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Theories of Justice Based on Symmetry

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The two issues of equity and efficiency are at the center of any economic analysis. While the concept of economic efficiency is now well understood, there is still considerable debate concerning the appropriate definition of economic equity.

It is only since Foley (1967) that an ordinal concept of equity--the concept of an envy-free allocation--has been available for economic analysis. Since then, Foley's criterion has been the object of a number of studies. Its various properties and limitations have been examined and further refinements have been offered. In this paper we will survey some of this literature on theories of equity based on considerations of symmetry.[1]

The paper is organized as follows. In section 1 we study the central criterion of an envy-free allocation and several variants and extensions of the criterion as well as its main competitors. Section 2 is devoted to several

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[1]A note on terminology is in order. What we call envy-free allocations have also been referred to as "equitable" allocations (e.g. Varian (1974),(1975), Champsaur and Laroque (1981)) and also as "fair" allocations (Pazner (1977), Crawford (1977)). Similarly what we call envy-free and efficient allocations have been called "fair" allocations by others (e.g. Schmeidler and Yaari (1971), Varian (1974), (1975), Champsaur and Laroque (1981).) We have tried to be reasonably consistent in our own choice of terminology but have not always indicated the exact choices used by other authors. To do so would have only caused confusion.

following definitions:

DEFINITIONS. Agent i <u>envies</u> agent j at the allocation x if he prefers agent j's bundle, x_j, to his own bundle x_i; in symbols: i envies j at x if $x_i <_i x_j$. An <u>envy-free</u> allocation is one in which no agent envies any other agent.

Since equal division is envy-free the concept is not vacuous. But are there envy-free allocations that are also efficient? The following theorem provides an important element of an answer to this question.

THEOREM. (Foley (1967), Schmeidler and Vind (1972), Kolm (1972)) <u>Assume preferences are non-satiated and let</u> (p,x) <u>be a Walrasian equilibrium with equal incomes. Then x is envy-free and pareto efficient.</u>

<u>Proof</u>. Efficiency follows from the standard argument. Suppose that the allocation is not envy-free so that some agent i envies some agent j. Since each consumer is maximizing on his budget set it must be that agent i cannot afford agent j's bundle. But all consumers have the same income, so what is affordable for j is also affordable for i. []

The above result shows that assumptions that guarantee the existence and the efficiency of Walrasian

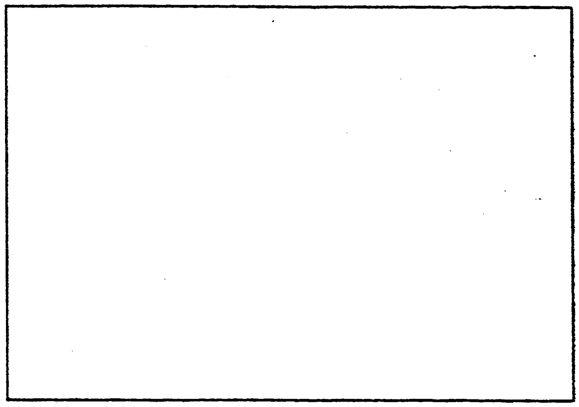

Figure 1. An envy-free allocation in an Edgeworth box

Several interesting facts concerning envy free allocations can be illustrated with this Edgeworth box apparatus. For example:

1. An allocation can be envy-free and efficient without necessarily pareto dominating equal division.

2. With convex preferences, any allocation Pareto-dominating equal division is envy-free.

3. Allocations may be in the core from equal division and yet not be envy-free, although this situation occurs in the two person case only if preferences are allowed to be non-convex. An example with convex preferences and three agents was constructed by Feldman and Kirman (1974).

These results and others of a similar vein ((Kolm (1972), Goldman and Sussangkarn (1978), Thomson (1982a)) indicate that the set of envy-free allocations does not have a particularly simple structure. (It is typically a disconnected union of closed sets).

There are several generalizations of the notion of envy-free allocations that have been proposed.

One class of generalizations is based on the number of one--and which agents are "worst off"--the agents that no one envies. This result provides an appealing interpretation of Rawls' (1971) criterion of maximin welfare. Rawls argues that the most desirable social states are those that maximize the welfare of the worst off individuals. This is generally interpreted as requiring some sort of interpersonal comparisons of utility. However if we consider the "worst off agents" to be the ones that no one envies, and we make them as well off as possible--that is, ensure that they are envied by no one, we are led naturally to the criterion of an envy-free allocation, at which there are no envied agents. This interpretation yields a concept of equity that does not involve interpersonal comparisons of utility and yet is still in the spirit of Rawls' original idea.

One can also use the construction involved in the proof of this result to define various orderings of the agents. Such orderings have been suggested by Varian (1976) and examined in greater detail by Feldman and Weiman (1979).

A second generalization of the concept of envy-free allocations is that of a coalitional envy-free allocation. We can ask for example if any group of agents prefers the aggregate consumption bundle of any other group of agents of the same size, in the sense that they could distribute the other group's bundle among themselves in a way that they all prefer to their current holdings. If it is possible to do this, it is natural to say that one group envies the other.

economics, we postpone their discussion to section 4.

2. Equality of opportunity

In the previous section we have considered various interpretations of the idea of equitable allocations. We turn now to notions based on the idea of equality of opportunities. This idea can be understood in several ways.

One way to give substance to it is to let each agent choose his consumption from a common choice set. (This is advocated for example by Archibald and Donaldson (1979).) Since everyone has access to the bundles chosen by others, the list of consumption bundles obtained from this process will necessarily be envy-free. The difficulty is of course to find a choice set C such that the choices of all the agents are jointly feasible; i.e. they constitute a feasible allocation. This requirement precludes that C be chosen once and for all. Instead one should have access to a whole family C of choice sets such that it can be guaranteed that for some member of C the aggregate feasibility condition can be met.

There is a well known family that has this property, namely the family of budget sets with identical endowments. It is on this family that Varian (1976) based his notion of opportunity-fair allocations. An allocation is opportunity-fair if it is efficient and each agent weakly prefers his own bundle to any bundle in any other agent's budget set, where the budget sets are determined by the to the set of all possible efficient allocations. The more specific the recommendation made by ϕ, the better it helps in the solution of the fair division problem.

The Walrasian correspondence satisfies all of these requirements, and in fact is often advocated on that basis either formally as in Schmeidler and Vind (1972) or informally as in Nozick (1974).

Another example of particular interest is the correspondence associating to every economy in the initial position ω the set of efficient allocations attainable through an envy-free trade, defined as follows. Given agent i's initial endowment ω_i and his final bundle x_i define $t_i = x_i - \omega_i$ to be agent i's net trade. Agent i's preference relation on final bundles x_i induces a preference relation on net trades t_i in the natural way: an agent prefers one net trade to another if he prefers the final bundle he ends up with in that net trade. A list of net trades is envy-free if no consumer prefers someone else's net trade to his own. Many properties of envy-free allocations have counterparts for net trades.

In particular, Schmeidler and Vind (1972), who introduced the concept, showed that Walrasian net trades are envy-free. They also proposed the stronger definition that each agent prefers his net trade to any sum of integer multiples of the net trades of the others, combining the equity notion behind the concept of an envy-free trade with a sort of strengthening of the anonymity requirement that

Pareto improving trade can lead from such an allocation to an efficient and envy-free allocation. Goldman and Sussangkarn (1980) also exhibit a three person economy in which a sequence of envy-free net trades from equal division leads to an allocation in the core from equal division which is not envy-free. Finally, Thomson (1982a) shows that there are economies admitting of envy-free allocations from which no envy-free and efficient allocation can be reached through an envy-free trade.

Are there natural consistency requirements that one can impose to avoid these kind of situations? Thomson (1980) argues that there are.

Given a transition principle ϕ he first defines the consistency of initial state and end state principles relative to ϕ as follows: given a pair (ω, x) of an initial allocation and of a final allocation obtainable from ω through the operation of the transition principle, x is declared ϕ-equitable if permutations of the initial endowments followed by the operation of ϕ leave x unaffected. Similarly, ω is declared equitable if permutations of its components followed by the operation of ϕ can always lead to the allocation x.

Unfortunately, such a consistency criterion is not enough to rule out final allocations which seem manifestly unjust, for example, allocations at which one agent consumes strictly more of all commodities than some other agent.

For instance, if ϕ is the core correspondence C, one can the corresponding transition principle by replacing in the formulation of the principle consumption bundles by trades and the preference relation over consumption bundles by the preference relation over net trades that it induces.

This allows us to complete the argument in an entirely internal way. Given some end state principle, we derive the corresponding transition principle, and from it we obtain in turn the associated end state principle. The final consistency requirement is that we end up where we started!

Are there examples for which this happens? Yes there are. Simply take no envy as both the end state and the transition principles. Or take an equal income Walrasian allocation as the end state principle and Walrasian trade as the transition principle. We are not aware of other examples, which suggests that the envy-free allocations and the equal income Walrasian allocations are truly central notions among the variety of possible definitions of equitable outcomes. This impression will be confirmed in other situations, particularly in section 5 concerned with implementation questions.

3. Economies with a large number of individuals

We have shown above that equal income Walrasian allocations are necessarily envy-free and efficient under very mild assumptions. In general, there are many other envy-free and efficient allocations. However, under certain assumptions to be discussed below, in "large" economies, is nonnegative everywhere and reaches its minimum value of 0 at s=t. Hence the derivative of g(s) (assuming it exists) vanishes at s=t which implies:

$$Dg(t) = Du(x(t),t)Dx(t) = 0$$

On the other hand x is efficient so that $Du(x(t),t) = \lambda(t)p$, for some price vector p. Substituting, we have:

$$\lambda(t)pDx(t) = 0$$

But this just says that the derivative of income with respect to t is zero. Hence it must be constant across consumers, and the result is proved.

The above argument required that both the utility function and the allocation be differentiable. It is easy to construct examples showing the necessity of assuming that the utility function be differentiable for this result to hold. The differentiability of the allocation itself is a bit more subtle, as we now discuss.

Varian (1976) established that his parameterization and a convexity assumption on preferences implied that an envy-free and efficient allocation would be necessarily continuous. Subsequently, Kleinberg (1978), (1980) showed by way of an example that continuity was not enough to guarantee the equal income property. However, he also showed that a minor strengthening of the regularity in general there will be many other coalitional-fair allocations. However, in large economies, the equal income Walrasian allocations are the only coalitional-fair allocations!

This result was originaly established by Vind (1972) using a measure space approach. Subsequently, Varian (1974) generalized Vind's theorem and provided a new argument using constructions similar to those used in the core convergence theorem of Debreu and Scarf. Yaari (1982) also investigated this issue.

4. Symmetry Theories Involving Production

The results of Section 1 concerning the existence and characterization of envy-free and efficient allocations in exchange economies are very appealing. It is important to consider how far they can be extended and in particular to ask how the no envy criterion works in economic situations involving production. We might well be worried not only about a fair division of a cake, but also about a simultaneous fair division of the labor involved in baking the cake!

If all consumers have the same preferences but different productive capabilities envy-free and efficient allocations continue to generally exist. (Varian (1974)). For in this situation the consumption bundles in an envy-free allocation must be indifferent in terms of the common preferences. Thus to satisfy efficiency, we need only find an allocation could afford to consume more of his own (less expensive) leisure.

Varian (1974) later suggested the concept of wealth-fairness. In a wealth-fair allocation it is required that each individual weakly prefers his consumption-output bundle to any other individual's consumption-output bundle. Thus an allocation at which one individual envies another's consumption of goods and leisure is not declared unfair unless the first individual is willing to match the other's production of output. Of course this definition is meaningful only if the aggregate output can be decomposed into outputs attributable to the individual agents.

A less restrictive definition can be obtained by requiring only that the envious individual produce output that has the same value at the supporting efficiency prices as the output of the other. However, the modified definition requires the existence of supporting prices and can therefore be applied only to efficient allocations.

If we compared a more able to a less able person's consumption bundles at a wealth-fair allocation, we would find that the more able person might well have a consumption that the less able person envies ; but the less able person could not object to it since he would be unwilling (or unable) to match the more able person's contribution to the social product.

It seems as though the less able are favored in a full-income-fair allocation and the more able are favored in a yields all the wealth-fair allocations according to the modified definition). The argument is again relatively straightforward: if I prefer your consumption-output bundle to my own--meaning that I am willing to produce output valued as highly as your output--then in equilibrium I would be able to earn as much as you earn. But then I could afford your consumption bundle in the first place.

Hence the full-income-fair notion totally corrects for the distribution of abilities, and the wealth-fair notion allows for no correction at all. Both of these views seem a bit extreme; are there any other intermediate cases that one might consider?

Daniel (1975) suggested that if we cannot expect to have no envy, we might at least hope for a kind of balanced envy. Using quite general assumptions on preferences (in particular, no convexity assumptions are needed) he demonstrated the existence of efficient allocations at which each person is envied by the same number of people that he envies.

An appealing feature of this kind of balanced envy criterion is that when envy-free allocations exist, they have this property. Otherwise very little is known about the set of allocations which satisfy the Daniel criterion; however it may be consistent with situations in which the number of occurrences of envy is very large, as emphasized by Pazner (1977).

We also have the notion of an egalitarian-equivalent in that it allows agents to gain from their own "labor ability set" while correcting for the distribution of other goods. Otsuki established that allocations according to labour exist under standard regularity conditions. However it is not clear that each person should be allowed the entire gains from his labor ability set as Otsuki seems to imply. After all, the initial distribution of ability is as ethically arbitrary as the initial distribution of material resources. (Or is it?)

Although the idea of enlarging the range of admissible comparisons to arbitrary (feasible or not) lists of consumption bundles is conceptually interesting and has the advantage of yielding existence results, it has not yet yielded a fully convincing criterion for distributive justice.

So as of this date there does not seem to be an entirely satisfactory concept of equity in the case of a production economy. Perhaps this reflects an inherent difficulty with notions of justice based on symmetry. They seem to work well when everyone is similar, but if there are too many things that differ across individuals the demands of equity and efficiency become difficult to reconcile.

5. Information and Incentives

Once an equity criterion has been selected, several questions of a more practical nature arise concerning the

We now turn to a discussion of how to implement equitable allocations. Given some criterion ψ let us call a <u>realization mechanism for ψ</u> a set of rules describing how information is (i) generated; (ii) exchanged between the agents themselves and perhaps an "auctioneer" (an outside agent whose purpose is to help in coordinating the agents' activities) and (iii) processed, so as to yield some, or perhaps all, allocations satisfying the criterion. As an example let ψ be the envy-free and efficient criterion and the realization mechanism for ψ be the Walrasian mechanism from equal division. As we showed in section 1 this will yield some, but not all, of the desired allocations.

It is interesting to note that the earliest papers devoted to problems of fair division centered on this realization question. The classical two-person divide and choose method (one agent divides and the other chooses) as well as the generalizations to n agents studied by Steinhaus (1948), Singer (1962), and Kuhn (1967) are operational methods of obtaining what these authors thought to be equitable outcomes, although the precise sense in which the outcomes could be described as equitable was left to Kolm (1972), Crawford (1977), Crawford and Heller (1979) and Samuelson (1980).

Given a realization mechanism for some criterion ψ, it is in general the case that if all but one agent follow the rules of behavior assigned to them by the mechanism the remaining agent will find it profitable not to behave as strategies adopted by the other agents, and in this case several positive results were established. The most important of these, due to Maskin (1977) is that a correspondence is Nash implementable only if it satisfies the following condition of monotonicity.

DEFINITION. Let e be an economy, x be an allocation in $\psi(e)$, and e' be an economy in which preferences are altered so that x does not fall in anyone's ordering. If x is necessarily in $\psi(e')$ then we say that ψ is <u>monotonic</u>.

Maskin shows that this definition is also sufficient for Nash implementation if there are at least three agents for any ψ satisfying "no veto power", a condition that, in economic environments with private goods, as studied here, is vacuously satisfied. (The condition says that if an allocation is at the top of all but one of the agents' preferences in the economy e then the last agent cannot prevent it from being in $\psi(e)$.) By the theorem, the correspondence that associates with each economy its set of envy-free and efficient allocation is Nash-implementable. The same is true of the Walrasian correspondence from equal division (ignoring boundary problems as we will here and in the following paragraphs.) On the other hand the egalitarian-equivalent and efficient correspondence are not monotonic as soon as there are more than two agents and are therefore not Nash-implementable.

Crawford (1977),(1979),(1980) and Moulin (1981). Crawford's starting point is the two person divide and choose method which he first shows to yield envy-free but possibly inefficient allocations (although they will be undominated by any other envy-free allocation.) He then shows that this drawback can be removed by having the divider present the chooser a choice between equal division and some other allocation x selected by him, the divider. This method yields an envy-free and efficient allocation but gives an advantage to the divider. If the role of the divider is auctioned off and the proceeds of the auction are equally distributed among the choosers, an egalitarian-equivalent and efficient allocation is obtained.

Unfortunately, the procedure may yield infeasible allocations out of equilibrium. Demange (1982) proposes a modification of Crawford's procedure that takes care of this problem.

Moulin (1981) studies the case of transferable utility economies (where preferences are representable by utility functions that are additively separable and linear in one commodity, the same for all agents) and proposes various procedures in the same spirit which yield egalitarian or Shapley-value allocations.

If a proposed equity criterion cannot be implemented by a game, then any mechanism that realizes it under truthful behavior will be manipulable in some manner. So the next question to consider is to what extent realization coinciding with the equilibrium allocations of the manipulation game associated with the Walrasian correspondence from equal division. Also, for most of the non-monotonic correspondences studied, it turns out that equal division is efficient according to any equilibrium list of strategies. If property P is satisfied then equal division will therefore appear to be efficient and it will seem that there is no problem of fair division to study.

6. Applications of Symmetry Theories

There have been several attempts to apply the concepts described above to practical policy issues. For example Baumol (1980), (1982) investigated how considerations of symmetry might influence policy choices involving natural resources, choice of rationing schemes, and peak load pricing. (Many of these issues will be discussed in his forthcoming book.) Crawford (1977), (1979), (1980) and Crawford and Heller (1979) considered a number of fair division schemes that can be used as arbitration devices. (These schemes were described briefly in the last section along with other work on the classical problem of "fair division.") It has often seemed to us that there would be many opportunities in the legal profession for applying such schemes.

Brock and Scheinkman (1976) analyzed the implications of the no-envy criterion for analyzing questions of intergenerational equity, while Sobel (1979) considered the some criterion such as a welfare function. Varian (1976) has proposed using an equal income allocation to "normalize" a parametric welfare function in order to make such comparisons. He suggests that one choose the parameters of some given welfare function so that the function reaches its maximum at an equal income allocation. The resulting welfare function can then be used to examine "second best" problems such as those described above.

7. Summary

We have examined several concepts of distributional justice based on considerations of symmetric treatment. In our opinion these concepts throw considerable light on important problems of distributional justice, but none can be said to be entirely satisfactory. Indeed, with a concept so complex and multifaceted as that of "justice", we can hardly expect that a single definition will be appropriate in all situations. However, we hope that the family of definitions described here can be used to examine the equity of distributional mechanisms and outcomes in a variety of frameworks.

In compiling this catalog of concepts of equity it has struck us how often equal income Walrasian allocations have arisen. Such allocations are known to be efficient under standard assumptions and to:

1)be envy-free;

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