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Cournot Equilibrium in Factor Markets

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In his classic work, *Researches into the Mathematical Principles of the Theory of Wealth* (1838), A. A. Cournot introduced two distinct theories of duopoly. The better known of these theories concerns the case of two sellers of the same product (mineral water). A less familiar analysis concerns the case of two monopolists whose outputs, (zinc and copper) are used in fixed proportions in the production of a final good (brass) which is produced competitively. In the former case, Cournot supposes that each monopolist sets his quantity and accepts the market determined price. In the latter case, each monopolist sets the price of the factor he controls and sells the quantity determined by derived demand from the competitively operated final goods industry.

Cournot's theory of duopolists producing identical goods was criticized by Bertrand (1883) and Edgeworth (1897) on the grounds that equilibrium as described by Cournot does not exist if firms assume constancy of their rivals' *prices* rather than *quantities*. Hotelling (1929) observed that the lack of a Cournot equilibrium in prices can be viewed as a consequence of the fact that when two firms produce an identical product, the demand for the output of either is discontinuous as a function of the other's price. He then demonstrates that where products are spatially differentiated, continuity is restored and Cournot equilibrium in prices can be found for an interesting class of models.

In this paper we study Cournot equilibria among monopolistic controllers of factor supplies where the production function for final goods is neoclassical and the final goods industry is competitive. This discussion unifies Cournot's two theories of duopoly as special cases of a more general theory and provides some perspective on the old question of the difference between Cournot equilibrium in prices and Cournot equilibrium in quantities. It is hoped that the analysis will contribute some insight into the workings of factor markets in which several complementary or substitute factors are unionized or otherwise non-competitively supplied. Finally we relate our analysis to the theory of non-cooperative equilibrium for classical "bilateral monopoly".

Production functions, cost functions and equilibrium

Consider an industry in which a single output is produced using n factors as inputs. There are constant returns to scale and production possibilities are represented in the usual way by a twice differentiable concave production function $f(x)$, where $x = (x_1, \ldots, x_n)$ is the vector of inputs. Where $w = (w_1, \ldots, w_n)$, let $\overset{\circ}{x}(w)$ be the cheapest vector of inputs, at the vector of wage rates, w, which is capable of producing one unit of output. Then $c(w) = w\overset{\circ}{x}(w)$ is the (unit) cost function for the industry.

Let f_i and c_i denote the ith partial derivatives of f and c. Let $D(p)$ be the demand function for the industry where p is the price of the industry output. Let $D^{-1}(q)$ be the inverse demand function, and let $\eta(p) = \frac{p}{D(p)} \frac{dD}{dp}$ be the elasticity of demand. We assume that the final goods industry operates competitively while the supply of each factor is monopolized by its own "union."

Competitive operation of the final goods industry requires that the following relations hold between the vector of wages, w, and the vector of quantities employed.

(1) $\quad D(c(w)) = f(x)$

(2a) $\quad c_i(w) = \frac{x_i}{f(x)}$

(2b) $\quad f_i(x) = \frac{w_i}{c(w)}$.

The first result holds since the competitive final goods industry must price at $c(w)$ and the market must clear. Result (2b) states that factors are paid their marginal value products and result 2a (sometimes called Shepherd's Lemma) is a well known consequence of the envelope theorem as applied to the definition of a cost function. 1/

In consequence of equations (1), (2a) and (2b), total revenue of factor i can be written either as a function, $R^i(w)$ of the vector of wage rates or as a function ${}^iR(x)$ of the quantities supplied. Thus from equations (1) and (2a) we deduce

(3a) $\quad w_i x_i = w_i c_i(w) D(c(w)) \equiv R^i(w)$.

From equations (1) and (2b) we deduce:

(3b) $\quad w_i x_i = x_i f_i(x) D^{-1}(f(x)) \equiv {}^iR(x)$.

It is convenient to transform all variables into logarithmic form. Thus for any positive vector $z = (z_1, \ldots, z_n)$, let $z^* = (z_1^*, \ldots, z_n^*) = (\ln z_1, \ldots, \ln z_n)$ and for any function $h(z)$ whose range is the positive real numbers, let
$h^*(z^*) \equiv \ln h(e^{z_1^*}, \ldots, e^{z_n^*}) \equiv \ln h(z)$.

Thus equations (1), (2a), and (2b) can be rewritten as:

$$(1^*) \qquad D^*(c^*(w^*)) = f^*(x^*)$$

$$(2^*) \qquad \frac{\partial f^*(x^*)}{dx_i^*} = \frac{\partial c^*(w^*)}{\partial w_i^*} = \frac{w_i x_i}{c(w)f(x)} \quad .$$

Since $c(w)f(x)$ is total revenue to the industry, we see from (2*) that $\frac{\partial f^*(x^*)}{\partial x_i^*}$ and $\frac{\partial c^*(w^*)}{\partial w_i^*}$ are both equal to the share of industry revenue accruing to factor i. Thus we can define $\theta^i(w^*) \equiv \frac{\partial c^*(w^*)}{\partial w_i^*}$ and ${}^i\theta(x^*) \equiv \frac{\partial f^*(x^*)}{\partial x_i^*}$. Then an alternative statement of (2*) is:

$$(2^{**}) \qquad \theta^i(w^*) = {}^i\theta(x^*) = \frac{w_i x_i}{c(w)f(x)}$$

From (2*) it is apparent that (3a) and (3b) can be replaced by:

$$(3a^*) \qquad R^{i*}(w^*) = \frac{\partial c^*}{\partial w_i^*} + c^*(w^*) + D^*(c^*(w^*))$$

and

$$(3b^*) \qquad {}^iR^*(x^*) = \frac{\partial f^*}{\partial x_i^*} + f^*(x^*) + D^{-1*}(f^*(x^*)).$$

We define a Cournot equilibrium in wages as a vector of wage rates, $\bar{w}$, such that for each $i, \bar{w}_i$ maximizes revenue to factor i on the assumption that the <u>wage rates</u> demanded by other unions do not respond to the wage rate set for factor i. More formally $\overline{w}$ is a Cournot equilibrium in wages if $R^i(\overline{w}) \geqq R^i(w)$ for all w such that $w_j = \overline{w}_j$ for all $j \neq i$.

In similar fashion we can define a Cournot equilibrium in quantities as a vector $\overline{x}$ of factor supplies such that revenue of factor i is maximized at $\overline{x}_i$ where it is assumed that the <u>quantities</u> supplied by the other unions are invariant to the quantity of factor i supplied. Thus a Cournot equilibrium in quantities is a vector $\overline{x}$ such that for each i, ${}^iR(\overline{x}) \geqq {}^iR(x)$ for all x such that $x_j = \overline{x}_j$ for all $j \neq i$.

It is clear from equations (3a*) and (3b*) that Cournot equilibrium in wages and Cournot equilibrium in quantities are formally dual to each other where we identify w^* with x^*, c^* with f^* and D^* with D^{-1*}. This generalizes an observation made by Sonnenschein (1968) who pointed out the formal duality between Cournot's two classical oligopoly models. For each theorem that we prove about one type of Cournot equilibrium, we will find a dual (but not identical) theorem about the other.

<u>Cournot equilibrium in wages</u>

Maximizing $R^i(w)$ with respect to w_i is equivalent to maximizing $R^{i*}(w^*)$ with respect to w_i^*. Thus first order necessary conditions for Cournot equilibrium are obtained by setting the partial derivatives of (3a*) equal to zero. At a Cournot equilibrium in wages, $\overline{w}$,

$$\text{(4a)} \qquad 0 = R_i^{i*}(\overline{w}^*) = \frac{\partial^2 c^*(\overline{w}^*)}{\partial w_i^{*2}} + \frac{\partial c^*(\overline{w}^*)}{\partial w_i^*} + \frac{\partial D^*(c^*(\overline{w}^*))}{\partial c^*} \quad \frac{\partial c^*(\overline{w}^*)}{\partial w_i^*}$$

Lemma la collects technical facts which will be used to transform equation 4a into a more useful expression. A proof is found in the Appendix.

Lemma 1 a

(i) $$\frac{dD^*(c^*(w^*))}{dc^*} = \eta(c(w)).$$

(ii) Where there are just two factors of production, and $\tilde{\sigma}(w)$ is the elasticity of substitution of the production function at $x = (c_1(w),\ldots,c_n(w))$,

$$\frac{\partial^2 c^*(w^*)}{\partial w_i^{*2}} \equiv \frac{\partial \theta^{i*}(w^*)}{\partial w_i^*} = (1-\theta^i(w))(1-\tilde{\sigma}(w))$$

for each i.

(iii) Where there are any number of factors and the production function has constant elasticity of substitution σ between all factors,

$$\frac{\partial^2 c^*(w^*)}{\partial w_i^{*2}} = \frac{\partial \theta^{i}*(w^*)}{\partial w_i^*} = (1 - \theta^i(w))(1-\sigma).$$

If a Cournot equilibrium in wages occurs at $\bar{w}$, let $\bar{\eta} = \eta(c(\bar{w}))$, $\bar{\theta}^i = \theta^i(\bar{w})$ and $\bar{\sigma} = \tilde{\sigma}(\bar{w})$. Where there are just two factors of production or the production function is c.e.s., Lemma 1a enables us to make substitutions in (4a) to obtain:

(5a) $$0 = R_i^{i*}(\bar{w}^*) = (1-\bar{\theta}^i)(1-\bar{\sigma}) + \bar{\theta}^i(1+\bar{\eta})$$

or equivalently:

(6a) $$0 = R_i^{i*}(\bar{w}^*) = 1-\bar{\sigma} + \bar{\theta}^i(\bar{\sigma}+\bar{\eta})$$

for each i.

From equation (6a) we deduce the following rather striking results.

<u>Proposition 1a</u>. If there are two factors of production and constant returns to scale, if $\bar{w}$ is a Cournot equilibrium in wages, then $\bar{\eta} = \bar{\sigma} - 2$. Furthermore, either $\bar{\sigma} = 1$ and $\bar{\eta} = -1$ or $\bar{\theta}^1 = \bar{\theta}^2 = \frac{1}{2}$.

<u>Proof</u>:

According to (6a),

$$1 - \bar{\sigma} + \bar{\theta}^1(\bar{\sigma} + \bar{\eta}) = 0$$

and

$$1 - \bar{\sigma} + \bar{\theta}^2(\bar{\sigma} + \bar{\eta}) = 0.$$

Adding these two equations and noticing that $\bar{\theta}^1 + \bar{\theta}^2 = 1$ leads us to conclude that $\bar{\eta} = \bar{\sigma} -2$. Therefore $\bar{\sigma} + \bar{\eta} = 2(\bar{\sigma} - 1)$. Thus if $\bar{\sigma} = 1$, then $\bar{\eta} = -1$ and if $\bar{\sigma} \neq 1$, then for $i = 1, 2$,

$$\bar{\theta}^i = \frac{\bar{\sigma}-1}{\bar{\sigma} + \bar{\eta}} = \frac{1}{2}.$$

Q.E.D.

A very similar proof establishes the following.

Proposition 2a. If there are n factors and the production function has constant returns to scale and constant elasticity of substitution, σ, then if $\bar{w}$ is a Cournot equilibrium in wages, $\bar{\eta} = (n-1)\sigma - n$. Furthermore, either $\sigma = 1$ and $\bar{\eta} = -1$ or $\bar{\theta}^1 = \bar{\theta}^2 = \ldots = \bar{\theta}^n = \frac{1}{n}$.

An intuitive rationalization of Propositions 1a and 2a is this. If $\sigma = 1$ for all w (i.e., if the production function is Cobb-Douglas) then the share θ^i of total revenue going to factor i is independent of w. Thus in order to maximize its revenue, each factor desires simply to maximize industry revenue. This happens when $\eta(w) = -1$. Thus given the wages charged by the other unions, a union will set its wage so that the unit cost in the industry, $c(w)$, is equal to the price at which industry revenue is maximized. If there is a constant elasticity of substitution $\sigma < 1$, then the higher its wages, the larger will be the share of total revenue enjoyed by factor i. This fact is just balanced by the rate of decline of industry revenue when $\eta = \sigma - 2 < -1$. Thus when $\sigma < 1$ unions will raise their wages until industry costs are so high as to make demand sufficiently elastic. Where $\sigma > 1$, lower wages result in a higher share for a factor. In equilibrium, $\eta = \sigma - 2 > -1$, so that the industry will be operating in the inelastic region of its demand curve. A union will not raise its wage so as to raise the price of output and thus industry total revenue, because if it does so its share of total revenue will be correspondingly reduced.

Propositions 1a and 2a are concerned with first order necessary conditions for Cournot equilibrium. We would like to know when these conditions are also sufficient for Cournot equilibrium in wages. Further we would like to be able to establish conditions for the existence and/or uniqueness of such an equilibrium.

These matters can be very neatly settled where production functions are c.e.s. and where the elasticity of demand for the final good declines (increases in absolute value) as its price increases. Alfred Marshall (1890) argues that this latter condition is the usual case. For this reason we call such a demand function Marshallian. We also have occasion to assume that as price goes from zero to a price large enough to shut off demand completely, the elasticity of demand declines strictly monotonically to minus infinity. This property holds for (downward sloping) linear demand curves and in fact for any demand curve which "touches both axes" with finite slope while $\frac{d\eta(p)}{dp} < 0$ for intermediate prices. We call a demand curve with this property, strongly Marshallian.

Definition. Let $D(p)$ be the demand for a commodity as a function of its own price and let $\eta(p) = \frac{dD(p)}{dp} \frac{p}{D(p)}$. The demand function is said to be Marshallian if $\eta(p) < 0$ whenever $D(p) > 0$ and if $\frac{d\eta(p)}{dp} \leq 0$ for all p. The demand function is said to be strongly Marshallian if it is Marshallian and if in addition for any $x \leq 0$, there exists exactly one value of p such that $\eta(p) = x$.

The next two propositions establish conditions for existence and uniqueness of Cournot equilibrium in wages.

Proposition 3a. If the production function is c.e.s. with constant returns to scale and the demand function is Marshallian, then the conditions of Proposition 2a are sufficient as well as necessary for $\bar{w}$ to be a Cournot equilibrium in wages.

Proof: From Lemma 2a of the Appendix we see that if demand is Marshallian, then $R_i^{i*}(\bar{w}*) = 0$ only if $R_{ii}^{i*}(\bar{w}*) < 0$. From Lemma 3 we see that this implies that $R_i^{i*}(\bar{w}*) \geqq R_i^{i*}(w*)$ for all $w*$ such that $w_j* = \bar{w}_j$ for $j \neq i$.[2/] Therefore $\bar{w}$ is a Cournot equilibrium in wages. Q.E.D.

Proposition 4a. If the demand function for the final good is strongly Marshallian and the production function is c.e.s. with elasticity of substitution σ, then

(i) there exists a unique Cournot equilibrium in wages if $\sigma \neq 1$ and $\sigma \leqq \frac{n}{n-1}$.

(ii) if $\sigma = 1$, there is a unique cost level $\bar{c}$ such that the set of Cournot equilibrium wage vectors is $\{w | c(w) = \bar{c}\}$.

(iii) if $\sigma > \frac{n}{n-1}$, there does not exist a Cournot equilibrium in wages.

Proof: Where $\sigma > 0$ and $\sigma \neq 1$, $f(x) = \left(\sum_{i=1}^{n} \alpha_i x_i^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$

where $\alpha_i > 0$ for each i. According to Proposition 2a, a necessary condition for $\bar{w}$ to be a Cournot equilibrium is that $\theta^i(w) = \frac{1}{n}$ for all i. By direct computation, we find that

$$\frac{\theta^i(w)}{\theta^j(w)} = \left(\frac{\alpha_i}{\alpha_j}\right)^{\sigma} \left(\frac{w_i}{w_j}\right)^{1-\sigma} .$$

Since

$$\frac{\theta^i(w)}{\theta^j(w)} = 1$$

and $\sigma \neq 1$, it must be that $\frac{\bar{w}_i}{\bar{w}_j} = \left(\frac{\alpha_i}{\alpha_j}\right)^{\frac{\sigma}{\sigma-1}}$. Thus relative wage rates in Cournot equilibrium are uniquely determined. Where $\sigma = 0$, there are fixed coefficients. Again equality of the $\bar{\theta}^i$'s determines the ratio of the wage rates. If $\sigma \leqq \frac{n}{n-1}$, then $(n-1)\sigma - n \leqq 0$. Since demand is assumed to be strongly Marshallian, there is exactly one level of costs $\bar{c}$ for which $\eta(\bar{c}) = (n - 1)\sigma - n$. Then since cost functions are homogeneous of degree one there can be one and only one vector $\bar{w}$ such that

$$\frac{\bar{w}_i}{\bar{w}_j} = \left(\frac{\alpha_i}{\alpha_j}\right)^{\frac{\sigma}{\sigma-1}}$$

for each i and j and such that $c(\bar{w}) = \bar{c}$. Therefore there is precisely one wage vector $\bar{w}$ which satisfies the necessary conditions of Proposition 2a. According to Proposition 3a these conditions are also sufficient. This proves Assertion (i). According to Proposition (2a), if $\sigma = 1$, then $\overline{w}$ is a Cournot equilibrium in wages if and only if $\eta(c(\overline{w})) = 1$. Since the demand function for the final good is strongly Marshallian, it follows that there is exactly one price, $\overline{c}$, for which $\eta(\overline{c}) = -1$. Thus the Cournot equilibria in wages are the members of $\{w | c(w) = \overline{c}\}$.

If $\sigma > \frac{n}{n-1}$, then $(n-1)\sigma - n > 0$. But a necessary condition for Cournot equilibrium in wages is $\overline{\eta} = (n-1)\sigma - n$. Since demand is Marshallian, $\eta(p) \leq 0$ for all p. Therefore the necessary condition can never be satisfied. This proves assertion (iii). Q.E.D.

Cournot equilibrium in quantities

To each of the results in the previous section corresponds a dual proposition concerning Cournot equilibrium in quantities. If $\overline{x}$ represents a Cournot equilibrium in quantities, then let $\overline{\sigma}$ be the elasticity of substitution of the production function at $\overline{x}$ and let $\overline{\eta} = \eta(\overline{p})$ where $D(\overline{p}) = f(\overline{x})$. Then we have the following:

Proposition 1b. If there are two factors of production and constant returns to scale, then if $\overline{x}$ is a Cournot equilibrium in quantities, $\frac{1}{\overline{\eta}} = \frac{1}{\overline{\sigma}} - 2$. Furthermore, either $\overline{\sigma} = 1$ and $\overline{\eta} = -1$ or ${}^{1}\overline{\theta} = {}^{2}\overline{\theta} = \frac{1}{2}$.

Proposition 2b. If there are n factors, constant returns to scale, and constant elasticity of substitution σ between all factors, and if $\overline{x}$ is a Cournot equilibrium in quantities, then

$\frac{1}{\bar{\eta}} = (n-1)\frac{1}{\sigma} - n$. Furthermore, either $\sigma = 1$ and $\bar{\eta} = -1$ or $^1\bar{\theta} = \ldots = {}^n\bar{\theta} = \frac{1}{n}$.

Proposition 3b. If the production function is c.e.s. with constant returns to scale and the demand function is Marshallian, then the conditions of Proposition 2b are sufficient as well as necessary for $\bar{x}$ to be a Cournot equilibrium in quantities.

Proposition 4b. If the demand function for the final good is strongly Marshallian and the production function is c.e.s. with elasticity of substitution, σ, then:

(i) there exists a unique Cournot equilibrium in quantities if $\sigma \neq 1$ and $\sigma \geqq \frac{n-1}{n}$.

(ii) if $\sigma = 1$, there is a unique quantity $\bar{q}$ such that the set of Cournot equilibrium quantity vectors is $\{x | f(x) = \bar{q}\}$.

(iii) if $\sigma < \frac{n-1}{n}$ there does not exist a Cournot equilibrium in quantities.

Each of these propositions can be proved directly by a construction parallel to that used for Propositions 1a-4a. A more elegant proof uses duality explicitly. The formal isomorphism of Cournot equilibirum in price and quantities enables us to establish direct analogs of Propositions 1a-4a by simply interchanging the entities which are identified in the two theories. When this is done, statements about the elasticity of substitution of the production function are replaced by statements about the elasticity of substitution of the cost function. Statements about the elasticity of demand are replaced by statements about the elasticity of inverse demand. Thus it is useful to introduce notation for each of these elasticities. Let $\tilde{\sigma}_c(x)$ be the elasticity of substitution of the cost function at $w = (f_1(x),\ldots,f_n(x))$. Let $\eta^{-1}(q) = \frac{d \ln D^{-1}(q)}{d \ln q}$ be the elasticity of the inverse demand function. Where $\bar{x}$ is a Cournot equilibrium in quantities, let $\bar{\sigma}_c = \tilde{\sigma}_c(\bar{x})$, $\overline{\eta}^{-1} = \eta^{-1}(f(\bar{x}))$ and ${}^i\bar{\theta} = {}^i\theta(\bar{x})$.

The desired analogs to Propositions 1a-4a are Propositions 1b'-4b' where the latter propositions are obtained from the former by interchanging the following words and symbols.

(i) "wages" and "quantities" ("w" and "x")

(ii) "cost function" and "production function" ("c" and "f")

(iii) "demand function" and "inverse demand function"

(iv) "σ" and "σ_c"

(v) "η" and "η^{-1}"

(vi) "θ^i" and "${}^i\theta$"

Thus, for example, Proposition 2b' reads:

"If there are n factors, and the cost function has constant returns to scale, and a constant elasticity of substitution σ_c, then if $\bar{x}$ is a Cournot equilibrium in wages, $\overline{\eta}^{-1} = (n-1)\sigma_c - n$. Furthermore, either $\sigma_c = 1$ and $\overline{\eta}^{-1} = -1$ or ${}^1\bar{\theta} = {}^2\bar{\theta} = \ldots = {}^n\bar{\theta} = \frac{1}{n}$."

and Proposition 3b' reads:

"If the cost function is c.e.s. with constant returns to scale and the inverse demand function is Marshallian, then the conditions of Proposition 2b' are sufficient as well as necessary for $\bar{x}$ to be a Cournot equilibrium in quantities."

We can deduce Propositions 1b-4b from Propositions 1b'-4b' as follows. Lemmas 4 and 5 of the Appendix inform us that if the production function is c.e.s., then so is the cost function and if the demand function is (strongly) Marshallian then so is the inverse demand function. From its definition it is apparent that a unit cost function must have constant returns to scale. Therefore the hypotheses of Propositions 1b-4b imply the hypotheses of Propositions 1b'-4b'. Also from Lemma 4 we find that where there are just two factors, $\bar{\sigma}_c = \frac{1}{\bar{\sigma}}$ and where there are n factors and the production function has constant elasticity of substitution σ, the cost function has constant elasticity of substitution $\sigma_c = \frac{1}{\sigma}$. According to Lemma 4, $\overline{\eta}^{-1} = \frac{1}{\bar{\eta}}$. Therefore the conclusions of 1b-4b follow from the conclusions of 1b'-4b'. Thus the former set of propositions are established as consequences of the latter.

Comparing Cournot equilibrium in wages and in quantities

In his analysis of monopolized complementary factors, Cournot assumed that the production function displayed fixed proportions. This is a special case of a c.e.s. production function with $\sigma = 0$. Propositions 2a and 5a inform us that in this case if there are n factors and if demand for the final good is strongly Marshallian, there will be a unique Cournot equilibrium in wages. In equilibrium, the factors share equally in industry revenues and the elasticity of demand for the final good is $-n$. This, of course, is the same result as that obtained by Cournot. Cournot does not analyze Cournot equilibrium in quantities for this case for the very good reason that such an equilibrium does not exist. In fact from Propositions 4a and 4b, it is apparent that for all $\sigma < \frac{n-1}{n}$, there exists a unique Cournot equilibrium in wages but Cournot equilibrium in quantities does not exist.

Classical Cournot oligopoly among n sellers of the same commodity is formally the same as the case of n sellers of factors which are perfect substitutes. (We could think of a production function in which the "output" is the services of the commodity and the factors are units of the commodity supplied by different firms.) This in turn is a limiting case of a c.e.s. production function as $\sigma \to \infty$. Propositions 2a and 2b and 4a and 4b extend to this case. The limiting version of Proposition 2b as σ approaches infinity states that where $\bar{p}$ is the Cournot equilibrium price for the final good, $\eta(\bar{p}) = -\frac{1}{n}$. Each factor sells the same amount as the others and receives the same total revenue. This again is the same result as that found by Cournot.

In their critical comments on Cournot's theory of oligopoly, Bertrand and Edgeworth observe that Cournot equilibrium in prices among producers of the same commodity does not exist. More generally, we see from Propositions 4a and 4b that where there are n suppliers of factors, demand is strongly Marshallian and production is c.e.s.

with elasticity $\sigma > \frac{n-1}{n}$, Cournot equilibrium in wages does not exist bu there is a unique Cournot equilibrium in quantities.

Where $\frac{n-1}{n} \leqq \sigma \leqq \frac{n}{n-1}$, both kinds of Cournot equilibrium exist and it is of some interest to compare them. Suppose that demand is strongly Marshallian and $\bar{\eta}$ and $\bar{\bar{\eta}}$ are the elasticities of demand for the final good at a Cournot equilibrium in wages and in quantities respectively. Then according to Propositions 2a and 2b, $\bar{\eta} = (n-1)\sigma - n$ and $\frac{1}{\bar{\bar{\eta}}} = (n-1)\frac{1}{\sigma} - n$. With a bit of rearrangement we have:

(7a) $$(\bar{\eta} + 1) = (n-1)(\sigma-1)$$

and

(7b) $$(\bar{\bar{\eta}} + 1) = -\frac{\bar{\bar{\eta}}}{\sigma}(n-1)(\sigma-1).$$

Thus it is clear that if $\sigma < 1$ then in both types of Cournot equilibrium, demand for the final good is elastic (that is, $\bar{\eta} < -1$ and $\bar{\bar{\eta}} < -1$). If $\sigma > 1$, then in both types of equilibrium, demand for the final good is inelastic. Finally if $\sigma = 1$, $\bar{\eta} = \bar{\bar{\eta}} = 1$.
From (7a) and (7b) we also deduce

(8) $$\frac{\bar{\eta}+1}{\bar{\bar{\eta}}+1} = -\frac{\bar{\bar{\eta}}}{\sigma} = 1 + \eta(\sigma-1).$$

Thus where both types of equilibrium exist, $\bar{\eta} > \bar{\bar{\eta}}$ if $\sigma > 1$ and $\bar{\eta} < \bar{\bar{\eta}}$ if $\sigma < 1$. Where demand is strongly Marshallian this implies that if $\sigma < 1$, Cournot equilibrium in wages leads to a higher price and lower output for the final good than Cournot equilibrium in quantities. If $\sigma > 1$, then Cournot equilibrium in quantities leads to the higher price and lower output.

Perhaps surprisingly, if both types of Cournot equilibrium exist, if demand is strongly Marshallian and production is c.e.s.

with $\sigma \neq 1$, Cournot equilibrium in quantities always leads to higher revenue for each factor than Cournot equilibrium in wages. To see this, recall that total revenue to an industry is an increasing function of price so long as demand is inelastic and a decreasing function of price when demand is elastic. But we have just shown that when $\sigma < 1$, demand is elastic, $\bar{\eta} < \bar{\bar{\eta}} < -1$ and $\bar{p} > \bar{\bar{p}}$. If $\sigma > 1$, demand is inelastic, $-1 < \bar{\bar{\eta}} < \bar{\eta}$ and $\bar{\bar{p}} > \bar{p}$. Thus in either case, Cournot equilibrium in quantities yields the higher revenue to the industry. Since in either kind of Cournot equilibrium factors share equally in industry revenue the assertion is proved.

The next proposition summarizes all of these results. Let $\bar{\eta}$ and $\bar{p}$, denote demand elasticity and price of the final good and let $\bar{R}$ denote total revenue to the industry in Cournot equilibrium in wages. Let $\bar{\bar{\eta}}$, $\bar{\bar{p}}$ and $\bar{\bar{R}}$ denote the corresponding magnitudes for Cournot equilibrium in quantities.

Proposition 6. If there are constant returns to scale and a constant elasticity of substitution, σ, in production and if the demand for the final good is strongly Marshallian,

(i) if $0 \leqq \sigma < \frac{n-1}{n}$ there exists a unique Cournot equilibrium in wages but no Cournot equilibrium in quantities.

(ii) if $\frac{n-1}{n} \leqq \sigma < n$, there exists unique Cournot equilibrium of each type, $\bar{\eta} < \bar{\bar{\eta}} < -1$, $\bar{p} > \bar{\bar{p}}$ and $\bar{R} < \bar{\bar{R}}$.

(iii) if $\sigma = 1$, there exist Cournot equilibria of each type. There is a single equilibrium price for the final good $\bar{p} = \bar{\bar{p}}$. Then, also $\bar{\eta} = \bar{\bar{\eta}}$ and $\bar{R} = \bar{\bar{R}}$.

(iv) if $1 < \sigma \leqq \frac{n}{n-1}$, there exists unique Cournot equilibrium of each type, $-1 < \bar{\bar{\eta}} < \bar{\eta}$, $\bar{\bar{p}} > \bar{p}$ and $\bar{R} < \bar{\bar{R}}$.

(v) if $\sigma > \frac{n}{n-1}$, there exists a unique Cournot equilibrium in quantities, but no Cournot equilibrium in wages.

Asymmetric Cournot Equilibrium

Where there are just two monopolized factors and the final goods industry is competitive, another interesting concept of equilibrium is that in which the first factor, "labor", chooses its wage so as to maximize total revenue on the assumption that the quantity supplied of the second factor, "capital", is invariant to its own wage decision. Also the capitalist chooses the quantity of capital which he will supply so as to maximize his revenue assuming that he will have no influence on the wage rate demanded by labor. It is worth noticing that where the final goods industry is competitive, a factor monopolist's choice of quantity also determines the wage of his factor and *vice versa*. Thus for our analysis, it is immaterial whether we suppose that the factor monopolist sets his own price or his own quantity. What does matter is whether he assumes the other factor sets an invariant factor price and meets derived demand or sets its quantity and accepts the value of its marginal product as a wage.

From equations 4a and 4b we know that if capital is maximizing its revenue on the assumption that labor's wage is fixed and labor is maximizing its revenue assuming that the quantity of capital is fixed, then in equilibrium (where labor is denoted as factor L and capital as factor K).

(9a) $(1-\bar{\sigma}) + \bar{\theta}^K(\bar{\eta} + \bar{\sigma}) = 0$

and

(9b) $(1 - \frac{1}{\bar{\sigma}}) + \bar{\theta}^L(\frac{1}{\bar{\eta}} + \frac{1}{\bar{\sigma}}) = 0.$

Where there are constant returns to scale $\bar{\theta}^L + \bar{\theta}^K = 1$. Using this fact we deduce that the above equations hold if and only if $\bar{\eta}\,\bar{\sigma} = -1$. Then $\bar{\theta}^L = \frac{1}{1+\sigma}$ and $\bar{\theta}^K = \frac{\sigma}{1+\sigma}$. Thus $\bar{\theta}^L \geqq \bar{\theta}^K$ if and only if $\sigma < 1$. (The reader can verify that this is true even in the extreme cases of perfect substitutes $\sigma = \infty$ and perfect complements, $\sigma = 0$.)

If the production function is c.e.s. and demand is strongly Marshallian, then it can be shown by arguments essentially the same as those used to prove propositions 4a and 4b that conditions (9a) and (9b) are sufficient as well as necessary for an asymmetric Cournot equilibrium. Since $\bar{\eta} = -\frac{1}{\sigma}$ is always a solution to these equations we have the following.

<u>Proposition 7</u>. If there are two factors of production, constant returns to scale and constant elasticity of substitution, σ, and if demand is strongly Marshallian, then there exists an asymmetric Cournot equilibrium where if $\bar{p}$ is the equilibrium price of the final good, $\eta(\bar{p}) = \bar{\eta}$ is the elasticity of demand in equilibrium and $\bar{\theta}^L$ and $\bar{\theta}^K$ are the equilibrium shares of total revenue going to

labor and capital, $\bar{\theta}^L = \frac{1}{1+\sigma}$ and $\bar{\theta}^K = \frac{\sigma}{1+\sigma}$. If $\sigma \neq 1$, this equilibrium is unique. If $\sigma = 1$, there is a unique price $\bar{p}$ of the final good and the set of equilibrium quantities supplied is $\{x | f(x) = D(\bar{p})\}$ while the set of equilibrium factor price vectors is $\{w | c(w) = \bar{p}\}$.

It is interesting that asymmetric Cournot equilibrium, in contrast to the symmetric types, exists for all values of σ. It can also be shown by straightforward algebraic manipulation that where both kinds of symmetric Cournot equilibrium exist, the asymmetric equilibrium price for the final good falls in between the other two.

Noncooperative bilateral monopoly.

Bilateral monopoly is a subject of considerable discussion in the literature on imperfect competition. Zeuthen (1930, page 89) suggests that there are two cases of interest. One case concerns "two entrepreneneurs supplying raw materials or services who are both sellers to a third non-monopolistic party." The second case, which is probably more widely discussed, concerns "two monopolistic enterprises standing face to face as buyer and seller." So far we have dealt only with Zeuthen's first case. In this section we discuss both kinds of bilateral monopoly in the hope that analysis of each will enrich understanding of the other.

The assumption that the final goods industry is competitive imposes a great deal of structure on the analysis of the first kind of bilateral monopoly. If, for example, the wage rate of each factor is set, then unit costs for the final good are determined. This in turn determines the competitive price and output of the final good and also the amount of each factor used in production. Dually once the quantity of each factor is set, total output and the competitive price of the final good are determined. The factor quantities and the price of the final good determine the values of marginal product and hence the wage rates. In either case there are only two independent magnitudes left to be determined by the non-competitive part of the theory. It is natural to assign "control" of one of these variables to each duopolist and to study the resultant "equilibrium" in which each monopolist chooses the "best possible" value for the variable it controls, given that it can not influence the other monopolist's choice of the value that it controls. Such an equilibrium is called a Cournot-Nash equilibrium.[3/] Each of the Cournot equilibria discussed above is of this type.

A rather subtle fact about Cournot-Nash equilibrium deserves emphasis. To assume that a Cournot monopolist has no influence on the selection of the variable controlled by another monopolist is not to say that the former has no influence on the overall behavior of the latter. Thus in the models discussed above, if one factor monopolist assumes that the other's wage is invariant to his own choice of a wage demand, he necessarily assumes that any influence of his own wage on the schedule of derived demand for the other's services will result in a change in the quantity supplied by the other factor. Similarly if the former assumes that the latter keeps his quantity constant, then he also assumes that any influence of his own supply decision on the other factor's marginal product will change the wage which the other accepts.

Bowley (1924) appears to have been the first to clearly formulate the bilateral monopoly problem of the second kind. There is a monopolistic producer of a final good (steel) who is the sole user of an input (iron ore) which is, itself, produced by a monopolist. Iron ore is assumed to be the only factor used in the production of steel and the production function is $q = f(x)$ where q is the output of steel and x is the quantity of iron ore used.[4/] The inverse demand function for steel is denoted $p = D^{-1}(\cdot)$. Let $C(x)$ be the total cost of producing x units of iron ore and let π be the price per unit at which iron ore is sold to the steel producer. Where x is the output of the iron ore industry, the output of the steel producer will be $f(x)$, and the price of steel will be $D^{-1}(f(x))$. Thus total revenue of the steel producer is $R(x) = f(x)D^{-1}(f(x))$ and his profit is $R(x) - \pi x$. Profits of the iron ore producer are $\pi x - C(x)$.

In this case the output of the iron ore producer determines the output of the steel producer which in turn determines the price of steel. Unlike the case with two factor monopolists, it is not possible for each monopolist to independently select the quantity of his sales. On the other hand, since the iron ore market is not competitive on either side, the price of iron ore is not determined by the quantity. Thus one could assign control of one of the variables, "iron ore price" and "iron ore quantity" to each of the monopolists.

Wicksell (1927) divides control of the variables in just this way. He assumes that the factor owner sets the price of the factor (iron ore) and the final goods (steel) producer controls the quantity of factor used. However, the non-cooperative equilibrium that

Wicksell describes is formally not a Cournot-Nash equilibrium but rather an equilibrium of the type that came to be known as a "leadership equilibrium" in the work of Stackelberg (1934) and Fellner (1949). In particular, Wicksell assumes that the iron ore producer controls the price of ore and acts as "leader". That is, he sets the price of ore knowing that the steel producer will choose a level of output and thus of input which is profit maximizing given the price of iron ore. Bowley (1928) points out that one could equally well make the steel producer the "leader" by assuming that the steel producer controls the price of iron ore and sets it in the expectation that the iron ore producer will adjust his price accordingly.

Neither Wicksell nor Bowley discuss the case where the "leader" chooses quantity and the "follower" chooses price. Nor do they discuss the cases of formal Nash equilibrium in price and quantity where one variable is assigned to each monopolist's control. There is a good reason why they do not discuss these cases. The reason is that if such equilibria exist at all, they have the unfortunate characteristic of zero output of both goods. If the ore producer must "lead" with a quantity of ore, which is invariant to the steel producer's choice of a price for ore, the steel producer will always choose a price of zero. Therefore the ore producer would choose to lead with a quantity of zero. A similar line of reasoning shows that if the steel producer were to "lead" with a specified quantity order for iron ore, while the ore producer "follows" with a choice of price, the only leadership equilibrium is one in which the quantity of iron ore is zero.

From the profit functions of the steel producer and the ore producer it is also apparent that there does not exist an interesting Cournot-Nash equilibrium in which one monopolist controls price and the other controls quantity. If the steel producer controls the price of ore, then if the quantity of ore is set at a positive level, the steel producer maximizes his profits by setting a zero price. But if the price is zero, the ore producer will maximize his profit with zero output. Thus there cannot be a non-trivial Cournot-Nash equilibrium for this assignment of control over the variables. A similar argument shows that there is no Cournot-Nash equilibrium with positive output if the ore producer is in charge of price and the steel producer controls quantity.

Despite these remarks it is possible to find interesting Cournot-Nash equilibria for this kind of bilateral monopoly. This can be done by parameterizing the system with a different pair of variables,one of which can be independently set by each monopolist, and which jointly determine all of the economic variables of interest.

To illustrate one way of doing so, consider the case where the production function takes the simple form $f(x) = x$ and where $C(x) =$ for all x. Define the steel producer's "mark-up" as $m = p - \pi$. When the variables, π and m, are determined, $p = \pi + m$ and $x = D(p)$ are also determined. Consider the Cournot-Nash equilibrium in which the iron ore producer chooses the price of iron ore, π, so as to maximize his profits assuming that the steel producer's mark-up, m, is invariant to the choice of π. Likewise, the steel producer choose

m to maximize his profits assuming that the ore price, π,

is invariant to his choice of m. Then the steel producer's profits are $m\,D(\pi + m)$ and the iron ore producer's profits are $\pi D(\pi + m)$. Cournot-Nash equilibrium occurs where $\pi = m$ and $\eta(\pi+m) = -2$.

It is interesting to notice that this solution is very similar to Cournot's solution for two monopolists of complementary factors and a competitive final goods industry. In both cases, the elasticity of demand for the final good at equilibrium is -2 and the two monopolists share industry revenues equally.

Extensions and Applications

Strong assumptions have left our models rather starkly pruned. This was done to display the analytical and conceptual structure as simply as possible and to allow convenient comparison with the equally spare models of Cournot and the other early writers. A reasonable next step would be to allow the models to foliate more abundantly.

We have assumed that the objective of the "unions" is simply to maximize the total revenue of the factors that they control. Clearly it would be useful to examine more general objective functions. A simple generalization is to allow each factor a non-zero "opportunity cost" in alternative uses. Then the union's objective function would be a constrained maximization problem where the union seeks to maximize its revenue subject to the constraint that its wage rate is at least as large as the opportunity cost of the factor. This leads to straightforward modifications of our earlier results. Factors which do not find this constraint binding will behave as before. Factors which in the absence of the constraint would maximize revenue at a wage lower than opportunity cost will now set this wage at the "competitive" opportunity cost. Somewhat more generally, it could be supposed that the unit cost of supplying the factors varies with the amount supplied. Still more generally, drawing inspiration from the behavior of actual labor unions, one might postulate that union decisions are the result of some not necessarily transitive internal decision process reflecting the power and interests of its members and leaders.

It would also be useful to extend results of this kind to the case of monopolies producing final goods which are demand-related but not perfect substitutes. Such an analysis would also extend Hotelling's interesting special case of spatial monopoly .

Finally models of this type could be allowed to proliferate into a full-grown jungle of non-competitive theories. Among the many possible extensions that could be made are the following. Some factors might be supplied by Cournot oligopolies rather than monopolies. One factor, perhaps capital, might behave as a Stackelberg leader who correctly perceives that the other factor monopolies will behave noncooperatively in the environment established by the leader's behavior. Some of the factors might behave cooperatively to maximize their joint revenue. Of particular interest would be a model in which commitments for rates of renumeration and factor supplies are made sequentially in the form of overlapping long term union contracts and the installation of durable and immobile capital goods.

This paper has been mainly concerned with the formal structure of models of monopolistic factor supply. It is hoped, however, that it will also serve a useful purpose by directing attention to some seemingly neglected applications of economic theory. Extensions of the theory we have discussed might provide helpful insights for explaining whether and at what terms separate "craft unions" in the same industry would unite as an industrial union. Some analytical underpinnings might be constructed for the suggestive but highly informal macroeconomic theories of a cost-push inflation generated by organized labor and oligopolistic industries. The fascinating issues raised by Adelman (1971) in his discussion of the relation between the crude oil cartel (OPEC) and the oligopolistic refinery industry might also be better understood with the aid of a more formal theory. While there remain serious questions about what is the appropriate model of imperfect competition to be applied in each of these instances, it would seem useful to find out what can be done with even very simple, but fully specified, formal models.

Appendix

Proof of Lemma 1a

Result (i) is immediate from the definitions.

To prove (ii), notice that

$$\frac{\theta^1(w)}{1-\theta^1(w)} = \frac{w_1 x_1}{w_2 x_2} \qquad \text{where} \qquad \frac{w_1}{w_2} = \frac{f_1(x)}{f_2(x)} .$$

Then $$\ln \frac{\theta^1(w)}{1-\theta^1(w)} = \ln\left(\frac{w_1}{w_2}\right) + \ln\left(\frac{x_1}{x_2}\right) .$$

By definition $$\tilde{\sigma}(w) = \frac{-d \ln\left(\frac{x_1}{x_2}\right)}{d \ln\left(\frac{w_1}{w_2}\right)} \quad \text{along the surface where} \quad \frac{w_1}{w_2} = \frac{f_1(x)}{f_2(x)} .$$

Therefore

$$\frac{\partial}{\partial w_1{}^*} \ln \frac{\theta^1(w)}{(1-\theta^1(w))} = (1-\tilde{\sigma}(w)) \frac{d \ln\left(\frac{w_1}{w_2}\right)}{d\, w_1{}^*} .$$

Computing these derivatives yields:

$$\frac{1}{\theta^1(w)(1-\theta^1(w))} \quad \frac{\partial \theta^1(w)}{\partial w_1{}^*} = (1 - \tilde{\sigma}(w))$$

or equivalently:

$$\frac{\partial \theta^{1*}(w)^*}{\partial w_1{}^*} = (1 - \tilde{\sigma}(w))(1 - \theta^1(w)) .$$

A parallel argument applies for $\dfrac{\partial \theta^{2*}(w^*)}{\partial w_2{}^*}$.

Result (iii) is obtained in straightforward fashion from similar computations. Q.E.D.

Lemma 2a

If production is c.e.s. and σ is the elasticity of substitution, then where $w = (w_1,\ldots,w_n)$ and $w^* = (\ln w_1,\ldots,\ln w_n)$.

(i) $R_i^{i*}(w^*) = (1-\sigma) + \theta^i(w)(\eta(c(w)) + \sigma)$.

If $R_i^{i*}(w^*) = 0$, then also:

(ii) $R_{ii}^{i*}(w^*) - (1-\theta^i(w))(1-\sigma)^2 + (\theta^i(w))^2 \dfrac{d\eta(c(w))}{dc} \cdot c(w)$.

(iii) $R_{ij}^{i*}(w^*) = \theta^i(w)(1-\sigma) + \theta^i(w)\theta^j(w) \dfrac{d\eta(c(w))}{dc} c(w)$.

Proof:

Result (i) is simply a restatement of equation (6ā) above. To get (ii), we differentiate (i) and obtain:

$$(7) \qquad R_{ii}^{i*}(w^*) = [\eta(c(w)) + \sigma] \frac{\partial\theta^i(w)}{\partial w_i^*} + \theta^i \frac{d\eta(c(w))}{dw_i^*} .$$

Since

$$\theta^i(w) = e^{\theta^{i*}(w^*)}, \quad \frac{\partial\theta^i(w)}{\partial w_i^*} = \theta^i(w)\frac{\partial\theta^{i*}(w^*)}{\partial w_i^*} = \theta^i(w)(1-\theta^i(w))(1-\tilde{\sigma}(w))$$

where the latter equality is a consequence of Lemma 1. Also, $\eta(c(w)) = e^{\eta^*(c^*(w^*))}$, so

$$\frac{\partial\eta(c(w))}{\partial w_i^*} = \eta(c(w))\frac{d\eta^*(c^*(w^*))}{dc^*} \quad \frac{\partial c^*(w^*)}{\partial w_i^*} = c(w)\frac{d\eta(c(w))}{dc} \theta^i(w).$$

Substituting these expressions into (5) we have:

$$(8) \qquad R_{ii}^{i*}(w^*) = [\eta(c(w)) + \sigma]\theta^i(1-\theta^i)(1-\sigma) + (\theta^i)^2 \frac{d\eta(c(w))}{dc} (c(w)).$$

But if $R_i^{i*}(w^*) = 0$, then from (i) it follows that $[\eta(w^*) + \sigma]\theta^i = \sigma - 1$. Substituting this into (6) yields (ii). To prove (iii) we proceed in essentially the same way, except that we use the fact that

$$\frac{\partial\theta^i(w)}{\partial w_j^*} = \theta^i\theta^j(1 - \sigma).$$

Q.E.D.

Lemma 3

Suppose that $R^i_i(w^*) = 0$ only if $R^i_{ii}(w^*) < 0$. Then if $R^i_i(\bar{w}^*) = 0$, it must be that $R^i(\bar{w}^*) > R^i(w^*)$ for all w^* such that $w^*_j = \bar{w}_j$ for $j \neq i$ and $w^*_i \neq \bar{w}_i$.

Proof: We use a simple, but often overlooked fact from calculus. This fact is: Let f be a continuously differentiable real valued function whose domain is a convex subset of the real numbers. If $f'(x) = 0$ only when $f''(x) < 0$, then if there is any point $\bar{x}$ such that $f'(\bar{x}) = 0$, that point is a unique global maximizer of f. To prove this fact, notice that if $f'(\bar{x}) = 0$, $f''(\bar{x}) < 0$ and $f(\bar{\bar{x}}) \geqq f(\bar{x})$, there must be a local minimum $\hat{x}$ in the interval between $\bar{x}$ and $\bar{\bar{x}}$. But if $\hat{x}$ is a local minumum, then $f'(\hat{x}) = 0$ and $f''(\hat{x}) \geqq 0$, which is contrary to our assumption. Now to prove the Lemma all one needs to do is apply this result to the function

$$f(w_i^*) \equiv R^i(\bar{w}_1^*, \ldots, w_i^*, \ldots, \bar{w}_n^*).$$

Q.E.D.

Lemma 4[5/]

Assume that there are constant returns to scale.

(i) If there are just two factors of production, let $\sigma(x)$ be the elasticity of substitution of the production function at the point x, and let $\sigma_c(w)$ be the elasticity of substitution of the cost function at the point w. Then

$$\sigma_c(f_1(x), f_2(x)) = \frac{1}{\sigma(x)} \quad \text{and}$$

$$\bar{\sigma}_c = \frac{1}{\bar{\sigma}} \quad \text{where} \quad \bar{\sigma}_c \equiv \tilde{\sigma}_c(\bar{x}) \equiv \sigma_c(f_1(\bar{x}),\ f_2(\bar{x})).$$

(ii) Where there are n factors of production, the production function has constant elasticity of substitution σ between all pairs of factors if and only if the cost function has constant elasticity of substitution $\frac{1}{\sigma}$ between all factors.

Proof: Since there are constant returns to scale, $\frac{f_1(x)}{f_2(x)}$ is determined by $\frac{x_1}{x_2}$. By definition,

$$\frac{1}{\sigma(x)} = \frac{-d\ \ln\left(\frac{f_1(x)}{f_2(x)}\right)}{d\ \ln\left(\frac{x_1}{x_2}\right)} .$$

The cost function must also display constant returns to scale and therefore $\frac{c_1(w)}{c_2(w)}$ is determined by $\frac{w_1}{w_2}$. By definition,

$$\sigma_c(f_1(x),f_2(x)) = -\frac{d\ \ln\left(\frac{f_1(x)}{f_2(x)}\right)}{d\ \ln\left(\frac{c_1(f_1(x),f_2(x))}{c_2(f_1(x),f_2(x))}\right)} .$$

But according to the duality theorem,

$$c_1(f_1(x),f_2(x)) \equiv \frac{x_1}{f(x)} \quad \text{and} \quad c_2(f_1(x),f_2(x)) \equiv \frac{x_2}{f(x)} .$$

Thus

$$\sigma_c(f_1(x),f_2(x)) = \frac{-d\ \ln\left(\frac{f_1(x)}{f_2(x)}\right)}{d\ \ln\left(\frac{x_1}{x_2}\right)} = \frac{1}{\sigma(x)} .$$

Therefore result (i) holds.

Result (ii) can be verified by direct computation. Q.E.D.

<u>Lemma 5</u>

(i) For any p such that $\eta(p) < 0$, $\eta^{-1}(D(p)) = \frac{1}{\eta(p)} < 0$. In particular if $f(\bar{x}) = D(\bar{p})$, $\overline{\eta^{-1}} \equiv \eta^{-1}(f(\bar{x})) = \frac{1}{\eta(\bar{p})} \quad \frac{1}{\bar{\eta}}$.

(ii) If the demand function is (strongly) Marshallian, then the cost function is (strongly) Marshallian.

Proof: Result (i) is a simple consequence of the inverse function theorem.

To prove Result (ii) we need to show that $\frac{d}{dq}\eta^{-1}(q) < 0$ whenever $q = D(p)$ for some p. According to to (i), $\eta^{-1}(D(p)) = \frac{1}{\eta(p)}$. Therefore

$$d^1 \frac{d}{dq}\eta^{-1}(D(p)) \cdot \frac{dD}{dp} = -\left(\frac{1}{\eta(p)}\right)^2 \frac{d\eta(p)}{dp} .$$

Since demand is assumed Marshallian, $\frac{dD}{dp} < 0$ and $\frac{d\eta(p)}{dp} < 0$ so that $\frac{d}{dq}\eta^{-1}(D(p)) < 0$. Using this fact the remainder of the proof of result (ii) consists of simple verification.

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Footnotes

1. This and related results are well summarized by Diewert (1974).
2. In fact Lemma 3 enables us to draw the stronger conclusion that the image sets of the "reaction correspondence", mapping the vector of wage rates $w_{)i(}$ of the other monopolists into the set of profit maximizing wage rates for i are the singleton sets $\{\bar{w}_i\}$ whenever there exists $\bar{w}_i{}^*$ such that $R_i^{i*}(\bar{w}_i{}^*, \bar{w}_{)i(}) = 0$. Thus we can speak of the reaction "function" rather than the reaction "correspondence".
3. Nash (1950) generalized the idea of Cournot equilibrium from specific market games to a broad class of abstract games.
4. To Hicks (1935) we owe the choice of the iron ore and steel industries as illustrations. Hicks' discussion of bilateral monopoly and related issues is still worth reading.
5. This result is reported by McFadden (1970).