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Strategy-Proofness and Singleton Cores in Generalized Matching Problems

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Abstract

We introduce and study the class of generalized matching problems. Two subclasses of this class are marriage problems (Gale and Shapley 1962) and the housing market (Shapley and Scarf 1974). We search for strategy-proof solutions to generalized matching problems. We show that if the core is a singleton and is stable for all problems then it is strategy-proof as a solution. We also show that on the class of problems with a non-empty core there exists a Pareto efficient, individually rational, and strategy-proof solution only if the core is a singleton for all problems. Furthermore if such a solution exists, it is the core.

Journal of Economic Literature Classification Numbers: C71, C78, D71, D78

Key Words: Matching Problems, Marriage Problems, Housing Market, Strategy-Proofness, Implementation, Core, Stable Matchings.

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1 Introduction

When one suggests alternative solutions to a public decision problem we usually find some of them more reasonable than the others. But how do we compare different solutions? What makes some of them more reasonable than the others? One consideration may be the extent to which a solution suggests efficient outcomes. Another may be the extent to which it is immune to manipulation. Or whether it is stable... This list could easily be extended to take all aspects of a solution into account. But one thing will be common in all items: They are all desirable properties we want to impose on a solution.

In this paper we search for solutions to various classes of *matching problems*[1]. For each of these classes we refer to solutions as *matching rules.* We search for solutions which are *efficient*, and *individually rational* in the sense that no agent is ever worse off than he would be on his own. In addition to these minimal requirements we also would like agents not to be able to profitably misrepresent their preferences. This property is known as *strategy-proofness*[2].

An extensively analyzed solution in both cooperative game theory and economic theory is the *core (correspondence).* An allocation is in the *core of a problem* if no coalition can improve on it with its own "resources." In general, there is no reason for such an allocation to exist for all problems and hence the core may not be well defined. But whenever it is, it is also efficient and individually rational. What about strategy-proofness? Is the core as a solution strategy-proof? This question is not well defined since strategy-proofness is a property of single-valued solutions. Nevertheless, in this paper we will identify a class of matching problems for which there is a very clear relation between the core and strategy-proofness.

Two extensively analyzed classes of matching problems are the *marriage problems* (Gale and Shapley 1962) and the *housing market* (Shapley and Scarf 1974). The results concerning strategy-proofness in marriage problems are quite disappointing. Roth (1982a) shows that there is no selection from the core of marriage problems

[1]For an exposition of game theoretic modelling and analysis of such problems, see Roth and Sotomayor (1990).

[2]Strategy-proofness was first analyzed in abstract social choice models where there are few or no restrictions on preferences. Gibbard (1973) and Satterthwaite (1975) show that, under minor conditions strategy-proofness is equivalent to dictatorship. In models with more structure (such as economic models) some positive results are available. See for example Barberà, Gül and Stachetti (1993), Barberà and Jackson (1994), Ching (1993,94), Moulin (1980,94), Moulin and Shenker (1992), Sönmez (1994a), Sprumont (1991) for some positive results and Sprumont (1994), Thomson (1994) for surveys on recent results of strategy-proofness.

which is strategy-proof. In fact, Alcalde and Barberà (1994) show that there is no matching rule which is efficient, individually rational, and strategy-proof. On the other hand the results pertaining to the housing market are much more encouraging. Roth (1982b) shows that the core of the housing market, which is shown to be single-valued by Roth and Postlewaite (1977), is strategy-proof. Moreover Ma (1994) shows that it is the only matching rule which is efficient, individually rational, and strategy-proof. What causes such different results in apparently similar classes of problems?

In this paper we search for foundations of the differences in these results. We do this by first showing how these two models can be obtained as special cases of a general class of matching problems. Then we study strategy-proofness in this class. The results we obtain explain the reasons for such different results in two models. We show that for any model in this class, as long as the core is well defined there exists an efficient, individually rational, and strategy-proof matching rule only if the core is single-valued. In fact, if such a rule exists it is the core itself. Furthermore, as long as the core is single-valued it is strategy-proof under a minor condition. We obtain the positive results of Roth (1982b), Ma (1994) for the housing market, and the negative results of Roth (1982a), Alcalde and Barberà (1994) for marriage problems, as direct applications of our general results.

An implication of our results is that the core is the key concept when one searches for strategy-proof matching rules which are efficient and individually rational. If there is any hope of having such a rule, it is the core. We believe this conclusion provides important non-cooperative support for the core, a cooperative solution.

2 Model

A **(generalized) matching problem** is a triple $G = (N, S, R)$. The first component N is a finite set of agents. The second component $S = (S_i)_{i\in N}$ is a list of subsets of N with $i \in S_i$ for all $i \in N$. Here S_i represents the set of possible assignments for agent i. The last component $R = (R_i)_{i\in N}$ is a list of preference relations. Let P_i denote the strict relation associated with the preference relation R_i for all $i \in N$. We consider the case where N, S are fixed and hence to define a matching problem it suffices to specify a preference profile.

The preference relation R_i of each agent $i \in N$ is a linear order on S_i. Let $\mathcal{R}_i$ be the class of all such preference relations for agent i. Let $\mathcal{R} = \prod_{i\in N} \mathcal{R}_i$. For all $R \in \mathcal{R}, T \subset N$, we denote the restriction of R to T by R_T, and the set $N \setminus T$ by $-T$. For all $i \in N$, we denote the set $N \setminus \{i\}$ by $-i$.

A **(generalized) matching** μ is a function from the set N into itself such that

1. $\forall i \in N \quad \mu(i) \in S_i$,
2. $\forall i \in N \quad |\mu^{-1}(i)| = 1$.

Note that μ is a bijection on N. For all $i \in N$, we refer to $\mu(i)$ as the **assignment** of i at μ. We denote the set of all matchings by $\mathcal{M}$. Let $\mu_I \in \mathcal{M}$ be defined by $\mu_I(i) = i$ for all $i \in N$. We exogenously specify a subset $\mathcal{M}^f$ of the set of matchings $\mathcal{M}$ as the set of **feasible matchings**. We always require that $\mu_I \in \mathcal{M}^f$.

Given a preference relation R_i of an agent $i \in N$, initially defined over S_i, we extend it to the set of feasible matchings $\mathcal{M}^f$ in the following natural way: agent i prefers the matching μ to the matching μ' if and only if he prefers his assignment under μ to his assignment under μ'. We slightly abuse the notation and also use R_i to denote this extension.

A matching $\mu \in \mathcal{M}^f$ is **individually rational** under R if $\mu(i) R_i i$ for all $i \in N$. We denote the set of all individually rational matchings under R by $\mathcal{I}(R)$.

A matching $\mu \in \mathcal{M}^f$ is **Pareto efficient** under R if there is no other matching $\mu' \in \mathcal{M}^f$ such that $\mu'(i) R_i \mu(i)$ for all $i \in N$ and $\mu'(j) P_j \mu(j)$ for some $j \in N$. We denote the set of all Pareto efficient matchings under R by $\mathcal{P}(R)$.

A matching $\mu' \in \mathcal{M}^f$ **dominates** the matching $\mu \in \mathcal{M}^f$ via a coalition $T \subseteq N$ under R if

1. $\forall i \in T \quad \mu'(i) \in T$,
2. $\forall i \in T \quad \mu'(i) R_i \mu(i)$,
3. $\exists j \in T \quad \mu'(j) P_j \mu(j)$.

In that case we say the coalition T **blocks** μ under R. A matching $\mu \in \mathcal{M}^f$ is in the **core** of the matching problem $R \in \mathcal{R}$ if it is not dominated by any matching. We denote the core of R by $\mathcal{C}(R)$.

A set of matchings $\mathcal{N} \subseteq \mathcal{M}^f$ is **stable** under $R \in \mathcal{R}$ if every matching in $\mathcal{M}^f \setminus \mathcal{N}$ is dominated by some matching in $\mathcal{N}$.[3] Note that any stable set under R is a superset of the core of R.

[3]In this paper we deal with the stability of singleton sets and hence the notion of stability coincides with von-Neumann Morgenstern stability.

A **(generalized) matching rule** is a correspondence $\varphi : \mathcal{R} \to \mathcal{M}^f$. A matching rule φ is **Pareto efficient** if $\varphi(R) \subseteq \mathcal{P}(R)$ for all $R \in \mathcal{R}$, and **individually rational** if $\varphi(R) \subseteq \mathcal{I}(R)$ for all $R \in \mathcal{R}$. The next two properties concern single-valued rules.

A matching rule φ is **strategy-proof** if for all $R \in \mathcal{R}$, for all $i \in N$, and for all $R'_i \in \mathcal{R}_i$

$$\varphi_i(R) R_i \varphi_i(R_{-i}, R'_i)$$

and it is **coalitionally strategy-proof** if for all $R \in \mathcal{R}$, for all $T \subseteq N$, and for all $R'_T \in \mathcal{R}_T$ there exists $i \in T$ such that

$$\varphi_i(R) R_i \varphi_i(R_{-T}, R'_T)$$

3 Results

Let $\mathcal{C}$ be the matching rule which selects the set of matchings in the core for each preference profile.[4] We will refer to the matching rule $\mathcal{C}$ as the core. Recall that, $N, S, \mathcal{M}^f$ are fixed. The next proposition, which is a variant of a theorem in Demange (1987), concerns the coalitional strategy-proofness of $\mathcal{C}$ whenever it is single-valued and stable.

Proposition 1: Let $|\mathcal{C}(R)| = 1$ and $\mathcal{C}(R)$ be stable for all $R \in \mathcal{R}$. Then $\mathcal{C}$ is coalitionally strategy-proof.

Proof: Let $|\mathcal{C}(R)| = 1$ and $\mathcal{C}(R)$ be stable for all $R \in \mathcal{R}$. Suppose $\mathcal{C}$ is not coalitionally strategy-proof. Then, there exist $R \in \mathcal{R}, T \subseteq N, R'_T \in \mathcal{R}_T$ such that

$$\forall i \in T \qquad \mathcal{C}_i(R_{-T}, R'_T) P_i \mathcal{C}_i(R)$$

Hence $\mathcal{C}(R_{-T}, R'_T) \neq \mathcal{C}(R)$. Let $\mu = \mathcal{C}(R), \mu' = \mathcal{C}(R_{-T}, R'_T)$.[5]

We have $|\mathcal{C}(R)| = 1$ and $\mathcal{C}(R)$ is stable for all $R \in \mathcal{R}$. Therefore μ dominates μ' under R. That is, there exists a coalition $U \subseteq N$ such that

$$\begin{array}{ll} \forall i \in U & \mu(i) \in U \\ \forall i \in U & \mu(i) R_i \mu'(i) \\ \exists j \in U & \mu(j) P_j \mu'(j) \end{array}$$

Therefore $T \cap U = \emptyset$. But then μ dominates μ' under (R_{-T}, R'_T) contradicting $\mu' = \mathcal{C}(R_{-T}, R'_T)$. *Q.E.D.*

[4] Note that $\mathcal{C}$ is not necessarily well defined for all $N, S, \mathcal{M}^f$.

[5] Whenever φ is a single-valued matching rule we write $\varphi(R) = \mu$ instead of $\varphi(R) = \{\mu\}$.

Remark 1: Demange (1987) introduces a notion of coalitional nonmanipulability for correspondences and shows that the core is coalitionally nonmanipulable as long as it is non-empty and satisfies the following weaker notion of stability for all problems: A set of allocations is weakly stable if every allocation outside the set is blocked by a coalition all of whose members prefer an allocation in the set to it. Proposition 1 is still valid if stability is replaced by weak stability. Demange's theorem reduces to this stronger version of Proposition 1 whenever the core is single-valued.

Remark 2: A sufficient condition for the core being a singleton and stable for each problem is that all matchings be feasible; that is $\mathcal{M}^f = \mathcal{M}$. In that case Gale's "top trading cycles" method can be used to obtain the unique matching in the core which is stable. The proof of this is a straightforward modification of a proof in Roth and Postlewaite (1977).

The core being single-valued is a very strong assumption. However the next theorem states that as long as the core is well defined there exists a Pareto efficient, individually rational, and strategy-proof matching rule only if the core is single-valued. Furthermore if such a matching rule exists, it is the core. Before stating and proving this theorem, we have the following lemma.

Lemma 1: If there exists a matching rule $\varphi : \mathcal{R} \to \mathcal{M}^f$ which is Pareto efficient, individually rational, and strategy-proof, then $|\mathcal{C}(R)| \leq 1$ for all $R \in \mathcal{R}$ and $\varphi(R) = \mathcal{C}(R)$ for all $R \in \mathcal{R}$ with $\mathcal{C}(R) \neq \emptyset$.

Proof: Let $\varphi : \mathcal{R} \to \mathcal{M}^f$ be Pareto efficient, individually rational, and strategy-proof. Let $R \in \mathcal{R}, \mu \in \mathcal{C}(R)$. Let $R' \in \mathcal{R}$ be such that for all $i \in N$

1. $j P'_i k \iff j P_i k \quad$ for all $j, k \in S_i \setminus \{i\}$,
2. $\mu(i) R'_i i \quad$ and $\quad \nexists j \in S_i \setminus \{i\}$ with $\mu(i) R'_i j R'_i i$.

Note that $\mu \in \mathcal{C}(R')$.

Claim 1: $\varphi(R') = \mu$.

Proof of Claim 1: Let $\nu \in \mathcal{M}^f$ be such that $\nu \in \mathcal{I}(R')$. Let $i_1 \in N$. Let $|N| = n$. Let $i_{k+1} = \nu(i_k)$ for all $k \in \{1, 2, \ldots, n\}$.

Suppose

$$i_2 P'_{i_1} \mu(i_1) \tag{1}$$

We will show that

$$\nu(i_k) \notin \{i_1, i_2, \ldots, i_k\} \qquad \text{for all } k \in \{2, 3, \ldots, n\}$$

by induction on k. Let us first show $\nu(i_2) \notin \{i_1, i_2\}$. We have $i_2 \notin \{i_1, \mu(i_1)\}$ by relation (1) and the construction of R'_{i_1}. Therefore $\nu(i_2) \neq i_2$ ($i_2 \neq i_1, \nu(i_1) = i_2$, and $|\nu^{-1}(i_2)| = 1$).

We either have $\nu(i_2) = i_1$ or $\nu(i_2) \neq i_1$. If the former holds $\nu \in \mathcal{I}(R')$ implies $i_1 P'_{i_2} i_2$ and hence

$$i_1 R'_{i_2} \mu(i_2) \tag{2}$$

by the construction of R'_{i_2}. But then the coalition $\{i_1, i_2\}$ blocks μ under R' by relations (1) and (2) contradicting $\mu \in \mathcal{C}(R')$. Therefore $\nu(i_2) \notin \{i_1, i_2\}$.

Next suppose $\nu(i_k) \notin \{i_1, i_2, \ldots, i_k\}$ for all $k \in \{2, 3, \ldots, l\}$ with $2 \leq l < n$. Then we have

$$\nu(i_k) = i_{k+1} \neq i_k \qquad \text{for all } k \in \{2, 3, \ldots, l\}$$

Therefore

$$\nu(i_k) = i_{k+1} P'_{i_k} i_k \qquad \text{for all } k \in \{2, 3, \ldots, l\}$$

as $\nu(R') \in \mathcal{I}(R')$, and hence

$$\nu(i_k) = i_{k+1} R'_{i_k} \mu(i_k) \qquad \text{for all } k \in \{2, 3, \ldots, l\} \tag{3}$$

by construction.

We have $i_{l+1} = \nu(i_l) \notin \{i_1, i_2, \ldots, i_l\}$. But $\nu(i_k) = i_{k+1}$ for all $k \in \{1, 2, \ldots, l\}$ and ν is a bijection therefore $\nu(i_{l+1}) \notin \{i_2, \ldots, i_{l+1}\}$. We either have $\nu(i_{l+1}) = i_1$ or $\nu(i_{l+1}) \neq i_1$. If the former holds $\nu \in \mathcal{I}(R')$ implies $i_1 P'_{i_{l+1}} i_{l+1}$ and hence

$$i_1 R'_{i_{l+1}} \mu(i_{l+1}) \tag{4}$$

by the construction of $R'_{i_{l+1}}$. But then the coalition $\{i_1, i_2, \ldots, i_{l+1}\}$ blocks μ under R' by relations (1), (3), and (4) contradicting $\mu \in \mathcal{C}(R')$. Therefore $\nu(i_{l+1}) \notin \{i_1, i_2, \ldots, i_{l+1}\}$. Hence $\nu(i_n) \notin \{i_1, i_2, \ldots, i_n\}$ by induction.

But we have

$$\begin{aligned} i_2 &= \nu(i_1) \notin \{i_1\} \\ i_3 &= \nu(i_2) \notin \{i_1, i_2\} \\ &\vdots \\ i_n &= \nu(i_{n-1}) \notin \{i_1, i_2, \ldots, i_{n-1}\} \end{aligned}$$

Therefore $i_j \neq i_k$ for all $j, k \in N$ with $j \neq k$ which implies $\{i_1, i_2, \ldots, i_n\} = N$. Thus, $\nu(i_n) \notin N$ contradicting $\nu \in \mathcal{M}^f$. Hence $\mu(i_1) R'_{i_1} \nu(i_1) = i_2$.

That is

$$\forall i \in N, \nu \in \mathcal{I}(R') \qquad \mu(i) R'_i \nu(i)$$

and therefore

$$\mathcal{P}(R') \cap \mathcal{I}(R') = \{\mu\}$$

which implies

$$\varphi(R') = \mu$$

completing the proof of Claim 1.

Claim 2: $\varphi(R) = \mu$.

Proof of Claim 2: We will show that

$$\varphi(R'_{-T}, R_T) = \mu \qquad \text{for all } T \subseteq N$$

by induction on the cardinality of T.

Let $i \in N$. Consider the preference profile (R'_{-i}, R_i). By strategy-proofness and Claim 1 we have

$$\begin{aligned} \varphi_i(R') = \mu(i) R'_i \varphi_i(R'_{-i}, R_i), \\ \varphi_i(R'_{-i}, R_i) R_i \varphi_i(R') = \mu(i), \end{aligned}$$

therefore $\varphi_i(R'_{-i}, R_i) = \mu(i)$. But μ is the only matching which is Pareto efficient and individually rational under (R'_{-i}, R_i) such that $\varphi_i(R'_{-i}, R_i) = \mu(i)$, therefore

$$\forall i \in N \qquad \varphi(R'_{-i}, R_i) = \mu$$

Next suppose

$$\varphi(R'_{-T}, R_T) = \mu \qquad \text{for all } T \subset N \text{ with } |T| = l < n \tag{5}$$

We will show that $\varphi(R'_{-T}, R_T) = \mu$ for all $T \subseteq N$ with $|T| = l + 1 \leq n$.

Let $T \subseteq N$ be such that $|T| = l + 1$. Let $i \in T$. Consider the preference profile (R'_{-T}, R_T). By strategy-proofness and relation (5) we have

$$\begin{aligned} \varphi_i(R_{T\setminus\{i\}}, R'_{(N\setminus T)\cup\{i\}}) = \mu(i) R'_i \varphi_i(R'_{-T}, R_T), \\ \varphi_i(R'_{-T}, R_T) R_i \varphi_i(R_{T\setminus\{i\}}, R'_{(N\setminus T)\cup\{i\}}) = \mu(i), \end{aligned}$$

therefore $\varphi_i(R'_{-T}, R_T) = \mu(i)$. That is we have

$$\forall i \in T \qquad \varphi_i(R'_{-T}, R_T) = \mu(i) \tag{6}$$

But μ is the only matching which is Pareto efficient and individually rational under (R'_{-T}, R_T) such that relation (6) holds therefore

$$\varphi(R'_{-T}, R_T) = \mu \qquad \text{for all } T \subseteq N \text{ with } |T| = l+1 \leq n$$

Therefore $\varphi(R) = \mu$ by induction completing the proof of Claim 2.

Suppose we also have $\nu \in \mathcal{C}(R)$ with $\nu \neq \mu$. Then by similar arguments we have $\nu = \varphi(R)$ contradicting $\nu \neq \mu$. Therefore $|\mathcal{C}(R)| \leq 1$ for all $R \in \mathcal{R}$. Furthermore $\varphi(R) = \mathcal{C}(R)$ whenever $\mathcal{C}(R) \neq \emptyset$ by Claim 2. *Q.E.D.*

Theorem 1: Suppose $\mathcal{C}(R) \neq \emptyset$ for all $R \in \mathcal{R}$ and $\varphi : \mathcal{R} \to \mathcal{M}^I$ be a matching rule which is Pareto efficient, individually rational, and strategy-proof. Then

1. $|\mathcal{C}(R)| = 1 \quad$ for all $R \in \mathcal{R}$,
2. $\varphi = \mathcal{C}$.

Proof: Follows from Lemma 1.

Theorem 1 deals with domains where the core is well defined. The next proposition deals with domains where the core is not necessarily well defined.

Proposition 2: Suppose $|\mathcal{C}(R)| > 1$ for some $R \in \mathcal{R}$. Then there is no matching rule which is Pareto efficient, individually rational, and strategy-proof.

Proof: Follows from Lemma 1.

4 Applications

4.1 Housing Market

Shapley and Scarf (1974) consider the following model: Each agent owns one indivisible good (say a house), and has preferences over the houses held by all agents in the economy. An allocation is a permutation of the houses among the agents.

Roth and Postlewaite (1977) show that there is a unique core allocation. They furthermore show that the core is stable. Roth (1982b) shows that the rule which selects the core allocation for each problem is strategy-proof. Bird (1984) shows that it is coalitionally strategy-proof. Ma (1994) recently shows that it is the only rule that is Pareto efficient, individually rational, and strategy-proof (and therefore

it is the only rule which is Pareto efficient, individually rational, and coalitionally strategy-proof).

In this section we obtain Roth's, Bird's, and Ma's results as a corollary to Proposition 1 and Theorem 1.

We can obtain the housing market as generalized matching problems as follows: Let $S_i = N$ for all $i \in N$ and $\mathcal{M}^I = \mathcal{M}$. That is, each agent ranks all agents (interpreted as ranking the houses) and a matching is any permutation of the houses. We have the following corollary.

Corollary 1 (Roth 1982b, Bird 1984, Ma 1990): The core in the context of the housing market is coalitionally strategy-proof. Furthermore it is the only rule which is Pareto efficient, individually rational, and strategy-proof.

Proof: Roth and Postlewaite (1977) shows that the core is a singleton for each problem and it is stable. Therefore the core is coalitionally strategy-proof, and hence strategy-proof due to Proposition 1. Uniqueness follows from Theorem 1.

4.2 Marriage Problems

Gale and Shapley (1962) introduce and study the following class of two-sided matching problems known as marriage problems. There are two finite disjoint sets of agents interpreted as a set of men and a set of women. Each man has a preference relation over the set of women and staying single. Similarly each woman has a preference relation over the set of men and staying single. An allocation is a matching of men and women. Gale and Shapley (1962) show that the core of a marriage problem is always non-empty.[6] They also show that it may not be a singleton. Roth (1982a) shows that there is no selection from the core which is strategy-proof. Alcalde and Barberà (1994) improve on this result and show that there is no matching rule which is Pareto-efficient, individually rational, and strategy-proof.[7] In this section we obtain Roth's and Alcalde and Barberà's results as a corollary to Theorem 1.

We can obtain marriage problems as generalized matching problems as follows: We partition N into two non-empty disjoint sets M and W. That is, $M \cup W =$

[6] See also Roth (1985).

[7] Kara and Sönmez (1993) weakens the incentive requirement and search for Nash implementable matching rules. They show that any solution which is Pareto efficient, individually rational, and Nash implementable is a supersolution of the core. Kara and Sönmez (1994) and Sönmez (1994b) generalize this result to many-to-one matching problems and generalized matching problems respectively.

$N, M \neq \emptyset, W \neq \emptyset$, and $M \cap W = \emptyset$. Let $S_m = W \cup \{m\}$ for all $m \in M$ and $S_w = M \cup \{w\}$ for all $w \in W$. Let

$$\mathcal{M}^f = \{\mu \in \mathcal{M} \mid \mu(\mu(i)) = i \quad \text{for all } i \in M \cup W\}$$

Note that unlike the housing market, we do not allow for all permutations of men and women. A man is assigned to a woman if and only if that woman is assigned to him. We have the following corollary to Theorem 1:

Corollary 2 (Alcalde and Barberà 1994, Roth 1982a): There is no matching rule in the context of marriage problems which is Pareto efficient, individually rational, and strategy-proof (and hence there is no strategy-proof selection from the core).

Proof: Gale and Shapley (1962) show that the core is always non-empty, yet it is not a singleton in general. Therefore Theorem 1 implies that there is no matching rule which is Pareto efficient, individually rational, and strategy-proof.

4.3 Roommate Problems

Consider the following class of problems known as roommate problems. There is a group of agents each of whom has strict preferences over all agents. An allocation is a partition of the set of agents into groups of size one and two. Here we are assigning either one or two persons to a room. We obtain roommate problems as generalized matching problems as follows: Let $S_i = N$ for all $i \in N$ and

$$\mathcal{M}^f = \{\mu \in \mathcal{M} \mid \mu(\mu(i)) = i \quad \text{for all } i \in N\}$$

Consider the following examples:

Example 1 (Shubik 1984):
$N = \{i, j, k\}$,

$$j P_i k P_i i$$
$$k P_j i P_j j$$
$$i P_k j P_k k$$

Note that in this problem staying single is each agent's last choice and each agent is someone else's first choice. Therefore whoever stays single in a matching will form a coalition to block this matching. Hence $\mathcal{C}(R) = \emptyset$.

Example 2:
$N = \{i, j, k, l\}$,

$$j P_i l P_i i P_i k$$
$$k P_j i P_j j P_j l$$
$$l P_k j P_k k P_k i$$
$$i P_l k P_l l P_l j$$

We have $\mathcal{C}(R) = \{\mu, \nu\}$ where $\mu(i) = j, \mu(j) = i, \mu(k) = l, \mu(l) = k$, and $\nu(i) = l, \nu(j) = k, \nu(k) = j, \nu(l) = i$.

Corollary 3: There is no matching rule in the context of roommate problems which is Pareto efficient, individually rational, and strategy-proof.

Proof: For Example 2 there are two allocations in the core and Proposition 2 implies that there is no matching rule which is Pareto efficient, individually rational, and strategy-proof.

5 Concluding Remarks and Related Literature

Strategy-proofness is a property motivated by the fact that agents will manipulate their preferences whenever they can gain by doing so. Its motivation is non-cooperative. On the other hand the core is one of the basic solution concepts in cooperative game theory. Our results show that in the context of matching problems it is possible to achieve strategy-proofness together with Pareto efficiency and individual rationality only by means of the core. This result provides a link between cooperative game theory and non-cooperative game theory and it gives important non-cooperative support to the core, a cooperative solution.

Do these results extend to other interesting economic domains? Although saying yes is premature at this point the literature seems quite consistent with possible generalizations. Ledyard (1977) obtains a result in a very similar spirit for exchange economies, and Sönmez (1994a) obtains analogous results for many-to-one matching problems. In what follows we discuss these papers and other relevant literature and the extent to which our results do or may extend to three interesting domains, exchange economies, public good economies, and many-to-one matching problems.

Hurwicz (1972) shows in his seminal work that there is no solution that is Pareto efficient, individually rational, and strategy-proof in the context of two-person two-good exchange economies. Motivated by this negative result Roberts and Postlewaite (1976), Barberà and Jackson (1995), and Ledyard (1977) follow three different approaches. Roberts and Postlewaite (1976) show that for a wide class of problems the gains from manipulating the Walrasian solution gets arbitrarily small as the economy gets large enough. Barberà and Jackson (1995) totally drop Pareto efficiency and characterize the class of individually rational and strategy-proof solutions with some additional very mild requirements. They also show that no solution in this class satisfy Pareto efficiency even in the case of large economies.[8] Ledyard (1977) shows that there exists a strategy-proof selection from the core only if the core is single-valued. One important question is whether it is possible to extend Ledyard's result in an analogous way to ours and show that there exists a solution that is Pareto efficient, individually rational, and strategy-proof only if the core is single-valued. If so Scarf (1962) and Debreu and Scarf (1963) core convergence results tie these three approaches and link Hurwicz's negative result to them. Another important question is whether there are interesting subclasses of large economies where the core (which coincides in this context with the Walrasian solution) is actually strategy-proof. What about coalitional strategy-proofness? This stronger property is obviously much more plausible than strategy-proofness in the context of large economies.

As far as public good economies are concerned, Ledyard and Roberts (1974) obtain an analogous result to Hurwicz's impossibility result. Because of the difficulties in obtaining core convergence result in this class, the only possible extension seems to be linking the Ledyard and Roberts's result to the multi-valuedness of the core.

Finally, in the context of many-to-one matching problems Sönmez (1994a) obtains analogous results and shows that there exists a solution that is Pareto efficient, individually rational, and strategy-proof only if the core is single-valued and if such a solution exists it is the core. He furthermore obtains the necessary and sufficient conditions to obtain such a solution.

[8]Here Barberà and Jackson do not impose any domain restriction that will lead to a single-valued core.

References

[1] Alcalde, J. and S. Barberà (1994). "Top Dominance and the Possibility of Strategy-Proof Stable Solutions to Matching Problems," *Economic Theory*, 4: 417-435.

[2] Barberà, S., F. Gül, E. Stachetti (1993). "Generalized Median Voter Schemes and Committees," *Journal of Economic Theory*, 61: 262-289.

[3] Barberà, S. and M. Jackson (1994). "A Characterization of Strategy-Proof Social Choice Functions for Economies with Pure Public Goods," *Social Choice and Welfare*, 11: 241-252.

[4] Barberà, S. and M. Jackson (1995). "Strategy-Proof Exchange," *Econometrica*, 63: 51-87.

[5] Bird, C. (1984). "Group Incentive Compatability in a Market with Indivisible Goods," *Economics Letters*, 14, 309-313.

[6] Ching, S. (1993). "Strategy-Proofness and Median Voters," University of Rochester mimeo.

[7] Ching, S. (1994). "An Alternative Characterization of the Uniform Rule," *Social Choice and Welfare*, 11: 131-136.

[8] Debreu, G. and H. Scarf (1963). "A Limit Theorem on the Core of an Economy," *International Economic Review*, 4: 235-246.

[9] Demange, G. (1987). "Nonmanipulable Cores," *Econometrica*, 55: 1057-1074.

[10] Gale, D. and L. Shapley (1962). "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 69: 9-15.

[11] Gibbard, A. (1973). "Manipulation of Voting Schemes: A General Result," *Econometrica*, 41: 587-601.

[12] Hurwicz, L. (1972). "On Informationally Decentralized Systems," In *Decision and Organization: A Volume in Honor of Jacob Marschak*, (R. Radner and C.B. McGuire eds.), North-Holland, Amsterdam.

[13] Kara, T. and T. Sönmez (1993). "Nash Implementation of Matching Rules," University of Rochester mimeo, forthcoming in *Journal of Economic Theory*.

[14] Kara, T. and T. Sönmez (1994). "Implementation of College Admission Rules," University of Rochester mimeo, forthcoming in *Economic Theory*.

[15] Ledyard, J. (1977). "Incentive Compatible Behavior in Core Selecting Organizations," *Econometrica*, 45: 1607-1621.

[16] Ledyard, J. and J. Roberts (1974). "On the Incentive Problem with Public Goods," Northwestern University discussion paper.

[17] Ma, J. (1994). "Strategy-Proofness and the Strict Core in a Market with Indivisibilities," *International Journal of Game Theory*, 23: 75-83.

[18] Moulin, H. (1980). "On Strategy-Proofness and Single Peakedness," *Public Choice*, 35: 437-455.

[19] Moulin, H. (1994). "Serial Cost Sharing of Excludable Public Goods," *Review of Economic Studies*, 61: 305-325.

[20] Moulin, H. and S. Shenker. (1992). "Serial Cost Sharing," *Econometrica*, 60: 1009-1037.

[21] Roberts, J. and A. Postlewaite (1976). "The Incentives for the Price-Taking Behavior in Large Economies," *Econometrica*, 44: 115-128.

[22] Roth, A. (1982a). "The Economics of Matching: Stability and Incentives," *Mathematics of Operations Research*, 7: 617-628.

[23] Roth, A. (1982b). "Incentive Compatibility in a Market with Indivisibilities," *Economics Letters*, 9: 127-132.

[24] Roth, A. (1985). "Common and Conflicting Interests in Two-Sided Matching Markets," *European Economic Review*, 27: 75-96.

[25] Roth, A. and A. Postlewaite (1977). "Weak versus Strong Domination in a Market with Indivisible Goods," *Journal of Mathematical Economics*, 4: 131-137.

[26] Roth, A. and M. Sotomayor (1990). *Two-Sided Matching: A Study in Game Theoretic Modeling and Analysis*, Cambridge University Press, London / New York.

[27] Satterthwaite, M. A. (1975). "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions," *Journal of Economic Theory*, 10: 187-216.

[28] Scarf, H. (1962). "An Analysis of Markets with a Large Number of Participants," *Recent Advances in Game Theory*, Princeton University Press, Princeton.

[29] Shapley, L. and H. Scarf (1974). "On Cores and Indivisibility," *Journal of Mathematical Economics*, 1: 23-28.

[30] Shubik, M. (1984). *A Game Theoretic Approach to Political Economy*, MIT Press, Cambridge.

[31] Sönmez, T. (1994a). "Strategy-Proofness in Many-to-One Matching Problems," University of Michigan working paper.

[32] Sönmez, T. (1994b). "Implementation in Generalized Matching Problems," University of Michigan working paper.

[33] Sprumont, Y. (1991). "The Division Problem with Single-Peaked Preferences: A Characterization of the Uniform Rule," *Econometrica*, 59: 509-519.

[34] Sprumont, Y. (1994). "Strategyproof Collective Choice in Economic and Political Environments," *Canadian Journal of Economics* 28, 68-107.

[35] Thomson, W. (1994). "Strategy-Proof Allocation Rules," University of Rochester mimeo.