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Divergence of Opinion in
Complete Markets

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Abstract. We consider an Arrow-Debreu model with different subjective probablities. In general asset prices will depend only on aggregate consumption and the distribution of subjective probabilities in each state of nature. If all agents have identical preferences then an asset with 'more dispersed' subjective probabilities will have a lower price than an asset with less dispersed subjective probabilities if risk aversion does not decline too rapidly. It seems likely that this condition is met in practice so that increased dispersion of beliefs will generally be associated with reduced asset prices in a given Arrow-Debreu equilibrium.

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There have been several recent investigations concerning the effect of heterogeneous probability beliefs on asset prices. Most of these investigations have taken place in the context of CAPM-like mean-variance models; see for example, Lintner (1969), Miller (1973), Williams (1977), Jarrow (1980), and Mayshar (1983).

Comparatively little has been done in the analysis of differences of opinion in an Arrow-Debreu contingent claims context. Rubinstein (1976) has accommodated divergence of opinion in the context of specific functional forms for utility functions and Breedon and Litzenberger (1978) have provided nice valuation formulas and characterizations of equilibria with commonly held probability beliefs. But little has been said about the *general* properties of dispersed beliefs in an Arrow-Debreu context.

In this paper I analyze the impact of divergence of opinion on asset prices in an Arrow-Debreu economy. The results serve to generalize the findings of Rubinstein and Breedon-Litzenberger to the case of different probability beliefs. Among the results I establish are:

1. In equilibrium asset prices depend only on aggregate consumption and the distribution of subjective probability beliefs.

2. Asset values are an increasing function of any one individual's probability beliefs.

3. An increase in the "spread" of the probability beliefs of investors may increase or decrease equilibrium asset values depending on the value of a parameter of the utility function. However, the most likely effect is to decrease the asset values.

1. The Arrow-Debreu Model

Suppose that there are n investors indexed by $i = 1, \ldots, n$. There are S states of nature indexed by $s = 1, \ldots, S$. Investor i has a vonNeuman-Morgenstern utility function for consumption in state s denoted by $u_i(c_{is})$. This function is assumed to be strictly increasing and strictly concave in consumption. We assume that there are given endowments of consumption in state s by consumer i denoted by $\bar{c}_{is}$.

Each consumer has a subjective probability distribution over the states of nature. We let π_{is} denote consumer i's probability that state s will occur. We assume that there are a set of Arrow-Debreu securities that pay off one unit of consumption if and only a given state of nature occurs. We let p_s denote the price of an Arrow-Debreu security that pays off in state s.

Each consumer chooses his portfolio of Arrow-Debreu securities by solving the following maximization problem:

$$\max \sum_{s=1}^{S} \pi_{is} u_i(c_{is})$$

$$\text{s.t.} \sum_{s=1}^{S} p_s c_{is} = \sum_{s=1}^{S} p_s \bar{c}_{is}$$

We assume that final consumption in each state of nature is nonnegative, but the net position in each Arrow-Debreu security for a given individual may be positive or negative. Thus there are no short sales restrictions of any sort. We assume that standard conditions are satisfied so that an Arrow-Debreu equilibrium will exist.

Any asset can be valued in terms of the Arrow-Debreu prices. For example, let an asset have a payoff of x_s in state of nature s. Then its equilibrium value must be:

$$v_x = \sum_{s=1}^{S} p_s x_s$$

Thus the results stated below apply for all assets, not just the Arrow-Debreu securities.

2. Analysis of the Model

In equilibrium each consumer maximizes his expected utility as given above. Hence his optimal consumption must satisfy the first order conditions:

$$\pi_{is} u_i'(c_{is}) = \lambda_i p_s \tag{1}$$

Since $u_i'(c_{is})$ is a strictly decreasing function, it has an inverse $f_i(\cdot)$. Thus we can write:

$$c_{is} = f_i(\lambda_i p_s / \pi_{is}) \tag{2}$$

Summing over all investors we have:

$$\sum_{i=1}^{n} c_{is} = c_s = \sum_{i=1}^{n} f_i(\lambda_i p_s / \pi_{is}) \tag{3}$$

For fixed values of (π_{is}) and (λ_i) the right hand side of this expression is a strictly decreasing function of p_s. Hence it has an inverse $F(\cdot, \pi_{1s}, \ldots, \pi_{ns})$. Applying this function to each side of this expression we have:

$$F(c_s, \pi_{1s}, \ldots, \pi_{ns}) = p_s \tag{4}$$

In any given Arrow-Debreu equilibrium the values of the λ_i terms are determined. Hence the above equation implies that the equilibrium contingent commodity prices are solely a function of ***aggregate*** consumption in each state and the distribution of beliefs about that state. Recording this fact for future reference:

FACT 1. In equilibrium the Arrow-Debreu price for consumption in state s depends only on aggregate consumption in that state and the distributions of subjective probabilities that the state will occur. The Arrow-Debreu price is a decreasing function of consumption in state s and an increasing function of π_{is} for each $i = 1, \ldots, n$.

Proof. Only the last sentence remains to be proved. That the prices are decreasing in consumption is obvious from the definition of F. In order to prove the second part we consider equation (3). Hold aggregate consumption and the λ_i terms fixed in this equation and increase π_{is}. Then p_s must increase to maintain the equality. $\square$

The first part of this fact is a generalization of Theorem 1 in Breedon and Litzenberger (1978). Extending of their discussion to this model we note that two states of nature that have the same aggregate consumption and the same distribution of probability beliefs have the same Arrow-Debreu prices; thus a set of Arrow-Debreu securities need only distinguish states with different value of aggregate consumption and different probability beliefs in order to support a given efficient pattern of consumption.

We now consider the impact of a change in the "spread" of the probability beliefs on asset prices. Consider two states of nature with the same value of aggregate consumption but with different probability beliefs. Suppose that the "average" probability over the investors is constant across the two states, but the "divergence of opinion" is higher in one state than the other. Which Arrow-Debreu price will be larger?

Intuition provides conflicting answers. One might argue that divergence of opinion about an asset's payoffs make the asset seem more risky. Hence divergence of opinion will decrease the value of an asset. On the other hand, one can argue that the market price is determined by the optimists, so that increasing the divergence of opinion is likely to increase an asset's price. Several of the CAPM type models mentioned in the introduction support this latter view.

In order to develop some intuition, we might consider a simple reservation price model. Suppose that each consumer is limited to purchasing at most one unit of a given asset in fixed supply, and let the reservation prices differ accross consumers. Then the simple supply-demand diagram depicted in Figure 1 gives us the equilibrium price.

Now suppose that the reservation prices become more dispersed; i.e. the pessimists think that the asset is worth less and the optimists think that it is worth more. This will tend to rotate the demand curve clockwise about the average reservation price. Hence the equilibrium price of the asset will increase or decrease as the supply of the asset is to the right or left of the pivot point.

Even in this simple model an increase in the diversity of opinion has an ambiguous effect on asset prices. Thus it seems unlikely that a definitive result is available in more general cases. However, the additional structure provided by the state independent vonNeuman-Morgenstern utility functions does allow to isolate the relevant parameter of the utility function that determines the effect of diversity on asset prices in a given equilibrium.

3. Diversity of opinion

Let us consider a fixed equilibrium and a particular state. We will assume that all consumers have the same vonNeuman-Morgenstern utility function $u(c_{is})$ with associated Arrow-Pratt measure of absolute risk aversion $r(c)$. We also introduce the *weighted* probabilities q_{is} for $i = 1, \ldots, n$ defined by:

$$q_{is} = \pi_{is}/\lambda_i$$

In general wealthier consumers will have lower marginal utilities of income so that their subjective probability beliefs will have a higher weight in the above expression. Using this notation we can rewrite equation (3) as:

$$c_s = \sum_{i=1}^{n} f(p_s/q_{is}) \tag{5}$$

We now note the following:

FACT 2. The function $f(p_s/q_{is})$ is an increasing function of q_{is}. It is a concave or convex function of q_{is} as $r'(c)$ is greater than or less than $-r^2$.
Proof. Suppose that $r'(c) > -r^2$. Then it is a straightforward calculation to show that:

$$\frac{u'u'''}{u''u''} < 2.$$

Using the fact that $f(u'(c)) \equiv c$, we can derive expressions for the derivatives of f in terms of the derivatives of u:

$$f' = 1/u''$$

$$f'' = -u'''/(u'')^3$$

These expressions in turn can be used to calculate the following derivatives:

$$\frac{\partial f}{\partial q_{is}} = -f'(p_s/q_{is})p_s/q_{is}^2 > 0$$

$$\frac{\partial^2 f}{\partial q_{is}^2} = \frac{p_s}{u'' q_{is}^3}[2 - \frac{u'}{u''}\frac{u'''}{u''}]$$

Combining this last expression with the first inequality we have the result. []

The fact that $f(p_s/q_{is})$ has a definite curvature allows us to use the standard techniques of Rothschild-Stiglitz (1970) to determine the effect of diversity of opinion on asset prices.

FACT 3. If $f(p_s/q_{is})$ is an increasing concave (convex) function of q_{is} then a mean-preserving spread in q_{is} must decrease (increase) the equilibrium value of p_s.
Proof. Refer to equation (5). Since $f(p_s/q_{is})$ is a concave function of q_{is} a mean-preserving spread in the distribution of q_{is} will decrease the value of the sum. If c is to remain fixed, this means that p_s must decrease. $\square$

Note that this Fact holds only for a 'cross sectional' comparison of asset prices in a *fixed* equilibrium. Thus if we have two states s and t with the same aggregate consumption but more dispersed beliefs in s than in t in the sense that the distribution of weighted beliefs in s is a mean-preserving spread of the one in t, then the Arrow-Debreu price for consumption in s will be less than the Arrow-Debreu price for consumption in t.

Facts 2 and 3 take together indicate that the crucial determinant effect of dispersion of opinion on asset prices is whether risk aversion declines 'too rapidly.' However, this condition in itself is not terribly transparent. There is an equivalent expression for the condition in terms of the Arrow-Pratt measure of *relative* risk aversion.

FACT 4. Let $\rho(c) = r(c)c$ be the coefficient of relative risk aversion. Then $r'(c) > -r^2$ if and only if the consumption elasticity of relative risk aversion is greater than $1-\rho$. That is: $\rho' c/\rho > 1-\rho$.
Proof. Differentiating $\rho = rc$ we have:

$$\rho' = r + r'c$$

Thus $r' > -r^2$ can be expressed as:

$$\frac{\rho' - r}{c} > -\frac{\rho^2}{c^2}$$

which reduces to:

$$rc - \rho' c < \rho^2$$

Or:

$$\rho - \rho^2 < \rho' c$$

$$1 - \rho < \frac{\rho' c}{\rho}$$

$\square$

Thus if we consider the family of constant relative risk averse utility functions, we see that the condition will be satisfied when $\rho > 1$. Since the empirical evidence indicates that ρ is at least 2, it seems that equilibrium asset prices should generally decrease with an increase in diversity of opinion.

It is an easy calculation to check other commonly used functional forms. For example quadratic utility and constant absolute risk aversion each imply $f(p_s, q_{is})$ is a concave function of q_{is}. Thus asset prices will decrease with an increase in diversity of opinion in both of these cases.

The most convenient general class of expected utility functions is the HARA, or linear risk tolerance class defined by:

$$-\frac{u'(c)}{u''(c)} = a + bc$$

FACT 5. If the representative utilty function is of the HARA class, then increasing the dispersion of opinion will decrease asset prices if and only if $b < 1$.
Proof. By Fact 2 and direct computation. $\square$

4. An Example with Constant Relative Risk Aversion

The family of constant relative risk averse utility functions provides a nice example. Here the first order conditions for utility maximization have the form:

$$\pi_{is} c_{is}^{-\rho} = \lambda_i p_s$$

Solving for c_{is} we have:

$$c_{is} = \left(\frac{\lambda_i}{\pi_{is}}\right)^{-1/\rho} p_s^{-1/\rho}$$

Summing this over $i = 1, \ldots, n$ and rearranging we have:

$$p_s^{1/\rho} c_s = \sum_{i=1}^{n} \left(\frac{\pi_{is}}{\lambda_i}\right)^{1/\rho}$$

For $\rho > 1$ the right hand side of this expression is a concave function of π_{is}/λ_i so that a mean preserving spread of $\pi_{is}/\lambda_i = q_{is}$ across individuals will decrease the right hand side. Hence p_s must decrease to maintain the equality.

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