in \mathbb{R}^n$ solves (L) if and only if there is a $\underline{y} \in \mathbb{R}^M_+$ such that $(\underline{x},\underline{y})$ is a saddle point of $L(\underline{x},\underline{y}) \equiv f(\underline{x}) + \underline{y} \cdot (\underline{b} - A\underline{x})$;

ii) if $\underline{x} \in \mathbb{R}^n$ solves (L), then there exists a $\underline{y} \in \mathbb{R}^M$ which solves (M), and conversely. Furthermore, $\underline{c} \cdot \underline{x} = \underline{y} \cdot \underline{b}$.

iii) if the constraint sets of (L) and of (M) are non-empty, then both problems have solutions.

iv) if $\underline{x}'$ is in the constraint set of (L) and $\underline{y}'$ is in the constraint set of (M) such that $\underline{c} \cdot \underline{x}' = \underline{b} \cdot \underline{y}'$, then $\underline{x}'$ solves (L) and $\underline{y}'$ solves (M).

Proof: Part i) follows directly from Theorems 5.4 and 5.5. The Lagrangian for (M) is $M(\underline{y},\underline{x}) = -\underline{b} \cdot \underline{y} + (\underline{y}A - \underline{c}) \cdot \underline{x}$. Note that $M(\underline{y},\underline{x}) = -L(\underline{x},\underline{y})$. By i), if $\underline{x}$ solves (L), there is a $\underline{y}$ such that $(\underline{x},\underline{y})$ is a saddle point of L, i.e., $(\underline{y},\underline{x})$ is a saddle point for $-M$. By i) again, $\underline{y}$ solves (M). Since $\underline{y} \cdot (\underline{b} - A\underline{x}) = 0$

and $(\underline{y}A - \underline{c}) \cdot \underline{x} = 0$ for the optimal $\underline{x}$ and $\underline{y}$ by Theorem 4.3, $\underline{y} \cdot \underline{b} = \underline{y} \cdot (A\underline{x}) = (yA) \cdot \underline{x} = \underline{c} \cdot \underline{x}$, and ii) follows.

To prove iii), let $\underline{y}°$ lie in the constraint set of (M) and $\underline{x}°$ in the constraint set of (L). Then,

$$\text{(N)} \qquad \underline{c} \cdot \underline{x}° \leq \underline{y}°A \cdot \underline{x}° = \underline{y}° \cdot A\underline{x}° \leq \underline{y}° \cdot \underline{b} \ .$$

Thus, the linear function f is bounded on the closed constraint set of Problem (L). Consequently, f achieves its maximum on this set. One argues similarly for (M).

To prove iv), let $\underline{x}'$ and $\underline{y}'$ be as in the hypothesis and let $\underline{x}°$ be any vector in the constraint set of (L). By (N),

$$\underline{c} \cdot \underline{x}° \leq \underline{y}' \cdot \underline{b} = \underline{c} \cdot \underline{x}'$$

i.e., $\underline{x}'$ maximizes $\underline{c} \cdot \underline{x}$ on the constraint set for (L). ∎

Note that by Theorem 5.3, the maximum value function $v(\underline{c},\underline{b})$ of problem (L) is a concave function of $(\underline{c},\underline{b})$.

Karlin (1959), Mangasarian (1969), and Intriligator (1971) have excellent discussions of duality theory. Mangasarian (1969) also gives an introduction to the study of non-linear duality. Kuhn (1976) discusses the origins of duality in mathematical programming.

§6. APPLICATIONS OF MATHEMATICAL PROGRAMMING TO ECONOMICS

Since one of the basic problems of economics is the allocation of scarce resources among competing groups, it is natural that much of mathematical economics deals with constrained maximization problems. In this chapter, we will examine some of the important programming problems that arise in economics, and we will try to use the theory of the last four chapters to gain some insights into these problems. Most theoretical books on mathematical economics study these and related problems. The reader should refer to Debreu (1959), Karlin (1959), Baumol (1961), Kuhn (1968), Intrilligator (1971), Malinvaud (1972), Silberberg (1978), and Varian (1978), for further discussion of such problems. Kuhn (1968) and Intrilligator (1971) base their entire presentations on programming methods.

6A Theory of the Consumer of Household

We first examine an individual consumer's (or family's) consumption decision. We suppose that there are n commodities with $1 < n < \infty$ and with $x_i \in \mathbb{R}$ denoting the amount of the i^{th} commodity. A consumption vector or commodity vector is an $\underline{x} = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$, listing the amount of each commodity to be consumed. To develop our theory, we make the following assumptions about our consumer and the set of available commodity vectors.

We assume that each commodity is perfectly divisible so that any non-negative quantity can be purchased. Thus, the commodity space, or space of all feasible commodity vectors, is

$$C = \{\underline{x} \in \mathbb{R}^n \mid x_i \geq 0\} .$$

We need not assume a bound on the availability of any commodity since budget restrictions will give us natural bounds.

We further assume that the tastes or preferences of the consumer are summarized by a complete pre-ordering $\prec$ on C. The consumer prefers commodity vector $\underline{y}$ to commodity vector $\underline{x}$ (or finds them equally preferable) if and only if $\underline{x} \prec \underline{y}$. We assume that this pre-ordering is continuous in that, for each $\underline{x} \in C$, $\{\underline{y} \in C \mid \underline{x} \prec \underline{y}\}$ and $\{\underline{y} \in C \mid \underline{y} \prec \underline{x}\}$ are both closed sets. By a theorem of Debreu (1959), there is a continuous function $u : C \to \mathbb{R}$ such that $\underline{x} \prec \underline{y}$ if and only if $u(\underline{x}) \leq u(\underline{y})$. The function u is called a utility function. (Note that infinitely many utility functions can represent the same preference ordering.)

We assume a fixed price vector $\underline{p} = (p_1,\ldots,p_n) \in \mathbb{R}^n$ with each p_i a positive number giving the unit price of the i^{th} commodity. The consumer has an initial wealth w in $\mathbb{R}_+$. In some problems, he has an initial commodity vector $\underline{x}^o \in C$, in which case his initial wealth is $w = \underline{p} \cdot \underline{x}^o$.

Finally, the consumer's goal is to select the commodity vector $\underline{x} \in C$ which is affordable yet maximizes his preference ordering among all affordable vectors in C. Mathematically, his problem is to find $\underline{x} \in \mathbb{R}^n$ such that $\underline{x}$ maximizes u subject to the constraints

$$(\Phi) \qquad 0 \leq x_i \text{ , for } i = 1,\ldots,n \text{ ; } \underline{p} \cdot \underline{x} \leq w \text{ .}$$

Note that since the constraint set is closed and bounded and u is continuous, problem (Φ) has a solution for each $\underline{p}$ and w.

Our first application of programming theory to this problem is to derive the norm that an interior optimal allocation the marginal rate of substitution of good i with respect to good j equals p_i/p_j. At commodity vector $\underline{x}^o \in C$, the marginal rate of substitution (MRS) of good j with respect to good i is $\frac{\partial u}{\partial x_i}(\underline{x}^o) \Big/ \frac{\partial u}{\partial x_j}(\underline{x}^o)$. It measures (at the infinitestimal level) the additional quantity of good j which would compensate the consumer for a one-unit loss of good i while keeping the consumer's utility constant. To see this, fix x_k^o for $k \neq i,j$ and write

$$u(x_1^o,\ldots,x_i,x_{i+1}^o,\ldots,x_j(x_i),\ldots,x_n^o) = u(\underline{x}^o)$$

to indicate how a change in x_i brings about a change in x_j at the same utility level. Taking the derivatives with respect to x_i and evaluating $\underline{x}^o$ yields

$$\frac{\partial u}{\partial x_i}(\underline{x}^o) + \frac{\partial u}{\partial x_j}(\underline{x}^o)\frac{dx_j}{dx_i}(\underline{x}^o) = 0 \quad \text{or}$$

$$\frac{dx_j}{dx_i} = -\frac{\partial u}{\partial x_i}(\underline{x}^o) \Big/ \frac{\partial u}{\partial x_j}(\underline{x}^o) .$$

The MRS is the slope of the consumer's indifference set, $\{y \mid u(y) = u(\underline{x}^o)\}$, at $\underline{x}^o$ in the i-j direction and measures the consumer's relative internal valuation of goods i and j. The optimality condition states that this internal valuation should equal the market's valuation, p_i/p_j.

Theorem 6.1 Suppose that $u : C \to \mathbb{R}$ is a C^1 utility function with the property that for each $\underline{x} \in C$ there is an i such that $\frac{\partial u}{\partial x_i}(\underline{x}) > 0$. Suppose that $\underline{p}$ is a positive price vector and that $\underline{x}^o \in C$ is a solution to problem (O) above. Then, there is a $\eta > 0$ in $\mathbb{R}$ such that

i) $\frac{1}{p_i}\frac{\partial u}{\partial x_i}(\underline{x}^o) \leq \eta$ for $i = 1,\dots,n$ with equality for those i with $x_i^o \neq 0$,

ii) thus, if $\underline{x}^o$ lies in the interior of C, $\frac{\partial u}{\partial x_i}(\underline{x}^o) > 0$ for all i and $\nabla u(\underline{x}^o) = \eta \underline{p}$,

iii) if x_i^o and x_j^o are non-zero, then $\frac{\partial u}{\partial x_j}(\underline{x}^o) > 0$,

$$\frac{1}{p_i}\frac{\partial u}{\partial x_i}(\underline{x}^o) = \frac{1}{p_j}\frac{\partial u}{\partial x_j}(\underline{x}^o), \text{ and } \frac{\partial u}{\partial x_i}(\underline{x}^o) \Big/ \frac{\partial u}{\partial x_j}(\underline{x}^o) = \frac{p_i}{p_j},$$

iv) $\underline{p} \cdot \underline{x}^o = w$ (all income is spent).

Conversely, if u is C^1 and pseudoconcave (e.g., u is C^2 and quasi-concave and some $\frac{\partial u}{\partial x_i}$ is positive at each $\underline{x}$) and if $\underline{x}^o$ satisfies i) and iv) for some $\eta > 0$, then $\underline{x}^o$ is a global solution of problem (O).

Figure 1 below illustrates Theorem 6.1 for an interior solution $\underline{x}^o$ of problem (O) when $n = 2$. The straight line through $(\frac{w}{p_1}, 0)$ and $(0, \frac{w}{p_2})$ is the price line $p_1x_1 + p_2x_2 = w$, which is perpendicular to the (dotted) price vector (p_1, p_2). The curved lines are the level sets of u with u increasing as x_1 and x_2 go to $+\infty$. Note that at the maximizer $\underline{x}^o$, $\nabla u(\underline{x}^o)$ is perpendicular to the price line and therefore parallel to (p_1, p_2) as ii) indicates.

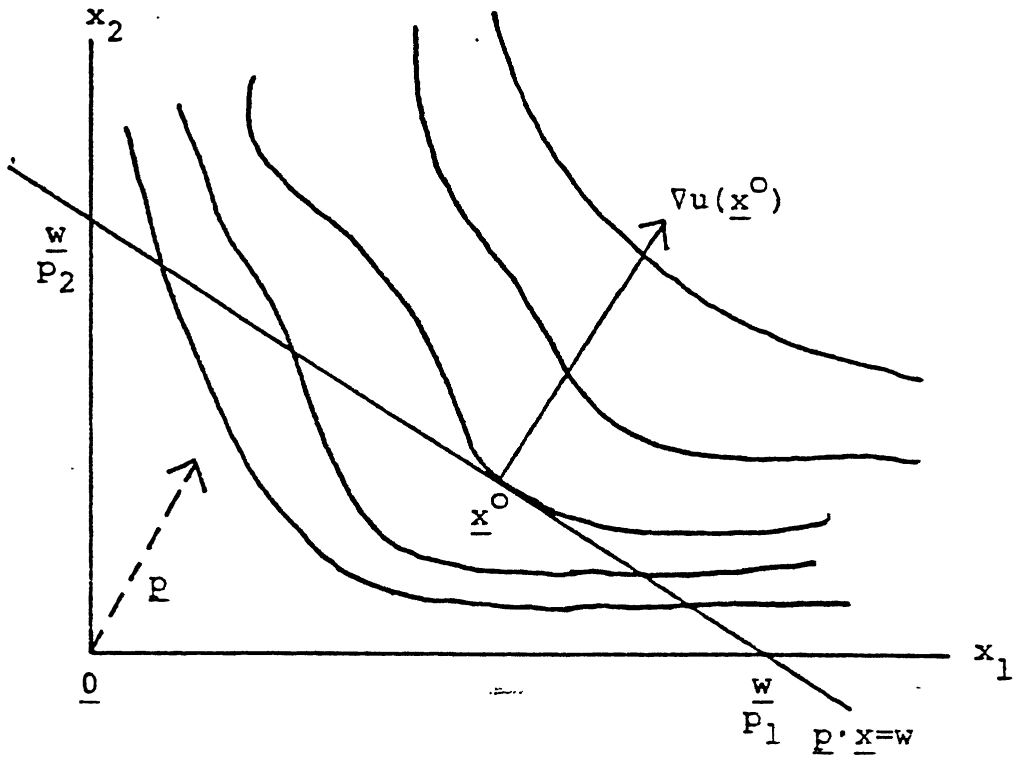

Figure 1

Proof: One merely applies the Kuhn-Tucker conditions of Theorem 4.3 to problem (O). Since the constraints are defined by linear functions, constraint qualifications KKT and (4.3.c) hold automatically. By Theorem 4.3, there are non-negative Lagrange multipliers $\lambda_1, \ldots, \lambda_n$, η such that for each i

$$\frac{\partial u}{\partial x_i}(\underline{x}^o) + \lambda_i - \eta p_i = 0$$

where $\lambda_i x_i^o = \eta(w - \underline{p} \cdot \underline{x}^o) = 0$. Since $\frac{1}{p_i} \frac{\partial u}{\partial x_i} (\underline{x}^o) - \eta = -\frac{\lambda_i}{p_i} \le 0$ and $\lambda_i x_i^o = 0$, i), ii) and iii) follow. Since $\frac{\partial u}{\partial x_i} (\underline{x}^o) > 0$ for some i and $p_i > 0$, $\eta > 0$ by i). Since $\eta(w - \underline{p} \cdot \underline{x}^o) = 0$, $w = \underline{p} \cdot \underline{x}^o$ as in iv).

The converse follows from Theorem 5.2. ∎

The correspondence which sends each price vector $\underline{p}$ and each initial wealth w to the corresponding optimal commodity vector or vectors (i.e., solutions of problem (O)) is called the <u>demand correspondence</u> and will be written as

$$(\underline{p}, w) \longmapsto \xi(\underline{p}, w) \in C .$$

If u is <u>strictly</u> concave, then ξ is a single valued function. Furthermore, if one makes the slightly stronger assumptions that u is C^2 and that $D^2u(\underline{x})$ is negative definite on C , then ξ is a C^1 function when it takes on values in the interior of C . One can use the Kuhn-Tucker equations (i) in Theorem 6.1 to compute the derivatives of ξ .

<u>Theorem 6.2</u> Suppose that u is a C^2 utility function on C with ξ the corresponding demand correspondence. Suppose that for some $\underline{x}^o$ in the interior of C , some price vector $\underline{p}^o$ and some wealth $w^o = \underline{p}^o \cdot \underline{x}^o$, $\underline{x}^o = \xi(\underline{p}^o, w^o)$. Suppose that $\frac{\partial u}{\partial x_i} (\underline{x}^o) > 0$

for some i and $D^2u(\underline{x}^o)(\underline{v},\underline{v}) < 0$ for all non-zero $\underline{v}$ such that $\underline{p}^o \cdot \underline{v} = 0$. (For example, $D^2u(x^o)$ may be negative definite).

Then, there are neighborhoods U of $\underline{x}^o$, V of $\underline{p}^o$, and W of $\underline{w}^o$ such that $\xi : V \times W \to U$ is a C^1 mapping. Furthermore, the multiplier η in Theorem 6.1 also equals $\frac{\partial u(\xi(p,w))}{\partial w}$ and therefore measures the sensitivity of the optimal value of u to change in the initial wealth w. (It is often called the marginal utility of money).

Proof: Choose a neighborhood U_1 of $\underline{x}^o$ such that for all $\underline{x} \in U_1$, $x_i > 0$ for all i and $\frac{\partial u}{\partial x_j}(\underline{x}) > 0$ for some j. Now apply Theorem 3.7 to theproblem of maximizing u under the constraints $\underline{x} \in U_1$ and $\underline{x} \cdot \underline{p} = w$. ∎

One can use some further optimization theory to derive more properties of ξ and its derivatives. Consider first the related problem of choosing the commodity bundle which achieves a fixed level of utility at minimum expenditure, i.e.,

($\mathbb{C}$') Minimize $\underline{p}\cdot\underline{x}$ subject to $u(\underline{x}) \geq u$ and $\underline{x} \geq \underline{0}$.

Let $z(\underline{p},u)$ be the minimizer of ($\mathbb{C}$'); z is called the compensated (or Hicksion) demand function since in its construction, income changes compensate for price changes to keep the consumer at a fixed level of utility.

In addition, consider the optimal value functions for problems ($\mathbb{C}$) and ($\mathbb{C}$'). The function $v(\underline{p},w) = u(\xi(\underline{p},w))$ is called the consumer's indirect utility function and $M(\underline{p},u) = \underline{p}\cdot z(\underline{p},u)$ is called the consumer's expenditure function. Note that M is a concave function of $\underline{p}$ by Theorem 5.3.b. These functions play a central role in modern consumer theory. We first list some elementary facts about them. The nonlinear programming problems $(\mathbb{C})$ and $(\mathbb{C}')$ are dual to each other in a natural way. Statements 4) and 5) in Theorem 6.3 are non-linear analogues of Theorem 5.5.

Theorem 6.3. Let ξ, z, v, and M be as in the above paragraph. Then

1) $\xi(\lambda\underline{p}, \lambda w) = \xi(\underline{p}, w)$ for all $\lambda > 0$ (homogeneity)
2) $z(\underline{p}, u) = \xi(\underline{p}, M(\underline{p}, u))$
3) $\xi(\underline{p}, w) = z(\underline{p}, v(\underline{p}, w))$
4) $u = v(\underline{p}, M(\underline{p}, u))$
5) $w = M(\underline{p}, v(\underline{p}, w))$.

Proof. 1) follows from the fact that $\underline{p}\cdot\underline{x} = w$ and $(\lambda\underline{p})\cdot\underline{x} = (\lambda\underline{w})$ are equivalent constraints. To prove 3), we show that if $\underline{x}^*$ solves $(\mathbb{C})$, then it solves $(\mathbb{C}')$ with $u = u(\underline{x}^*)$. By the Saddle Point Theorem 5.5,

$$(*) \qquad u(\underline{x}) + \lambda^*(w-\underline{p}\cdot\underline{x}) \le u(\underline{x}^*)+\lambda^*(w-\underline{p}\cdot\underline{x}^*)$$

for all $\underline{x} \geq 0$ where λ^* is the multiplier in $(\mathbb{C})$ corresponding to $\underline{x}^*$. Let $\underline{x}'$ be an arbitrary bundle in the constraint set of $(\mathbb{C}')$, i.e.,

$$(**) \qquad u(\underline{x}') \geq u(\underline{x}^*) \quad \text{and} \quad \underline{x}' \geq 0 \ .$$

Then,

$$u(\underline{x}^*)+\lambda^*(w-\underline{p}\cdot x') \le u(\underline{x}')+\lambda^*(w-\underline{p}\cdot x')$$
$$\le u(x^*)+\lambda^*(w-\underline{p}\cdot x^*)$$

by (**) and (*) . These imply that $\underline{p}\cdot x' \geq \underline{p}\cdot\underline{x}^*$, and so $\underline{x}^*$ is a solution of $(\mathbb{D}')$.

The proof of 2) is similar; 4) and 5) follow directly from 2) and 3) by evaluation. []

We can now compute some properties of ξ and its derivatives.

Theorem 6.4. Assume that $\xi, z, v,$ and M are C^1 functions. (See, for example, the hypotheses of Theorem 6.2.) Then,

1) $$\xi_i(\underline{p},w) = -\frac{\partial v(\underline{p},w)}{\partial p_i} \bigg/ \frac{\partial v(\underline{p},w)}{\partial w} \qquad \text{(Roy's Identity)}$$

2) $$\frac{\partial \xi_j}{\partial p_i}(\underline{p},w) = \frac{\partial z_j(p,v(p,w))}{\partial p_i} - \frac{\partial \xi_j(p,w)}{\partial w}\cdot \xi_i$$

(Slutsky Equation)

3) The matrix of "substitution terms" $\left(\left(\frac{\partial z_j(p,u)}{\partial p_i}\right)\right)$.

$$= \left(\left(\frac{\partial \xi_j}{\partial p_i}(p,w) + \xi_i \frac{\partial \xi_j}{\partial w}(p,w)\right)\right)$$ is a symmetric, negative semi-definite matrix.

4) In particular, $\frac{\partial z_i}{\partial p_i}(p,u) \leq 0$ and $\frac{\partial \xi_i}{\partial p_i} + \xi_i \frac{\partial \xi_i}{\partial w} \leq 0$.

Proof: 1) follows from differentiating 4) in the statement of Theorem 6.3 with respect to p_i :

$$0 = \frac{\partial v}{\partial p_i} + \frac{\partial v}{\partial w}\cdot\frac{\partial M}{\partial p_i} .$$

But $\frac{\partial M}{\partial p_i}(p,u) = \xi_i$, since by the Envelope Theorem 3.8.b,

$$(***) \quad \frac{\partial M(p,u)}{\partial p_i} = \frac{\partial}{\partial p_i}[\underline{x}\cdot\underline{p} + \mu(u(\underline{x})-u(\underline{x}^*))] = x_i^* = z_i(\underline{p},u) = \xi_i(\underline{p},M(\underline{p},u)).$$

Conclusion 2) follows from differentiating equation 2) in Theorem 6.3 with respect to p_i :

$$\frac{\partial z_j}{\partial p_i}(p,u) = \frac{\partial \xi_j}{\partial p_i}(\underline{p},w) + \frac{\partial \xi_j}{\partial w}(\underline{p},w) \cdot \frac{\partial M}{\partial p_i}(p,u).$$

Then, apply (***) and rearrange terms. To prove the symmetry in 3), recall from (***) that $\frac{\partial M}{\partial p_i}(\underline{p},u) = z_i(\underline{p},u)$. So,

$\frac{\partial z_i}{\partial p_j} = \frac{\partial^2 M}{\partial p_j \partial p_i} = \frac{\partial^2 M}{\partial p_i \partial p_j} = \frac{\partial z_j}{\partial p_i}$. Since M is concave by Theorem 5.3.b, its Hessian is negative-semi-definite. Finally 4) follows from the fact that the 1×1 principal minors of a negative semi-definite matrix must be non-postive. []

The Slutsky equation in 2) of Theorem 6.4 is an important relationship between the two demand function ξ and z . If we write this equation as

$$\Delta x_j \cong \frac{\partial \xi_j}{\partial p_i}\Delta p_i = \frac{\partial z_j}{\partial p_i}(p,u) \cdot \Delta p_i - \frac{\partial \xi_j}{\partial w} \cdot x_i \cdot \Delta p_i ,$$

we see that the change in demand Δx_j due to a change in price Δp_i decomposes into two separate effects: the <u>substitution effect</u>

$\frac{\partial z_j}{\partial p_i}(\underline{p},u) \cdot \Delta p_i$, during which utility is held constant, and the <u>income effect</u> $\frac{\partial \xi_j}{\partial w} \cdot x_i \cdot \Delta p_i$, in which $x_i \Delta p_i$ represents the change in income.

It turns out that conclusion 3) in Theorem 6.4 provides a necessary and sufficient condition for an observed demand function $\xi(\underline{p},w)$ to arise from utility maximization. See Samelson (1950) and Hurwicz and Uzawa (1971).

6B. <u>Theory of the Firm or Producer</u>.

We turn now to an analysis of the economic behavior of a firm. A firm uses inputs such as materials, labor, and land to produce outputs which it sells to households or other firms. Given the price and supply of each input, the price and demand of each output, and the technological relations between input and output, the firm must decide how much to produce and how much input to use in this production in order to meet its economic objectives.

Suppose that the firm in question produces a single commodity from n inputs. Let x_i denote the quantity of i^{th} input, $\underline{x} = (x_1,\ldots,x_n)$ the resulting <u>input vector</u>, and $y \in \mathbb{R}$ the amount of output produced. We assume that there is a <u>production function</u> $f : \mathbb{R}^n \to \mathbb{R}$, where $f(\underline{x})$ denotes the maximum output for each input vector $\underline{x}$.

In order to examine the most general situations, let $p_1(y)$ and $p_2(\underline{x})$ denote the inverse demand functions for output and input, respectively, i.e., $p_1(y)$ is the unit price a firm can charge if its level of output is y and $p_2(\underline{x}) \in \mathbb{R}^n_+$ is the input price vector which the firm will pay if it needs input vector $\underline{x}$. For a firm in perfect competition, p_1 and p_2 are constant; for a monopolist firm, p_2 is constant but p_1 is not and the firm can control the price of its product by varying production amounts; for a monopsonistic firm, p_1 is constant but p_2 is not and the firm can influence the price of an input by varying its purchases of the input.

Let us assume first that our firm wants to maximize its profit, $\Pi = p_1(y)y - p_2(\underline{x}) \cdot \underline{x}$, where $y = f(\underline{x})$. We can use the results of chapter two to find a necessary condition for such a maximum, namely that the marginal revenue product equals the marginal cost of each input. The marginal cost of input k is, of course, $\frac{\partial}{\partial x_k} p_2(\underline{x}) \cdot \underline{x}$. The marginal revenue product of input k is the marginal revenue, $\left.\frac{dp_1(y)y}{dy}\right|_{y = f(\underline{x})}$, times the marginal product of input k, $\frac{\partial f}{\partial x_k}(\underline{x})$. To derive this norm, one merely sets the first derivative of Π with respect to x_k equal to zero.

If one considers the case of a firm in perfect competition where $p_1 = p$ and $\underline{p}_2 = w$ are constant, the above norm becomes

$$\text{(P)} \qquad p\,\Delta f(\underline{x}^{\circ}) = \underline{w}\ .$$

If one assumes further that the production function is concave, then one learns from Theorem 2.3 that (P) is also a sufficient condition for $\underline{x}^{\circ}$ to be an input which maximizes profit. In this case, the firm is operating at the optimal input level if an additional unit of output will bring in as much revenue as it costs to produce.

Furthermore, as in section 6.A, one can define $\chi(p,\underline{w}) = \underline{x}^o$ to be the solution of (P) for a fixed p and w . This correspondence is called the <u>input demand correspondence</u>. If f is strictly concave, χ is a single-valued function, which is homogeneous of degree zero. If f is C^2 and $D^2f(x)$ is negative definite for all x , then χ is C^1 . The function $F(p,\underline{w}) = f(\chi(p,\underline{w}))$ is called the <u>output supply function</u>, a component in the usual demand/supply analysis.

One can derive conditions on the derivatives of these functions by using techniques similar to those in Theorems 6.3 and 6.4 of the previous section. Let $\pi^*(p,\underline{w}) = \pi(\chi(p,\underline{w}))$ be the optimal profit function. By the Envelope Theorem 3.8.a,

$$(\S) \qquad \frac{\partial \pi^*(p,\underline{w})}{\partial p} = y = F(p,\underline{w}) \quad \text{and} \quad \frac{\partial \pi^*(p,\underline{w})}{\partial w_i} = -x_i = -\chi_i(p,\underline{w}) \; .$$

The latter leads to the <u>reciprocity condition</u>:

$$\frac{\partial \chi_j}{\partial w_i}(p,\underline{w}) = -\frac{\partial^2 \pi^*}{\partial w_i \partial w_j} = -\frac{\partial^2 \pi^*}{\partial w_j \partial w_i} = \frac{\partial \chi_i}{\partial w_j}(p,\underline{w}) \; , \text{ i.e.,}$$

the effect of a change in the wage of the i^{th} input on the demand for the j^{th} input is the same as the effect of a change in the wage of the j^{th} input on the demand for the i^{th} input. The first equation in (§) leads to:

$$\frac{\partial \chi_i}{\partial p}(p,\underline{w}) = -\frac{\partial^2 \pi^*}{\partial p \partial w_i} = -\frac{\partial^2 \pi^*}{\partial w_i \partial p} = -\frac{\partial F}{\partial w_i}(p,\underline{w}) \; ;$$

so an increase in the output price raises the demand for input i if and only if an increase in the wage of input i reduces the optimal output. Finally, by Theorem 5.3.b, $\pi^*(p,w)$ is a convex function, and so its Hessian is positive semi-definite. In particular, this means that $\frac{\partial x_i}{\partial w_i} = -\frac{\partial^2 \pi^*}{\partial w_i^2}$ must be negative for all i ; an increase in the wage of an input always leads to a reduction in its demand. See Varian (1978) and Silberberg (1978) for further discussion.

Let us now change the problem a little. Suppose that the firm in question has its policy determined by a manager whose objective is to maximize sales, i.e., revenue, without letting the profit drop below some fixed level. (See Baumol (1961) for a complete discussion of such firms and Kuhn (1968) for the following mathematical analysis). To make things even more interesting, let us add an advertising cost $a \in \mathbb{R}_+$ to this problem. Let $R(y,a)$ denote the firm's revenue when the level of production is $y \in \mathbb{R}_+$ and the advertising cost is $a \in \mathbb{R}_+$. Let $C(y)$ denote the cost of manufacturing y units of output. We will assume not only that C and R are C^1 functions but also that $C'(y) > 0$ (increased production implies increased costs) and $\frac{\partial R}{\partial a} > 0$ (increased advertising brings in increased revenues). Our programming problem is to maximize $R(y,a)$ subject to the constraints $y \geq 0$, $a \geq 0$, and

$$\Pi \equiv R(y,a) - C(y) - a \geq m .$$

Assume that (y^o, a^o) is an optimal solution with $y^o > 0$. In addition, assume that some constraint qualification is valid at (y^o, a^o), e.g., R may be concave and C convex. Then, there are non-negative multipliers μ^o and v^o such that the Lagrangian

$$L(y, a) = R(y, a) + \mu^o a + v^o[R(y, a) - C(y) - a - m]$$

has a critical point at (y^o, a^o). In other words,

$$\text{(Q)} \quad \frac{\partial L}{\partial y}(y^o, a^o, \lambda^o, \mu^o, v^o) = (1 + v^o)\frac{\partial R}{\partial y} - v^o C'(y^o) = 0 \quad \text{and}$$

$$\text{(R)} \quad \frac{\partial L}{\partial a}(y^o, a^o, \lambda^o, \mu^o, v^o) = (1 + v^o)\frac{\partial R}{\partial a} + \mu^o - v^o = 0 .$$

Since $\frac{\partial R}{\partial a} > 0$ and $v^o \geq 0$, $\mu^o - v^o < 0$ in (R). Since $\mu^o \geq 0$, v^o must be strictly positive. Therefore, $\Pi(y^o, a^o) = m$; the profit realized is the minimal profit allowed. Since $v^o > 0$ and $C'(y^o) > 0$ in (Q), $\frac{\partial R}{\partial y}(y^o, a^o) > 0$ and marginal revenue is positive at the optimum level. On the other hand, the marginal profit, $\frac{\partial \Pi}{\partial y}$, is negative at (y^o, a^o) since

$$\begin{aligned}(1 + v^o)\frac{\partial \Pi}{\partial y}(y^o, a^o) &= \frac{\partial L}{\partial y}(y^o, a^o) - C'(y^o) \\ &= 0 - C'(y^o) < 0 .\end{aligned}$$

Consequently, output y^o is greater than the output in the profit-maximizing situation.

Finally, by Theorem 3.7, the multiplier v^0 can be interpreted as the marginal loss in maximal revenue with respect to the limit on profit.

Just as we added advertising cost to our study of a sales maximizing firm, so the economist can use programming principles to determine the effect of such items as sales taxes and regulatory constraints on the optimal behavior of a firm. For example, see Averch and Johnson (1962) for an analysis of how a "fair rate of return" regulatory constraint could alter the behavior of a monopolist firm.

§6.C Activity Analysis

In this section, we will apply the linear duality theory, discussed at the end of chapter five, to the important problem of the activity analysis of production. In this model, a firm in a competitive economy produces k different outputs from m different resources or inputs. Furthermore, different combinations of inputs can be used to produce the same combination of outputs, but these transformations are organized into n processes or activities, where $1 \leq n < \infty$. The j^{th} activity, for example, combines the k inputs in fixed proportions into the m outputs in fixed proportions at some non-negative level or intensity, $z_j \geq 0$.

The firm's technology is then described by an $m \times n$ matrix

$A = ((a_{ij}))$ and a $k \times n$ matrix $B = ((b_{ij}))$, where $a_{ij} > 0$ is the amount of the j^{th} input used in operating the j^{th} activity at unit intensity and $b_{ij} > 0$ is the amount of the i^{th} output produced when process j runs at unit intensity. If the firm conducts all its activities at the same time with the j^{th} activity at level $z_j \geq 0$ for $j = 1, \ldots, n$, then it transforms the input vector $\underline{x} = A\underline{z} \in \mathbb{R}^m_+$ into the output vector $\underline{y} = B\underline{z} \in \mathbb{R}^k_+$.

Let $p_i > 0$ denote the fixed market price for the i^{th} output, $i = 1, \ldots, k$: and let $q_i > 0$ denote the fixed market price for the i^{th} input, $i = 1, \ldots, m$. Thus, $\underline{p} = (p_1, \ldots, p_k)$ and $\underline{q} = (q_1, \ldots, q_m)$ are the corresponding price vectors in an economy of perfect competition. Let b_i denote the available stock of the i^{th} resource or input, with $\underline{b} = (b_1, \ldots, b_k)$.

If the firm's director wants to maximize profits, he must solve the following linear programming problem:

Find an activity vector $\underline{z}$ in $\mathbb{R}^m$ such that $\underline{z}$ maximizes $\underline{p} \cdot \underline{y} - \underline{q} \cdot \underline{x}$ subject to

$$\underline{x} = A\underline{z}\ ,\ \underline{y} = B\underline{z}\ ,\ \underline{x} \leq b\ ,\ z \geq \underline{0}\ .$$

If we substitute the equality constraints into the profit function, the problem becomes:

Find $\underline{z} \in \mathbb{R}^n$ such that $\underline{z}$ maximizes

$$\underline{p} \cdot B\underline{z} - \underline{q} \cdot A\underline{z}$$

subject to $A\underline{z} \leq \underline{b}$ and $\underline{z} \geq 0$.

Finally, if we let $\underline{r} = B^t\underline{p} - A^t\underline{q}$ in $\mathbb{R}^n$, r_j denotes the value or profit of the output achieved by operating the j^{th} activity at unit level. We then want to choose $\underline{z}$ to

$$\text{(S)} \qquad \text{maximize } \underline{r} \cdot \underline{z} \text{ subject to } A\underline{z} \leq \underline{b} \text{ and } \underline{z} \geq \underline{0} .$$

One can, of course, use the simplex algorithm to solve this linear programming problem; but let us see what we can learn about the problem and its solution from our programming theory. The Lagrangian is

$$L(\underline{z}, \underline{\lambda}) = \underline{r} \cdot \underline{z} + \underline{\lambda} \cdot (\underline{b} - A\underline{z}) .$$

The Kuhn-Tucker necessary and sufficient conditions for a solution are that we find a $\underline{\lambda} \geq \underline{0}$ in $\mathbb{R}^m$ and a $\underline{z} \geq \underline{0}$ in $\mathbb{R}^n$ such that

$$\underline{r} - A^t\underline{\lambda} \leq \underline{0} \ , \ \underline{z} \cdot (\underline{r} - A^t\underline{\lambda}) = 0 \ , \ \underline{\lambda} \cdot (\underline{b} - A\underline{z}) = \underline{0}$$

If we have such a $\underline{\lambda}$, then by Theorem 3.9, λ_i can be regarded as the infinitesimal change in maximal profit as the amount of the i^{th} resource that is available increases. It can therefore be interpreted as the firm's internal valuation of the i^{th} resource and is usually called the firm's imputed or <u>shadow price</u> of this input.

This naturally leads us to consider the dual problem, as discussed in section 5. The dual problem to (S) is to

(S′) find $\underline{\lambda} \geq \underline{0}$ in $\mathbb{R}^m$ such that $\underline{\lambda}$ minimizes $\underline{\lambda} \cdot \underline{b}$ subject to the constraints $\underline{\lambda}A \geq \underline{r}$.

If $\underline{\lambda}$ is the shadow price vector described above, $\underline{\lambda} \cdot \underline{b}$ is the total value which the firm sets on its resources in stock (in its internal price system). Examining the constraint in (S′) , one notices that the j^{th} component of $\underline{\lambda}A = A^t\underline{\lambda}$ is the total value of the output as a result of operating activity j at unit intensity in the internal price system $\underline{\lambda}$ and that r_j is the actual value of this output (in the external price system). Therefore, in solving (S′) , the firm tries to determine a valuation or internal price system on its resources so that the value of its resources will be minimized under the constraint that when the firm operates any activity at unit level the total value of the resulting output in the internal valuation must be at least as large as its total value in the market's price system. Karlin (1959) summarizes this constraint by stating that "internal prices cannot be set to get more value from a product than you put into it".

Let $\underline{z}^* \in \mathbb{R}^n_+$ be the activity vector which solves (S) and let $\underline{\lambda}^*$ be the shadow price vector which solves (S′). By Theorem 5.5, $\underline{z}^* \cdot \underline{r} = \underline{\lambda}^* \cdot \underline{b}$, i.e., the optimal total profit in the activity analysis problem equals the minimal total (internal) value of the resources in stock. This equation is closely related to the macroeconomic norm that at an equilibrium the value of the final goods produced (national product) must equal the cost of the primary factors of production (national income).

An alternative but related interpretation of the dual problem is the viewpoint of a competitor who wants to buy the resources of the producer (possibly believing that he can use them more efficiently). He offers to pay the producer the amount λ_i for each unit of resource i. The constraint $\lambda A \geq r$ assures the original producer that "the amount offered is at least as much as he could obtain from any production schedule". (Gale (1960).) The competitor tries to minimize the total cost of his purchase subject to the above assurance to the producer. By (N), the producer has nothing to lose and may even gain if his competitor misses the optimal buy-out price.

Finally, in programming problems where the primal problem describes the search for the best joint strategy in a decentralized economic system, the dual problem often can be interpreted as a central planner's viewpoint of the same system.

Finally, we can use the activity analysis model to gain further insight into the economists' enthusiasm for the saddle point approach to concave programming. Generalizing (S), consider the problem of maximizing $f(\underline{x})$ subject to $\underline{x} \geq 0$ and $g(\underline{x}) \leq \underline{b}$. Assume that $\underline{x} \geq \underline{0}$ represents the activity level of a firm's operations, f is a C^1 concave function representing the value of the firm's output for any given activity level, $\underline{b}$ is a constant vector which measures the amount of primary resources that are available, and $g(\underline{x})$ is a measure of the amount of these resources used when the activity vector is $\underline{x}$.

The Lagrangian function for this problem is $L(\underline{x}, \underline{\lambda}) = f(\underline{x}) + \underline{\lambda}[\underline{b} - g(\underline{x})]$, and, as mentioned above, $\underline{\lambda}$ can be viewed as the vector of shadow prices for the primary resources. Thus, L is the combined value of the firm's outputs and the unused balance of primary resources. Suppose there is an $\underline{x}' \geq 0$ with $g(\underline{x}') < \underline{b}$, and that $\underline{x}^o$ maximizes f subject to $g(\underline{x}) \leq \underline{b}$ and $\underline{x} \geq \underline{0}$. Then, by Theorem 5.5, there is a $\underline{\lambda}^o \geq \underline{0}$ such that L has a saddle point at $(\underline{x}^o, \underline{\lambda}^o)$, i.e.,

$$L(\underline{x}, \underline{\lambda}^o) \leq L(\underline{x}^o, \underline{\lambda}^o) \leq L(\underline{x}^o, \underline{\lambda}) \quad \text{for all} \quad \underline{x} \geq 0, \underline{\lambda} \geq 0 .$$

(By Theorem 5.4, $\underline{x}^o$ solves problem (S) if $(\underline{x}^o, \underline{\lambda}^o)$ is a saddle point of L .) The existence of $(\underline{x}^o, \underline{\lambda}^o)$ expresses an equilibrium between the value of the output and the prices of the available resources and is a basic step in the theory of equilibria for production economics.

For further discussion on the activity analysis problem the reader is referred to Koopmans (1951), Karlin (1959), Charnes-Cooper (1961), Varaiya (1972), and Silverberg, (1978).

§7. VECTOR MAXIMIZATION

7.A Preliminaries

In the applications of chapter six, we studied an individual consumer trying to maximize his utility function under budgetary constraints and a single firm striving to maximize its profits or sales while producing a single output from a stock of available resources. The next step is to examine economies where a number of consumers compete among themselves for goods and services and where firms producing a number of products decide on optimal output vectors. To treat such problems as a large number of independent maximization problems would be to ignore not only the boundedness of the stock of available goods and resources but also the interactions between the various components of the economy. More importantly, such a treatment will usually lead to a mathematical problem with an empty solution set. We therefore introduce the more natural notion of a vector maximum or Pareto optimum for situations where a number of different participants are trying to meet their independent objectives.

Definition. Let C be a subset of $\mathbb{R}^n$. Let $u_1,\ldots,u_a$ be real-valued functions on C. Then, $u = (u_1,\ldots,u_a)$ has a vector maximum or Pareto optimum at $\underline{x}° \in C$ if there is no $\underline{x} \in C$ such that $u_i(\underline{x}°) \leq u_i(\underline{x})$ for all i and $u_j(\underline{x}°) < u_j(\underline{x})$ for some j, i.e., such that $u(\underline{x}°) \leq u(\underline{x})$ but $u(\underline{x}°) \neq u(\underline{x})$ in the usual partial ordering on $\mathbb{R}^a$.

A number of recent papers have proven necessary conditions and sufficient conditions for vector maximization on constrained sets without using any of the non-linear programming results surveyed in chapters 3,4, and 5. In this chapter, we will show how many of these vector maximization theorems do indeed follow easily from the scalar maximization theorems we have studied. The following theorem is the key step in this process.

Theorem 7.1. Let C be a subset of $\mathbb{R}^n$. A necessary and sufficient condition that $\underline{u}: \mathbb{R}^n \to \mathbb{R}^a$ have a Pareto optimum at $\underline{x}^\circ$ on C is that $\underline{x}^\circ$ maximizes each u_i on the constraint set $C_i \equiv \{\underline{x} \in C \mid u_j(\underline{x}) - u_j(\underline{x}^\circ) \geq 0; j = 1,\ldots,a; j \neq i\}$.

Proof: Suppose that u has a Pareto optimum on C at $\underline{x}^\circ$. If $\underline{x}$ does not maximize u_k on C_k, then there is an $\underline{x} \in C$ such that $u_j(\underline{x}) \geq u_j(\underline{x}^\circ)$ for all $j \neq k$ and $u_k(\underline{x}) > u_k(\underline{x}^\circ)$, contradicting the Pareto optimality of $\underline{x}^\circ$.

Conversely, suppose that $\underline{x}^\circ$ maximizes each u_k on C_k. If $\underline{x}^\circ$ is not a Pareto optimum on C, there is an $\underline{x} \in C$ and a k such that $u_i(\underline{x}) \geq u_i(\underline{x}^\circ)$ for all i and $u_k(\underline{x}) > u_k(\underline{x}^\circ)$, contradicting the maximality of $\underline{x}^\circ$ for u_k on C_k. ∎

Although this result is probably well known to many who work in this area, I have not found an explicit statement of it in the literature. Some authors, such as El-Hodiri (1971) and Wan (1975), have noted and used parts of this theorem in their work.

7.B Necessary Conditions for Optimality

In this section, we'll use the results of chapters 3 and 4 to derive necessary conditions for $\underline{x}^\circ \in \mathbb{R}^n$ to be a Pareto optimum. Throughout this section, $C_{g,h}$ will denote the constraint set $\{\underline{x} \in \mathbb{R}^n | g_i(\underline{x}) \geq 0,\ i = 1,\ldots,M; h_j(\underline{x}) = \underline{0}, j = 1,\ldots,N\}$.

Theorem 7.2. Suppose that $u_1,\ldots,u_a,g_1,\ldots,g_M,h_1,\ldots,h_N$ are C^1 functions on $\mathbb{R}^n$. Suppose that $\underline{x}^\circ \in C_{g,h}$ is a Pareto optimum for $u = (u_1,\ldots,u_a)$ on $C_{g,h}$. Then, there exist scalars $\alpha_1,\ldots,\alpha_a,\lambda_1,\ldots,\lambda_M,\mu_1,\ldots,\mu_N$ such that

$$
\text{(T)} \quad \begin{cases} (\underline{\alpha},\underline{\lambda},\underline{\mu}) \neq \underline{0} \ ; \\ \alpha_i \geq 0 \ ,\ i = 1,\ldots,a;\ \lambda_j \geq 0,\ j = 1,\ldots,M \ ; \\ \lambda_j g_j(\underline{x}^\circ) = 0 \ ,\ j = 1,\ldots,M \ ; \\ \sum_1^a \alpha_i Du_i(\underline{x}^\circ) + \sum_1^M \lambda_j Dg_j(\underline{x}^\circ) + \sum_1^N \mu_k Dh_k(\underline{x}^\circ) = \underline{0} \ . \end{cases}
$$

Proof. Since $\underline{x}^\circ$ is a Pareto optimum of u on $C_{g,h}$, $\underline{x}^\circ$ maximizes u_1 on the set $\{\underline{x}^\circ \in C_{g,h} | u_j(x) - u_j(\underline{x}^\circ) \geq 0 \ ,\ j = 2,\ldots,a\}$ By F.John's result (Theorem 4.1), there exist $\alpha_1 \geq 0;\ \alpha_2,\ldots,\alpha_a$, $\lambda_1,\ldots,\lambda_M$ non-negative; and $\mu_1,\ldots,\mu_N$ such that $(\underline{\alpha},\underline{\lambda},\underline{\mu}) \neq 0$ in $\mathbb{R}^{a+M+N}$, $\lambda_i g_i(\underline{x}^\circ) = 0$, and

$$\alpha_1 Du_1(\underline{x}^\circ) + \sum_2^a \alpha_i D[u_i - u_i(\underline{x}^\circ)](\underline{x}^\circ) + \sum_1^M \lambda_j Dg_j(\underline{x}^\circ) + \sum_1^N \mu_k Dh_k(\underline{x}^\circ) = \underline{0} .$$

But $D[u_i - u_i(\underline{x}^\circ)](\underline{x}^\circ) = Du_i(\underline{x}^\circ)$. ∎

Another proof of Theorem 7.2 may be found in DaCunha and Polak (1967). As before, Theorem 7.2 says very little unless one can guarantee that all of the α_i are non-zero. Thus, we need to make some assumptions on $u, g,$ and h so that we can apply our theorems on constraint qualifications.

Theorem 7.3. Suppose that $u_1,\ldots,u_a,g_1,\ldots,g_M,h_1,\ldots,h_N$ are C^1 functions on $\mathbb{R}^n$. Suppose that $\underline{x}^\circ \in C_{g,h}$ and that u has a Pareto optimum on $C_{g,h}$ at $\underline{x}^\circ$. Suppose that $u, g,$ and h satisfy one of the following hypotheses, where $u^{(i)} \equiv (u_1,\ldots,u_{i-1}, u_{i+1},\ldots,u_a): \mathbb{R}^n \to \mathbb{R}^{a-1}$:

a) $D(u^{(i)}, g_E, h)(\underline{x}^\circ)$ has maximal rank for each $i = 1,\ldots,a$.

b) Let $A_1 = \{i \mid u_i$ is pseudo-convex in some neighborhood of $\underline{x}^\circ\}$ $A_2 = \{1,\ldots,a\} - A_1$, $E_1 = \{j \in E \mid g_j$ is pseudo-convex in some neighborhood of $\underline{x}^\circ\}$, and $E_2 = E - E_1$. Suppose that h is linear and there is a $\underline{v} \in T_{\underline{x}^\circ}\mathbb{R}^n$ such that $Du_{A_1}(\underline{x}^\circ)\underline{v} \geq 0$, $Du_{A_2}(\underline{x}^\circ)\underline{v} > 0$, $Dg_{E_1}(\underline{x}^\circ)\underline{v} \geq 0$, $Dg_{E_2}(\underline{x}^\circ)\underline{v} > \underline{0}$, and $Dh(\underline{x}^\circ)\underline{v} = 0$.

c) u and g are pseudo-convex and h is linear,

d) h is affine; u and g are concave on some convex neighborhood U of $\underline{x}^\circ$; and for each $i \in \{1,\ldots,a\}$ there is an

$\underline{x}^i \in \mathbb{R}^n$ such that $u^{(i)}(\underline{x}^i) > u^{(i)}(\underline{x}^\circ)$ and $g(\underline{x}^i) > \underline{0}$, $h(\underline{x}^i) = 0$.

e) Suppose whenever $Du^{(i)}(\underline{x}^\circ)\underline{v} \geq 0$, $Dg_E(\underline{x}^\circ)\underline{v} \geq 0$, and $Dh(\underline{x}^\circ)\underline{v} = 0$ for some i and some $\underline{v} \in T_{\underline{x}^\circ}\mathbb{R}^n$, there is a C^1 path $a:[0,\varepsilon) \to \mathbb{R}^n$ with $a(0) = \underline{x}^\circ$, $a'(0) = \underline{v}$, $u^{(i)}(a(t)) \geq u^{(i)}(\underline{x}^\circ)$ $g(a(t)) \geq \underline{0}$, and $h(a(t)) = \underline{0}$.

f) (Kuhn-Tucker(1951)): For each $i = 1,2,\ldots,a,$ there is no vector $\underline{v}$ such that

$$\begin{aligned} &Du_i(\underline{x}^\circ)\underline{v} > 0 \\ &Du_j(\underline{x}^\circ)\underline{v} \geq 0 \text{ , for all } j \neq i \\ &Dg_E(\underline{x}^\circ)\underline{v} \geq \underline{0} \text{ ,} \\ &Dh(\underline{x}^\circ)\underline{v} = \underline{0} \text{ .} \end{aligned}$$

g) (Geoffrion (1968)): There exists a scalar M such that, for each i , we have

$$\frac{u_i(\underline{x}) - u_i(\underline{x}^\circ)}{u_j(\underline{x}^\circ) - u_j(\underline{x})} \leq M$$

for some j such that $u_j(\underline{x}) < u_j(\underline{x}^\circ)$ whenever $\underline{x} \in C_{g,h}$ and $u_i(\underline{x}) > u_i(\underline{x}^\circ)$.

Then, there are scalars $\alpha_1,\ldots,\alpha_a,\lambda_1,\ldots,\lambda_M,\mu_1,\ldots,\mu_N$ such that (T) of Theorem 7.2 holds, where the α_i are all strictly positive.

<u>Proof</u>: For hypotheses a) through e) , fix $i \in \{1,\ldots,a\}$. By Theorems 3.3, 4.3, and 4.4, there are $\beta_1^i,\ldots,\beta_a^i,\lambda_1^i,\ldots,\lambda_M^i$ $\mu_1^i,\ldots,\mu_N^i$ such that

$\beta^i_j \geq 0$ for all $j = 1,\ldots,a; \beta^i_i = 1$;

$\lambda^i_j \geq 0$ and $\lambda^i_j g_j(\underline{x}°) = 0$ for all $j = 1,\ldots,M_j$ and

$$\sum_{j=1}^{a} \beta^i_j Du_j(\underline{x}°) + \sum_{m=1}^{M} \lambda^i_m Dg_m(\underline{x}°) + \sum_{k=1}^{N} \mu^i_k Dh^k(\underline{x}°) = 0 .$$

Let $\alpha_j = \sum_{i=1}^{a} \beta^i_j \geq 1$, $\lambda_m = \sum_{i=1}^{a} \lambda^i_m \geq 0$, and $\mu_k = \sum_{i=1}^{a} \mu^i_k$.

For hypothesis f) , apply Farkas' Lemma (see section 4.B) for each i with $A = (Du^{(i)}(\underline{x}°), Dg_E(\underline{x}°), Dh(\underline{x}°), -Dh(\underline{x}°))$. By hypothesis f), whenever $A\underline{v} \geq \underline{0}$, $-Du_i(\underline{x}°)\underline{v} \geq 0$. So, there exist $\beta^i_1,\ldots,\lambda^i_1,\ldots,\mu^i_1,\ldots,\mu^i_N$ as in the preceding paragraph.

For hypothesis g), see Geoffrion (1968). ∎

Kuhn and Tucker call a vector maximum which satisfies hypothesis f) in Theorem 7.3 a <u>proper</u> solution of the vector maximum problem. Geoffrion (1968) calls a vector maximum which satisfies hypothesis g) a <u>properly efficient solution</u> of the vector maximum problem. Both of these papers indicate that at a Pareto optimum which is not proper one can find paths which allow first-order gains for some of the u_i's and only second-order losses for the other u_i's. See also Klinger (1967) .

We will use some of the hypotheses of Theorem 7.3 when we study some more economics applications in Chapter 8.

7.C Second Order Sufficient Conditions

One can now easily combine Theorem 7.1 and the results of section 3 to prove the following strong second order sufficiency condition for Pareto optimization. Weinberger (1974), Smale (1975b), Wan (1975b), and deMelo (1975) have proven similar results using other methods.

Theorem 7.4. Let $u_1,\ldots,u_a,g_1,\ldots,g_M,h_1,\ldots,h_N\colon \mathbb{R}^n \to \mathbb{R}^1$ be C^2 functions. Suppose that $\underline{x}° \in C_{g,h} \equiv \{\underline{x} \in \mathbb{R}^n | g(\underline{x}) \geq \underline{0}, h(\underline{x}) = \underline{0}\}$. Suppose there exist multipliers $\underline{\alpha} \geq \underline{0}$ in $\mathbb{R}^a$, $\underline{\lambda} \geq \underline{0}$ in $\mathbb{R}^M$, and $\underline{\mu} \in \mathbb{R}^N$ such that $\lambda_i g_i(\underline{x}°) = 0$ for all i

and if $$L \equiv \sum_1^a \alpha_i u_i + \sum_1^M \lambda_j g_j + \sum_1^N \mu_k h_k$$

then $$DL(\underline{x}°) = \underline{0} \text{ and}$$

$$D^2L(\underline{x}°)(\underline{v},\underline{v}) < 0 \text{ for all non-zero } \underline{v} \text{ such that}$$

$\alpha_i Du_i(\underline{x}^o)\underline{v} = 0$ and $Du_i(\underline{x}^o)\underline{v} \geq 0$ for $i = 1,\ldots,M$; $\lambda^i Dg_i(\underline{x}^o)\underline{v} = 0$ and $Dg_i(\underline{x}^o)\underline{v} \geq 0$ for each $i \in E$; and $Dh(\underline{x}^o)\underline{v} = 0$.

Then, $\underline{x}°$ is a strict local Pareto optimum for $\underline{u}$ in $C_{g,h}$.

Proof: By Theorem 7.1, we need only show that $\underline{x}°$ maximizes each u_i on $u^{(i)} - u^{(i)}(\underline{x}°) \geq \underline{0}, g \geq \underline{0}, h = \underline{0}$. We will work with $i = 1$ for simplicity of notation and use Theorem 3.4. Of course, we choose the $\alpha_1,(\alpha_2,\ldots,\alpha_a),\underline{\lambda},\underline{\mu}$ of our hypothesis for the multipliers in our scalar maximization problem.

$$\text{Letting } L' = \alpha_1 u_1 + \sum_2^a \alpha_i (u_i - u_i(\underline{x}^\circ)) + \sum_1^M \lambda_j g_j + \sum_1^N \mu_k h_k ,$$

we see that $DL'(\underline{x}^\circ) = DL(\underline{x}^\circ) = \underline{0}$. Now, choose non-zero $\underline{v}$ so that $Du_1(x^O)\underline{v} \geq 0$, so that $\alpha_i D(u_i - u_i(\underline{x}^O))(\underline{x}^O)\underline{v} = 0$ and $D(u_i - u_i(\underline{x}^O))(\underline{x}^O)\underline{v} \geq 0$ for $i = 2,\ldots,a$, so that $\lambda_i Dg_i(\underline{x}^O)\underline{v} = 0$ and $Dg_i(\underline{x}^O)\underline{v} \geq 0$ for each $i \in E$, and so that $Dh(\underline{x}^O)\underline{v} = 0$. Since $DL(\underline{x}^O) = 0$,

$$-\alpha_1 Du_1(\underline{x}^\circ)\underline{v} = \sum_2^a \alpha_i Du_i(\underline{x}^\circ)\underline{v} + \sum_1^M \lambda_j Dg_j(\underline{x}^\circ)\underline{v}$$

$$+ \sum_1^N \mu_k Dh_k(\underline{x}^\circ)\underline{v} = 0 .$$

By hypothesis, $D^2L'(\underline{x}^\circ)(\underline{v},\underline{v}) = D^2L(\underline{x}^\circ)(\underline{v},\underline{v}) < 0$. By Theorem 3.4, u restricted to $\{u^{(1)} \geq u^{(1)}(\underline{x}^\circ), g \geq \underline{0}, h = \underline{0}\}$ has a strict local maximum at $\underline{x}^\circ$. Since this is clearly true for all $i > 1$ also, $\underline{u}$ restricted to $g \geq \underline{0}, h = \underline{0}$ has a strict local Pareto optimum at $\underline{x}^\circ$ by Theorem 7.1.∎

As before, one can strengthen the sufficiency test of Theorem 7.4 by allowing the multipliers to depend on the vector $\underline{v}$ to be tested. See Ben-tal (1980) and Weinberger (1974).

See example 1 after Theorem 7.6 below for a calculation of a Pareto optimum based on Theorem 7.4.

7.D First Order Sufficient Conditions

In many applications in economics, the u_i's, g_j's, and h_k's which arise naturally are concave or convex. For example, let $u(x_1,x_2)$ denote a consumer's utility function in an economy with

two commodities. If commodities one and two are desirable ones and about equally so, then the natural assumption that the consumer would prefer to have some of each commodity rather than lots of one commodity and little or none of the other leads to the usual hypothesis that the u_j's are concave or at least quasi-concave functions. In fact, the desire of consumers to achieve balanced distributions of the goods in question - hopefully, by trading with other consumers - is a concept at the core of the theory of microeconomics.

In this section, we use Theorem 7.1 and the results of section 5 to describe sufficient conditions for optimality when the functions involved are concave or almost concave.

Theorem 7.5. Suppose $u_1,\dots,u_a,g_1,\dots,g_M,h_1,\dots,h_N\colon \mathbb{R}^n \to \mathbb{R}^1$ are C^1 functions with $g(\underline{x}^\circ) \geq \underline{0}$, $h(\underline{x}^\circ)=\underline{0}$. Suppose that

i) the u_i's are pseudoconcave at $\underline{x}^\circ$, e.g., $\nabla u_i(x^O) \neq 0$ and u_i quasiconcave at x_O.

ii) the g_j's are quasi-concave at $\underline{x}^\circ$, and

iii) the h_k's are quasi-concave and quasi-convex at $\underline{x}^\circ$ (e.g., linear).

If there exist multipliers $\underline{\alpha} \geq 0$ in $\mathbb{R}^a$, $\underline{\lambda} \geq 0$ in $\mathbb{R}^M$, $\underline{\mu} \in \mathbb{R}^N$ such that $\alpha_i > 0$ for $i = 1,\dots,a$,

$$\lambda_j g_j(\underline{x}^\circ) = 0 \quad \text{for} \quad j = 1,\dots,M, \quad \text{and}$$

$$D\Big[\sum_1^a \alpha_i u_i + \sum_1^M \lambda_j g_j + \sum_1^N \mu_k h_k\Big](\underline{x}^\circ) = 0 \ ,$$

then $\underline{u}$ restricted to $C_{g,h}$ has a global Pareto optimum at $\underline{x}^\circ$.

Proof: The proof of Theorem 7.5 is similar to that of Theorem 7.4. By Theorem 7.1, we need only show that each u_i attains its maximum at $\underline{x}^\circ$ when the constraint set is $u^{(i)} - u^{(i)}(\underline{x}^\circ) \geq 0$, $g \geq 0, h = 0$. To demonstrate this, one applies Theorem 5.2.∎

In using Theorem 7.5, one should keep in mind the hierarchies of concavity as described in Theorem 1.9. One is tempted to try to generalize Theorem 7.5 to the case where the u_i's are quasi-concave. However, if $u_1(x_1,x_2) = x_1^3$ and $u_2(x_1,x_2) = x_2^3$, u_1 and u_2 are quasi-concave and

$$1 \cdot Du_1(0,0) + 1 \cdot Du_2(0,0) = \underline{0} \quad .$$

But $(0,0)$ is not a Pareto optimum for (u_1,u_2).

Nor can one generalize Theorem 7.5 to the case where some of the α_i's are zero. For, let $u_1(x_1,x_2) = x_1, u_2(x_1,x_2) = -x_1$, and $u_3(x_1,x_2) = x_2$. The u_i's are all linear and therefore concave. If one chooses multipliers $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = 0$, then the origin (in fact, any point) is a critical point of the corresponding Lagrangian. However, if $x_2' > x_2$, then (x_1,x_2') is superior to (x_1,x_2) for all x_1.

Thus, Theorem 7.5 is just about the strongest first order sufficient condition possible. It is a bit stronger than some similar results in the literature, e.g., Kuhn and Tucker (1951), Karlin (1959), Geoffrion (1968), and Smale (1976).

There are two other aspects of concave Pareto optimization that should be mentioned because of their important place in the past and present theory of microeconomics. The first involves the classical treatment of Pareto optimization problems (e.g., see section 8.C and Samuelson (1947)) whereby one tried to reduce such a problem to a single maximization problem by working with a weighted sum of the u_i's .

Theorem 7.6. Suppose that $u_1,\ldots,u_a,g_1,\ldots,g_M,h_1,\ldots,h_N: \mathbb{R}^n \to \mathbb{R}^1$ are C^1 functions and that $\underline{x}^\circ \in C_{g,h}$, i.e., $g(\underline{x}^\circ) \geq \underline{0}$, $h(\underline{x}^\circ) = \underline{0}$. Suppose that the u_i's are concave, the g_j's are quasi-concave, and the h_k's are linear. If $\underline{u}$ restricted to $C_{g,h}$ has a local Pareto optimum at $\underline{x}^\circ$, then there exist multipliers $\alpha_1,\ldots,\alpha_a \geq 0$, not all zero, such that $\underline{x}^\circ$ maximizes $\sum_1^a \alpha_i u_i$ (globally) on $C_{g,h}$. If, in addition, u,g, and h satisfy one of hypothese a) to g) of Theorem 7.3 at $\underline{x}^\circ$, then one can choose all the α_i's to be strictly positive.

Proof: Since $\underline{u}$ restricted to $C_{g,h}$ has a local Pareto optimum at $\underline{x}^\circ$, there exists non-zero $(\underline{\alpha},\underline{\lambda},\underline{\mu}) \in \mathbb{R}^a \times \mathbb{R}^M \times \mathbb{R}^N$ such that $\alpha_i \geq 0$ for all i , $\lambda_j \geq 0$ for all j , and

$$\text{(U)} \qquad D\Big[\sum_1^a \alpha_i u_i + \sum_1^M \lambda_j g_j + \sum_1^N \mu_k h_k\Big](\underline{x}^\circ) = \underline{0} \ .$$

But $\Sigma\, \alpha_i u_i$ is concave. Applying Theorem 5.2, one sees that $\sum_1^a \alpha_i u_i$ must have a global maximum in $C_{g,h}$ at x° .

If one of the hypotheses of Theorem 7.3 holds,then we can choose all the α_i's to be positive in (U) and therefore in the theorem. However, we still need to find a non-zero $\underline{\alpha}$ in the general case in order to give this theorem some content.

To do this, we must use the fact that disjoint convex sets can be separated by a hyperplane. See Chapter 3 in Mangasarian (1969) or Appendix B in Karlin (1959).

Suppose that for all non-zero choices of $(\underline{\alpha},\underline{\lambda},\underline{\mu})$ as above, $\underline{\alpha} = \underline{0}$. It follows that 1) $D(\Sigma \lambda_j g_j + \Sigma \mu_k h_k)(\underline{x}^\circ) = \underline{0}$ for all such $(\underline{\alpha},\underline{\lambda},\underline{\mu})$, 2) there is no non-zero $\underline{\alpha} \geq 0$ with $\sum_i \alpha_i Du_i(\underline{x}^\circ) = \underline{0}$, and 3) $Du_i(\underline{x}^\circ) \neq 0$ for all i. Let $U = \{\underline{x} \in \mathbb{R}^n | u_i(\underline{x}) > u_i(\underline{x}^\circ)$ for all $i\}$ and let $C_{g,h}$ denote the constraint set as usual. By Gordan's Lemma (see section 4.A), 2) implies that there is a non-zero vector $\underline{v} \in T_{x_0}\mathbb{R}^n$ with $Du_i(\underline{x}^\circ)v > 0$ for all i. Thus, U is non-empty and $\underline{x}^\circ$ is in its closure. Also, since $\underline{x}^\circ$ is a Pareto optimum, $\underline{x}^\circ + t\underline{v} \notin C_{g,h}$ for all $t > 0$ and $C_{g,h}$ does not contain an open neighborhood of $\underline{x}^\circ$.

Since $\underline{u}$ restricted to $C_{g,h}$ has a Pareto optimum at $\underline{x}^\circ$, U and $C_{g,h}$ are disjoint convex sets. By the above mentioned separation theorems, there exists a hyperplane H that separates U and $C_{g,h}$.

Suppose that $f: \mathbb{R}^n \to \mathbb{R}^1$ is a C^1 pseudoconcave function with $\nabla f(\underline{x}^\circ)$ perpendicular to H and lying in the half-space of U. We claim that f restricted to $C_{g,h}$ has a global maximum at $\underline{x}^\circ$. For, let $\underline{x}' \in C_{g,h}$. Since $Df(\underline{x}^\circ)(\underline{x}' - \underline{x}^\circ) = \nabla f(\underline{x}^\circ) \cdot (\underline{x}' - \underline{x}^\circ) \leq 0$, and since f is pseudoconcave, $f(\underline{x}') \leq f(\underline{x}^\circ)$ and our claim is verified.

If $\nabla u_i(\underline{x}^\circ)$ were normal to H for some i, then u_i restricted to $C_{g,h}$ would have a maximum at $\underline{x}^\circ$ and we would be done. Thus, we can assume that each $\nabla u_i(\underline{x}^\circ)$ is non-zero and is not perpendicular to H. Let $P: T_{x_0}\mathbb{R}^n \to H$ be the standard projection along the perpendicular to H . We know that $P(\nabla u_i(\underline{x}^\circ))$ is not zero for all i . If there were a non-zero vector $\underline{w} \in T_{\underline{x}^\circ}\mathbb{R}^n \cap H$ such that $P(\nabla u_i(\underline{x}^\circ)) \cdot \underline{w} > 0$ for all i , then $\nabla u_i(\underline{x}^\circ) \cdot \underline{w} > 0$, for all i a contradiction to the fact that U lies on one side of H . Gordan's Lemma now implies that there exists a non-zero $(\alpha_1,\ldots,\alpha_a)$ with

$$\alpha_i \geq 0 \quad \text{for all} \quad i \quad \text{and} \quad \sum_1^a \alpha_i P(\nabla u_i(\underline{x}^\circ)) = \underline{0} \quad .$$

The linearity of P gives $P(\sum_1^a \alpha_i \nabla u_i(\underline{x}^\circ)) = \underline{0}$, i.e., $\sum_1^a \alpha_i \nabla u_i(\underline{x}^\circ)$ is perpendicular to H . By the claim of the proceeding paragraph, $\sum_1^a \alpha_i u_i$ restricted to $C_{g,h}$ has a maximum at $\underline{x}^\circ$. ∎

Example 1. Smale (1975a) gives an example to show that Theorem 7.6 is not true if the u_i are not concave. Let

$$u_1(x,y) = y - x^2 + y^3 \ , \ u_2(x,y) = \frac{-y}{x^2 + 1} \quad .$$

Since $Du_1(0,0) = (0,1)$ and $Du_2(0,0) = (0,-1)$, $D[\lambda_1 u_1 + \lambda_2 u_2](0,0) = \underline{0}$ if and only if $\lambda_1 = \lambda_2 = \lambda$. Since $D^2[\lambda u_1 + \lambda u_2](0,0) =$

$$\begin{bmatrix} -2\lambda & 0 \\ 0 & 0 \end{bmatrix} \quad ,$$

which is negative definite on the kernel of $D(u_1,u_2)(0,0)$, Theorem 7.4 tells us that $(0,0)$ is a local Pareto optimum of (u_1,u_2) . (Keep in mind that λ must be positive.) However,

$$\lambda(u_1 + u_2)(x,y) = \lambda[x^2(\frac{y}{x^2+1}) + y^3]$$

is a strictly increasing function on the line $x = 0$, and certainly does not have a maximum at $(0,0)$.

Example2. The following simple example shows that one cannot always expect to find all positive α_i's in Theorem 7.6. Let $u_1(x_1,x_2) = -x_1^2 - x_2^2$ and $u_2(x_1,x_2) = x_1$. Since u_1 has a global maximum at $(0,0)$, (u_1,u_2) has a Pareto optimum at $(0,0)$. But,

$$\alpha_1 Du_1(0,0) + \alpha_2 Du_2(0,0) = (\alpha_2,0)$$

equals zero if and only if $\alpha_2 = 0$.

The converse to Theorem 7.6 is a classical result, whose simple proof we will leave to the reader. Note that no continuity or convexity assumptions are needed.

Theorem 7.7. Let $u_1,\ldots,u_a\colon \mathbb{R}^n \to \mathbb{R}$ be functions and let X be a subset of $\mathbb{R}^n$. If $\underline{x}° \in X$ and if there exist $\alpha_1,\ldots,\alpha_a$ all strictly positive such that $\Sigma\, \alpha_i u_i$ restricted to X has a local (global) maximum at $\underline{x}°$, then $(u_1,\ldots,u_a)$ restricted to X has a local (global) Pareto optimum at $\underline{x}°$. If $\underline{x}° \in X$ and if there exist a non-zero $(\alpha_1,\ldots,\alpha_a)$, with $\alpha_i \geq 0$ for all i, such that $\sum_1^a \alpha_i u_i$ restricted to X has a strict (local) maximum at $\underline{x}°$, then $(u_1,\ldots,u_a)$ has a strict (local) Pareto optimum at $\underline{x}°$.

7.E Saddle Point Formulations

The other important approach to concave Pareto optimization is the saddle point formulation. In trying to optimize $(u_1,\ldots,u_a)$: $\mathbb{R}^n \to \mathbb{R}^a$ over the constraint set $C_g \equiv \{\underline{x} \in \mathbb{R}^n | g_i(\underline{x}) \geq 0, i = 1,\ldots,$ economists often go right to the Lagrangian $L: \mathbb{R}^n \times \mathbb{R}^a \times \mathbb{R}^M \to \mathbb{R}$; where

$$L(\underline{x},\underline{\alpha},\underline{\lambda}) = \sum_1^a \alpha_i u_i + \sum_1^M \lambda_j g_j \quad .$$

A saddle point for L is an $(\underline{x}°,\underline{\alpha}°,\underline{\lambda}°)$ such that $\underline{\alpha}° \geq \underline{0}$, $\underline{\alpha}° \neq \underline{0}$, $\underline{\lambda}° \geq \underline{0}$, and for all $\underline{x}$ and all $\underline{\lambda} \geq \underline{0}$

$$L(\underline{x},\underline{\alpha}°,\underline{\lambda}°) \leq L(\underline{x}°,\underline{\alpha}°,\underline{\lambda}°) \leq L(\underline{x}°,\underline{\alpha}°,\lambda) \quad .$$

If $(\underline{x}°,\underline{\alpha}°,\underline{\lambda}°)$ is a saddle point with $\alpha_i° > 0$ for $i = 1,\ldots,a$, then it is called a strong saddle point.

The following theorem summarizes the relationship between strong saddle points and Pareto optima.

Theorem 7.8. Let $u_1,\ldots,u_a,g_1,\ldots,g_M: \mathbb{R}^n \to \mathbb{R}^1$ be C^1 functions. Let $L: \mathbb{R}^n \times \mathbb{R}^a \times \mathbb{R}^M$ be

$$L(\underline{x},\underline{\alpha},\underline{\lambda}) = \sum_1^a \alpha_i u_i(\underline{x}) + \sum_1^M \lambda_j g_j(\underline{x}) \quad .$$

A) If $(\underline{x}°,\underline{\alpha}°,\underline{\lambda}°)$ is a strong saddle point for L , then u restricte to C_g has a Pareto optimum at $\underline{x}°$.

B) If the u_i's and g_j's are concave, if $\underline{x}° \in C_g$, and if any of hypotheses a) to g) of Theorem 7.3 hold, then u restricted to C_g has a Pareto optimum at $\underline{x}^\circ$ if and only if there is an $(\underline{\alpha}^\circ, \underline{\lambda}^\circ) \in \mathbb{R}^a \times \mathbb{R}^M$ such that $(\underline{x}^\circ, \underline{\alpha}^\circ, \underline{\lambda}^\circ)$ is a strong saddle point for L.

Proof: By Theorem 5.4, hypothesis A) implies that $\underline{x}^\circ \in C_g$ and that $\underline{x}^\circ$ maximizes $\sum_1^a \alpha_i^\circ u_i$ on C_g. By Theorem 7.7, u restricted to C_g has a Pareto optimum at $\underline{x}^\circ$. To prove B), one combines Theorem 7.6 and Theorem 5.5.∎

Of course, one would like to replace the phrase "strong saddle point" by the phrase "saddle point" in Theorem 7.8. It is easy to see that this is impossible for part A). However, following Bergstrom (notes), one can make the following modification to part B).

Theorem 7.9. Let $u_1, \ldots, u_a, g_1, \ldots, g_M : \mathbb{R}^n \to \mathbb{R}^1$ be C^1 concave functions. Let $L(\underline{x}, \underline{\alpha}, \underline{\lambda}) = \underline{\alpha} \cdot \underline{u} + \underline{\lambda} \cdot \underline{g}$ be the corresponding Lagrangian. Suppose that $\underline{x}^\circ \in C_g$, that there is an $\underline{x}^*$ with $g(x^*) \geq \underline{0}$ and that there is a conflict of goals at $\underline{x}^\circ$, i.e., for each proper subset K of $\{1, \ldots, a\}$, there is an $\underline{x} \in C_g$ such that $u_i(\underline{x}) \geq u_i(\underline{x}^\circ)$ for all $i \in K$ and $u_j(\underline{x}) > u_j(\underline{x}^\circ)$ for some $j \in K$. Then, u restricted to C_g has a Pareto optimum at $\underline{x}^\circ$ if and only if there is an $(\underline{\alpha}^\circ, \underline{\lambda}^\circ) \in \mathbb{R}^a \times \mathbb{R}^M$ such that $(\underline{x}^\circ, \underline{\alpha}^\circ, \underline{\lambda}^\circ)$ is a saddle point for L.

Proof: One shows that when there is a conflict of goals, a saddle point is a strong saddle point. Suppose $(\underline{x}^\circ, \underline{\alpha}^\circ, \underline{\lambda}^\circ)$ is a saddle point, but not a strong one. Let $K = \{i \mid \alpha_i^\circ > 0\}$, a proper subset of $\{1,\ldots,a\}$. But there is an $\underline{x}' \in C_g$ such that $u_i(\underline{x}') \geq u_i(\underline{x}^\circ)$ for all $i \in K$ and $u_j(\underline{x}') > u_j(\underline{x}^\circ)$ for some $j \in K$. Then, $L(\underline{x}', \underline{\alpha}^\circ, \underline{\lambda}^\circ) > L(\underline{x}^\circ, \underline{\alpha}^\circ, \underline{\lambda}^\circ)$, a contradiction which implies that $(\underline{x}^\circ, \underline{\alpha}^\circ, \underline{\lambda}^\circ)$ is a strong saddle point.

Part A) of Theorem 7.8 now yields half of Theorem 7.9. To prove the other half, suppose that $\underline{u}$ restricted to C_g has a Pareto optimum at $\underline{x}^\circ$. By Theorem 7.6, there is a non-zero $\underline{\alpha}^\circ \geq \underline{0}$ such that $\underline{x}^\circ$ maximizes $\sum_1^a \alpha_i^\circ u_i$ on C_g. By Theorem 5.5, there is a $\underline{\lambda}^\circ \geq 0$ such that $(\underline{x}^\circ, \underline{\alpha}^\circ, \underline{\lambda}^\circ)$ is a saddle point, and therefore a strong saddle point, for L.

The basic references for saddle points in concave vector maximization problems are Kuhn-Tucker (1951) in the finite-dimensional case and Hurwicz (1958) in the infinite-dimensional case.

7.F Pareto Optima Via Differential Topology

The field of differential topology has made important contributions to the qualitative, global study of critical points and maxima of scalar-valued functions under non-degenerate constraints, i.e., on manifolds. For example, see Milnor (1963). Smale (1973) in a series of papers entitled Global Analysis and Economics applied the techniques of differential topology and of singularity theory (e.g., Golubitsky and Guillemin (1973)) to the study of vector maxima. Corresponding to the usual critical set of a real valued function, Smale (1973, 1974b) defined the critical Pareto set Θ to be the set of feasible points which satisfy the first order necessary conditions for optimality, i.e.,

$\Theta = \{\underline{x} \in C_{g,h} |$ there exists non-zero $(\underline{\alpha}, \underline{\lambda}, \underline{\mu}) \in \mathbb{R}^a \times \mathbb{R}^M \times \mathbb{R}^N$ such that $\underline{\alpha} \geq \underline{0}$, $\underline{\lambda} \geq \underline{0}, \underline{\lambda} \cdot g(\underline{x}) = 0$ and

$$\sum_1^a \alpha_i Du_i(\underline{x}) + \sum_1^M \lambda_j Dg_j(\underline{x}) + \sum_1^N \mu_k Dh_k(\underline{k}) = \underline{0}\} .$$

Working with constraint sets which are compact manifolds, i.e., bounded sets described by non-degenerate equality constraints in the sense of chapter 3, Smale (1973) argued that, for an open dense subset of the set $C(M, \mathbb{R}^a)$ of all smooth mappings from an m-dimensional manifold M into $\mathbb{R}^a$, $\Theta - \partial\Theta$ is an (a-1)-dimensional manifold and $\partial\Theta$, the boundary of Θ, is a finite union of lower dimensional manifolds. (There are some dimensional requirements on the magnitude of m relative to a — requirements that are always met in the economic applications.) The proof of this result was completed and extended by deMelo (1975). Wan (1976) has shown that for most mappings the set of local Pareto optima sit in M in a similar way.

Let us see why it is natural for the set of Pareto optima of a mapping $u : \mathbb{R}^n \to \mathbb{R}^a$ to be an (a-1)-dimensional subset. We are assuming that our usual constraint space $C_{g,h}$ is (locally) $\mathbb{R}^n$. Suppose $\underline{x}^o$ is a non-degenerate Pareto optimum for u, i.e.,

$\sum_1^a \alpha_i^o Du_i(\underline{x}^o) = \underline{0}$ for some positive $\alpha_1^o, \ldots, \alpha_a^o$, $\sum_1^a \alpha_i^o D^2u_i(\underline{x}^o)$ is negative definite on the nullspace of $Du(\underline{x}^o)$, $Du_i(\underline{x}^o) \neq \underline{0}$ for all i , and the rank of $Du(\underline{x}^o)$ is $(a-1)$. Choose a neighborhood U of $\underline{x}^o$ in $\mathbb{R}^n$ and a neighborhood V of α^o in $\mathbb{R}^a_+$ such that for all $(\underline{x},\underline{\alpha}) \in U \times V$: i) rank $Du(x) \geq a-1$, ii) each $Du_i(\underline{x}) \neq \underline{0}$, and $\sum_1^a \alpha_i D^2 u_i(x)$ is negative definite on the nullspace of $Du(x)$. By Theorem 7.4, $\underline{x}' \in U$ will be a local Pareto optimum for u if and only if rank $Du(\underline{x}') = a-1$, i.e., if all the $a \times a$ minors of $Du(\underline{x}')$ have zero determinant. (It follows from i) and ii) that if $\underline{x}' \in U$ and rank $Du(\underline{x}') = a-1$, there exists positive $\alpha_1', \ldots, \alpha_a'$ near $\alpha_1^o, \ldots, \alpha_a^o$ so that $\sum \alpha_i' Du_i(\underline{x}') = \underline{0}$) . Since there are $(r - a + 1)$ independent $(a \times a)$ minors in $Du(\underline{x})$, $\underline{x}' \in V$ must be a zero of a system of $(r - a + 1)$ equations to be a local Pareto optimum. If these equations are independent at $\underline{x}^o$ (as they usually are), then the local Pareto set in V will have dimension $r - (r - a + 1) = a - 1$.

Under the classical monotonicity and strict concavity assumptions of welfare economics, the set of Pareto optima is homeomorphic to the standard (a-1)-dimensional simplex. See Arrow-Hahn (1971) or Smale (1976a). However, even with all these concavity assumptions, the set of optima need not be conves if $a>1$, as the example at the end of the next chapter shows. (See Figure 2). Of course, this set is affine in the linear vector maximization problem. See Koopmans (1951) and Charnes-Cooper (1961).

Simon and Titus (1975), also using tools of differential topology and working with non-degeneracy hypotheses that occur in economics problems, showed how to reduce a vector maximization problem to a <u>single</u> scalar maximization problem (in contrast to Theorem 7.1) where the functions involved are non-linear but are not concave so that Theorem 7.5 cannot be applied. The following theorem summarizes their results in this direction.

Theorem 7.10. Let $u_1,\ldots,u_a,h_1,\ldots,h_N\colon \mathbb{R}^n \to \mathbb{R}^1$ be C^1 functions with $h(\underline{x}^\circ) = \underline{0}$. Suppose that i) $Dh(\underline{x}^\circ)$ has maximal rank $N \leq a$, ii) for each i, $D(u_i,h)(\underline{x}^\circ)$ has maximal rank, and iii) rank $D(u,h)(\underline{x}^\circ) \geq N + a - 1$. Then the following are equivalent:

a) u restricted to $h^{-1}(\underline{0})$ has a local Pareto optimum at $\underline{x}^o$;

b) $\underline{x}^\circ \in \Theta$, the critical Pareto set; and for some $i \in \{1,\ldots,a\}$ $\underline{x}^\circ$ maximizes u_i on the constraint set

$$U^i_{\underline{x}^\circ} \equiv \{\underline{x} \in \mathbb{R}^n \mid u_j(\underline{x}) = u_j(\underline{x}^\circ), j \neq i \text{ , and } h(\underline{x}) = \underline{0}\} ,$$

an $(n+1-a-N)$-dimensional submanifold of $h^{-1}(\underline{0})$.

We omit the proof of Theorem 7.10 since it involves techniques of differential topology. In the hypotheses, condition i) implies that $h^{-1}(\underline{0})$ is a manifold around $\underline{x}^\circ$, condition ii) means that no $u_i|h^{-1}(\underline{0})$ has a critical point at $\underline{x}^\circ$, and condition iii) asserts that the corank of $Du(\underline{x}^\circ)$ on $h^{-1}(\underline{0})$ must be at most one. If $a = 2$, condition iii) holds for all $\underline{x}^\circ \in h^{-1}(\underline{0})$ for an open dense set of mappings from $\mathbb{R}^n$ to $\mathbb{R}^2$. (See Golubitsky-Guillemin (1973)). Saari and Simon (1977) have shown that, if one searches for Pareto optima using Theorem 7.10, one finds large open subsets of mappings u for which degenerate maxima of $u_i|U^i$ arise naturally. More specifically, when $a \geq 3$, there are open sets of mappings u:

$\mathbb{R}^n \to \mathbb{R}^a$ which have Pareto optima that do not pass the second order sufficiency test of Theorem 7.4. This contrasts with the situation for scalar maximization where most mappings from $\mathbb{R}^n$ to $\mathbb{R}^1$ have only non-degenerate critical points (see Golubitsky-Guillemin (1973)) and with the situation for $a = 2$ where, for most mappings from $\mathbb{R}^n$ to $\mathbb{R}^2$, all the Pareto optima fulfill the second order condition of Theorem 7.4 (see Wan (1975a) and Saari-Simon (1977)).

O.Lange (1942) carried out one of the earliest systematic studies of Pareto optima in economics using techniques of calculus. He defined a "maximum of total welfare" of a utility mapping $u: \mathbb{R}^n \to \mathbb{R}^a$ as an $\underline{x}° \in \mathbb{R}^n$ that maximizes each u_i subject to the (a-1) equality constraint $u_j = u_j(\underline{x}°)$ for $j \neq i$. By taking all the u_j's to be equal, it is apparent that, in general, Lange' notion is different from that of the vector maxima in this chapter. However, in the next section, we will use Theorem 7.10 to show that Lange's notion is equivalent to the usual one in an economic setting with the classical monotonicity and concavity assumptions.

§8. VECTOR MAXIMIZATION IN ECONOMICS

8.A Pareto Optima in Welfare Economics

In section 6.A, we formalized the theory behind a consumer's desire to select a most preferred commodity vector from the set of all feasible and affordable commodity vectors. We now examine the situation where there are A consumers in an economy with n goods, $1 \leq A$, $n < \infty$. Assume that the k^{th} consumer has a smooth utility function $u^k : C \to \mathbb{R}$ and an initial commodity vector $\underline{z}^k \in C \equiv \{\underline{x} \in \mathbb{R}^n \mid \underline{x} \geq \underline{0}\}$. There is still a fixed positive price vector $\underline{p} \in \mathbb{R}^n_+$; and the initial wealth of the k^{th} consumer is $w^k = \underline{p} \cdot \underline{z}^k$. (Note that superscripts are being used to index consumers, while subscripts are used to label commodities.)

Let $\xi^k(\underline{p}, \underline{z}^k)$ denote the k^{th} consumer's demand correspondence, i.e., the solution set for the problem of maximizing $u^k(\underline{x})$ subject to $\underline{0} \leq \underline{x}$ in $\mathbb{R}^n$ and $\underline{p} \cdot \underline{x} \leq \underline{p} \cdot \underline{z}^k$. For simplicity of notation, we will assume that each ξ^k is a single-valued function. However, much of the theory of this section holds for demand correspondences as well as for demand functions, provided the reader substitutes set inclusion for equality in the relevant equations below.

Assume now that we are dealing with a closed economy in that the total amount of each commodity remains fixed during the consumer's interactions. Thus, if $\underline{b} = \sum_1^A \underline{z}^k$, our <u>state space</u> is

$$\Omega \equiv \{\underline{X} = (\underline{x}^1, \ldots, \underline{x}^A) \in (\mathbb{R}^n_+)^A \mid \underline{x}^k \geq 0 \text{ for each } k \text{ and } \sum_1^A \underline{x}^k = \underline{b}\},$$

an $(nA - n)$-dimensional affine subspace of $(\mathbb{R}^n)^A$. We will call an element of Ω a commodity bundle. The A utility functions can be considered as functions on Ω by writing

$$U^k(\underline{x}^1, \ldots, \underline{x}^A) = u^k(\underline{x}^k) \ , \quad \text{for} \quad k = 1, \ldots, A \ .$$

Finally, these A utility functions can be combined to form the utility mapping

$$U = (U^1, \ldots, U^A) : \Omega \to \mathbb{R}^A \ .$$

In this simple setting, an economy is an initial commodity bundle $(\underline{z}^1, \ldots, \underline{z}^A)$, a utility mapping U, and a price system $\underline{p}$.

There are a couple of natural ways of expressing an optimum or equilibrium in such an economy. There is, of course, the notion of a Pareto optimum (PO) or Pareto-optimal bundle $\underline{X}$ in Ω for the utility mapping U, i.e., $\underline{X}$ is a PO for U if there is no $\underline{Y} \in \Omega$ such that $U(\underline{Y}) \geq U(\underline{X})$ and $U(\underline{Y}) \neq U(\underline{X})$ in $\mathbb{R}^A$. There is the similar concept of a local Pareto optimum (LPO).

Our first goal is to use the theorems of chapter 7 to write necessary conditions and sufficient conditions for a commodity bundle to be an LPO. We would also like to know whether or not we can find strictly positive Lagrange multipliers and whether we can use Theorem 7.10 to find LPO's. Theorem 8.1 below collects the necessary conditions for an LPO, while Theorem 8.2 deals with the sufficient conditions.

<u>Theorem 8.1</u> Let $\underline{b}$ be a positive vector in $\mathbb{R}^n$ and let

$\Omega \equiv \{\underline{X} = (\underline{x}^1, \ldots, \underline{x}^A) \in (\mathbb{R}^n)^A \mid \text{each } \underline{x}^k \geq \underline{0} \text{ and } \sum_1^A \underline{x}^k = \underline{b}\}$.

Let $u^1, \ldots, u^A : \mathbb{R}^n_+ \to \mathbb{R}$ be C^1 utility functions. Suppose that for each $k \in \{1, \ldots, A\}$ and for each $\underline{x}$ with $\underline{0} \leq \underline{x} \leq \underline{b}$, $\frac{\partial u^k}{\partial x_i}(\underline{x}) > 0$ for some i, and that whenever $x_j = 0$, $\frac{\partial u^k}{\partial x_j}(\underline{x}) > 0$.

a) Suppose that $Y = (\underline{y}^1, \ldots, \underline{y}^A) \in \Omega$ is an LPO for $U : \Omega \to \mathbb{R}^A$. Then, there exist non-negative multipliers $\alpha^1, \ldots, \alpha^A$, not all zero, and a non-zero vector $\underline{\gamma} \in \mathbb{R}^n$ such that

$$\text{(V)} \quad \begin{aligned} &\alpha^k \nabla u^k(\underline{y}^k) \leq \underline{\gamma}, \text{ for } k = 1, \ldots, A \\ &\text{with } \alpha^k \frac{\partial u^k}{\partial x_j}(\underline{y}^k) = \gamma_j, \text{ whenever } y_j^k \neq 0. \end{aligned}$$

b) Let $\underline{Y}$ be as in a) with the added hypothesis that no (vector) component $\underline{y}^k$ of $\underline{Y}$ is zero. Suppose $\frac{\partial u^k}{\partial x_j}(\underline{x}) > 0$ for all k and j and for all $\underline{x}$ with $\underline{0} \leq \underline{x} \leq \underline{b}$. Then, there exist $\alpha^1, \ldots, \alpha^A, \gamma_1, \ldots, \gamma_n$, <u>all positive</u>, such that (V) holds.

c) If $\underline{Y}$ is an LPO in the interior of Ω, then there are <u>positive</u> multipliers $\alpha^1, \ldots, \alpha^A$ and a non-zero vector $\underline{\gamma}$ in $\mathbb{R}$ such that

i) $\alpha^k \nabla u^k(\underline{y}^k) = \underline{\gamma}$, for $k = 1, \ldots, A$;

ii) $\sum_{1}^{A} \alpha^k DU^k(\underline{Y})\underline{V} = \underline{0}$, for all $\underline{V} = (\underline{v}^1, \ldots, \underline{v}^A) \in (\mathbb{R}^n)^A$

such that $\sum_{1}^{A} \underline{v}^k = \underline{0}$, i.e., $\underline{V} \in T_{\underline{Y}}\Omega$;

iii) at $\underline{Y}$, the marginal rate of substitution of good i for good j is the same for all consumers, i.e., if

$$\frac{\partial u^1}{\partial x_j}(\underline{y}^1) \neq 0 , \quad \frac{\partial u^k}{\partial x_i}(\underline{y}^k) \Big/ \frac{\partial u^k}{\partial x_j}(\underline{y}^k) = \partial u^m / \partial x_i (\underline{y}^m) \Big/ \frac{\partial u^m}{\partial x_j}(\underline{y}^m)$$

for all $k, m \in \{1, \ldots, A\}$ and all $i, j \in \{1, \ldots, n\}$;

iv) if α^1 is set equal to 1 , the other α^k 's are uniquely determined.

Proof: This theorem is a reasonably straightforward application of the results of chapter seven to our economic model. One simple method of handling $U : \Omega \to \mathbb{R}^A$ is to remove the equality constraint that defines Ω by letting $(\underline{x}^1, \ldots, \underline{x}^{A-1})$ be independent coordinates for Ω with $\underline{x}^A = \underline{b} - \sum_{1}^{A-1} \underline{x}^k$. This is the approach taken by Simon-Titus (1975). We will use an approach more in line with the techniques of earlier chapters of this paper. See Smale (1974b, 1976) for a similar approach.

The Lagrangian for this optimization problem is

$$L(\underline{x}^1, \ldots, \underline{x}^A, \alpha^1, \ldots, \alpha^A, \underline{\mu}^1, \ldots, \underline{\mu}^A, \underline{\gamma}) =$$

$$\sum_1^A \alpha^k U^k(\underline{X}) + \sum_1^A \underline{\mu}^k \cdot \underline{x}^k + \underline{\gamma} \cdot (\underline{b} - \sum_1^A \underline{x}^k) \quad .$$

Setting the derivatives of L with respect to $\underline{x}^k$ equal to zero and evaluating at $\underline{Y}$ yields

$$\text{(W)} \qquad \alpha^k \nabla u^k(\underline{y}^k) + \underline{\mu}^k - \underline{\gamma} = \underline{0} \text{ in } \mathbb{R}^n \ .$$

If $\underline{Y}$ is an LPO, Theorem 7.2 states that there exists a non-zero $(\alpha^1, \ldots, \alpha^A, \underline{\mu}^1, \ldots, \underline{\mu}^A, \underline{\gamma})$ that solves (W) for $k = 1, \ldots, A$, where each α^k and μ_i^k is non-negative and each $\mu_i^k y_i^k$ is 0. Now, (V) follows from (W) since each $\underline{\mu}^k \geq \underline{0}$.

Suppose every α^k is zero. For any $i \in \{1, \ldots, n\}$, there is a $j \in \{1, \ldots, A\}$ such that $y_i^j \neq 0$ and, consequently,

$$0 = \alpha^j \frac{\partial u^j}{\partial x_i}(\underline{x}^j) = \gamma_i \ .$$

So, $\alpha^1 = \ldots = \alpha^A = 0$ implies $\gamma_1 = \ldots = \gamma_n = 0$, which in turn implies that each $\underline{\mu}^k$ is $\underline{0}$ in (W). This contradiction to the fact that $(\underline{\alpha}, \underline{\mu}^1, \ldots, \underline{\mu}^A, \underline{\gamma}) \neq \underline{0}$ shows that some α^k is positive.

To prove b), let $\underline{Y}$ be an LPO with each $\underline{y}^k \neq \underline{0}$. By part a), some α^j is non-zero, say α^1. Since

$$\alpha^1 \frac{\partial u^1}{\partial x_j}(\underline{y}^1) \leq \gamma_j \quad \text{and} \quad \frac{\partial u^1}{\partial x_j}(\underline{y}^1) > 0 \quad \text{for all } j \ ,$$

$\gamma_j > 0$ for $j = 1, \ldots, n$. Let $k \in \{1, \ldots, A\}$. By hypothesis, some $y_i^k \neq 0$; and therefore $\alpha^k \frac{\partial u^k}{\partial x_i}(\underline{y}^k) = \gamma_i$. Since γ_i and $\frac{\partial u^k}{\partial x_i}(\underline{y}^k)$ are positive, so is α^k .

To prove c), note that i) follows from (W) since each $\underline{\mu}^k$ is $\underline{0}$ for an interior LPO. If any α^k were 0 , then $\underline{\gamma}$ would be $\underline{0}$. Since no $\nabla u^k(\underline{y}^k)$ is $\underline{0}$, $\underline{\gamma} = 0$ implies that each $\alpha^k = 0$ — a contradiction.

Suppose $\frac{\partial u^1}{\partial x_j}(\underline{y}^1)$ is positive. If one sets $\alpha^1 = 1$, then $\underline{\gamma} = \nabla u^1(\underline{y}^1)$ and $\alpha^k = \frac{\partial u^k}{\partial x_j}(\underline{y}^k) \Big/ \frac{\partial u^1}{\partial x_j}(\underline{y}^k)$ for any j , i.e., each α^k is uniquely determined. If $\frac{\partial u^1}{\partial x_i}(\underline{y}^1) \neq 0$ also,

$$\frac{\alpha^k}{\alpha^m} = \frac{\partial u^k}{\partial x_j}(\underline{y}^k) \Big/ \frac{\partial u^m}{\partial x_j}(\underline{y}^m) = \frac{\partial u^k}{\partial x_i}(\underline{y}^k) \Big/ \frac{\partial u^m}{\partial x_i}(\underline{y}^m).$$

; and part iii) in c) follows.

To prove ii), let $\underline{V} = (\underline{v}^1, \ldots, \underline{v}^A) \in (\mathbb{R}^n)^A$ with $\sum_1^A \underline{v}^k = \underline{0}$.

Then,

$$\sum_1^A \alpha^k DU^k(Y)\underline{V} = \sum_1^A \alpha^k Du^k(\underline{y}^k)\underline{v}^k$$

$$= \sum_1^A \underline{\gamma} \cdot \underline{v}^k = \underline{\gamma} \cdot \sum_1^A \underline{v}^k$$

$$= \underline{0} \text{ . } \blacksquare$$

REMARK: The hypotheses that for each feasible $\underline{x}$ some $\frac{\partial u^k}{\partial x_j}(\underline{x}) > 0$ and that $\frac{\partial u^k}{\partial x_j}(\underline{x}) > 0$ whenever $x_j^k = 0$ are basic economics assumptions. They state that each consumer would always like to consume more of some commodity and that he would like to have at least a little of each commodity. Without some such mild desirability assumptions on the commodities in our economy, interaction among the consumers might not take place.

Theorem 8.2 (Sufficient Conditions). Let $\underline{b}$, Ω, and $U : \Omega \to \mathbb{R}^A$ be as in Theorem 8.1. Assume again for each $\underline{X} \in \Omega$ and for each $k \in \{1, \ldots, A\}$ that some $\frac{\partial u^k}{\partial x_j}(\underline{x}^k) \neq 0$.

a) If $\underline{Y}$ is in the interior of Ω, then $\underline{Y}$ is an LPO if and only if i) there exists positive $\alpha^1, \ldots, \alpha^A$ and non-zero $\underline{\gamma}$ such that $\alpha^k \nabla u^k(\underline{y}^k) = \underline{\gamma}$ for $k = 1, \ldots, A$, and

ii) $\underline{Y}$ maximizes some U^k on the submanifold $\{\underline{X} \in \Omega \mid U^j(\underline{X}) = U^j(\underline{Y}), j = 1, \ldots, A, j \neq k\}$.

b) Let $\underline{Y} \in \Omega$, and suppose $u^1, \ldots, u^A$ are quasi-concave. If there exist positive $\alpha^1, \ldots, \alpha^A$ such that (V) holds (or if $\underline{Y}$ is in the interior of Ω, such that i), ii), or iii) of Theorem 8.1.c holds), then $\underline{Y}$ is a PO for U.

c) Let $Y \in \Omega$. If there exist non-negative $\alpha^1, \ldots, \alpha^A$ such that (V) holds and such that

$$\sum_1^A \alpha^k D^2 U^k(Y)(\underline{V}, \underline{V}) = \sum_1^A \alpha^k D^2 u^k(\underline{y}^k)(\underline{v}^k, \underline{v}^k) < 0$$

for all non-zero $V = (\underline{v}^1, \ldots, \underline{v}^A) \in (\mathbb{R}^n)^A$ such that $\sum_1^A \underline{v}^k = \underline{0}$ and $Du^k(\underline{y}^k)\underline{v}^k = \underline{0}$, $k = 1, \ldots, A$, then $\underline{Y}$ is a strict LPO for U.

Proof: Part a) follows directly from Theorem 7.10, part b) from Theorem 7.5, and part c) from Theorem 7.4. One computes easily that the hypotheses of Theorem 7.10 are satisfied with $h(\underline{X}) = \underline{b} - \sum_1^A \underline{x}^k$ and

$$D(U, h)(\underline{X}) = \begin{pmatrix} Du^1(\underline{x}^1) & 0 & \cdots & 0 \\ 0 & Du^2(\underline{x}^2) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & Du^A(\underline{x}^A) \\ -I_n & -I_n & & -I_n \end{pmatrix}.$$

8.B PARETO OPTIMA AND PRICE EQUILIBRIA

The notion of a Pareto optimum, while natural in our economic model, ignores the economy's price system and the consumer's demand functions. A natural notion of equilibrium which includes these is the competitive equilibrium. Let $\underline{Y}$ be an initial commodity bundle in Ω, and let $\underline{p}$ be the prevailing price system. With these initial conditions, the k^{th} consumer will demand commodity vector $\xi^k(\underline{p}, \underline{y}^k)$. The total demand of the A consumers is the vector

$$\sum_{k=1}^{A} \xi^k(\underline{p}, \underline{y}^k) \in \mathbb{R}^n_+ ,$$

and the commodity bundle demanded is

$$\Xi(\underline{p}, \underline{Y}) = (\xi^1(\underline{p}, \underline{y}^1), \ldots, \xi^A(\underline{p}, \underline{y}^A)) \in (\mathbb{R}^n)^A .$$

If $\Xi(\underline{p}, \underline{Y})$ is in Ω, i.e., if the total demand vector equals the total supply vector,

$$\sum_1^A \xi^k(\underline{p}, \underline{y}^k) = \sum_1^A \underline{y}^k = \underline{b} ,$$

then we say that $\underline{p}$ is an <u>equilibrium price</u> for $\underline{Y}$ and that $(\underline{p}, \Xi(\underline{p}, \underline{Y}))$ is a <u>competitive equilibrium</u> (with respect to $\underline{Y}$). Often, one defines the <u>excess demand vector</u>

$$Z(\underline{p}, \underline{Y}) = \sum_1^A \xi^k(\underline{p}, \underline{y}^k) - \sum_1^A \underline{y}^k$$

and notes that $\underline{p}$ is an equilibrium price for $\underline{Y}$ if and only if $Z(\underline{p}, \underline{Y}) = \underline{0}$ in $\mathbb{R}^n$. By Theorem 6.1. iv,

$$\underline{p} \cdot Z(\underline{p}, \underline{Y}) = 0 \qquad \text{for all } \underline{Y} \in \Omega$$

if U satisfies the usual monotonicity assumption. The equation is usually known as <u>Walras' Law</u>.

If $(\underline{p}, \underline{\Xi})$ is a competitive equilibrium, then $\underline{\Xi}$ represents a solution to A independent maximization problems - but a solution with economic relevance since $\underline{\Xi} \in \Omega$. A natural question is : how does $\underline{\Xi}$ relate to the vector maximization problems we discussed in the previous section? Theorems 8.3 and 8.4 below, often called the <u>Fundamental Theorems of Welfare Economics</u>, answer this question by stating that a competitive equilibrium is always a Pareto optimum and that a Pareto optimum can always be realized as a competitive equilibrium for some price vector $\underline{p}$.

The latter statement solves a major dilemma. If $\underline{Y} \in \Omega$, there is usually a multi-dimensional set of LPO's which are Pareto-superior to $\underline{Y}$. The economist, who would like to have some natural way of choosing a meaningful LPO from this set, can proceed as follows. He first finds a price system $\underline{p}^*$ which is an equilibrium price for $\underline{Y}$. To prove the existence of such a $\underline{p}^*$ and also to compute it, economists use Walras' Law and the Brouwer or Kakutani fixed point theorem to find a zero of $\underline{p} \in Z(\underline{p}, \underline{Y})$. (See Debreu (1959), Dierker (1974), and Malinvaud (1972), for example, for proofs of the existence of $\underline{p}^*$. See Scarf (1973) and Smale (1976b) for methods of computing $\underline{p}^*$.) Theorem 8.3 then assures the economist that the corresponding competitive equilibrium $(\underline{p}^*, \underline{\Xi})$ with respect to $\underline{Y}$ lies in the set of PO's.

<u>Theorem 8.3</u> Let $\underline{b}$ be a positive vector in $\mathbb{R}^n$ and let

$\Omega = \{\underline{X} = (\underline{x}^1, \dots, \underline{x}^A) \in \mathbb{R}^{n^A} \mid \text{each } \underline{x}^k \geq \underline{0} \text{ and } \sum_1^A \underline{x}^k = \underline{b}\}$. Let

$u^1, \ldots, u^A : \mathbb{R}^n_+ \to \mathbb{R}$ be C^1 utility mappings such that

$$\frac{\partial u^k}{\partial x_j}(\underline{x}) > 0 \quad \text{for all } k, j \text{ and all } \underline{x} .$$

Let $\underline{p}$ be a positive price vector in $\mathbb{R}^n$ and let

$$\hat{Y} = (\xi^1(\underline{p}, \underline{y}^1), \ldots, \xi^A(\underline{p}, \underline{y}^A)) .$$

If $\hat{\underline{Y}} \in \Omega$, i.e. $(\underline{p}, \hat{\underline{Y}})$ is a competitive equilibrium for $\underline{Y}$, then $\hat{\underline{Y}}$ is a PO for U , that is Pareto-superior to $\underline{Y}$.

Proof: Suppose that $\hat{\underline{Y}}$ is not a PO for U , i.e., that there exists $Z \in \Omega$ and a non-empty subset S_1 of $\{1, \ldots, A\}$ such that

$$U^k(Z) > U^k(\hat{Y}) \quad \text{for } k \in S_1 ,$$

$$U^k(Z) = U^k(\hat{Y}) \quad \text{for } k \in S_2 \equiv \{1, \ldots, A\} - S_1 .$$

Since each $\hat{y}^k$ maximizes u^k on $\{\underline{x} \in \mathbb{R}^n_+ \mid \underline{p} \cdot \underline{x} \leq \underline{p} \cdot \underline{y}^k\}$, it follows that $u^k(\hat{y}^k) \geq u^k(y^k)$ for each k and $\underline{p} \cdot \underline{z}^k > \underline{p} \cdot \underline{y}^k$ for $k \in S_1$.

We claim that $\underline{p} \cdot \underline{z}^k \geq \underline{p} \cdot \underline{y}^k$ for all $k \in S_2$. For, if $\underline{p} \cdot \underline{z}^m < \underline{p} \cdot \underline{y}^m$ for some $m \in S_2$, then $\underline{z}^m$ also maximizes u^m on $\{\underline{x} \in \mathbb{R}^n_+ \mid \underline{p} \cdot \underline{x} \leq \underline{p} \cdot \underline{y}^m\}$. This contradicts Theorem 6.2.iv which states that $\underline{p} \cdot \underline{z}^m = \underline{p} \cdot \underline{y}^m$ for all $\underline{z}^m$ in $\xi^m(\underline{p}, \underline{y}^m)$.

Consequently, $\sum_1^A \underline{p} \cdot \underline{z}^k > \sum \underline{p} \cdot \underline{y}^k$ or

$$\underline{p} \cdot \sum_1^A \underline{z}^k > \underline{p} \cdot \sum_1^A \underline{y}^k ,$$

which contradicts $\sum_1^A \underline{y}^k = \sum_1^A \underline{z}^k = \underline{b}$. ∎

The above proof of Theorem 8.3 is adapted from Malinvaud (1972). Smale (1974b) proves this result by using Theorems 6.1 and 7.3 to show that when $(\underline{p}, \underline{Y})$ is a competitive equilibrium, $\underline{Y}$ is a critical Pareto point (as in section 7.F). Then, concavity assumptions are needed to show that $\underline{Y}$ is a PO. Theorem 8.4 states the converse of Theorem 8.3.

Theorem 8.4 Let $\underline{b}$, Ω, $\underline{p}$, and $u^1, \ldots, u^A$ be as in the hypothesis of Theorem 8.3. Suppose further that each u^k is quasi-concave and that $\underline{Y}$ is a PO for U . Then, there is a positive price vector $\underline{p}$ in $\mathbb{R}^n$ such that $(\underline{p}, \underline{Y})$ is a competitive equilibrium on Ω .

Proof: By Theorem 8.1.a, b, there is a positive vector $\underline{\gamma} \in \mathbb{R}^n$ and non-negative $\alpha^1, \ldots, \alpha^A$ with

$$\alpha^k > 0 \text{ , if } \underline{y}^k \neq \underline{0} \text{ , and}$$

$$\alpha^k \nabla u^k(\underline{y}^k) \leq \underline{\gamma} \text{ , for } k = 1, \ldots, A .$$

Let $\underline{p}$ be the positive vector $\underline{Y}$. For each k such that $y^k \neq 0$, we have

$$\nabla u^k(\underline{y}^k) \leq \frac{1}{\alpha^k}\underline{p} \quad \text{and}$$

$$\frac{\partial u^k}{\partial x_j}(\underline{y}^k) = \frac{1}{\alpha^k} p_j \,, \text{ if } \; y_j^k \neq 0 \,.$$

For such k, $\underline{y}^k$ maximizes the pseudoconcave function u^k on $\{\underline{x} \in \mathbb{R}^n_+ \mid \underline{p} \cdot \underline{x} \leq \underline{p} \cdot \underline{y}^k\}$ by Theorem 6.1. But this statement holds trivially for those k for with $\underline{y}^k = \underline{0}$ since in this case, the constraint set $\{\underline{x} \in \mathbb{R}^n_+ \mid \underline{p} \cdot \underline{x} \leq \underline{p} \cdot \underline{y}^k\}$ contains only the zero vector. Therefore, $(\underline{p}, \underline{Y})$ is a competitive equilibrium. ∎

The model we have been describing in chapter eight is a simple one since it does not include firms, production, shares, etc. However, it is a straightforward matter to bring all these concepts into our model and to define Pareto optimum and competitive equilibrium in this more general framework. One then proves the same fundamental theorems relating these two types of optima, using the same techniques but keeping track of a few more constraints and multipliers. See the excellent presentations in Debreu (1959), Karlin (1959), Intriligator (1971), Malinvaud (1972), and Smale (1976).

There is another setting where theorems comparing Pareto optimal situations with price equilibria are important - the activity analysis model introduced in section 6.C. In this case, an output vector $\underline{y} = (y_1, \ldots, y_m)$ is called efficient (instead of Pareto-optimal) if there is no feasible output vector $\underline{z}$ such that $\underline{z} \geq \underline{y}$ and $\underline{z} \neq \underline{y}$. The feasible output vectors are those which can be produced by some activity vector, i.e., $\{\underline{y} \in \mathbb{R}^m \mid \underline{y} = B\underline{x}$ for some $\underline{x} \in \mathbb{R}^n_+\}$. Using linear analysis similar to the marginal analysis of Theorems 8.3 and 8.4, one shows that the equilibrium outputs for the activity analysis problem of section 6.C are efficient and that every efficient, feasible output vector is an equilibrium solution for some price vector $\underline{p}$. For further readings in this area, see Koopmans (1951), Karlin (1959), and Charnes-Cooper (1961).

8.C SOCIAL WELFARE FUNCTIONS

As we discussed earlier, Theorem 8.3 provides an effective method for selecting an economically important element from the set of PO's that are Pareto superior to a given $\underline{Y} \in \Omega$. Another method that has classically been used for this selection process involves a social welfare or social utility function, i.e., a real valued function S on $\mathbb{R}^A$ (in an economy with A consumers) with the property that $S(\underline{a}^1) \geq S(\underline{a}^2)$ whenever $\underline{a}^1 \geq \underline{a}^2$. The function $\Sigma : \Omega \to \mathbb{R}$ defined by $\Sigma(\underline{X}) = S(U^1(\underline{X}), \ldots, U^A(\underline{X}))$ gives a complete ordering to the states in Ω in contrast to the partial ordering that $U : \Omega \to \mathbb{R}^A$ bestows on Ω. In principle, by maximizing Σ, one can now make a choice among the Pareto optimal bundles.

For example, one can give the k^{th} consumer a weight (or measure of importance) $c_k > 0$ and let

$$S(a_1, \ldots, a_A) = \sum_{1}^{A} c_k a_k .$$

By Theorem 7.7, a maximizer of Σ is a Pareto optimal element of Ω. By Theorem 7.6, one can find all the PO's this way by proper choice of $c_1, \ldots, c_A$ if the u^k's are concave - but not if the u^k's are not concave, as Example 2 after Theorem 7.6 shows.

Thus, social welfare functions were often used to reduce concave vector maximization problems to more comfortable scalar maximization problems. Because they attach importance to the actual values of the utility functions and judge among the various consumer's gains in utility, social welfare functions are used less enthusiastically than they were thirty years ago. For further readings on social welfare functions, see Samuelson (1947) Arrow (1951), and Malinvaud (1972).

8.D EFFICIENT PORTFOLIOS

We close with a different but very interesting application of the theory of vector maximization in economics - an investor's selection of an optimal portfolio of securities. This problem is discussed in detail in Markowitz (1952; 1959) and summarized in Karlin (1959).

Assume that an investor desires to select a portfolio of securities. If there are n different securities involved, let $x_1 \geq 0$

denote the percentage of the investor's assets that will be invested in security i. The state space is $S = \{\underline{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0$ for all i and $\sum_1^n x_i = 1\}$.

The investor's first task is to appraise the future performances of the n securities. If he computes that r_{it} is the anticipated return at time t for each dollar invested in security i and d_{it} is the rate of return on security i at time t discounted back to the present, then he computes the discounted return of one unit of security i as

$$R_i = \sum_t r_{it} d_{it} .$$

In this case, the discounted anticipated return from security vector $\underline{x} \in S$ is $R(\underline{x}) = \sum_1^n R_i x_i$.

If the investor decided to maximize R on S, he would clearly select only the security (or securities) for which R_i is maximal. However, such a choice goes against the axiom that a wise investor should diversify his holdings to take into consideration the inaccuracies in his expectations and the fluctuations in the various sectors of the market.

To get around this dilemma, let us regard the R_i's as normal random variables and suppose that the investor computes some fixed probability beliefs $\{\mu_1, \ldots, \mu_n, \sigma_{11}, \sigma_{12}, \ldots, \sigma_{nn}\}$ concerning the expected returns. Here, μ_i is the mean return on the i^{th} security and σ_{ij} is the covariance between R_i and R_j, i.e., the expected value of $(R_i - \mu_i)(R_j - \mu_j)$. Now, the mean return $E(\underline{x})$ for $\underline{x} \in S$ is

$$E(\underline{x}) = \sum_1^n \mu_i x_i ,$$

and its variance is $V(\underline{x}) = \sum_{i,j} \sigma_{ij} x_i x_j$. Since $E(\underline{x})$ is a measure of the "return" of security vector $\underline{x}$ and $V(\underline{x})$ is a measure of the "risk" involved in choosing $\underline{x}$, The investor will want to maximize both E and $-V$ on S. It makes sense to define an <u>efficient portfolio</u> vector $\underline{x}$ as a PO of $(E, -V) : S \to \mathbb{R}^2$.

If one assumes that V is positive definite, then there are a number of ways to compute efficient portfolios. Since E and $-V$ are both concave, the investor can give a positive weight a to E and another positive weight b to $-V$ and then maximize $aE - bV : S \to \mathbb{R}$. By Theorem 7.7, such maximizers in S will be efficient, and by Theorem 7.6 all efficient portfolios can be found this way. Alternatively, the investor can use Theorem 7.10 and maximize E on any level set of V or minimize V on the constant hyperplanes of E, provided that the gradient of E and the gradient of V point in the same direction at such solutions. By using this type of analysis, one can easily check that the set of efficient portfolio vectors is a (possibly broken) line segment on S which runs from the minimizer of $V|S$ to the boundary of S and which can be parameterized by values of E .

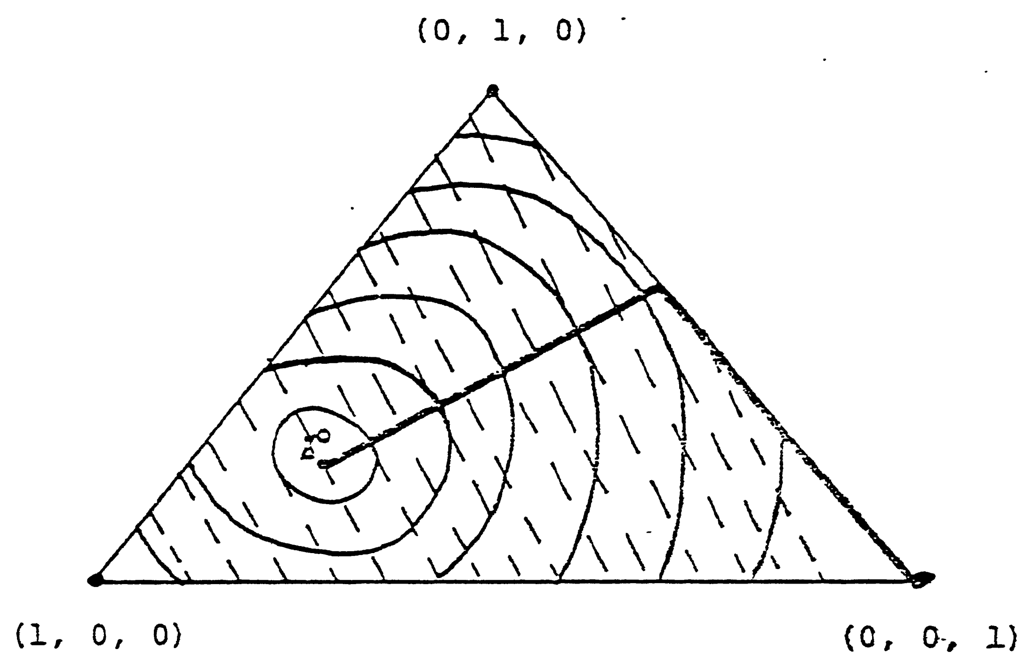

Figure 2

In figure 2, we have diagrammed an example for $n = 3$. The concentric ellipses are the level sets of $V|S$ and the dotted lines are the level sets of $E|S$. The vector $\underline{x}^o$ is a minimizer of $V|S$ and $(0, 0, 1)$ is a maximizer of $E|S$. The heavy solid line from $\underline{x}^o$ to $(0, 0, 1)$ is the set of efficient portfolios. For more details, we refer the reader to Markowitz (1952, 1959).

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