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Sequential Provision of Public Goods

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Abstract. I consider the private provision of public goods when agents are able to make sequential contributions rather than simultaneous contributions. In the case of two agents with quasilinear utility, a quite complete analysis is possible. If the agent who likes the public good least gets to move first, the amount of the public good supplied will be the same as in the Nash equilibrium, but if the agent who likes the public good most moves first, less of the public good may be supplied. If the agents bid for the right to move first, the agent who values the public good *least* will always outbid the other agent. In general, each agent would prefer to subsidize the other agent's contributions. If each agent chooses the rate at which they subsidize the other agent, the subsidizes that support the Lindahl allocation are the unique equilibrium subsidies. For general utility functions, I show that the subgame perfect equilibrium always results in less of the public good being supplied than does the Nash equilibrium.

Keywords. public goods, sequential games, Stackelberg equilibrium, Nash equilibrium.

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Sequential Provision of Public Goods

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Several authors have examined the private provision of public goods in the context of a simultaneous move game. The Nash equilibria in these games turn out to have several surprising and interesting properties. For details see Warr (1983), Cornes and Sandler (1986) and Bergstrom, Blume and Varian (1986).

As far as I know, no one has examined the *Stackelberg* equilibria in such games. For example, suppose that agents decide on their contributions to a public good sequentially, so that later agents know the contributions of earlier agents when they make their own decisions. In this sort of game, the earlier agents are able to *commit* to their contributions; such commitment is not possible in a simultaneous move game.

Admanti and Perry (1988) analyze a game in which agents alternate contributions to joint project. However, in their game the project is either completed or not, and no benefits are generated from a partially completed project. In this paper, by contrast, the focus is on the amount of the public good—the project—that is generated by sequential game. There is also a literature on sequential entry in oligopoly that contains some aspects of the public good problem I examine here; see, for example, McLean and Riordan (1989).

It turns out that the ability to commit to a contribution exacerbates the free rider problem. Our main theorem establishes that the total amount of the public good provided in a sequential game is typically smaller than the amount provided in a simultaneous move game. Along the way, we establish several other interesting results concerning equilibria in sequential contribution games.

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1. An Example with Quasilinear Utility

It is instructive to start with a simple example. Suppose that there are two agents. Agent i is endowed with wealth w_i. Each agent divides his wealth between private consumption, $x_i \geq 0$, and a contribution to a public good, $g_i \geq 0$. The total amount of the public good is $G = g_1 + g_2$.

Each agent's utility function is linear in his private consumption and increasing and concave in G, so that the utility of agent i is given by

$$u_i(G) + x_i = u_i(g_1 + g_2) + w_i - g_i.$$

We will say that agent i *likes the public good more* than agent j if $u_i'(G) > u_j'(G)$ for all $G \geq 0$. This says that agent i has a uniformly higher marginal-willingness-to-pay than agent j for the public good.

Let $\bar{g}_i$ be the amount of the public good that maximizes agent i's utility function if the other agent contributes zero. Note that if agent i likes the public good more than agent j, it follows that $\bar{g}_i > \bar{g}_j$.

We assume that $w_i > \bar{g}_i$ so that the wealth constraint is never binding. It is therefore convenient to drop w_i, since it is an inessential constant in each agent's utility function.

The Reaction Function

Let us derive the reaction function for agent 2. The first-order condition if agent 2 contributes a positive amount is

$$u_2'(g_1 + g_2) = 1.$$

Letting $G_2(g_1)$ be agent 2's reaction function, we must have

$$u_2'(g_1 + G_2(g_1)) = 1.$$

It follows that

$$G_2(g_1) = \bar{g}_2 - g_1.$$

Recall that $\bar{g}_2$ is defined to be the amount that agent 2 contributes when $g_1 = 0$.

However, this derivation is correct only when agent 2 contributes a positive amount to the public good. Since $g_2 \geq 0$, we must have

$$G_2(g_1) = \max\{\bar{g}_2 - g_1, 0\}.$$

This "kink" in the reaction function is what makes the analysis interesting.

The Nash Equilibrium

A Nash equilibrium is a point (g_1, g_2) such that

$$g_1 = G_1(g_2)$$
$$g_2 = G_2(g_1).$$

Given the simple forms of the reaction functions, we can illustrate the equilibrium in Figure 1. In the case depicted, agent 1 likes the public good more than agent 2; here, the *unique* Nash equilibrium is for agent 1 to contribute the entire amount of the public good. Agent 2 is a complete free rider. If both agents have the same tastes for the public good, the reaction functions coincide and there is a whole range of equilibrium contributions, although there is still a unique equilibrium amount of the public good.

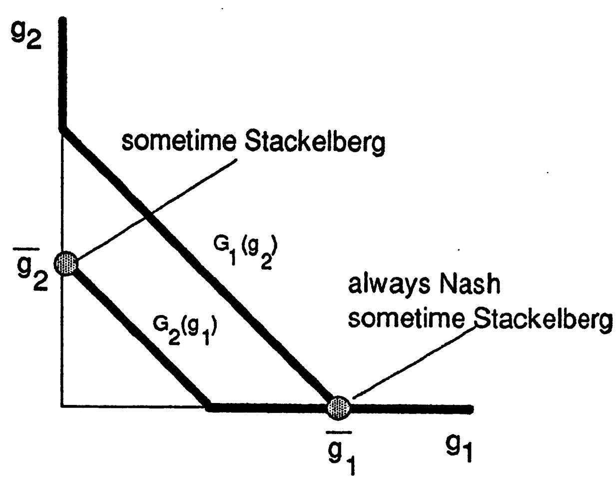

Figure 1. Nash equilibrium. The player who likes the public good the most contributes everything and the other player contributes nothing.

The Stackelberg Equilibrium

We assume that agent 1 moves first. In order to investigate the Stackelberg equilibrium, it is useful to plot the utility of agent 1 as a function of his contribution:

$$\begin{aligned} V_1(g_1) &= u_1(g_1 + G_2(g_1)) - g_1 \\ &= u_1(g_1 + \max\{\bar{g}_2 - g_1, 0\}) - g_1. \end{aligned}$$

It is easy to see that this function has the form

$$V_1(g_1) = \begin{cases} u_1(\bar{g}_2) - g_1 & \text{for } g_1 \leq \bar{g}_2 \\ u_1(g_1) - g_1 & \text{for } g_1 \geq \bar{g}_2. \end{cases}$$

This function is depicted in Figure 2.

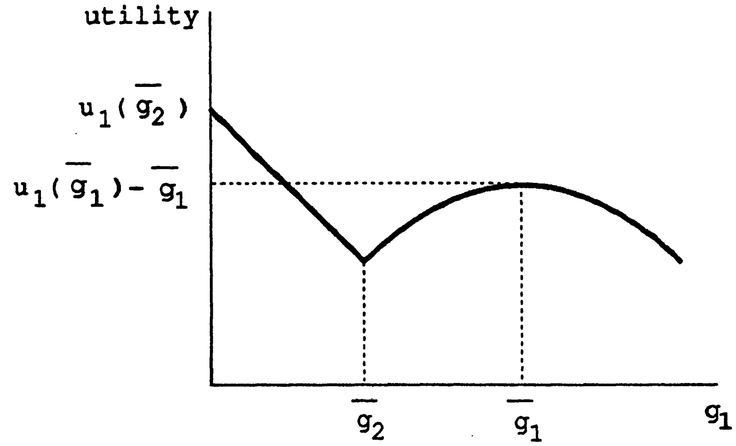

Figure 2. This is the utility function of the first contributor as a function of his gift. In the case illustrated, agent 1's optimal action is to contribute zero, but if the "hump" were higher, he would want to contribute $\bar{g}_1$.

It is clear by inspection that there are two possible optima: either the first agent contributes zero or $\bar{g}_1$. In order to determine which one of these possibilities is appropriate consider two cases.

Case 1. The agent who likes the good least is the first contributor. In this case, the optimal contribution by the first agent is zero. This is true since

$$u_1(\bar{g}_2) > u_1(\bar{g}_1) > u_1(\bar{g}_1) - \bar{g}_1.$$

Case 2. The agent who likes the public good the most is the first contributor. In this case, either outcome can occur. The easiest way to see this is by example. Suppose that agent i's utility for the public good is $u_i(G) = a_i \ln G$. If only agent i contributes he will contribute a_i. Normalize $a_1 = 1$. If agent 1 contributes the entire amount of the public good, he gets a utility of $\ln 1 - 1$. If he free-rides on agent 2's contribution, he gets $\ln a_2$. Agent 1 will prefer to contribute when

$$\ln 1 - 1 > \ln a_2,$$

which implies that $a_2 < 1/e \approx .37$. ∎

In general if the agents have tastes that are very similar, then the first agent will free ride on the second's contribution. However, if the first agent likes the public good *much* more than the second, then the first agent may prefer to contribute the entire amount of the public good himself.

Referring to Figure 1 we see that there are two possible Stackelberg equilibria: one is the Nash equilibrium, in which the agent who likes the good most contributes everything. The other Stackelberg equilibrium is where the agent who likes the good *least* contributes everything. This equilibrium cannot arise as a Nash equilibrium since the threat to contribute nothing is not credible in the simultaneous move game.

Note that it is always advantageous to move first. This is true since there are only two possible outcomes, $\bar{g}_1$ and $\bar{g}_2$ and the first mover gets to pick the one he prefers. That is, he gets to compare $u_1(\bar{g}_1) - \bar{g}_1$ to $u_1(\bar{g}_2)$.

Also note that the sum of the utilities is higher at the higher level of the public good. If you want to ensure that the higher level of the public good is provided, then you should make sure that the person who likes the good *least* moves first. Of course, if you want to minimize the level of the public good provided, you should ensure that the person who likes the good *most* moves first.

2. Bidding for the Right to Move First

Since the first mover always has an advantage in this game, we might consider auctioning off the right to move first. If the first mover likes the public good much more than the second, then he will provide the entire amount of the public good anyway, so it is no advantage to him to be first mover. The advantage to the first mover only arises when the players have similar tastes for the public good. In this case, each player would prefer to move first and free ride on the others contribution.

Consider, then, the case where agent 1 likes the public good a bit more than agent 2, so that $\bar{g}_1 > \bar{g}_2$, but not too much more. That is agent 1 prefers to free ride than to contribute the good himself, so that

$$u_1(\bar{g}_2) > u_1(\bar{g}_1) - \bar{g}_1.$$

Agent 1 can get $u_1(\bar{g}_2)$ by moving first and free riding. If he moves second, he must contribute and he gets $u_1(\bar{g}_1) - \bar{g}_1$. The amount that he would be willing to bid to be first, b_1, thus solves the equation

$$u_1(\bar{g}_2) - b_1 = u_1(\bar{g}_1) - \bar{g}_1,$$

or

$$b_1 = u_1(\bar{g}_2) - u_1(\bar{g}_1) + \bar{g}_1.$$

Similarly, the second agent's bid would be

$$b_2 = u_2(\bar{g}_1) - u_2(\bar{g}_2) + \bar{g}_2.$$

The difference between these two bids is

$$b_1 - b_2 = [u_1(\bar{g}_2) - u_1(\bar{g}_1)] + [u_2(\bar{g}_2) - u_2(\bar{g}_1)] - [\bar{g}_2 - \bar{g}_1]. \quad (1)$$

It seems plausible to suppose that the agent who values the public good more would be willing to pay more to move first. However, this is exactly wrong! Under our assumptions, the agent who values the public good *least* is willing to pay more for the first-mover position.

To see this, note that concavity of the utility functions gives us the following inequalities:

$$u_1(\bar{g}_2) - u_1(\bar{g}_1) \leq u_1'(\bar{g}_1)[\bar{g}_2 - \bar{g}_1]$$

$$u_2(\bar{g}_2) - u_2(\bar{g}_1) \leq u_2'(\bar{g}_1)[\bar{g}_2 - \bar{g}_1].$$

Substituting these into (1) we have

$$b_1 - b_2 \leq [u_1'(\bar{g}_1) + u_2'(\bar{g}_2) - 1](\bar{g}_2 - \bar{g}_1).$$

Since $u_2'(\bar{g}_2) = u_1'(\bar{g}_1) = 1$, this simplifies to

$$b_1 - b_2 \leq \bar{g}_2 - \bar{g}_1 < 0,$$

where the last inequality follows since $\bar{g}_1 > \bar{g}_2$.

It follows that $b_1 < b_2$. That is, the agent who likes the public good least will be willing to pay more in order to move first. As we've seen, this will ensure that the largest amount of the public good will be provided.

Essentially each agent is paying for the right to free ride on the other agent. The agent who likes the public good the most gets less of the public good than he would contribute on his own, while the agent who likes the public good the least gets more than he would contribute on his own. Hence the agent who likes the public good the least is willing to pay more for the right to move first than the agent who likes the public good the most.

3. Subsidizing the Other Agent

Suppose that a subsidy is offered to one of the agents in a simultaneous-move public goods contribution game. Roberts (1987) and Bergstrom (1989) have shown that typically each agent would prefer that the *other* agent receives the subsidy. In the Roberts-Bergstrom framework, the subsidy is paid for by a lump-sum tax on both agents.

The application of this observation in our framework is rather nice. The general idea is easiest to see in the case where the agents have identical utility functions for the public good, $u(G)$.

Consider, first, what happens in without any subsidies. In this case, the Nash equilibrium amount of the public good is determined by the condition

$$u'(G^n) = 1.$$

Any set of contributions (g_1^n, g_2^n) such that $g_1^n + g_2^n = G^n$ constitute a Nash equilibrium.

Suppose now that agent 1 offers to subsidize agent 2 with a subsidy s_2. This means that the cost to agent 2 if he contributes g_2 is $(1 - s_2)g_2$, while agent 1 must pay the subsidy amount of $s_2 g_2$.

Suppose that agent 1 offers a tiny subsidy to agent 2. Then in the unique Nash equilibrium agent 2 will contribute the entire amount of the public good himself. Let this amount be denoted by $G(s_2)$; it is defined by the equation

$$u_2'(G(s_2)) = 1 - s_2.$$

Agent 1's utility from offering this subsidy is

$$u_1(G(s_2)) - s_2 G(s_2).$$

Originally agent 1 had utility $u_1(G^n) - g_1^n = u_1(G(0)) - g_1^n$. Hence the increase in utility from offering the subsidy is

$$u_1(G(s_2)) - u_1(G(0)) + g_1^n - s_2 G(s_2).$$

As s_2 approaches zero, this expression converges to g_1^n. Hence as long as agent 2 is not contributing the entire amount originally, so that $g_1^n > 0$, it follows that there is a small enough subsidy s_2 such that agent 1's utility strictly increases if he offers it to agent 2.

The intuition is the following: if agent 1 offers a very tiny subsidy to agent 2, agent 2 will end up contributing the entire amount of the public good in the resulting Nash equilibrium. But a tiny subsidy hardly costs agent 1 anything. Hence he is always better off offering the subsidy.

Note that this is a stronger result that proved by Roberts (1987) and Bergstrom (1989). He showed that each agent preferred that the other agent be subsidized if the subsidy was paid by an *equal* lump sum tax. We have shown that each agent prefers to subsidize the other agent even if he must pay the *entire* amount of the subsidy himself. However, they were concerned with more general form of preferences than those examined here.

4. Equilibrium Subsidies

We have seen that each agent will prefer to subsidize the other agent in our contribution game. It is natural to ask for the *equilibrium* level of subsidization. Suppose that each agent simultaneously names a rate at which he is willing to subsidize the other agent. Then, given these subsidy rates, the agents play a simultaneous contribution game. What are the equilibrium value of the subsidies?

In order to investigate this question, let us first examine the Pareto efficient level of provision of the public good. This is the level of the public good, $G = g_1 + g_2$, that solves the problem

$$\max_{g_1, g_2} u_1(g_1 + g_2) + u_2(g_1 + g_2) - g_1 - g_2.$$

Hence the Pareto efficient amount of the public good satisfies

$$u_1'(G^e) + u_2'(G^e) = 1. \tag{2}$$

This is just the familiar condition that the sum of the marginal willingnesses-to-pay must equal marginal cost.

Returning to the contribution game, suppose that agent i has his contributions subsidized by the amount s_i. Then if the subsidies are chosen to satisfy the conditions

$$\begin{aligned} u_1'(G^e) &= 1 - s_1^e \\ u_2'(G^e) &= 1 - s_2^e, \end{aligned} \tag{3}$$

the Pareto efficient level of the public good will be provided. The subsidy rates here are effectively playing the role of Lindahl prices; exploiting this analogy, we will refer to the subsidies defined in equation (3) as the *Lindahl subsidies.*

Note that (2) and (3) together imply

$$s_1^e + s_2^e = 1. \tag{4}$$

Now suppose that the subsidy rates are chosen independently by the agents, each agent recognizing that once the subsidy rates are chosen they will play a Nash contribution game. What subsidy rates will be chosen in equilibrium?

Theorem 1. *The unique subgame perfect equilibrium of the subsidy setting game involves setting the subsidy rates to be the Lindahl subsidies.*

Proof. Here we show that the efficient subsidies are equilibrium subsidies. The proof that the equilibrium is unique involves investigating several cases and is given in the appendix.

Suppose that the subsidies are the Lindahl subsidies. Then there is a whole range of Nash equilibria, namely any pair (g_1^n, g_2^n) such that $g_1^n + g_2^n = G^e$. This is because any such pair will satisfy the appropriate first-order conditions given in (3). Suppose we have settled on some such equilibrium (g_1^n, g_2^n). We need to show that neither agent would want to increase or decrease the subsidy that they offer to the other agent.

1. ***Would agent 1 want to increase his subsidy to agent 2?***

If agent 1 increases his subsidy to s_2 from s_2^e, agent 2 will end up contributing the entire amount of the public good. Let $G(s_2)$ be the amount that results as a function of the subsidy rate. Of course $G(s_2^e) = G^e$.

By concavity of the utility function

$$u_1(G(s_2)) - u_1(G^e) \leq u_1'(G^e)[G(s_2) - G^e].$$

Using the fact that $u_1'(G^e) = 1 - s_1^e = s_2^e$, and that $G^e = g_1^n + g_2^n$ we can write this inequality as

$$\begin{aligned} u_1(G(s_2)) - s_2^e G(s_2) &\leq u_1(G^e) - (1 - s_1^e)G^e \\ &\leq u_1(G^e) - (1 - s_1^e)g_1^n - s_2^e g_2^n. \end{aligned} \tag{5}$$

Using the fact that $s_2 > s_2^e$, we have

$$u_1(G(s_2)) - s_2 G(s_2) \leq u_1(G(s_2)) - s_2^e G(s_2). \tag{6}$$

Combining (5) and (6), we have

$$u_1(G(s_2)) - s_2 G(s_2) \leq u_1(G^e) - (1 - s_1^e)g_1^n - s_2^e g_2^n.$$

The left-hand side of this inequality is the utility that agent 1 gets if he chooses $s_2 > s_2^e$. The right-hand side is the utility he gets at the original Nash equilibrium. Hence, agent 1 will not wish to increase the subsidy rate that he offers to agent 2.

2. *Would agent 1 want to decrease his subsidy to agent 2?*

If agent 1 decreases his subsidy to agent 2, agent 1 will end up contributing the entire amount of the public good. Indeed, he will contribute the same amount that agent 2 was contributing. This will give him a utility of $u_1(G^e) - (1 - s_1^e)G^e$ and we need to compare this to his original utility, $u_1(G^e) - (1 - s_1^e)g_1^n - s_2^e g_2^n$.

Note that

$$(1 - s_1^e)G^e = (1 - s_1^e)(g_1^n + g_2^n) = (1 - s_1^e)g_1^n + s_2^e g_2^n.$$

Hence

$$u_1(G^e) - (1 - s_1^e)G^e = u_1(G^e) - (1 - s_1^e)g_1^n - s_2^e g_2^n.$$

Hence agent 1's utility does not increase if he decreases his subsidy to agent 2.

The arguments for agent 2 are parallel. Hence neither agent would want to change their subsidy rates to the other agent, and the pair (s_1^e, s_2^e) is a Nash equilibrium. ∎

The remarkable thing about this game is that it yields the Lindahl equilibrium as a Nash equilibrium. Hurwicz (1979) investigates the problem implementing the Lindahl equilibrium as a Nash equilibrium for general preferences, but he requires a much more complicated "penalty" for deviating from the equilibrium choice. Also, the Hurwicz outcome function is not feasible away from equilibrium.

These difficulties do not arise in our framework. In the context of quasilinear utility, the penalty for deviation from the Lindahl prices is "automatic" in that deviations from the Lindahl prices immediately imply that one agent free rides on the other.

Clarke (1971) and Groves (1976), (1979) describe a demand revealing process that yields the Pareto optimal amount of the public good. However the Clarke-Groves mechanisms are not generally Pareto efficient in that they may require some payment of the private good. The mechanism that we propose is fully efficient.

In Varian (1989) I describe a related mechanism that works for n agents with general utility functions; i.e., preferences are not restricted to the quasilinear form. The related mechanism also has two stages. In the first stage each agent announces a "price" for each other agent. In the second stage, each agent contributes to the public good, and the cost of his contribution depends on the price determined in the first stage. Each agent also bears an additional cost which depends on the other agents' contributions and on the dispersion of the prices. I show that the only subgame perfect equilibria of this game are Lindahl allocations.

5. General Utility Functions

The quasilinear case is very special. Suppose that we now consider a general utility function $u_i(G, x_i)$ where G is the level of the public good and x_i is the private consumption of agent i. As before, we assume that utility is a differentiable, concave function.

We first derive the form of the reaction function. Agent 2's maximization problem is

$$\max_{x_2, g_2} u_2(g_1 + g_2, x_2)$$
$$\text{such that } g_2 + x_2 = w_2$$
$$g_2 \geq 0.$$

We can add g_1 to each side of the constraints in this problem and use the definition $G = g_1 + g_2$ to rewrite this problem as

$$\max_{x_2, G} u_2(G, x_2)$$
$$\text{such that } G + x_2 = w_2 + g_1$$
$$G \geq g_1.$$

In this interpretation, agent 2 is in effect choosing the level of the public good, subject to the constraint that the level that he chooses is at least as large as that contributed by agent 1.

Following Bergstrom, Blume and Varian (1986) we note that this problem is simply a standard consumer demand problem except for the inequality constraint. Let $f_2(w)$ be agent 2's Engel function for the public good as a function of his wealth. This is the function that gives the optimal level of G for agent 2, holding prices fixed at (1,1) and letting wealth vary. It follows that

$$G = g_1 + g_2 = \max\{f_2(w_2 + g_1), g_1\}.$$

Subtracting g_1 from each side of this equation, we have the reaction function

$$G_2(g_1) = \max\{f_2(w_2 + g_1) - g_1, 0\}.$$

The reaction function for agent 1 has the simple interpretation that agent 1 will contribute the amount of the public good that he would demand if his wealth were $w_1 + g_2$ *minus* the amount contributed by the other agent, or zero, whichever is larger.

This expression for the reaction function is useful since we know quite a bit about the behavior of Engel functions. For example, it is quite natural to make the following assumption:

Assumption. Both the public and the private good are strictly normal goods at all levels of wealth. It follows that $1 > f_i'(w) > 0$.

Given this assumption it is easy to see the general shape of the reaction function. Consider the reaction function for agent 2. When $g_1 = 0$, agent 2 will contribute $f_2(w_2)$. As g_1 increases, the contribution of agent 2 will decrease, less than one-for-one. If $f_2'(w)$ is bounded away from zero, then there will be some contribution by agent 1, g_1^c, at which $f_2(w_2 + g_1^c) - g_1^c = 0$ and agent 1 will contribute nothing. This amount, g_1^c, is the amount where agent 1's contribution just crowds out agent 2's contribution.

We summarize some properties of the reaction function in the following fact, the proof of which follows immediately from the assumption.

Fact 1. *The reaction function $G_2(g_1)$ is a nonincreasing function. It will be strictly decreasing when it is not equal to zero. The function $H(g_1) = g_1 + G_2(g_1)$ is a strictly increasing function.*

As before, we can use this reaction function to calculate the Nash equilibria and the Stackelberg equilibrium. A Nash equilibrium is a solution (g_1^n, g_2^n) to the following equations

$$g_1^n = \max\{f_1(w_1 + g_2^n) - g_2^n, 0\}$$

$$g_2^n = \max\{f_2(w_2 + g_1^n) - g_1^n, 0\}.$$

A Stackelberg equilibrium is a pair $(g_1^s, G_2(g_1^s))$ in which g_1^s solves

$$\max_{g_1} \; u_1(g_1 + \max\{f_2(w_2 + g_1) - g_1, 0\}, w_1 - g_1).$$

Our main interest is in comparing the solutions of these two sets of equations. This comparison is made simpler by noting that Bergstrom, Blume, and Varian (1986) have proved that under the normality assumption we have made there is a *unique* Nash equilibrium. There will also be one Stackelberg equilibrium for each ordering of the agents.

6. A Cobb-Douglas Example

A useful example to fix ideas is the Cobb-Douglas case:

$$u_i(G, x_i) = a_i \ln(g_1 + g_2) + (1 - a_i)\ln(w_i - g_i),$$

where $0 < a_i < 1$. In this case the agent spends a constant fraction a_i of his wealth on the public good so that the reaction function for agent i takes the form

$$G_i(g_j) = \max\{a_i(w_i + g_j) - g_j, 0\}.$$

Here the slope of the reaction function is $a_i - 1$ up to the point $g_j^c = a_i w_i/(1 - a_i)$, the point where agent j's contribution just crowds out agent i's contribution. After this point, agent i contributes zero. It is worth noting that the reaction function will have this form for *any* homothetic utility function.

Nash equilibria are the solutions to

$$g_1 = \max\{a_1(w_1 + g_2) - g_2, 0\}$$
$$g_2 = \max\{a_2(w_2 + g_1) - g_1, 0\}.$$

The Stackelberg equilibrium is the solution to

$$\max_{g_1} \; a_1 \ln(g_1 + \max\{a_2(w_2 + g_1) - g_1, 0\}) + (1 - a_1)\ln(w_1 - g_1).$$

Straightforward computations show that the interior solutions to these equations imply equilibrium levels of the public good of

$$G^n = \frac{a_1 a_2(w_1 + w_2)}{a_1 + a_2 - a_1 a_2}$$
$$G^s = a_1 a_2(w_1 + w_2).$$

Note that in this example, $G^n > G^s$ since $a_i < 1$.

7. Results for General Utility Functions

We have three sets of results. The first set of results concerns who contributes and who free rides. The second set of results concerns the effect of redistributions of wealth. The third set of results concerns how the amount of the public good provided in the Stackelberg equilibrium compares to the amount provided in the Nash equilibrium.

Recall that $\bar{g}_i$ denotes the optimal contribution of agent i if the other person contributes nothing and that g_1^c is defined by $f_2(w_2+g_1^c) = g_1^c$; this is the level of g_1 at which agent 1's contribution just crowds out agent 2's contribution.

Free Riding

Fact 2. *If $\bar{g}_1 < g_1^c$ then both agents must contribute in the Stackelberg equilibrium.*

Proof. Evaluate the right derivative of agent 1's utility function at g_1^c. We have

$$\frac{\partial u_1(g_1^c, w_1 - g_1^c)}{\partial G} - \frac{\partial u_1(g_1^c, w_1 - g_1^c)}{\partial x_1} < 0.$$

The inequality follows since the derivative equals zero at $\bar{g}_1$, and $g_1^c > \bar{g}_1$. (Recall that $u_1(g_1, w_1 - g_1)$ is a concave function.) It follows that agent 1's utility will increase if he contributes less than g_1^c, even if he is the only one to contribute. The fact that the other agent will also contribute can only increase the first agent's utility. Hence the Stackelberg equilibrium must involve contributions by both agents. ∎

Fact 3. *If there is a Nash equilibrium with $g_1^n = 0$, then this is also a Stackelberg equilibrium.*

Proof. By definition of Nash equilibrium, agent 2 is on his reaction curve, so we only need to show that agent 1 is optimized. If agent 1 contributes 0, then agent 2 must contribute $\bar{g}_2$. Let $g_1 > 0$ be any other possible contribution by agent 1. Then we have :

$$u_1(\bar{g}_2, w_1) > u_1(g_1 + \bar{g}_2, w_1 - g_1) > u_1(g_1 + G_2(g_1), w_1 - g_1).$$

The first inequality follows from the Nash assumption. The second inequality follows since $G_2(0) = \bar{g}_2$ and $G_2(g_1)$ is a nonincreasing function. ∎

Fact 4. *Suppose that we have a Stackelberg equilibrium* (g_1^s, g_2^s)*. Let* $(\Delta w_1, \Delta w_2)$ *be a redistribution of wealth such that* $g_i \geq \Delta w_i$ *for* $i = 1, 2$*. Then the Stackelberg equilibrium after this redistribution is* $(g_1^s + \Delta w_1, g_2^s + \Delta w_2)$ *and the total amount of the public good remains unchanged.*

Proof. Note that the requirement that $g_i \geq \Delta w_i$ implies that the assumptions can only be satisfied when each person is contributing a positive amount. The first-order condition for an optimum is

$$\frac{\partial u_1(g_1 + g_2, w_1 - g_1)}{\partial G} f_2'(w_2 + g_1) - \frac{\partial u_1(g_1 + g_2, w_1 - g_1)}{\partial x_1} = 0.$$

Now suppose that each agent changes his contribution by the amount of his wealth change so that $\Delta g_i = \Delta w_i$ for $i = 1, 2$. Note that since $\Delta w_1 + \Delta w_2 = 0$ we must have $\Delta g_1 + \Delta g_2 = 0$.

Under this rule none of the arguments of any of the functions in the first-order condition change. The conclusion follows immediately. ∎

Warr (1983) and Bergstrom, Blume, and Varian (1986) show that essentially the same result holds in an (interior) Nash equilibrium. Bergstrom, Blume, and Varian (1986) also investigate the boundary cases in some detail. In the two-agent context we are investigating here the analysis of the boundary cases are quite straightforward.

Fact 5. *Suppose that person 1 is contributing and person 2 is not. Then a redistribution from 2 to 1 will increase the amount of the public good, while a redistribution from 1 to 2 can decrease or increase the amount of the public good.*

Proof. A distribution from 2 to 1 increases the amount of the public good since $f_1(w_1)$ is an increasing function. Since g_2 is equal to zero it will remain zero at lower levels of wealth.

A redistribution from 1 to 2 will decrease the level of the public good for small redistributions by the monotonicity of $f_1(w_1)$. But when w_1 gets small enough relative to w_2, it can easily happen that person 2 starts to contribute, thereby increasing the amount of the public good. ∎

Fact 6. *Suppose that person 2 is contributing and person 1 is not. Then a transfer from 1 to 2 will increase the level of the public good, while a transfer from 2 to 1 can increase or decrease the level of the public good.*

Proof. A transfer from 1 to 2 will increase the level of the public good by the monotonicity of $f_2(w_2)$, and a larger contribution by person 2 will never induce agent 1 to begin contributing.

A small transfer of wealth from 2 to 1 will decrease the level of the public good, but a larger transfer may induce 1 to start contributing. ∎

Comparison to the Nash Equilibrium

Our main result has to do with the comparison of the Nash and Stackelberg equilibria.

Theorem 2. *The amount of the public good contributed by agent 1 in the Stackelberg equilibrium is never larger than the amount provided by agent 1 in the Nash equilibrium. That is,* $g_1^s \leq g_1^n$.

Proof. The proof is by contradiction. Assume the theorem is not true so that $g_1^s > g_1^n$. We consider two cases, first when agent 2 contributes zero in the Nash equilibrium, and second, where agent 2 makes a positive contribution in the Nash equilibrium.

Case 1. $g_2^n = 0$.

Since $G_2(g_1^n) = 0$ and $G_2(g_1)$ is a nonincreasing function, we also have $G_2(g_1^s) = 0$. Now, by the Nash assumption,

$$u_1(g_1^n, w_1 - g_1^n) > u_1(g_1^s, w_1 - g_1^s) = u_1(g_1^s + G_2(g_1^s), w_1 - g_1^s).$$

It follows that agent 1's utility must decrease when moving from g_1^n to g_1^s which means that g_1^s cannot be a Stackelberg equilibrium.

Case 2. $G_2(g_1^n) > 0$.

First, we observe that since $G_2(g_1)$ is strictly decreasing in this case, the assumption that $g_1^s > g_1^n$ implies

$$G_2(g_1^s) < G_2(g_1^n) = g_2^n.$$

Rearranging this inequality,

$$\epsilon = g_2^n - G_2(g_1^s) > 0. \tag{7}$$

Second, we know that agent 1 must get at least as large a utility at the Stackelberg equilibrium as at the Nash equilibrium so:

$$u_1(g_1^n + g_2^n, w_1 - g_1^n) = u_1(g_1^n + G_2(g_1^n), w_1 - g_1^n) < u_1(g_1^s + G_2(g_1^s), w_1 - g_1^s). \tag{8}$$

The fact that utility is increasing in the amount of the public good and that $\epsilon > 0$ implies:

$$u_1(g_1^s + G_2(g_1^s), w_1 - g_1^s) < u_1(g_1^s + G_2(g_1^s) + \epsilon, w_1 - g_1^s). \tag{9}$$

Combining (8) and (9), and using (7) for the definition of ϵ,

$$u_1(g_1^n + g_2^n, w_1 - g_1^n) < u_1(g_1^s + g_2^n, w_1 - g_1^s).$$

But this contradicts the assumption that (g_1^n, g_2^n) is a Nash equilibrium. ∎

Corollary. *The total amount of the public good in the Stackelberg equilibrium is less than or equal to the total amount provided in the Nash equilibrium.*

Proof. According to Fact 1, the function $H(g_1) = g_1 + G_2(g_1)$ is an increasing function. Therefore,

$$H(g_1^n) = g_1^n + G_2(g_1^n) = g_1^n + g_2^n \geq g_1^s + G_2(g_1^s) = H(g_1^s).$$

The corollary follows. ∎

8. Incomplete Information

Our analysis has concerned the case where each agent knows the preferences and wealth of the other agent. One could also consider a model with *incomplete* information in which one or both of the agents does not know these things for certain.

The second agent reacts passively, making his optimal choice given the first agent's contribution. Hence it is irrelevant whether or not he knows anything about the first agent. The only interesting uncertainty concerns the first agent's knowledge of the second agent's type.

Consider the quasilinear model examined earlier. In this case all that is relevant from agent 1's point of view is the value of $\bar{g}_2$. Regard $\bar{g}_2$ as a random variable with some probability density function $f(\bar{g}_2)$ and suppose that agent 1 seeks to maximize expected utility.[1]

Suppose that $\bar{g}_2$ is distributed between 0 and ∞. The expected utility of agent 1 is given by

$$v_1(g_1) = \int_0^{\infty} [u(g_1 + \max\{\bar{g}_2 - g_1, 0\}) - g_1] f(\bar{g}_2)\, d\bar{g}_2,$$

which can be written as

$$v(g_1) = \int_0^{g_1} [u(g_1) - g_1] f(\bar{g}_2)\, d\bar{g}_2 + \int_{g_1}^{\infty} [u(\bar{g}_2) - g_1] f(\bar{g}_2)\, d\bar{g}_2.$$

Letting $F(g)$ be the cumulative distribution function for f, we can write this as

$$v(g_1) = [u(g_1) - g_1]F(g_1) + \int_{g_1}^{\infty} [u(\bar{g}_2) - g_1] f(\bar{g}_2)\, d\bar{g}_2.$$

Differentiating this expression with respect to g_1 we have

$$v_1'(g_1) = [u_1(g_1) - g_1]f(g_1) + [u_1'(g_1) - 1]F(g_1) - [u_1(g_1) - g_1]f(g_1) - \int_{g_1}^{\infty} f(\bar{g}_2)\, d\bar{g}_2.$$

Simplifying gives us

$$v_1'(g_1) = u_1'(g_1)F(g_1) - 1.$$

[1] We assume that the von Neumann-Morgenstern utility function takes the quasilinear form. This is restrictive, but seems necessary for a simple analysis.

Note that when $g_1 = 0$, the probability that $\bar{g}_2$ is less than this is zero, so that $v_1'(0) = -1$. If g_1 is large enough so that agent 1 is certain that $\bar{g}_2$ is less than g_1, then $v_1'(g_1) = u_1'(g_1) - 1$. Hence, agent 1's utility as a function of his gift is similar to the shape depicted in Figure 2. Depending on the beliefs of agent 1 about agent 2's maximum contribution, agent 1 will either choose to free ride, or to contribute an amount g_1^* that satisfies the condition

$$v_1'(g_1^*) = u_1'(g_1^*)F(g_1^*) - 1 = 0.$$

This marginal condition is quite intuitive. If agent 1 decides to contribute a bit more of the public good, he will get $u_1'(g_1^*)$, but only if agent 2 has $\bar{g}_2 < g_1^*$. Otherwise, agent 1 will get no incremental utility from his contribution—since his contribution would just crowd out some of the public good that agent 2 would have given anyway. Hence the *expected* marginal utility of agent 1's contribution is $u_1'(g_1^*)$ times the probability that $\bar{g}_2 < g_1$, which is just $u_1'(g_1^*)F(g_1^*)$. The optimal contribution is determined by the condition that this expected marginal utility must equal the (certain) marginal cost of the contribution.

How does this amount compare to $\bar{g}_1$, which is what agent 1 would contribute under certainty? Note that $v_1'(\bar{g}_1) = u_1'(\bar{g}_1)F(\bar{g}_1) - 1 = F(\bar{g}_1) - 1$. As long as there is some possibility that agent 2 will have $\bar{g}_2 > \bar{g}_1$, we will have $F(\bar{g}_1) < 1$ and $v_1'(\bar{g}_1)$ will be negative. Hence, agent 1's utility increases if he contributes less than $\bar{g}_1$. Assuming a concave shape for $v_1(g_1)$ this implies that the equilibrium contribution in the presence of uncertainty is less than the contribution under certainty.

Intuitively, the possibility that agent 2 may value the good more than agent 1 leads agent 1 to reduce his contribution to the public good, hoping to free ride on agent 2's contribution.

9. Summary

We have examined some sequential games involving contributions to a public good. If preferences are quasilinear, then:

1. The sequential equilibrium of the contribution game will provide the same or less of the public good than the simultaneous move game. This is also true for general utility functions.

2. The player who likes the public good least will bid the most to move first.

3. Each player would like to subsidize the other player's contributions. If both players choose subsidy rates and then play the voluntary contribution game, a Lindahl equilibrium is the unique subgame perfect equilibrium of this two-stage game.

4. The equilibrium of the sequential move game is independent of small redistributions of wealth.

5. If the first agent is uncertain about the type of the second agent, he will tend to contribute less to the public good.

Appendix. Proof of uniqueness of equilibrium

Consider any subsidies (s_1^*, s_2^*) other than the efficient subsidies. We need to show that there is some action by one of the agents that will increase his utility. Given (s_1^*, s_2^*), there will be some amount $G^* = g_1^* + g_2^*$ of the public good provided, and agent 1 will get utility

$$u_1(G^*) - (1 - s_1^*)g_1^* - s_2^* g_2^*. \tag{10}$$

There are several cases to consider.

Case 1. The equilibrium amount of the public good is less than the efficient amount: $G^* < G^e$ and both agents are contributing.

Since G^* provides less than the efficient amount of the public good,

$$u_1'(G^*) + u_2'(G^*) > 1,$$

which means that

$$1 - s_1^* + 1 - s_2^* > 1,$$

or

$$1 - s_1^* - s_2^* > 0. \tag{11}$$

Suppose that agent 1 increases his subsidy on agent 2 to some amount $s_2 > s_2^*$. Then in the contribution stage of the game, agent 2 will contribute the entire amount of the public good, which we denote by $G(s_2)$.

Agent 1 will then get utility $u_1(G(s_2)) - s_2 G(s_2)$. The difference between this utility and the utility at (s_1^*, s_2^*) given in equation (10) is:

$$u_1(G(s_2)) - s_2 G(s_2) - u_1(G^*) + (1 - s_1^*)g_1^* + s_2^* g_2^*. \tag{12}$$

Let s_2 approach s_2^*. Since $G(s_2)$ is a continuous function of s_2, expression (12) converges to

$$-s_2^* G^* + (1 - s_1^*)g_1^* + s_2^* g_2^*. \tag{13}$$

Substituting $G^* = g_1^* + g_2^*$ and simplifying, we have

$$(1 - s_1^* - s_2^*)g_1^*.$$

Using equation (11) and the fact that $g_1^* > 0$, we have

$$(1 - s_1^* - s_2^*)g_1^* > 0.$$

Hence, a subsidy level s_2 that is slightly larger than s_2^* will increase the utility of agent 1.

Case 2. The equilibrium amount of the public good is less than the efficient amount and one agent contributes zero.

Suppose that agent 1 is contributing a positive amount and agent 2 is contributing zero. This means that $u_2'(G^*) \leq 1 - s_2^*$. Let agent 1 increase his subsidy of agent 2 up to the point where agent 2 is just willing to contribute. That is, agent 1 increases agent 2's subsidy to s_2^{**}, where s_2^{**} satisfies

$$u_2'(G^*) = 1 - s_2^{**}.$$

In doing this, agent 1's utility doesn't change. Now let agent 1 increase the subsidy to some amount s_2 slightly beyond s_2^{**}, and let s_2 approach s_2^{**}. According to equation (13), the limit of the change in utility is given by

$$-s_2^{**}G^* + (1 - s_1^*)g_1^* + s_2^* g_2^*. \tag{14}$$

Since $g_2^* = 0$, and $g_1^* = G^*$, this expression becomes

$$(1 - s_1^* - s_2^{**})G^* > 0.$$

The last inequality follows since we have an inefficiently small amount of the public good, so that $u_1'(G^*) + u_2'(G^*) < 1$.

Case 3. The equilibrium amount of the public good is more than the efficient amount: $G^* > G^e$ and both agents are contributing.

In this case, the argument we used to derive equation (11) implies that

$$1 - s_1^* - s_2^* < 0. \tag{15}$$

Suppose that agent 1 considers cutting the subsidy to agent 2 to some s_2 slightly smaller than s_2^*. If this is done, agent 2 will contribute zero. Since agent 1's subsidy hasn't changed, he will contribute the entire amount of the public good G^*, giving him utility $u(G^*) - (1 - s_1^*)G^*$. The increase in agent 1's utility is given by

$$u_1(G^*) - (1 - s_1^*)G^* - u_1(G^*) + (1 - s_1^*)g_1^* + s_2^* g_2^*.$$

Using the fact that $G^* = g_1^* + g_2^*$, we can simplify this expression to

$$(s_1^* + s_2^* - 1)g_1^* > 0,$$

where the inequality follows from equation (15).

Case 4. The equilibrium amount of the public good is more than the efficient amount: $G^* > G^e$ and one agent contributes zero.

Suppose that agent 1 contributes zero. Then agent 1 has utility

$$u_1(G(s_2^*)) - s_2^* G(s_2^*) = u_1(G^*) - s_2^* G^*. \tag{16}$$

Since we have an inefficiently large amount of the public good being provided,

$$u_1'(G^*) + u_2'(G^*) < 1.$$

Since agent 2 is contributing, we can write

$$u_1'(G^*) + 1 - s_2^* < 1,$$

or,

$$u_1'(G^*) - s_2^* < 0. \tag{17}$$

Differentiate equation (16) with respect to s_2 and evaluate the derivative at s_2^*:

$$[u_1'(G^*) - s_2^*]G'(s_2^*) - G(s_2^*).$$

The bracketed term is negative by equation (17). The derivative $G'(s_2^*)$ is positive since increasing the subsidy on an agent who is contributing must increase his contribution. Hence the sign of the entire expression is negative.

This means that agent 1's utility must decrease if he increases his subsidy of agent 2 by a small amount. Conversely, agent 1's utility will increase if he *decreases* his subsidy of agent 2 by a small amount, which is what we wanted to show. ∎

References

Bergstrom, T., L. Blume, and H. Varian (1986) "On the Private Provision of Public Goods," *Journal of Public Economics*, 29,25–49.

Bergstrom, T. (1989), "Love and Spaghetti, The Opportunity Cost of Virtue," *Journal of Economic Perspectives*, 3:2, 165–173.

Admanti, A. and M. Perry (1988), "Joint Projects without Commitment," Graduate School of Business Research Paper Series, Stanford University.

Clarke, E. (1971), "Multipart Pricing of Public Goods," *Public Choice*, 11, 17–33.

Cornes, R. and T. Sandler (1986), *The Theory of Externalities, Public Goods, and Club Goods,* Cambridge University Press, Cambridge, England.

Groves, T. (1976), "Information, Incentives, and the Internalization of Production Externalities," in Lin, S. A. (ed.), *Theory and Measurement of Economic Externalities*, Academic Press, New York.

Groves, T. (1979), "Efficient Collective Choice when Compensation is Possible," *Review of Economic Studies*, 46, 227–241.

Hurwicz, L. (1979), "Outcome Functions Yielding Walrasian and Lindahl Allocations at Nash Equilibrium Points," *Review of Economic Studies*, 46:143, 217–226.

McClean, R. and M. Riordan (1989), "Industry Structure with Sequential Technology Choice," *Journal of Economic Theory,* 47, 1–21.

Roberts, R. (1987), "Financing Public Goods," *Journal of Political Economy*, 95, 420–437.

Varian, H. (1989) "A Mechanism that Implements Efficient Allocations in the Presence of Externalities and Public Goods," University of Michigan CREST Working Paper.

Warr, P. (1983) "The Private Provision of a Public Good is Independent of the Distribution of Income," *Economic Letters*, 13, 207–11.

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