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Revealed Preference With a Subset of Goods

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Suppose that you observe n choices of k goods and prices when the consumer is actually choosing from a set of $k+1$ goods. Then revealed preference theory puts essentially no restrictions on the behavior of the data. This is true even if you also observe the quantity demanded of good $k+1$, or its price. The proofs of these statements are not difficult.

Suppose that we are given n observations on a consumer's choices of k goods, (p_i, x_i), where p_i and x_i are nonnegative k-dimensional vectors. Under what conditions can we find a utility function $u : R^k \to R$ that rationalizes these observations? That is, when can we find a utility function that achieves its constrained maximum at the observed choices?

This is, of course, a classical question of consumer theory. It has been addressed from two distinct viewpoints, the first known as *integrability theory* and the second known as *revealed preference theory*. Integrability theory is appropriate when one is given an entire demand *function* while revealed preference theory is more suited when one is given a finite set of demand observations, the case described above. In the revealed preference case, it is well known that some variant of the Strong Axiom of Revealed Preference (SARP) is a necessary and sufficient condition for the data (p_i, x_i) to be consistent with utility maximization.

Now suppose that we are also given n observations on another chosen good, (z_i), $i = 1, \ldots, n$ where z_i is a nonnegative scalar, but we do not have a price series to accompany these observations. We now ask when will there exist a utility function $u : R^{k+1} \to R$ that rationalizes the data (p_i, x_i, z_i) for $i = 1, \ldots, n$? Equivalently, we can ask when we can find a series of scalar prices (q_i) such that the entire data set (p_i, q_i, x_i, z_i) is consistent with utility maximization?

This question is of considerable interest, since we typically can observe only a subset of the goods chosen by a consumer. For example, we would expect that the planned consumption of future goods would enter

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the utility function, and these are generally not observed. Similarly, contingent consumption plans are also not observed.

The variables (z_i) may also be interpreted as a "demographic" variable such as household size or household location. Such variables are often used in applied demand analysis to control for taste differences. Then we are asking when we can find a family of utility functions, $u(x,z)$, parameterized by z, such that for each fixed z_i the data (p_i, x_i) satisfy the restrictions imposed by demand theory. I will return to this interpretation below.

The first interpretation, that of a missing price, has been addressed by Polemarchakis (1983) in the intertemporal context, using the machinery of integrability theory. He shows that there are essentially no observable restrictions on demand functions in this context. Here we examine these issues using the methods of revealed preference theory and reach a similar conclusion. However, if one is willing to place bounds on the expenditure on the unobserved good, then we show that demand theory does impose some restrictions on the observed behavior.

1. OBSERVED QUANTITIES, UNOBSERVED PRICES

Let us first describe the form of the revealed preference conditions that we will use. If all prices and goods are observed, a necessary and sufficient condition for these choices to be consistent with utility maximization is that the data satisfy the Generalized Axiom of Revealed Preference (GARP).

DEFINITION. *An observation x_i is* directly revealed preferred *to a bundle x (written $x_i R^0 x$) if $p_i x_i \geq p_i x$. An observation is* revealed preferred *to a bundle x (written $x_i R x$) if there is some sequence such that $x_i R^0 x_j \ldots x_k R^0 x$. A set of data (p_i, x_i), $i = 1, \ldots n$ satisfies the* Generalized Axiom of Revealed Preference *(GARP) if $x_i R x_j$ implies $p_j x_j \leq p_j x_i$.*

Further information on these concepts may be found in Varian (1982) which in turn is based on earlier work by Afriat (1967) and Diewert (1973). GARP is a generalization of the Strong Axiom of Revealed Preference (SARP) that allows for different quantity vectors to be observed for a single price vector. It is therefore appropriate for examining cases where preferences may be weakly convex, rather than strictly convex preferences as is required by SARP.

It is now easy to answer the question posed above. We simply ask when we can construct a price series (q_i) for $i = 1, \ldots n$ such that the entire data set (p_i, q_i, x_i, z_i) satisfies GARP. As it turns out this can be done simply by choosing large enough values for q_i so that the expenditure on the z–good "swamps" the revealed preference comparison. The details are given in the following theorem.

THEOREM 1. *Let (p_i, x_i), $i = 1, \ldots, n$ be a set of data and let (z_i) be a set of n positive scalars. For each i, let E_i be subset of the data that satisfies $z_t = z_i$ for all t in E_i. Then the only restriction imposed by the maximization hypothesis is that the data (p_i, x_i) satisfy GARP in each subset E_i. In particular, if $z_i \neq z_j$ for all i and j, so that $E_i = \{z_i\}$, then the maximization model imposes no restrictions whatsoever on the observed choices.*

Proof. First we show that the condition is necessary. This follows easily from the fact that $x_s R^0 x_t$ implies that $(x_s, z_i)R^0(x_t, z_i)$ whatever the price q_i. Thus a violation of GARP involving observations in E_i will necessarily create a violation of GARP with the additional good z_i.

In order to prove sufficiency, we will construct a set of prices (q_i) such that the entire data set satisfies GARP. For each i choose q_i such that

$$q_i > \max_{j \notin E_i} \left\{ \frac{p_i(x_j - x_i)}{z_i - z_j}, 1 \right\}. \tag{1}$$

Within each subset E_i we are assured that the data satisfy GARP. What about across subsets? I claim that $(x_i, z_i)R^0(x_j, z_j)$ if and only if $z_i > z_j$, when z_i and z_j are in different subsets E_i and E_j. There are two cases:

(1) $z_i > z_j$. Cross multiplying (1) we have:

$$p_i(x_i - x_j) + q_i(z_i - z_j) > 0$$

which means that $(x_i, z_i)R^0(x_j, z_j)$.

(2) $z_i < z_j$. Cross multiplying (1) again gives:

$$p_i(x_i - x_j) + q_i(z_i - z_j) < 0$$

which means that it is not the case that $(x_i, z_i)R^0(x_j, z_j)$.

Given these choices for q_i, is it possible that the set of data (p_i, q_i, x_i, z_i) could violate GARP? We know that there are no violations within the subsets E_i, so any violations must involve observations from different subsets. But if (x_i, z_i) and (x_j, z_j) are in different subsets, we know that $(x_i, z_i)R(x_j, z_j)$ if and only if $z_i > z_j$ by construction. Thus a violation of GARP would imply $z_i > z_j$ and $z_i < z_j$, which is a contradiction. ∎

There are several remarks worth making about this theorem. First, if z_i were a vector, we could simply choose a vector q_i with zeros (or small numbers) in every component but one. The above construction would still work. Secondly, the numbers z_i provide a complete preference ordering for the subsets E_i. Within each subset, the data are partially ordered by the revealed preference order.

If the variables (z_i) are thought of as demographic variables we can use Afriat's theorem, as described in Varian (1982), to construct a piecewise linear utility function, $u(x, z)$, that will rationalize the data in the required sense. Thus, as long as we don't have any violations of revealed preference for fixed values of the demographic variables, the maximization hypothesis puts no restrictions on the behavior of the choice data.

2. Observed Prices, Unobserved Quantities

The above theorem raises the question of what happens if we observe prices for the omitted good but not quantities? In this case we get no restrictions of any sort.

THEOREM 2. *Let (p_i, x_i) $i = 1, \ldots n$ be a set of data and let (q_i) $i = 1, \ldots, n$ be a set of positive prices. Then there always exists a set of quantities (z_i) $i = 1, \ldots, n$ such that the data (p_i, q_i, x_i, z_i) satisfies GARP.*

Proof. Choose $z_1 = 0$ and successively define

$$z_{i+1} > \max\left\{\frac{p_{i+1}x_i - p_{i+1}x_{i+1} + q_{i+1}z_i}{q_{i+1}}, 1\right\} \qquad \text{for } i = 1, \ldots, n-1.$$

Then for all $i = 1, \ldots, n-1$ we have:

$$p_{i+1}x_{i+1} + q_{i+1}z_{i+1} > p_{i+1}x_i + q_{i+1}z_i$$

so that each observation $i+1$ is revealed preferred to observation i. Thus the data must satisfy GARP. ∎

Note that the data can be reordered in any way desired so that any preference ordering is consistent with the data. Furthermore, if neither q_i nor z_i is observed there are clearly no restrictions whatsoever on the data (p_i, x_i).

3. Bounding the Expenditure on the Omitted Good

In the constructions given above we have essentially made the expenditure on the omitted good so large that it has "swamped" the revealed preference comparisons. If we are willing to bound the expenditure on this good, we can get some restrictions on the subset of choices.

THEOREM 3. *Let (p_i, x_i), $i = 1, \ldots, n$ be a set of data and (z_i), $i = 1, \ldots, n$ be an omitted good. Suppose that we postulate that the maximum expenditure possible on the omitted good is bounded by (e_i), $i = 1, \ldots, n$. Then the data (p_i, x_i, e_i) are consistent with utility maximization if and only if there is a positive solution (U_i, λ_i, d_i) to the following set of linear inequalities:*

$$U_i \leq U_j + \lambda_j p_j(x_i - x_j) + d_j(z_i - z_j)$$

$$d_j z_j \leq \lambda_j e_j.$$

Proof. Choose $q_i = d_i/\lambda_i$ and rewrite the inequalities in the following form:

$$U_i \leq U_j + \lambda_j p_j(x_i - x_j) + \lambda_j q_j(z_i - z_j)$$

$$q_j z_j \leq e_j.$$

But these are simply the Afriat inequalities which have been shown to be a necessary and sufficient condition for utility maximization in Afriat (1967), Diewert (1973), and Varian (1982). ∎

In order for this result to be of interest, we have to show that the inequality conditions given in Theorem 3 are not vacuous. A simple example will suffice.

Suppose for example that we have two observations on three goods with the following specifications: $x_1 = (1,2)$, $x_2 = (2,1)$, $p_1 = (1,2)$, $p_2 = (2,1)$, $z_1 = 1$, and $z_2 = 2$. It is easy to see that these observations violate the Generalized Axiom of Revealed Preference. Suppose that we try to patch things up by choosing prices (q_1, q_2) so that observation 2 is not revealed preferred to observation 1. This implies that:

$$5 + 2q_2 \leq 4 + q_2$$

which yields

$$q_2 \leq -1$$

which is impossible.

Since that doesn't work, let's try to assure that observation 1 is not revealed preferred to observation 2. In this case we have:

$$5 + q_1 \leq 4 + 2q_1$$

which implies that $q_1 \geq 1$. Thus the expenditure on z_1 must be at least 1 in order to satisfy the revealed preference restrictions. It follows that if the bound on the expenditure on the omitted good is less than 1, the data can not be consistent with GARP.

If (q_i) is observed but (z_i) is not observed, a similar set of inequalities can be constructed, but they are now nonlinear. This is also the case if total expenditure is observed, but neither q_i nor z_i is observed.

Theorem 3 suggests a way to check for "significant violations" of revealed preference. Suppose that we have a set of data (p_i, x_i, z_i) that violates revealed preference. It would be convenient to have an "index" of the degree of violation of revealed preference. One way to do this would be to find the smallest value $\bar{e}$ such that the inequalities

$$U_i \leq U_j + \lambda_j p_j(x_i - x_j) + d_j(z_i - z_j)$$

$$d_j z_j \leq \lambda_j \bar{e}$$

have a positive solution. The number $\bar{e}$ tells us how much the expenditure would have to be on the z–good in order to satisfy the restrictions of utility maximization. If the z–good is a demographic variable, $\bar{e}$ would give us an index of how important it would have to be for it to account for the taste differences necessary to describe the data. Since the inequalities described in Theorem 3 are linear in the unknown variables, checking for feasibility does not pose undue computational difficulties.

4. CONCLUSION

If the utility function is assumed to have some special structure such as separability—where $u(x, z)$ has the form $U(v(x), z)$—it is well known that maximization does impose restrictions on the data (p_i, x_i). These restrictions are summarized in Varian (1983). However, without this assumption of special structure, there are essentially no restrictions imposed by the maximization model on a subset of the choice data.

I take this to be a negative result, similar in spirit to the Sonnenschein–Mantel–Debreu results described in Shaefer and Sonnenschein (1983), although obviously not as deep. The sad fact of the matter is that the restrictions imposed by the optimization hypothesis only apply when we have observed the entire choice set. Hence the normal sorts of tests for consistency of observed choice must be interpreted instead as tests for separability of the observed choices from other variables in the utility function rather than test of maximization *per se*.

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ESTIMATING RISK AVERSION FROM ARROW-DEBREU PORTFOLIO CHOICE

BY HAL R. VARIAN[1]

This paper derives necessary and sufficient conditions for Arrow-Debreu choices of contingent consumption to be compatible with the maximization of a state independent expected utility function that exhibits increasing or decreasing absolute risk aversion, or increasing or decreasing relative risk aversion. The conditions can be used to bound different measures of risk aversion based on a single observation of Arrow-Debreu portfolio choice.

THE EXPECTED UTILITY hypothesis forms the basis for much of our understanding of investor behavior under uncertainty. It is commonly agreed that a well-behaved expected utility function should be an increasing and concave function of wealth, or, equivalently, that its first derivative should be positive and its second derivative should be negative. It is also widely accepted that the Arrow-Pratt measure of absolute risk aversion should be declining with wealth. There is much less agreement about the behavior of the Arrow-Pratt measure of *relative* risk aversion, although some investigators have argued that it should increase with wealth.

In this note I derive necessary and sufficient conditions for choices of contingent consumption across states of nature to satisfy various hypotheses about the behavior of these measures of risk aversion. If the portfolio choice behavior of the consumer is consistent with the conditions I derive, then the conditions can be used to bound the Arrow–Pratt measures of absolute and relative risk aversion. The conditions are derived using methods of the "nonparametric approach" to optimizing behavior introduced by Afriat (1967) and extended by Diewert (1973), Diewert and Parkan (1978), and Varian (1982), (1983a). Applications of these methods to choice under uncertainty include Dybvig and Ross (1982), Green and Srivastava (1983), and Varian (1983b).

1. THE MAXIMIZATION PROBLEM

Consider an investor who chooses a pattern of consumption across states of nature to solve the following problem:

$$\max \sum_{s=1}^{S} \pi_s u(c_s)$$

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$$\text{s.t.} \quad \sum_{s=1}^{S} q_s c_s = \sum_{s=1}^{S} q_s \bar{c}_s \tag{1}$$

Here π_s is the probability that state s will occur, $\bar{c}_s$ is the endowment of consumption in state s, and q_s is the price for an Arrow-Debreu contingent commodity that pays off one unit of consumption if state s occurs. We will suppose that $u(c)$ is a strictly increasing, strictly concave, twice differentiable function.

The first-order conditions for this problem are:

$$\pi_s u'(c_s) = \lambda q_s \quad s = 1, \ldots, S$$

We can always choose an affine transformation of the expected utility function so that $\lambda = 1$. Thus, we can define $p_s = q_s/\pi_s$ and rewrite the first order conditions as:

$$u'(c_s) = p_s \quad s = 1, \ldots, S$$

The ratios of state prices to probabilities are assumed to be observable, so that the numbers (p_s, c_s) for $s = 1, \ldots, S$ represent the potentially observable consumption data associated with the choice problem we are studying.

We will say we can *rationalize* this choice behavior if we can find a once differentiable increasing concave function $v(c)$ that satisfies the appropriate first-order conditions. Since the first-order conditions are sufficient conditions for the maximization of a concave function, this will guarantee that (c_s) actually solves the given maximization problem.

Breeden and Litzenberger (1978), Dybvig and Ross (1982) and Green and Srivastava (1983) show that a necessary and sufficient condition for a utility function to exist that rationalizes some choices (p_s, c_s) is that c_s is a decreasing function of p_s. If we number the states of nature so that $c_1 < c_2 < \ldots < c_S$ we can express this condition as simply requiring that $p_1 > p_2 > \ldots > p_S$. From now on we will assume that the states have been numbered in this way and that the observed consumption values satisfy this condition. The question that we pose is: what further conditions must be satisfied if the observed choices are to be compatible with various hypotheses about the behavior of risk aversion?

2. ABSOLUTE RISK AVERSION

Let $r(c) = -u''(c)/u'(c)$ be the Arrow-Pratt measure of absolute risk aversion. We observe that decreasing (increasing) absolute risk aversion is equivalent to the requirement that $\log u'(c)$ is a convex (concave) function. This follows since

$$\frac{d \log u'(c)}{dc} = \frac{u''(c)}{u'(c)} = -r(c).$$

Differentiating both sides of this identity, and using the assumption that $r'(c) < 0$, we have

$$\frac{d^2 \log u'(c)}{dc^2} = -r'(c) > 0.$$

Since $\log u'(c)$ is a convex function, it must satisfy the following inequalities:

$$\log u'(c_{s+1}) \geq \log u'(c_s) + \frac{d \log u'(c_s)}{dc}[c_{s+1} - c_s] \tag{2}$$

$$\log u'(c_{s-1}) \geq \log u'(c_s) + \frac{d \log u'(c_s)}{dc}[c_{s-1} - c_s] \tag{3}$$

Substituting $p_s = u'(c_s)$, $r(c_s) = -d \log u'(c_s)/dc$, and recalling that $c_{s-1} < c_s < c_{s+1}$, we can rearrange these inequalities to give us the following *Ratio Condition*:

$$\frac{\log p_s - \log p_{s+1}}{c_{s+1} - c_s} \leq r(c_s) \leq \frac{\log p_{s-1} - \log p_s}{c_s - c_{s-1}} \tag{4}$$

The Ratio Condition gives us an observable bound on risk aversion at each choice of contingent consumption. It has a simple geometric interpretation given in Figure 1. Here we have plotted $\log p_s$ versus c_s and connected the dots. (The other constructions in Figure 1 will be explained below.) It follows easily from risk aversion that the resulting piecewise linear function must be downward sloping. The Ratio Condition implies that the slopes of the line segments connecting each dot must be becoming flatter as consumption increases. That is, the piecewise linear function in Figure 1 depicted by the bold line must be a convex function. If risk aversion is increasing then the piecewise linear function must be concave.

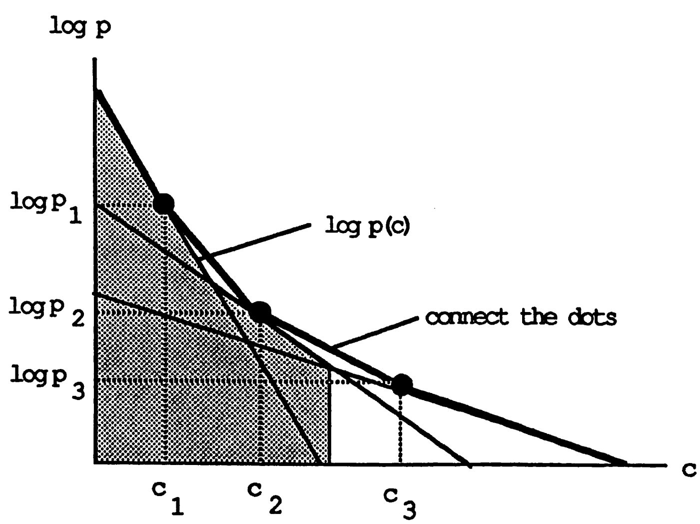

FIGURE 1.—Geometric interpretation of the Ratio Condition.

Thus the Ratio Condition gives us a *necessary* condition for decreasing absolute risk aversion. However, it turns out that the condition is sufficient as well.

That is, if the inequalities defined by the Ratio Condition are satisfied, then it will always be possible to construct an increasing, concave von Neumann-Morgenstern utility function $v(c)$ that exhibits decreasing absolute risk aversion and which generates the choices (c_s) as optimizing choices.

In order to prove this, let us suppose that the inequalities given in the Ratio Condition can be satisfied so that we can choose a set of numbers (r_s) that satisfies the inequalities

$$\frac{\log p_s - \log p_{s+1}}{c_{s+1} - c_s} \leq r_s \leq \frac{\log p_{s-1} - \log p_s}{c_s - c_{s-1}} \tag{5}$$

Geometrically, this simply means that we can choose a set of slopes (r_s) that lie between the slopes given by piecewise linear function depicted in Figure 1 at each observation.

Now define the function given by the lower envelope of these lines:

$$\log p(x) = \min_s \{\log p_s - r_s(x - c_s)\}.$$

Since we have assumed that $c_{s-1} < c_s < c_{s+1}$ this function will be differentiable at $x = c_s$ for $s = 1, \ldots, S$, and its derivative will be given by

$$\frac{d \log p(c_s)}{dc} = \frac{p'(c_s)}{p(c_s)} = -r_s.$$

Now define the function

$$v(c) = \int_0^c p(x)\, dx = \int_0^c \exp[\log p(x)]\, dx.$$

This is simply a monotonic transformation of the area under $\log p(x)$, as depicted in Figure 1. We now observe the following:

1. The first derivative of $v(c)$ at c_s is p_s.
2. The second derivative of $v(c)$ at c_s is given by

$$v''(c_s) = p'(c_s) = -r_s p_s < 0.$$

Thus $v(c)$ is a concave function. It follows that the satisfaction of the first-order conditions is a sufficient condition for the observed choices to actually solve the maximization problem given in (1). That is, the function $v(c)$ rationalizes the choices (p_s, c_s).

3. The absolute risk aversion of $v(c)$ at c_s is given by

$$r(c_s) = -\frac{v''(c_s)}{v'(c_s)} = r_s,$$

and the numbers (r_s) form a decreasing sequence by construction.

This completes the argument. It follows that the Ratio Conditions are a necessary and sufficient condition for the observed Arrow-Debreu portfolio choices to satisfy decreasing absolute risk aversion. Of course, the entire argument works *mutatis mutandis* for the case of increasing absolute risk aversion.

3. RELATIVE RISK AVERSION

Let us turn now to the case of relative risk aversion, given by

$$\rho(c) = -\frac{u''(c)c}{u'(c)}.$$

Essentially the same kind argument works, but there are a few twists, so it is worthwhile spelling out the details.

We first observe that the chain rule gives us

$$\frac{d\log u'(c)}{dc} = \frac{d\log u'(c)}{d\log c}\frac{d\log c}{dc}.$$

Taking the derivatives,

$$\frac{u''(c)}{u'(c)} = \frac{d\log u'(c)}{d\log c}\frac{1}{c}.$$

It follows that

$$\rho(c) = -\frac{d\log u'(c)}{d\log c}. \tag{6}$$

Applying the chain rule once more to (6), we have

$$\rho'(c) = -\frac{d^2\log u'(c)}{d(\log c)^2}\frac{d\log c}{dc}.$$

It follows that increasing (decreasing) relative risk aversion is equivalent to $\log u'(c)$ being a concave (convex) function of $\log c$. We will treat the case of increasing relative risk aversion, but the other case simply involves switching around the inequalities.

Concavity implies the following inequalities

$$\log u'(c_{s+1}) \leq \log u'(c_s) + \frac{d\log u'(c_s)}{d\log c}[\log c_{s+1} - \log c_s] \tag{7}$$

$$\log u'(c_{s-1}) \leq \log u'(c_s) + \frac{d\log u'(c_s)}{d\log c}[\log c_{s-1} - \log c_s] \tag{8}$$

Manipulation and substitution along the lines of that performed above yields

$$\frac{\log p_{s-1} - \log p_s}{\log c_s - \log c_{s-1}} \leq \rho(c_s) \leq \frac{\log p_s - \log p_{s+1}}{\log c_{s+1} - \log c_s}. \tag{9}$$

We will refer to this as the *Relative Ratio Condition.*

The Relative Ratio Condition has the same geometric interpretation as before if we plot $\log p_s$ against $\log c_s$. And, as before, it is also a sufficient condition: given an increasing series of numbers (ρ_s) that satisfy the relative ratio condition, it is possible to construct a utility function that will generated the observed choices. The construction is given by

$$v(c) = \int_0^c p(x)\,dx.$$

$$\text{where } \log p(x) = \min_s\{\log p_s - r_s(\log x - \log c_s)\}.$$

The demonstration that this construction actually works is left as an exercise for the reader.

4. A CONTINUOUS STATE SPACE

The conditions given above easily generalize to an economy with a continuum of possible consumption values. Let $p(c)$ be the Arrow–Debreu price for contingent consumption c. From the first-order conditions for maximization, we know that:

$$\log p(c) = \log u'(c)$$

Differentiating twice with respect to c, we have

$$\frac{d^2 \log p(c)}{dc^2} = -r'(c). \tag{10}$$

It follows that decreasing absolute risk aversion, for example, implies that $\log p(c)$ is a convex function. Similar conditions can be stated for increasing absolute risk aversion, increasing or decreasing relative risk aversion, and so on.

Equation (10) shows that the derivative of $\log p(c)$ gives absolute risk aversion directly, so the bounds given earlier are irrelevant in the case of a continuum of consumption values. Indeed, the ratios given on the left and right-hand sides of (4) are simply the definitions of the left and right derivatives of $\log p(c)$ as c_{s-1} and c_{s+1} approach c_s.

5. AGGREGATION ACROSS INDIVIDUALS

Up until now the analysis has only applied to a single consumer. However, the conditions easily generalize to aggregate consumption, since curvature is preserved under addition. Suppose, for example, that we have n consumers, each of whom has a concave utility function that exhibits decreasing absolute risk aversion.

Using a superscript to denote consumer i, the lower bound in equation (4) implies that

$$\frac{1}{r^i(c_s^i)}[\log p_s - \log p_{s+1}] \le c_{s+1}^i - c_s^i.$$

Summing over all consumers $i = 1, \ldots, n$, letting C_s denote aggregate consumption in state s, and rearranging we have

$$\frac{\log p_s - \log p_{s+1}}{C_{s+1} - C_s} \le \left(\sum_{i=1}^{n} \frac{1}{r^i(c_s^i)} \right)^{-1}.$$

The gives us a lower bound on a particular average of absolute risk aversion, and an upper bound can be derived in a similar manner. It follows by inspection that if every consumer has decreasing absolute risk aversion then the graph of $\log p_s$ against C_s will have to have the same general shape as that depicted in Figure 1; that is, it must be a decreasing, convex function.

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Preliminary Draft

On the Goodness-of-Fit of Revealed Preference Conditions

by

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Abstract. Revealed preference analysis provides a definitive method to test for optimizing behavior. However, it has been criticized because it fails to allow for approximate satisfaction of optimizing behavior. In this note, I describe some possible solutions to this problem.

Keywords. Nonparametric, demand, revealed preference

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On the Goodness-of-Fit of Revealed Preference Conditions

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There are two approaches to the analysis of consumer choice behavior. Parametric analysis proceeds by postulating a functional form for a utility function, deriving the associated demand equations, and estimating the parameters of the resulting system of equations. The resulting estimates can be used to test the maximization hypothesis, forecast demand, or do welfare analysis. Nonparametric analysis uses revealed preference techniques to achieve the same ends.

Samuelson (1938) and Houthakker (1950) were the first to develop the implications of the revealed preference idea for economic theory, but Afriat (1967) was the first to pursue its implications for empirical demand analysis. Subsequently Diewert (1973), Diewert and Parkan (1978), and Varian (1982a), (1982b) extended Afriat's analysis in a number of directions. More recently, several authors such as Browning (1984), Bronars (1985), Deaton (19??) Green and Srivastava (1985), (1986), Houtman and Maks (1985), Landsburg (1981), Manser and McDonald (1986), and Swofford and Whitney (1986) have contributed to nonparametric analysis.

The aspect of nonparametric analysis that I wish to examine in this note has to do with the goodness-of-fit of the utility maximization model—what does it mean to say that some consumer behavior is "almost" consistent with maximization? The particular answer I give to this question has some novel implications for parametric demand analysis as well.

1. Goodness of Fit

Consider the violation of the Weak Axiom of Revealed Preference depicted in Figure 1. Here we have x^t revealed preferred to x^s and x^s revealed preferred to x^t. However, the size of the violation is not large: a small perturbation of the budget line through either observation would eliminate the problem. Hence we might want to consider this an insignificant violation of the maximization model.

The notion of what is or is not significant implicitly relies on a statistical model of how the data were generated and what are the possible sources of error. In Varian (1985) I examined how one might formalize the concept of significant violations in the context of measurement error. Here I want to examine a different approach to the idea of "almost maximizing" behavior that was first described by Afriat (1967). For the time being, I will describe the goodness-of-fit measure without referring to a statistical model of error generation, but that is clearly the next step on the research agenda.

Suppose that we have some demand data (p^t, x^t) where p^t is a vector of prices and x^t a vector of quantities for $t = 1, \ldots, T$. For a given set of numbers (e^t), $t = 1, \ldots, T$, with $0 \leq e^t \leq 1$, define an extension of the standard direct revealed preference relation by

$$x^t \; R_e^0 \; x^s \text{ if and only if } e^t p^t x^t \geq p^t x^s.$$

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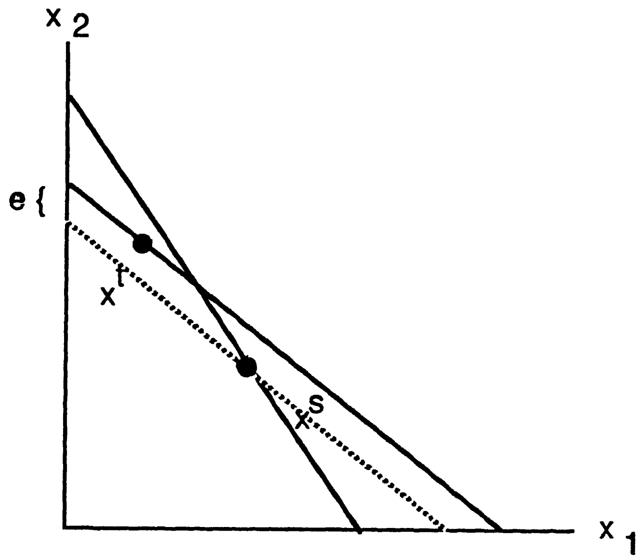

Figure 1. A violation of the Weak Axiom. However, a small perturbation of the budget line through x^t would remove the violation.

If $e^t = 1$ this is the standard revealed preference relation; if $e^t = 0$ the relation is vacuous in the sense that observation t can not be revealed preferred to any other observation. As e^t varies from 1 to 0 the number of observations revealed preferred to other observations monotonically decreases.

The number e^t can be thought of as how much less the potential expenditure on a bundle x^s has to be before we will to consider it worse than the observed choice x^t. If e^t is .9, for example, we will only count bundles whose cost is less than 90% of an observed choice x^t as being revealed worse than x^t. Said another way: if e^t is .9 and x^s would cost only 5% less than x^t, we would not consider this a significant enough different to conclude that x^t was preferred by the consumer to x^s. We are allowing the consumer a "margin of error" of $(1 - e^t)$.

Given an arbitrary set of data (p^t, x^t), let us choose a set of efficiency indices (e^t) that are as close as possible to 1 in some norm. If the data satisfy the revealed preference conditions exactly, then we can choose $e^t = 1$ for all $t = 1, \ldots, T$. If we choose $e^t = 0$ for all $t = 1, \ldots, T$, then the data vacuously satisfy the revealed preference conditions, since no observation is revealed preferred to any other. Thus for any reasonable norm, there will be some set of (e^t) that is as close as possible to 1 that will summarize "how close" the observed choices are to maximizing choices.

In Afriat's (1967) original treatment of this idea, he considered choosing a single e that applied to all observations, rather than a different e^t for each observation. We refer to this as a *single index* model as opposed to the *multiple index* model described above. The advantage of Afriat's proposal is that it is much easier to compute a single index e than the multiple indices (e^t).

Houtman and Maks (1985) suggest the following binary search. Start with $e = 1$ and test for violations of revealed preference using Warshall's algorithm as described in Varian (1982a). If the data fail to satisfy the strong axiom, try $e = 1/2$. If $e = 1/2$ doesn't work, try $e = 1/4$. If $e = 1/2$ *does* work, try $e = 3/4$, and so on. After n revealed preference tests, you are within $1/2^n$ of the actual efficiency index.

Computing the set of efficiency indices that are as close as possible to 1 in some norm is substantially more difficult. If we choose a quadratic norm, for example, we would have so solve a problem such as:

$$E = \min_{(e^t)} \sum_{t=1}^{T} (e^t - 1)^2 \tag{1}$$

subject to the constraint that the revealed preference relation R_e satisfies the Strong Axiom. This approach is significantly more demanding from a computational perspective.

2. A Characterization of the Efficiency Indices

There is a characterization of the set of (e^t) that minimize some norm that will be useful in what follows. In order to describe it, we need some formal definitions.

As above, define the relation R_e^0 by $x^t\ R_e^0\ x$ iff $e^t p^t x^t \geq p^t x$, and let R_e be the transitive closure of this relation. Then define $GARP_e$ to mean

$$x^s\ R_e\ x^t \text{ implies } e^t p^t x^t \leq p^t x^s.$$

If $e^t = 1$ for all t then this reduces to the standard definition of GARP.

Here is another way to state this definition: if some data (p^t, x^t, e^t) satisfy $GARP_e$, then

$$\text{for all } x^s\ R_e\ x^t \text{ we have } e^t p^t x^t \leq p^t x^s.$$

This statement can be transformed to

$$e^t \leq \frac{p^t x^s}{p^t x^t} \text{ for all } x^s\ R_e\ x^t.$$

If we attempt to choose a set of (e^t) that are on the average as close as possible to 1, then it must be that

$$e^t = \min_{x^s R_e x^t} \frac{p^t x^s}{p^t x^t}. \tag{2}$$

Note that this is not really an "operational" way to determine e^t, since e^t is implicitly involved in the relation R_e. Nevertheless, the characterization is still useful, as we will see below.

3. Parametric Methods

The characterization of (e^t) described in the last section is useful because it can be extended easily to a novel way to estimate parametric demand systems. Suppose that one is willing to postulate that some observed demand behavior was generated by the maximization of a particular parametric utility function $u(x, \beta)$, where β is a vector of parameters.

Let $\succeq$ be the preferences generated by the utility function $u(x, \beta)$. Then it is natural to define a parametric generalization of Afriat's efficiency index by

$$i^t = \min_{x \succeq x^t} \frac{p^t x}{p^t x^t}.$$

All we have done is to replace the partial order R_e by the total order $\succeq$.

Using some constructs from duality theory allows for an easier statement of this definition. The money metric utility function $m(p, x)$ is defined to be

$$m(p, x) = \min_y \; py$$
$$\text{s.t. } y \succeq x.$$

In words, the money metric utility function measures the minimum expenditure at prices p the consumer would need to be as well off as he would be consuming the bundle x. For more on the money metric utility function see Samuelson (1974), King (1982), and Varian (1984). If utility is parameterized by β, then the money metric utility function depends on the same parameters and we write $m(p, x, \beta)$.

In terms of the money metric utility function, we can restate the definition of the efficiency index as

$$i^t = \frac{m(p^t, x^t, \beta)}{p^t x^t}.$$

In words, $m(p^t, x^t, \beta)$ gives the minimum expenditure necessary to achieve utility $u(x^t)$ while $p^t x^t$ gives the expenditure actually observed. Therefore, the consumer is "wasting" a fraction $1 - i^t$ of his money.

An index of the degree of violation of maximization in the data set could be given by

$$I = \sum_{t=1}^{T} \left(\frac{m(p^t, x^t, \beta)}{p^t x^t} - 1 \right)^2.$$

This definition is directly analogous to equation (1).

The discussion to this point has proceeded under the assumption that β was known. But what if β is unknown? Then we would like to have an estimate of β—an estimate that provides the best fit to the maximizing model. A natural estimate is to find that value of β that minimizes the degree of violation of maximizing behavior as measured by the index I. This makes the average value of e^t as close as possible to 1, using the sum-of-squared-error norm. I believe that this estimator has several desirable characteristics.

First, it uses a sensible *economic* norm for goodness-of-fit. Conventional estimators of demand parameters use the sum-of-squared errors of the observed and predicted quantities demand, or some variant on this. But this has little *economic* content; a large difference between predicted and observed demand can easily be consistent with a small difference in utility. This is depicted in Figure 2. Here the observed choice is far from the predicted choice in Euclidean distance, but quite close in terms of money metric utility. The model is a bad fit in terms of Euclidean distance, but a good fit in the sense that the consumer really isn't that far from maximizing behavior in terms of money metric utility.

Second, the minimized value of the objective function gives a meaningful economic measure of how close the observed choices are to maximizing choice for the particular parametric form involved. If the average value of e^t is .95 for example, then it is meaningful to say that the observed choice behavior was 95 percent as efficient as maximizing behavior.

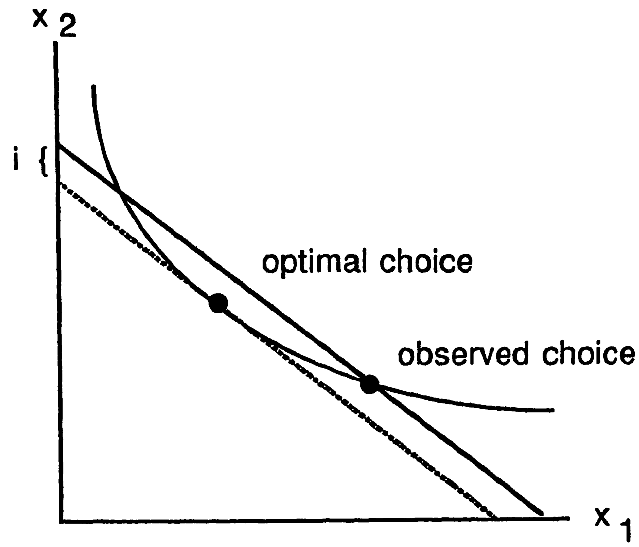

Figure 2. This is a good fit in terms of money metric utility although it is a bad fit in terms of demand behavior.

Third, the mechanics of the estimation problem are much simpler than they are using the conventional approach. Economic theory imposes the restriction that a money metric utility function must be an increasing, linearly homogeneous, and concave function of prices. These constraints are not terrible difficult to impose on the maximization problem. By contrast theory implies that a system of demand equations must have a symmetric negative semidefinite Slutsky substitution matrix. Imposing this restriction involves imposing nonlinear cross equation restrictions on a system of equations. In general this is a difficult thing to do.

In summary: the Afriat efficiency index offers not only a sensible and easily computable measure of the degree of violation of maximizing behavior in a nonparametric revealed preference context, but it also suggests a novel way to estimate unknown parameters in the parametric approach to demand estimation.

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