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Sequential Provision of Public Goods

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Abstract. I consider the private provision of public goods when agents are able to make sequential contributions rather than simultaneous contributions. I show that this tends to exacerbate the free-rider problem in the sense that the amount of the public good provided under sequential contribution is always less than under simultaneous contribution.

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Sequential Provision of Public Goods

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Several authors have examined the private provision of public goods in the context of a simultaneous move game. The Nash equilibria in these games turn out to have several surprising and interesting properties. For details see Warr (1983), Cornes and Sandler (1986) and Bergstrom, Blume and Varian (1986).

As far as I know, no one has examined the *Stackelberg* equilibria in such games. For example, suppose that agents decide on their contributions to a public good sequentially, so that later agents know the contributions of earlier agents when they make their own decisions. In this sort of game, the earlier agents are able to *commit* to their contributions; such commitment is not possible in a simultaneous move game.

It turns out that this ability to commit to a contribution exacerbates the free rider problem. Our main theorem establishes that the total amount of the public good provided in a sequential game is typically smaller than the amount provided in a simultaneous move game. Along the way, we establish several other interesting results concerning equilibria in sequential games.

1. An Example with Quasilinear Utility

It is instructive to start with a simple example. Suppose that there are two agents. Agent i is endowed with wealth w_i. Each agent divides his wealth between private consumption, $x_i \geq 0$, and a contribution to a public good, $g_i \geq 0$. The total amount of the public good is $G = g_1 + g_2$.

Each agent's utility function is linear in his private consumption and increasing and concave in G, so that the utility of agent i is given by

$$u_i(G) + x_i = u_i(g_1 + g_2) + w_i - g_i.$$

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We will say that agent i *likes the public good more* than agent j if $u_i'(G) > u_j'(G)$ for all $G \geq 0$. Let $\bar{g}_i$ be the amount of the public good that maximizes agent i's utility function if the other agent contributes zero. Note that if agent i likes the public good more than agent j, it follows that $\bar{g}_i > \bar{g}_j$.

We assume that $w_i > \bar{g}_i$ so that the wealth constraint is never binding. It is therefore convenient to drop w_i as it is an inessential constant in each agent's utility function.

The Reaction Function

Let us derive the reaction function for agent 1. The first-order condition if agent 1 contributes a positive amount is

$$u_1'(g_1 + g_2) = 1.$$

Letting $G_1(g_2)$ be agent 1's reaction function, we must have

$$u_1'(G_1(g_2) + g_2) = 1.$$

It follows that

$$G_1(g_2) = \bar{g}_1 - g_2.$$

Recall the $\bar{g}_1$ is defined to be the amount that 1 contributes when $g_2 = 0$.

However, this derivation is correct only when agent 1 contributes a positive amount to the public good. Since $g_1 \geq 0$, we must have

$$G_1(g_2) = \max\{\bar{g}_1 - g_2, 0\}.$$

This "kink" in the reaction function is what makes the analysis interesting.

The Nash Equilibrium

A Nash equilibrium is a point (g_1, g_2) such that

$$g_1 = G_1(g_2)$$
$$g_2 = G_2(g_1).$$

Given the simple forms of the reaction functions, we can illustrate the equilibrium in Figure 1. In this case the *unique* Nash equilibrium is for the person who likes the public good more to contribute the entire amount of the public good. The other agent is a complete free rider.

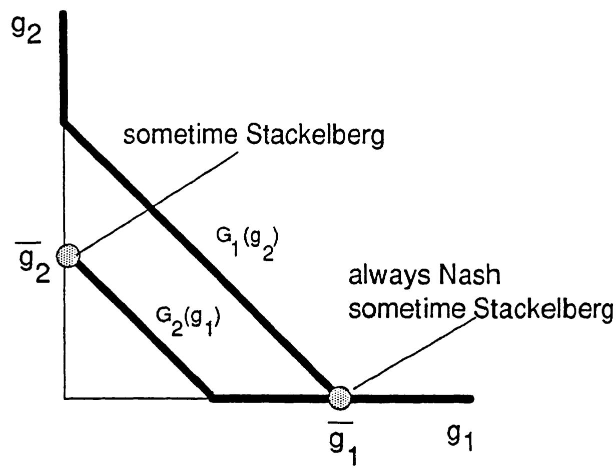

Figure 1. Nash equilibrium. The player who likes the public good the most contributes everything and the other player contributes nothing.

The Stackelberg Equilibrium

In order to investigate the Stackelberg equilibrium, it is useful to plot the utility of agent 1 as a function of his contribution:

$$\begin{aligned} V_1(g_1) &= u_1(g_1 + G_2(g_1)) - g_1 \\ &= u_1(g_1 + \max\{\bar{g}_2 - g_1, 0\}) - g_1. \end{aligned}$$

It is easy to see that this function has the form

$$V_1(g_1) = \begin{cases} u_1(\bar{g}_2) - g_1 & \text{for } g_1 \leq \bar{g}_2 \\ u_1(g_1) - g_1 & \text{for } g_1 \geq \bar{g}_2. \end{cases}$$

This function is depicted in Figure 2.

It is clear by inspection that there are two possible optima: either the first agent contributes zero or $\bar{g}_1$. In order to determine which one of these possibilities is appropriate consider two cases..

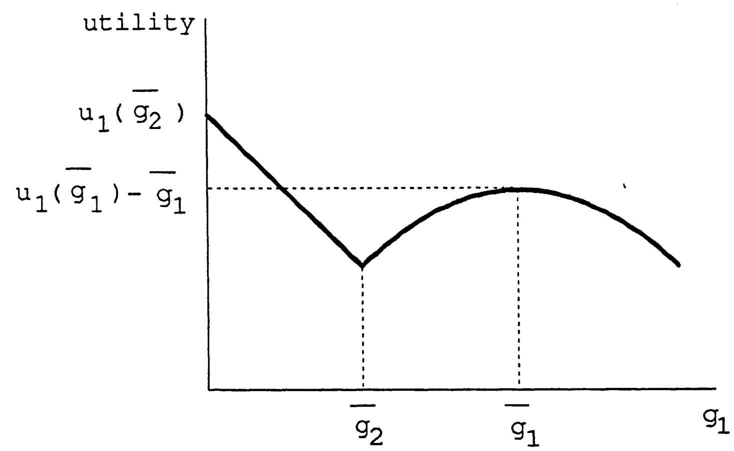

Figure 2. Indirect utility. This is the utility function of the first contributor as a function of his gift.

Case 1. The agent who likes the good least is the first contributor. In this case, the optimal contribution by the first agent is zero. This is true since

$$u_1(\bar{g}_2) > u_1(\bar{g}_1) > u_1(\bar{g}_1) - g_1.$$

Case 2. The agent who likes the public good the most is the first contributor. In this case, either outcome can occur. The easiest way to see this is by example. Suppose that $u_i(G) = a_i \ln G$. If only agent i contributes he will contribute a_i. Normalize $a_1 = 1$. Then agent 1 will prefer to contribute only if

$$\ln a_i < \ln 1 - 1 = -1,$$

which implies that $a_i < 1/e \approx .37$. ∎

In general if the agents have tastes that are very similar, then the first agent will free ride on the second's contribution. However, if the first agent likes the public good *much* more than the second, then the first agent may prefer to contribute the entire amount of the public good himself.

Referring to Figure 1 we see that there are two possible Stackelberg equilibria: one is the Nash equilibrium, in which the agent who likes the good most contributes everything. The other Stackelberg equilibrium is where the agent who likes the good *least* contributes everything. This equilibrium cannot arise as a Nash equilibrium since the threat to contribute nothing is not credible in a simultaneous move game.

Note that the sum of the utilities is higher at the higher level of the public good. If you want to ensure that the higher level of the public good is provided, then you should make sure that the person who likes the good *least* moves first.

2. The General Case

The quasilinear case is very special and it is worthwhile examining how far the results generalize. Suppose that we now consider a general utility function $u_i(G, x_i)$ where G is the level of the public good and x_i is the private consumption of person i. We assume that utility is a differentiable, concave function.

Person 1's maximization problem is

$$\begin{aligned} &\max_{x_1, g_1} u_1(g_1 + g_2, x_1) \\ &\text{such that } g_1 + x_1 = w_1 \\ &g_1 \geq 0. \end{aligned}$$

We can add g_2 to each side of the constraints in this problem and use the definition $G = g_1 + g_2$ to rewrite this problem as

$$\begin{aligned} &\max_{x_1, G} u_1(G, x_1) \\ &\text{such that } G + x_1 = w_1 + g_2 \\ &G \geq g_2. \end{aligned}$$

In this interpretation, agent 1 is in effect choosing the level of the public good, subject to the constraint that the level that he chooses is at least as large as that contributed by agent 2.

Following Bergstrom, Blume and Varian (1986) we note that this problem is simply a standard consumer demand problem except for the inequality constraint. Let $f_1(w)$ be agent 1's Engel function for the public good as a function of his wealth. This is simply the function that gives the optimal level of G for agent 1, holding prices fixed at (1,1) and letting wealth vary. It follows that

$$G = g_1 + g_2 = \max\{f_1(w + g_2), g_2\}.$$

Subtracting g_2 from each side of this equation, we have the reaction function

$$G_1(g_2) = \max\{f_1(w_1 + g_2) - g_2, 0\}.$$

The reaction function for agent 1 has the simple interpretation that agent 1 will contribute the amount of the public good that he would demand if his wealth were $w_1 + g_2$ *minus* the amount contributed by the other agent, or zero, whichever is larger.

This expression for the reaction function is useful since we know quite a bit about Engel curves. For example, it is quite natural to make the following assumption:.

Assumption. Both the public and the private good are strictly normal goods at all levels of wealth. It follows that $1 > f_1'(w) > 0$.

Given this assumption it is easy to see the general shape of the reaction function. When $g_2 = 0$, agent 1 will contribute $f_1(w_1)$. As g_2 increases, the contribution of agent 1 will decrease but at less than one-for-one. If $f_1'(w)$ is bounded away from zero, then there will be some g_2^c at which $f_1(w_1 + g_2^c) - g_2^c = 0$ and agent 1 will contribute nothing.

The notable feature of this reaction curve as compared to the one in the quasilinear case is that it is 1) generally nonlinear, and 2) has a slope that is flatter than -1.

As before, we can use this reaction function to calculate the Nash equilibria and the Stackelberg equilibrium. A Nash equilibrium is a solution (g_1^n, g_2^n) to the following equations

$$g_1 = \max\{f_1(w_1 + g_2) - g_2, 0\}$$

$$g_2 = \max\{f_2(w_2 + g_1) - g_1, 0\}.$$

A Stackelberg equilibrium is a pair $(g_1^s, G_2(g_1^s))$ in which g_1^s solves

$$\max_{g_1} V_1(g_1) = u_1(g_1 + \max\{f_2(w_2 + g_1) - g_1, 0\}, w_1 - g_1).$$

Our main interest is in comparing the solutions of these two sets of equations. This comparison is made simpler by noting that Bergstrom, Blume, and Varian (1986) have proved that under the normality assumption we have made there is a *unique* Nash equilibrium. There will also be one Stackelberg equilibrium for each ordering of the agents.

3. A Cobb-Douglas Example

A useful example to fix ideas is the Cobb-Douglas case:

$$u_i(G, x_i) = a_i \ln(g_1 + g_2) + (1 - a_i)\ln(w_i - g_i),$$

where $0 < a_i < 1$. In this case the agent spends a constant fraction a_i of his wealth on the public good so that the reaction function for agent i takes the form

$$G_i(g_j) = \max\{a_i(w_i + g_j) - g_j, 0\}.$$

Here the slope of the reaction function is $a_i - 1$ up to the point $\hat{g}_j = a_i w_i/(1 - a_i)$, and zero thereafter. It is worth noting that the reaction function will have this form for *any* homothetic utility function.

Nash equilibria are the solutions to

$$g_1 = \max\{a_1(w_1 + g_2) - g_2, 0\}$$

$$g_2 = \max\{a_2(w_2 + g_1) - g_1, 0\}.$$

The Stackelberg equilibrium is the solution to

$$\max_{g_1} \ a_1 \ln(g_1 + \max\{a_2(w_2 + g_1) - g_1, 0\}) + (1 - a_1)\ln(w_1 - g_1).$$

Straightforward computations show that the interior solutions to these equations imply equilibrium levels of the public good of

$$G^n = \frac{a_1 a_2 (w_1 + w_2)}{a_1 + a_2 - a_1 a_2}$$

$$G^s = a_1 a_2 (w_1 + w_2).$$

Note that in this example, $G^n > G^s$ since $a_i < 1$.

4. Results

We have three sets of results. The first set of results concerns who contributes and who free rides. The second set of results concerns the effect of redistributions of wealth. The third set of results concerns how the amount of the public good provided in the Stackelberg equilibrium compares to the amount provided in the Nash equilibrium.

Recall that $\bar{g}_i$ denotes the optimal contribution of agent i if the other person contributes nothing and that g_1^c is defined by $f_2(w_2 + g_1^c) = g_1^c$; this is the level of g_1 at which agent 1's contribution just crowds out agent 2's contribution.

Free Riding

Fact 1. *If $\bar{g}_1 < g_1^c$ then person 2 must contribute.*

Proof. Evaluate the right derivative of agent 1's utility function at g_1^c. We have

$$\frac{\partial u_1(g_1^c, w_1 - g_1^c)}{\partial G} - \frac{\partial u_1(g_1^c, w_1 - g_1^c)}{\partial x_1} < 0.$$

The inequality follows since the derivative equals zero at $\bar{g}_1$, and $g_1^c > \bar{g}_1$. It follows that agent 1's utility will increase if he contributes less than g_1^c, even if he is the only one to contribute. The fact that the other agent will contribute can only increase the first agent's utility.∎

Fact 2. *If there is a Nash equilibrium with $g_1^n = 0$, then this is also a Stackelberg equilibrium.*

Proof. By definition of Nash equilibrium, agent 2 is on his reaction curve, so we only need to show that agent 1 is optimized. This follows from the following string of inequalities:

$$u_1(\bar{g}_2, w_1) > u_1(g_1 + \bar{g}_2, w_1 - g_1) > u_1(g_1 + G_2(g_1), w_1 - g_1).$$

The first inequality follows from the Nash assumption. The second inequality follows since $G_2(0) = \bar{g}_2$ and $G_2(g_1)$ is a nonincreasing function. ∎

Fact 3. *Suppose that we have a Stackelberg equilibrium* (g_1^s, g_2^s). *Let* $(\Delta w_1, \Delta w_2)$ *be a redistribution of wealth such that* $g_i \geq \Delta w_i$ *for* $i = 1, 2$. *Then the Stackelberg equilibrium after this redistribution is* $(g_1^s + \Delta w_1, g_2^s + \Delta w_2)$ *and the total amount of the public good remains unchanged.*

Proof. Note that the requirement that $g_i \geq \Delta w_i$ implies that the assumptions can only be satisfied when each person is contributing a positive amount. The first-order condition for an optimum is

$$\frac{\partial u_1(g_1 + g_2, w_1 - g_1)}{\partial G} f_2'(w_2 + g_1) - \frac{\partial u_1(g_1 + g_2, w_1 - g_1)}{\partial x_1} = 0.$$

Now suppose that each agent changes his contribution by the amount of his wealth change so that $\Delta g_i = \Delta w_i$ for $i = 1, 2$. Note that since $\Delta w_1 + \Delta w_2 = 0$ we must have $\Delta g_1 + \Delta g_2 = 0$.

Under this rule none of the arguments of any of the functions in the first-order condition change. The conclusion follows immediately. ∎

Warr (1983) and Bergstrom, Blume, and Varian (1986) show that essentially the same result holds in an (interior) Nash equilibrium. Bergstrom, Blume, and Varian (1986) also investigate the boundary cases in some detail. In the two-agent context we are investigating here the results are quite straightforward.

Fact 4. *Suppose that person 1 is contributing and person 2 is not. Then a redistribution from 2 to 1 will increase the amount of the public good, while a redistribution from 1 to 2 can decrease or increase the amount of the public good.*

Proof. A distribution from 2 to 1 increases the amount of the public good since $f_1(w_1)$ is an increasing function. Since g_2 is equal to zero it will remain zero at lower levels of wealth.

A redistribution from 1 to 2 will decrease the level of the public good for small redistributions by the monotonicity of $f_1(w_1)$. But when w_1 gets small enough relative to w_2, it can easily happen that person 2 starts to contribute, thereby increasing the amount of the public good. ∎

Fact 5. *Suppose that person 2 is contributing and person 1 is not. Then a transfer from 1 to 2 will increase the level of the public good, while a transfer from 2 to 1 can increase or decrease the level of the public good.*

Proof. A transfer from 1 to 2 will increase the level of the public good by the monotonicity of $f_2(w_2)$, and a larger contribution by person 2 will never induce agent 1 to begin contributing.

A small transfer of wealth from 2 to 1 will decrease the level of the public good, but a larger transfer may induce 1 to start contributing. ∎

Comparison to the Nash Equilibrium

Our main result has to do with the comparison of the Nash and Stackelberg equilibria.

Theorem. *The amount of the public good contributed by agent 1 in the Stackelberg equilibrium is never larger than the amount provided by agent 1 in the Nash equilibrium. That is, $g_1^s \leq g_1^n$.*

Proof. Evaluate agent 1's utility function at the Nash equilibrium:

$$v_1(g_1^n) = u_1(\max\{f_2(w_2 + g_1^n), g_1^n\}, w_1 - g_1^n).$$

Consider two cases.

Case 1. $g_1^n \geq f_2(w_2 + g_1^n)$

Note that if we increase g_1 the inequality continues to be satisfied, since f_2 has a slope of less than 1. This means that once agent 2 has stopped contributing, any increase in g_1 will not induce him to contribute more. This means that $g_1^n = g_1^c$, so that any increase in agent 1's contribution must reduce his utility. It follows that $g_1^s \leq g_1^n$.

Case 2. $g_1^n < f_2(w_2 + g_1^n)$

Now take the right derivative of $v_1(g_1^n)$ and evaluate it at the Nash equilibrium. We have

$$\frac{\partial u_1(g_1^n+g_2^n, w_1-g_1^n)}{\partial G} f_2'(w_2+g_1^n) - \frac{\partial u_1(g_1^n+g_2^n, w_1-g_1^n)}{\partial x_1} < 0.$$

The inequality is due to the fact that

$$\frac{\partial u_1(g_1^n+g_2^n, w_1-g_1^n)}{\partial G} - \frac{\partial u_1(g_1^n+g_2^n, w_1-g_1^n)}{\partial x_1} = 0$$

at the Nash equilibrium and $f_2'(w_2+g_1^n) < 1$.

It follows that either g_1^s must be smaller than g_1^n *or* $g_1^s = \bar{g}_1$ and $g_2^s = 0$. We will establish that the second case cannot occur.

Let g_1^m be the value of g_1 that maximizes agent 1's utility on the region $\{g_1 : f_2(w_2 + g_1) \geq g_1\}$. If g_1^m is *not* a global optimum, then

$$\begin{aligned} u_1(\bar{g}_1+g_2^n, w_1-\bar{g}_1) &>_1 u_1(\bar{g}_1, w_1-\bar{g}_1) \\ &>_2 u_1(g_1^m+G_2(g_1^m), w_1-g_1^m) \\ &\geq_3 u_1(g_1^n+g_2^n, w_1-g_1^n). \end{aligned}$$

Inequality 1 follows from the fact that utility is increasing in G. Inequality 2 follows from the assumption that g_1^m is not a global optimum. Inequality 3 follows from the fact that g_1^m optimizes utility over a region that contains g_1^n.

Inspecting the first and last terms we see that we contradict the assumption that we have a Nash equilibrium. It follows that this case cannot occur and we are left with $g_1^s \leq g_1^n$. ∎

Corollary. *The total amount of the public good in the Stackelberg equilibrium is less than or equal to the total amount provided in the Nash equilibrium.*

Proof. The function $G_2(g_1)$ has a slope of less than 1. Since $g_1^s \leq g_1^n$,

$$g_1^s + g_2^s = g_1^s + G_2(g_1^s) \leq g_1^n + G_2(g_1^n) = g_1^n + g_2^n.$$

∎

5. Incomplete Information

The above analysis has concerned the case where each agent knows the preferences and wealth of the other agent. One could also consider a model with *incomplete* information in which one or both of the agents does not know these things for certain.

The second agent reacts passively, making his optimal choice given the first agent's contribution. Hence it is irrelevant whether or not he knows anything about the first agent. The only interesting uncertainty concerns the first agent's knowledge of the second agent's type.

Consider the quasilinear model examined earlier. In this case the Stackelberg equilibrium was either $(\bar{g}_1, 0)$ or $(0, \bar{g}_2)$. Hence all that is relevant from agent 1's point of view is the distribution of $\bar{g}_2$. Regard $\bar{g}_2$ as a random variable with some distribution in the population and suppose that agent 1 seeks to maximize expected utility. Then agent 1 will choose to contribute zero if

$$Eu_1(\bar{g}_2) < u_1(\bar{g}_1) - \bar{g}_1$$

and otherwise agent 1 will contribute $\bar{g}_1$.

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