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Lectures on Public Economics

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These lectures were delivered to a lively assortment of undergraduate honors students and masters students on various occasions at the Australian National University and at the University of Canterbury in Christchurch, New Zealand. I am grateful to the economics departments in both places for their genial hospitality.

A Primitive Public Economy

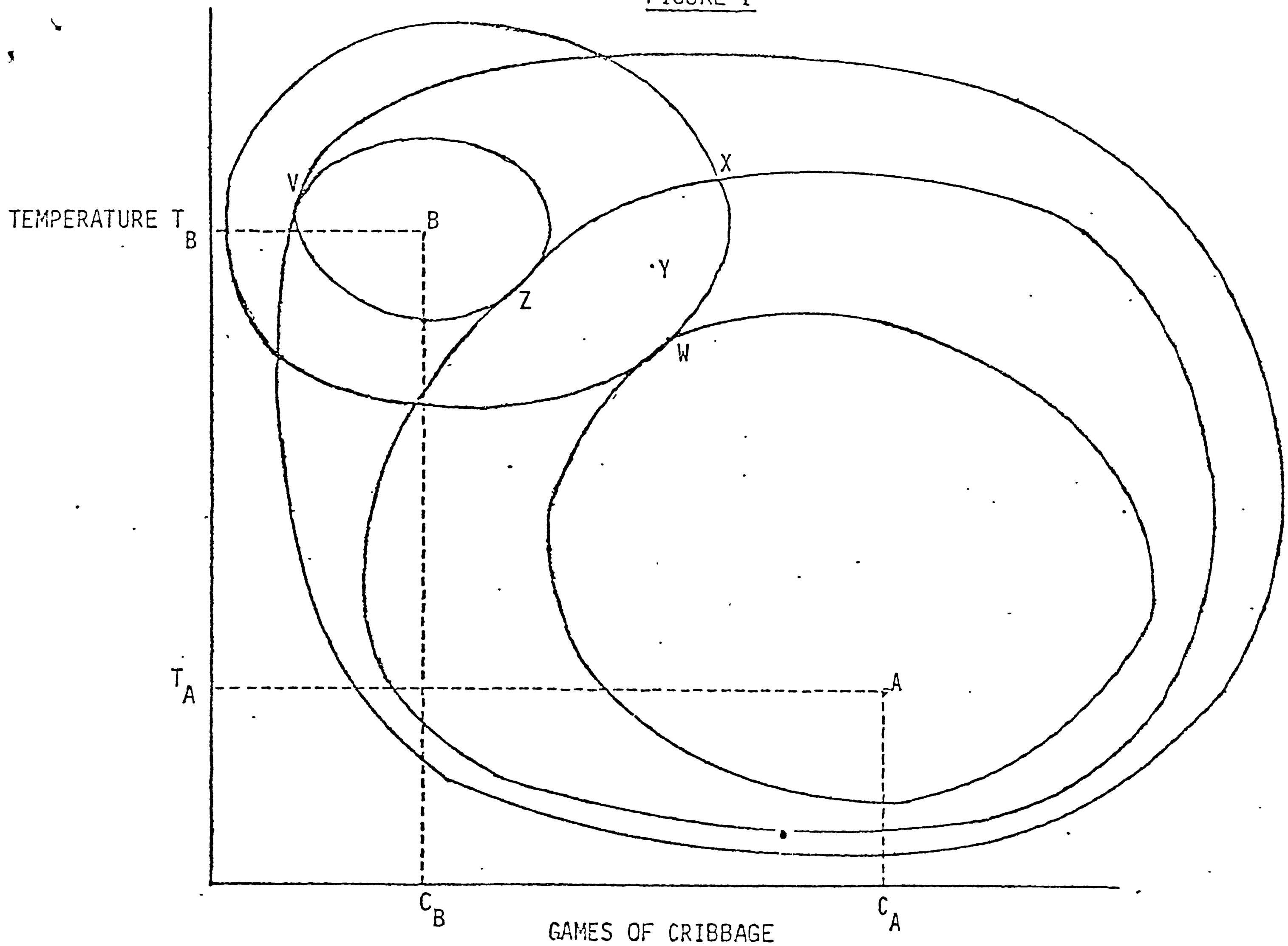
Anne and Bruce are roommates. They are interested in only two things.... the temperature of their room and playing cribbage together. Each has a favourite room temperature and a preferred number of games of cribbage per week. The further the temperature rises above or falls below Anne's favourite temperature the less happy she is. Similarly changes in the direction away from her favourite amount of cribbage make her less happy. Bruce's preferences have the same qualitative character as Anne's. But, what makes our story economically interesting is that Anne and Bruce differ in their favourite temperatures and in their preferred number of games of cribbage. Since they both must experience the same level of each, they will have to reach some kind of agreement in the presence of conflicting interests. We will begin our study of public economics with an analysis of efficient conduct of the Anne-Bruce household.

Our understanding of the case of Anne and Bruce will be aided by the use of a diagram. In Figure 1, the points $A = (C_A, T_A)$ and $B = (C_B, T_B)$ represent Anne's and Bruce's favourite combinations of cribbage and temperature. The closed curves encircling A are indifference loci for Anne. She regards all points on such a curve as equally good, while she prefers points on the inside of any such curve to points on the outside. In similar fashion, the closed curves encircling B are Bruce's indifference curves.

At this point it is useful to introduce a bit of vocabulary. We shall speak of each combination of a room temperature and a number of games of cribbage as a *situation*. If everybody likes situation α as well as situation β and someone likes α better, we say that α is *Pareto superior* to β. A situation is said to be *Pareto optimal* if there are no possible situations that are *Pareto* superior to it. Thus if a situation is not Pareto optimal, it should be possible to obtain unanimous consent for a beneficial change.

If the existing situation is Pareto optimal, then there is pure conflict of interest in the sense that any benefit to one person can come only at the cost of harming another.

FIGURE I



Our task is now to find the set of Pareto optimal situations, *chez Anne and Bruce*. Consider a point like X in Figure 1. This point is not Pareto optimal. Since each person prefers his inner indifference curve to his outer ones, it should be clear that the situation Y is preferred by both Anne and Bruce to X. Now if we take any point representing a possible situation, the qualitative nature that we have assumed for preferences requires that Anne and Bruce each have exactly one indifference curve passing through that point. Either these two indifference curves cross or they are tangent (or the point is on the boundary of the diagram). If they cross at a point, then, by just the sort of reasoning used for the point X, we can show that this point can not be Pareto optimal. Therefore the Pareto optimal points must either be points at which Anne's indifference curves are tangent to Bruce's or must be on the boundary of the diagram. (For the time being we will confine our attention to "interior" Pareto optima). In Figure 1, all of the Pareto optimal points are points of tangency between Anne's and Bruce's indifference curves. Thus the points Z and W are Pareto optimal. In fact there are many more Pareto optima which would be found if we drew more indifference curves and found more tangencies The set of such Pareto optima is depicted by the line BA in Figure 1. Notice that there are some points of tangency such as V, which are not Pareto optimal. It should be clear that, for example, the situation A is Pareto superior to V.

Let us define a person's *marginal rate of substitution* between temperature and cribbage in a situation to be the slope of his indifference curve as it passes through that situation. From our discussion above, it should be clear that at an interior Pareto optimum, Anne's marginal rate of substitution between temperature and cribbage, must be the same as Bruce's. If we compare a Pareto optimal tangency like Z with a non-optimal tangency like V we notice a second necessary condition for an interior Pareto optimum. At Z, Anne wants more cribbage and a lower temperature while Bruce wants less cribbage and a higher temperature. At V, although their marginal rates of substitution are the same, both want more cribbage and a lower temperature. Thus a more complete necessary condition for a Pareto optimum is that their marginal rates of substitution be equal and their preferred directions of change be opposite.

In order to generalize our theory to more people and more commodities, we need more powerful tools. The toolkit in which we find them is the calculus of functions of several variables. We will first analyze our (by now) cosy little household with these tools. After that we will demonstrate the great power of the tools we have acquired for analysis of public problems of all kinds.

A Lagrangean Approach to Playing House

One way of describing a Pareto optimum is to say that each Pareto optimum solves a constrained maximization problem where we fix Bruce's utility at some level and then maximize Anne's utility subject to the constraint that Bruce receives at least his fixed level of utility. We should, in principle, be able to generate the entire set of Pareto optimal situations by repeating this operation fixing Bruce on different indifference levels.

Suppose that Anne's utility function is $U^A(C, T)$ and Bruce's utility function is $U^B(C, T)$. To find one Pareto optimum, pick a level of utility $\bar{U}^B$ for Bruce and find (C, T) to maximize $U^A(C, T)$ subject to the constraint that $U^B(C, T) \geqq \bar{U}^B$. A convenient tool for the study of problems of maximization subject to constraints is the "method of Lagrange multipliers". The fact that we need to know is the following:

Let $f(\cdot)$ and $g^1(\cdot), \ldots, g^k(\cdot)$ be differentiable real valued functions of n real variables. Then (subject to certain regularity conditions) a necessary condition for $\bar{x}$ to yield an interior maximum of $f(\cdot)$ subject to the constraints that $g^i(x) \leqq 0$ for all i is that there exist real numbers $\lambda^1 \geqq 0, \ldots, \lambda^k \geqq 0$, such that the "Lagrangean" experession

$$L(x, \lambda_1, \ldots, \lambda_n) \equiv f(x) + \sum_{j=1} \lambda^j g^j(x)$$

has each of its partial derivatives equal to zero at $\bar{x}$.

Here we will not study the "certain regularity conditions" alluded to above. In most applications they are satisfied. However we will present an alternative version of the Lagrangean condition which is *always* true. This is:

> If $f(\cdot)$ and $g^1(\cdot)$, ... $g^k(\cdot)$ are differentiable, a necessary condition for an interior $\bar{x}$ to maximize $f(\cdot)$ subject to $g^j \leqq 0$ for each j is that the gradient vectors (the vectors of partial derivatives) at $\bar{x}$ for f and the g^j's be linearly dependent.

Returning to Anne and Bruce, a Pareto optimum is found by finding $(\bar{C}, \bar{T})$ and λ such that the partial derivatives of

$$U^A(\bar{C}, \bar{T}) + \lambda[\bar{U}^B - U^B(\bar{C}, \bar{B})]$$

with respect to C and T are both zero.

This tells us that:

$$(1) \quad \frac{\partial U^A(\bar{C}, \bar{T})}{\partial C} - \lambda \frac{\partial U^B(\bar{C}, \bar{T})}{\partial C} = 0$$

$$(2) \quad \frac{\partial U^A(\bar{C}, \bar{T})}{\partial T} - \lambda \frac{\partial U^B(\bar{C}, \bar{T})}{\partial C} = 0$$

Using (1) and (2) we can deduce

$$(3) \quad \frac{\dfrac{\partial U^A(\bar{C}, \bar{T})}{\partial C}}{\dfrac{\partial U^A(\bar{C}, \bar{T})}{\partial T}} = \frac{\dfrac{\partial U^B(\bar{C}, \bar{T})}{\partial C}}{\dfrac{\partial U^B(\bar{C}, \bar{T})}{\partial T}}.$$

Thus we see that Anne's marginal rate of substitution between cribbage and temperature must be the same as Bruce's at any Pareto optimal point.

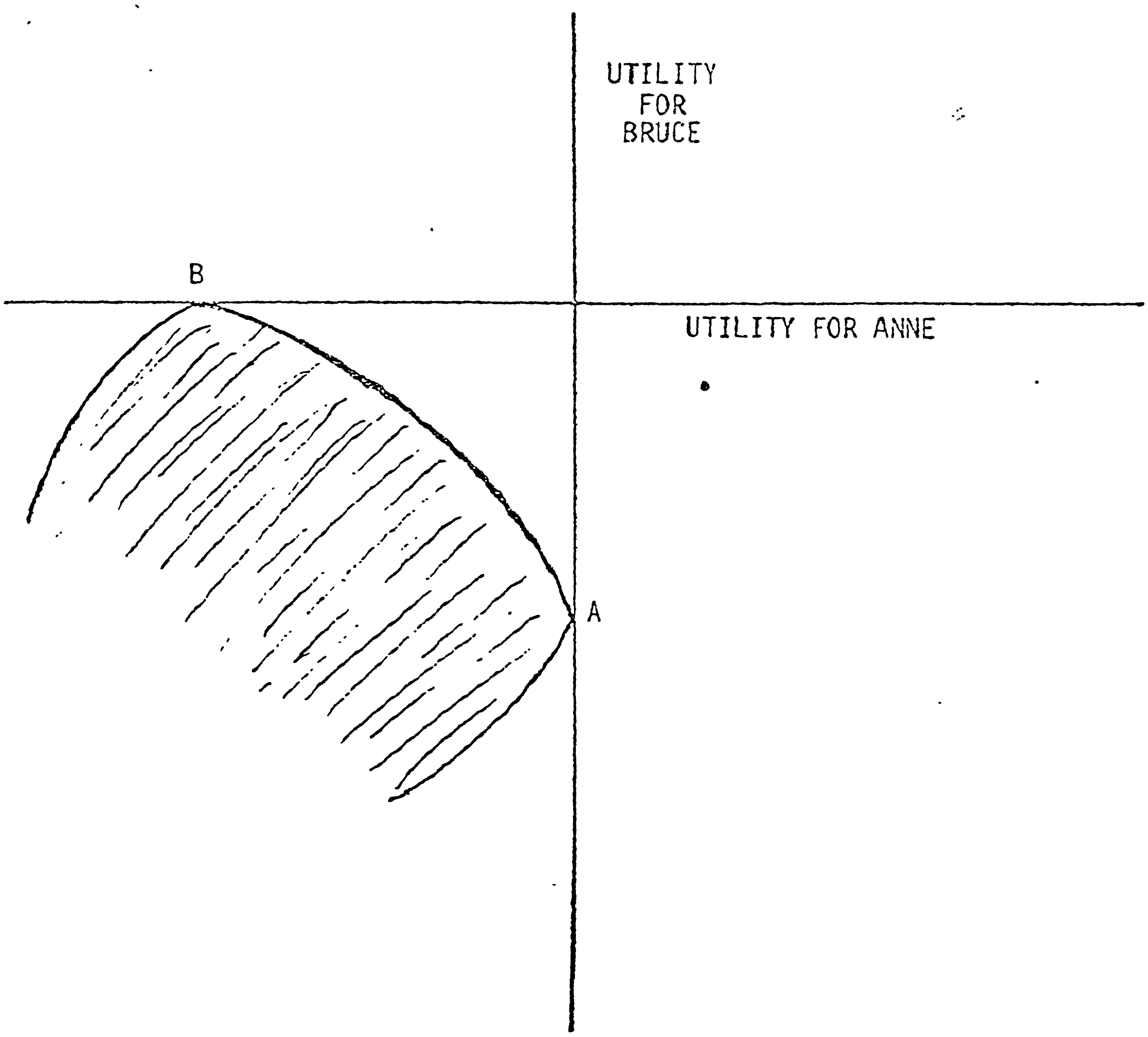
Notice that the term $\bar{U}^B$ does not enter equation (3). This condition must hold regardless of the level, $\bar{U}^B$, at which we set Bruce's utility. In general there will be many solutions of (3) corresponding to different points on the locus of Pareto optimal points in Figure 1 or equivalently to different levels of $\bar{U}^B$.

Recall also that we must have $\lambda \geqslant 0$. Therefore from (1) and (2) we see that the marginal utilities of cribbage and temperature must have opposite signs for Anne and Bruce at a Pareto optimal point.

We have now uncovered all of the optimality conditions that we saw from the diagram, but using Lagrangeans. This may not seem like a big gain. But what we will soon discover is that we now have a tool that will enable us easily to analyze cases that are much too complicated for graphs.

Before we leave Anne and Bruce to their own devices, we will ask them to introduce one more notion that we will find useful. This is the "utility possibility frontier". In Figure 2 we put utilities for Anne and Bruce on the axes. We shade in the set of utility combinations for Anne and Bruce

FIGURE 2



that are achievable by means of possible situations. For example the point B in Figure 2 represents the utilities achieved from Bruce's favourite position (B in Figure 1). Likewise A represents the utilities achieved from Anne's favourite situation. The line segment AB in Figure 2 which is the northeast boundary of the utility possibility set is known as the "utility possibility frontier". Points on this line represent utilities achievable from Pareto optimal allocations. In general, a utility possibility frontier might be either convex or concave but it must slope downhill. Otherwise it would not represent the set of Pareto optimal distributions of utility.

EXERCISES

1. Suppose that Anne's preferences are represented by the utility function $U^A(C, T) = -[(C-20)^2 + (T-25)^2]$ and Bruce's preferences are represented by the utility function $U^B(C, T) = -[(C-10)^2 + (T-15)^2]$.

A. Sketch their indifference curves on a diagram.

B. Is the situation (10, 15) Pareto optimal?

C. Find the set of all Pareto optimal situations. (Hint: While Lagrangean analysis solves this problem nicely, it could also be solved by plain plane geometry).

D. Find the set of situations that is Pareto superior to (9, 14).

E. Find a situation in which Anne's indifference curve is tangent to Bruce's but which is not Pareto optimal. Explain what is going on.

F. If situation α is Pareto optimal and situation β is not Pareto optimal, must α be Pareto superior to β? Explain.

G. Suppose that Charles moves in with Anne and Bruce. The three of them learn three-handed cribbage and lose all interest in two-handed cribbage. Anne's and Bruce's utilities are as before, while Charles' utility function is:

$$U^C(C, T) = -[(C-20)^2 + (T-15)^2].$$

Find the new set of Pareto optimal situations.

H. Forget Charles.

Suppose that Anne and Bruce lose interest in cribbage, but are still concerned about two things ... the room temperature and breakfast time. In principle, breakfast could be served at any time on a 24 hour clock. On the tube from the center of a roll of toilet paper, or some similar material draw indifference curves for Anne and Bruce. (Use two colours of ink). Show the locus Pareto optimal situations.

I. Suppose the only two issues of concern are breakfast time and dinner time. Explain how you could use a doughnut to diagram indifference curves and Pareto optimal points. What can you tell us about the set of Pareto optimal points?

J. Find the utility possibility frontier for Anne and Bruce in problem A.

K. Explain why the utility possibility frontier must in general slope down.

L. In Figure 2 the boundary of the utility possibility set slopes uphill to the west of point B. Explain why this is so.

Charles and Diana live on a green island. They are interested in only two things ... sherry and monuments. They both like sherry and monuments, the more, the better. Neither Charles nor Diana is allowed to work, but they do have a fixed amount of money worth \$W. They can spend this money either on sherry which costs p_x per litre or on monuments which cost p_y each. Now sherry, once purchased, can be drunk either by Charles or by Diana, while a monument is enjoyed by both. A good like sherry which can be consumed by one person or the other but not by both is known as a *private good*. A good like monuments which is not divided among the different consumers but which may affect the utility of more than one person is known as a *public good*. In the tale of Anne and Bruce, there were two public goods and no private goods. Here we have one public good and one private good. To fully describe an allocation of resources on the island we need to know not only the total output of private goods and of public goods, but also how the private good is divided between Charles and Diana.

We proceed to analyze conditions for efficient resource allocation on the green island. Let X_C and X_D be the amounts of sherry consumed by Charles and Diana respectively. Let Y be the number of monuments that they obtain. Since sherry and monuments are purchased at prices p_x and p_y, sugject to a budget of \$W, the set of possible resource allocations for Charles and Diana is described by the set of triples:

$$\left\{(X_C, X_D, Y) \mid p_x(X_C + X_D) + p_y Y \leqslant W\right\}$$

In general, Charles' utility function might depend on Diana's sherry consumption as well as on his own and on the number of monuments. He might, for example, like her to consume sherry because he likes her to be happy or he might not like her to overindulge and embarrass him. Thus we would want him to have a utility function of the form:

$$U_C(X_C, X_D, Y)$$

For our first pass at the problem, however, let us simplify matters by assuming that both Charles and Diana are totally selfish about sherry.

That is, neither cares how much or little the other consumes. If this is the case, then their utility functions would have the form:

$$U_C(X_C, Y) \text{ and } U_D(X_D, Y) .$$

The allocation problem of Charles and Diana is mathematically a bit more complicated than that of Anne and Bruce. There are three decision variables instead of two and there is a feasibility constraint as well as the two utility functions. Therefore it is more difficult to represent the whole story on a graph. In fact, at least at first, doing this is so complicated as to obscure rather than clarify matters. It is, however, quite easy to find interesting conditions for Pareto optimality using Lagrangean methods. In fact, as we will show, these conditions can also be decuded by a bit of careful "literary" reasoning.

We begin with the Lagrangean approach. At a Pareto optimum it should be impossible to find a feasible allocation that makes Charles better off without making Diana worse off. Therefore, Pareto optimal allocations can be found by setting Diana at an arbitrary (but possible) level of utility, $\overline{U}_D$ and maximizing Charles' utility subject to the constraint that $U_D(X_D, Y) \geqslant \overline{U}_D$ and the feasibility constraint. Formally then we seek a solution to the constrained maximization problem:

Choose X_C, X_D and Y to maximize:

$$U_C(X_C, Y)$$

subject to:

$$U_D(X_D, Y) \geqslant \overline{U}_D) \quad \text{and}$$

$$p_x(X_C + X_D) + p_y Y \leqslant W \quad .$$

The Lagrangean for this problem is:

$$L(X_C, X_D, Y, \lambda_1, \lambda_2) = U_C(X_C, Y)$$

$$+ \lambda_1(U_D(X_D, Y) - \overline{U}_D$$

$$+ \lambda_2(W - p_x(X_C + X_D) - p_y Y) \qquad (1)$$

Setting the partial derivatives of $L(.)$ equal to zero we have:

$$\frac{\partial U_C}{\partial X_C} - \lambda_2 p_x = 0 \tag{2}$$

$$\lambda_1 \frac{\partial U_D}{\partial X_D} - \lambda_2 p_x = 0 \tag{3}$$

$$\frac{\partial U_C}{\partial Y} + \lambda_1 \frac{\partial U_D}{\partial Y} - \lambda_2 p_y = 0 \tag{4}$$

From (2) it follows that:

$$\lambda_2 = \frac{1}{p_x} \frac{\partial U_C}{\partial X_C} \tag{5}$$

From (3) and (5) it follows that:

$$\lambda_1 = \frac{\partial U_C}{\partial X_C} \div \frac{\partial U_D}{\partial X_D} \tag{6}$$

Use (5) and (6) to eliminate λ_1 and λ_2 from (4). Divide the resulting expression by $\frac{\partial U_C}{\partial X_C}$ and you will obtain:

$$\frac{\frac{\partial U_C}{\partial Y}}{\frac{\partial U_C}{\partial X_C}} + \frac{\frac{\partial U_D}{\partial Y}}{\frac{\partial U_D}{\partial X_D}} = \frac{p_y}{p_x} \tag{7}$$

This is the fundamental "Samuelson condition" for efficient provision of public goods. Stated in words, (7) requires that the *sum* of Charles' and Diana's marginal rates of substitution between monuments and sherry must equal the cost of an extra unit of monument relative to an extra unit of sherry.

Let us now try to deduce this condition by literary methods. The rate at which either person is willing to exchange a marginal bit of sherry for a marginal bit of monument is just his marginal rate of substitution. Thus the left side of equation (7) represents the amount of sherry that Charles would be willing to give up in return for an extra bit of monument plus the amount that Diana is willing to forego for an extra bit of monument. If the left side of (7) were greater than $\frac{p_y}{p_x}$ then they could both be made better off, since the total amount of sherry that they are willing to give up for an extra bit of monument is greater than the total amount, $\frac{p_y}{p_x}$, of sherry that they would have to give up in order to be able to afford an extra bit of monument. Similarly if the left hand side of (7) were less than $\frac{p_y}{p_x}$, it would be possible to make both better off by purchasing fewer monuments and giving each person more sherry. Therefore an allocation can be Pareto optimal only if (7) holds.

A Family of Special Cases - Transferable Utility

Suppose that Charles and Diana have utility functions that have the special functional form:

$$U_C(X_C, Y) = X_C + f_C(Y) \quad \text{and}$$

$$U_D(X_D, Y) = X_D + f_D(Y) \quad \text{where}$$

the functions f_C and f_D have positive first derivatives and negative second derivatives. Then the indifference curves of each person will be qualitatively like Figure 1.

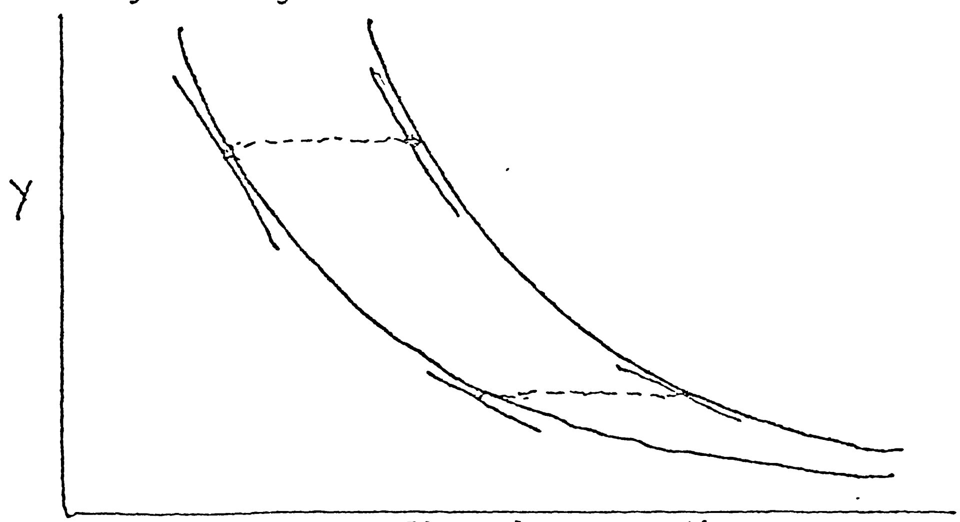


Figure 1

The assumptions about the derivatives of f_C and f_D ensure that the curves slope down and are convex toward the origin. The assumption that $U(X_C, Y)$ is linear in X_C implies that each indifference curve is simply a vertical translation of any other and that indifference curves are parallel in the sense that for given Y, the slope of the indifference curve at (X, Y) is the same for all X.

Since $\frac{\partial U^C}{\partial X^C} = 1$, it is easy to see that Charles' marginal rate of substitution between monuments and sherry is simply $\frac{\partial U_C}{\partial Y} = f_C'(Y)$. Likewise Diana's marginal rate of substitution is just $f_D'(Y)$. Therefore the necessary condition stated in equation (7) takes the special form

$$f_C'(Y) + f_D'(Y) = \frac{p_y}{p_x} \tag{10}$$

Since we have assumed that f_C'' and f_D'' are both negative, the left side of (8) is a decreasing function of Y. Therefore, given $\frac{p_y}{p_x}$, there can be at most one value of Y that satisfies (8). The following diagram may be helpful.

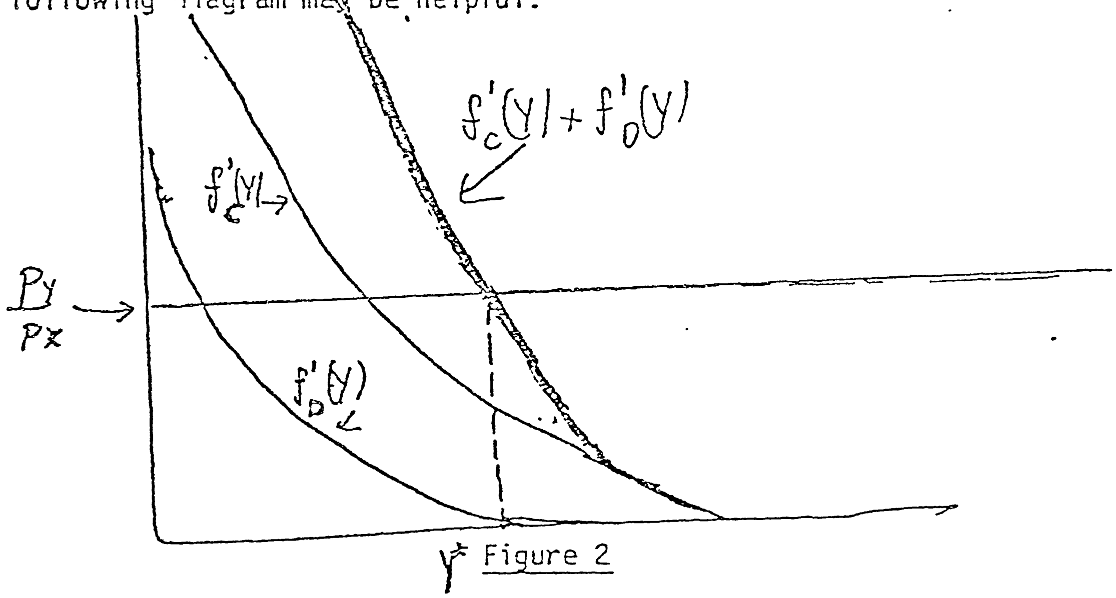


Figure 2

The curves $f_C'(Y)$ and $f_D'(Y)$ represent Charles' and Diana's marginal rates of substitution between monuments and sherry. (In the special case treated here marginal rates of substitution are independent of the other variables X_C and X_D.) The curve $f_C'(Y) + f_D'(Y)$ is obtained by summing the individual m.r.s. curves *vertically* (rather than summing horizontally as one does with demand curves for private goods).

The only value of Y that satisfies condition (8) is Y*, where the summed m.r.s. curve crosses the level $\frac{p_y}{p_x}$.

As we will show later, the result that the optimality condition (8) uniquely determines the amount of public goods is special to models of the kind discussed in this section. In fact this class of models is so special and so convenient that it has earned itself a special name. Models in which there is some commodity consumed by everyone in which each consumer's utility function is linear are said to have *transferable utility*. If there is transferable utility, the utility possibility frontier will be a straight line at all points on the frontier achievable with positive consumptions of the good in which utility is linear.

An Example Without Transferable Utility

Suppose that Charles and Diana both have Cobb-Douglas utility functions. In particular:

$$U_C(X_C, Y) = X_C^2 Y$$

$$U_D(X_D, Y) = X_D Y$$

Suppose that $p_x = p_y = 1$.

We calculate the marginal rates of substitution to find that condition (7) in this special case reduces to

$$2\frac{X_C}{Y} + \frac{X_D}{Y} = 1 \quad (9)$$

or equivalently:

$$2X_C + X_D = Y \quad (10)$$

We see that (10) does not yield a unique solution for Y as did (8). How much public good is to be provided depends on how we choose to divide the private good. Equivalently the amount of public good selected depends upon which point on the utility possibility frontier we choose.

Since $p_x = p_y = 1$, the feasibility constraint for our example reduces to:

$$X_C + X_D + Y = W \qquad (11).$$

Equations (10) and (11), together give us two linear equations in the three unknowns X_C, X_D and Y. Thus there is in general a one dimensional family of solutions to these two equations. If we specify in advance, say, the ratio of X_C to X_D, then Y is completely determined. For example if $X_C = X_D$, then (10) implies that $Y = 3X_C$ and (11) implies that $5X_C = W$ so that $Y = \frac{3}{5} W$. If, instead, $X_C = 2X_D$, then $Y = 5X_C$ and from (11), $8X_C = W$ so that $Y = \frac{5}{8} W$.

On Social Welfare Functions

In the last example we saw that the Pareto efficient amount of public good depends in general on how the private goods are divided. Different points on the utility possibility frontier correspond to different distributions of private goods and in general to different amounts of public goods.

Suppose that a benevolent dictator were placed in charge of choosing an allocation of resources between Charles and Diana. The dictator's income, we will assume, is constitutionally fixed so that he cannot take sherry or monuments away from Charles and Diana. His utility depends on the allocation of resources, but it has the special property that the happier Charles and Diana are, the happier he is. Thus the dictator's utility function takes the special form

$$U^*(X_C, X_D, Y) = W(U_C(X_C, Y), U_C(X_D, Y)$$

where $\frac{\partial W}{\partial U_C} > 0$ and $\frac{\partial W}{\partial U_D} > 0$ for all U_C and U_D. The dictator will choose X_C, X_D and Y so as to maximize $W(U_C(X_C, Y), U_D(X_C, Y)$ subject to the constraint, $p_x(X_C + X_D) + p_y Y = W$. Since $W(.)$ is an increasing function of U_C and U_D, it follows that the dictator will choose a Pareto optimal allocation. In fact, if we wish we can separate his choice problem into three parts.

(i) Construct the utility possibility frontier.

(ii) Find the point $(\overline{U}_C, \overline{U}_D)$ on the utility possibility frontier that maximizes $W(U_C, U_D)$.

(iii) Find the allocation of resources that yields the utility combination $(\overline{U}_C, \overline{U}_D)$.

The theory we have just outlined is a pretty good theory of benevolent dictatorships. One difficulty with the theory is that many of the organizations and governments that we want to study are not dictatorships. One point of view which could be taken is that there is a "social welfare function", formally the same as the function, W, but representing not the will of a dictator but rather a "social ethic" or a "justice" or the "will of God".

Implicit in many discussions of social choice is the following kind of reasoning. (i) It has been shown that the optimal amount of public goods cannot be determined without knowing the distribution of income. (ii) In order to decide on the best distribution of income we need a social welfare function. (iii) Therefore economists must either find a social welfare function or despair of giving intelligent advice to policy-makers. Some economists have been led by this reasoning to press the search for a social welfare function, no matter how unpromising the search may seem. Others have concluded that since there is no likelihood of a social welfare function ever being found, there can be no respectable intellectual foundation for economics as a "policy sicence". Perhaps the majority of economists muddle ahead, advising anyone who will listen to them. Among these, the intellectually sensitive may occasionally express discomfort about the logic of what they do.

Much of this concern is, I believe, misplaced. Although premise (i) of the argument is true as we have shown and premise (ii) is approximately true (though somewhat overstated), the conclusion (iii) does not follow from (i) and (ii). There is much useful advice that economics can give even if it is unable to determine a "best" allocation of resources. To see this point, let us consider a stylized example. An economist discovers that a certain public project would cost one million dollars and that the sum of everyone's willingness to pay for the project, given their current incomes is ten million dollars. If he knew everybody's preferences it would not be difficult in principle for him to devise a tax scheme that would pay for the project and leave everyone better off. (Even if he had only estimates of individual preferences he could probably devise a scheme that makes almost everybody better off.) The only difficulty is that he has an embarrassment of possible solutions. There are many different ways to collect one million dollars from a population willing to pay ten million dollars so that everyone benefits more from the project than he is taxed. Even without a social welfare function, the economist can recommend that the project is justified by the Pareto criterion if it is paid for by any one of a family of tax schemes that he suggests. This is important and useful advice. What the economist cannot do without something like a social welfare function is specify which of these alternative tax schemes is "best". For to do so he would have to decide on the "best" way to divide the surplus obtained from the project. Of course the economist may have his own ideas about the best way to divide the spoils. He could suggest what he would do if he were dictator. (I am aware of no historical instances in which an economist was allowed to be dictator.)

The appropriate lesson to be drawn from this discussion is that economics could use the Pareto criterion in important and useful ways even if it offered little or nothing in the way of guidance about choice among Pareto optimal points. In later lectures we will argue that, while there is no hope for finding "the social welfare function", economic reasoning can provide some useful guidance about choosing among alternative Pareto optimal points.

PROBLEMS

1. Generalize the analysis of this lecture by using Lagrangean analysis to extend the results to the case of several people, and several goods of each kind.

2. Prove the assertions made in the two sentences following Figure 1.

3. Suppose that $U(X_C, Y) = X_C + 2\sqrt{Y}$ and $U_C(X_D, Y) = X_C + \sqrt{Y}$. Suppose that $p_y = p_x = 1$. Determine, if possible, the amount of Y to be produced if the output is to be Pareto optimal and both persons are to have positive consumption of private goods.

4. In problem 3, find the Pareto optimal allocations (if any) in which one or the other persons consumes no private good. Hint: At "boundary solutions" of this type the summed marginal rate of substitution conditions need not apply. The problem can still be solved without any advanced mathematics. Careful thinking will do the trick.

5. For problem 3, find the set of possible utility combinations and the utility possibility frontier.

6. Show that if there is transferable utility as discussed in the text of the lecture, then the utility possibility frontier is linear at utility combinations achieved with positive consumption of sherry by both persons.

7. <u>Diagrams for Charles and Diana</u>

Reduce the dimensionality of the problem and eliminate explicit use of the feasibility constraint by using the feasibility constraint to eliminate Y. Since

$$p_x(X_C + X_D) + p_y Y = W, \qquad \text{we can write}$$

$$Y = W - \frac{p_x}{p_y}(X_C + X_D)$$

Define $U_C{}^*(X_C, X_D) \equiv U_C(X_C, W - \frac{p_x}{p_y}(X_C + X_D))$ and

$$U_D{}^*(X_C, X_D) \equiv U_D(X_D, W - \frac{p_x}{p_y}(X_C + X_D)) \quad .$$

Notice that $U_C{}^*(.)$ and $U_D{}^*(.)$ depend both on X_C and X_D. The problem has thus been transformed into a problem that is formally similar to the problem of Anne and Bruce. What do the indifference curves for $U_C{}^*$ and $U_D{}^*$ look like? Draw a diagram similar to that done in Lecture 1 for the transformed utilities. Find optimality conditions and show that they are the same as those found in Lecture 2.

8. Prove that the dictator discussed in the lecture would choose a Pareto optimal allocation.

Pollution

Ed smokes. Fiona, his neighbor, hates smoke. Ed and Fiona both love beans. Neither cares how many beans the other eats. Ed can get tobacco for free. Both have fixed incomes that can be used to buy beans. Ed's utility function is

$$U^E(S, B_E)$$

and Fiona's utility function is

$$U^F(S, B_F)$$

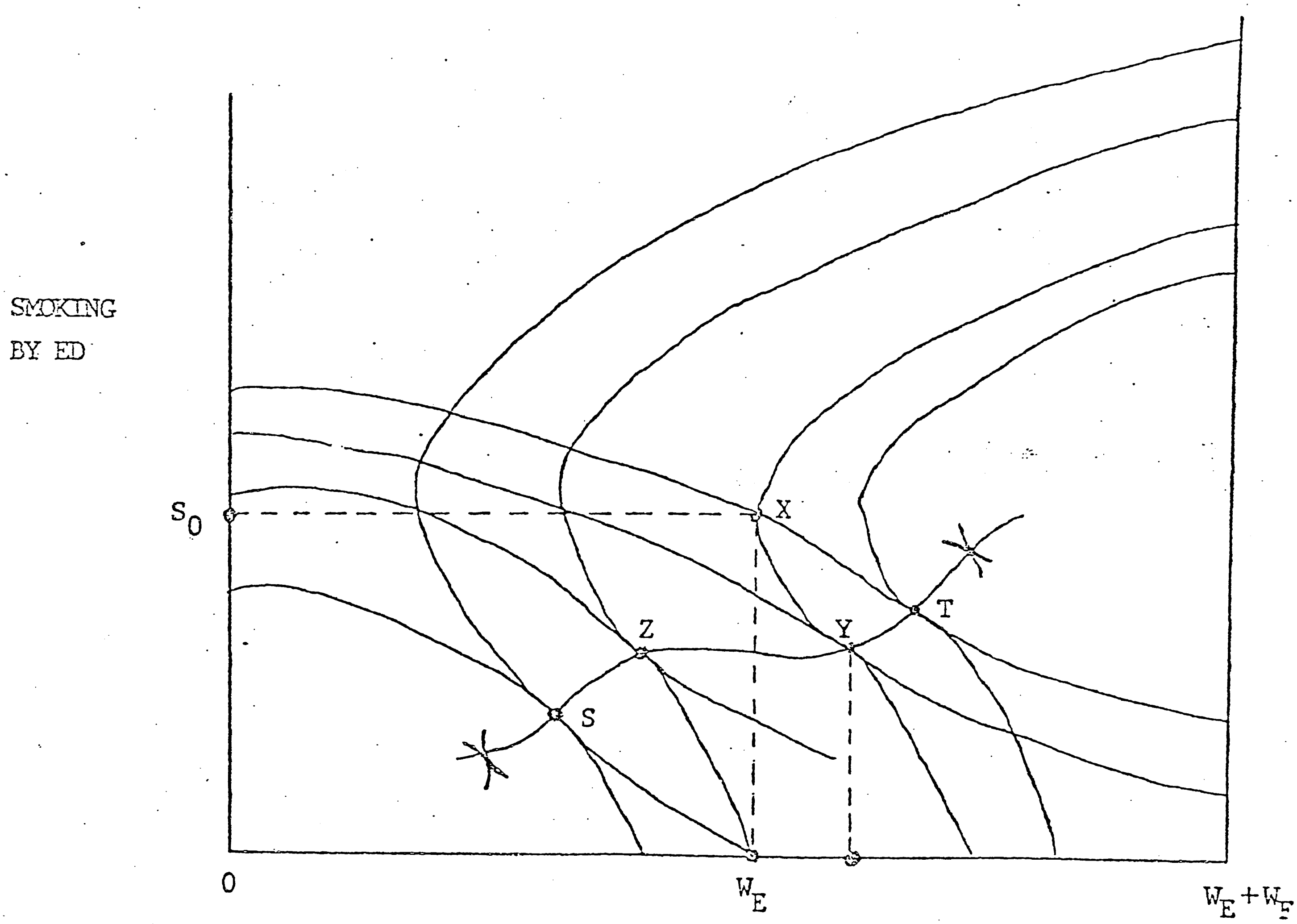
where S is the amount of smoking that Ed does and B_E and B_F are the amounts of beans consumed by Ed and Fiona respectively.

The set of allocations available to Ed and Fiona consists of all the triples (S, B_E, B_F) such that

$$B_E + B_F = W_E + W_F$$

where W_E and W_F are the wealths of Ed and Fiona, measured in terms of the numeraire, beans. Figure 1 represents the preferences of Ed and Fiona among possible allocations.

FIGURE 1



A point on the graph in Figure 1 represents an allocation in the following way. The horizontal distance of the point from the left side of the graph is beans for Ed. The distance from the right side is beans for Fiona. The vertical distance from the bottom of the graph is the total amount of smoking by Ed. Each point on the graph represents a feasible allocation since the sum of Ed's and Fiona's beans will always be $W_E + W_F$ and since we have assumed that there is no resource constraint on Ed's smoking. Ed's indifference curves are the curves bulging out from the right side. They bend back on themselves because we suppose that, even for Ed, too much smoking is unpleasant. Fiona's curves slope downwards away from the point 0. This gives her convex preferences and a preference for more beans and less smoke.

If there were no restrictions on smoking and no bargains were made between Ed and Fiona, then Ed and Fiona would each spend their own wealth on their own beans and Ed would smoke an amount, S_0. But the allocation

$$X = (S_0, W_E, W_F)$$

is not Pareto optimal. This can be seen by noticing that any point inside the football-shaped region whose tip is X designates a feasible allocation that is Pareto superior to X. They would both be made better off if Fiona would give Ed some of her beans in return for which Ed would smoke less. It is easy to see that the Pareto optimal allocations are points of tangency between Ed's and Fiona's curves. Those Pareto optimal allocations which are better for both Ed and Fiona than the allocation X are represented by the points on the line segment, YT. If Ed has a legal right to smoke as much as he likes and if Fiona and Ed bargain to reach a Pareto optimal point, the outcome would be somewhere on YT.

Alternatively, there might be a law that forbids Ed to smoke without Fiona's consent. If no deal were struck, the allocation would have no smoking, Ed consuming W_E and Fiona consuming W_F. Again we see from

Figure 1 that this allocation is not Pareto optimal. Both parties would benefit if Ed gave Fiona some beans in return for permission to smoke. The Pareto optimal allocations that are Pareto superior to the no-smoking allocation are represented by the line SZ.

The set of all Pareto optimal allocations includes the entire line ST as well as points of tangency beyond S and T. We notice that the optimal amount of smoke will not be entirely independent of the point on the curve ST that is chosen. More importantly, we notice that private bargains can be struck to reach optimality regardless of whether Ed has a right to smoke as much as he wants or whether Fiona has a right to prevent him from smoking altogether. What the law says that one has a right to do is not necessarily what will be done. Rather it determines the starting point from which bargaining can proceed. The situation is exactly analogous to the specification of the initial allocation of ownership in the theory of private goods in pure exchange. This point is made and illustrated with fascinating legal examples by Ronald Coase in his article "The Problem of Social Cost", *Journal of Law and Economics,* 1960.

LECTURE 4

Congestion

Only two activities interest the citizens of Hot Rod, Indiana. These are driving cars and eating Big Macs. Each individual, i, in Hot Rod has an initial wealth, W_i. Big Macs cost $1 each. The cost of the fuel used per unit of driving is p_F. Driving also causes congestion. Let H be the amount of highways in Hot Rod and let

$$D = \sum_{j=1}^{n} D_j$$

be the total amount of driving by the n citizens of Hot Rod. The level of congestion is a function C(D, H). Denote $\frac{\partial C}{\partial D}$ by C_D and $\frac{\partial C}{\partial H}$ by C_H. We assume that $C_D > 0$ and $C_H < 0$. Preferences of the ith Hot Rodder are represented by a utility function:

$$(1) \qquad U_i = U_i(D_i, C(\sum_{j=1}^{n} D_j, H), M_i)$$

where D_i and M_i are respectively the amount of driving and the number of Big Macs consumed by i. We assume that U_i is an increasing function of D_i and M_i and a decreasing function of C. The set of feasible allocations in Hot Rod is described as the set of vectors of the form $(D_1, \ldots, D_n, H, M_1, \ldots, M_n)$ such that total expenditures on Big Macs, fuel, and highways in Hot Rod add to the total wealth of its citizens. If the price of a Big Mac is $1, a unit of fuel costs p_F, and a unit of highway costs the community p_H, this budget constraint is:

$$(2) \qquad \sum_{i=1}^{n} M_i + p_F \sum_{i=1}^{n} D_i + p_H H \leq \sum_{i=1}^{n} W_i$$

We could solve directly for the necessary conditions for Pareto optimality, using Lagrangean analysis. Alternatively we could notice that formally this model is equivalent to a model in which there are n+1 public goods and one private good. Each person's driving enters everyone's utility function. So does the level of highway expenditures. Hence there are n+1 public goods where anyone's driving is a public good. (Of course one person's driving is a public good which is disliked by everyone other than the driver, but the theory we have developed works as well for public "bads" as for public goods.) No-one but i cares about i's Big Mac consumption. Thus, Big Macs are private goods.

Recalling the Samuelson conditions for Pareto efficiency, we have:

$$(3) \qquad \sum_{j=1}^{n} \left(\frac{\partial U_j}{\partial H} \div \frac{\partial U_j}{\partial M_j}\right) = p_H$$

and for all i:

$$(4) \qquad \sum_{j=1}^{n} \frac{\partial U_j}{\partial D_i} \div \frac{\partial U_j}{\partial M_j} = p_F \,.$$

Condition (3) says that the sum of the marginal rates of substitution between highways and private goods must equal the marginal cost of highways. Condition (4) asserts that for each i, the sum of the marginal rates of substitution for the public good "driving by person i" must equal the marginal resource cost of driving by i. Condition (4) could be written equivalently as:

$$(5) \qquad \frac{\partial U_i}{\partial D_i} \div \frac{\partial U_i}{\partial M_i} = p_F - \sum_{j \neq i} \left(\frac{\partial U_j}{\partial D_i} \div \frac{\partial U_j}{\partial M_j}\right)$$

for all i. Since for $j \neq i$, $\frac{\partial U_j}{\partial D_i} < 0$, we see that (5) requires that i's marginal rate of substitution between driving and Big Mac's must equal the fuel cost p_F plus the sum of the amounts that would have to be paid each other citizen to compensate for a marginal bit of driving by i.

Where congestion costs enter through a congestion function of the form indicated in (1), we can exploit the special structure of the problem to deduce stronger results. Given the special form of (1), we can write (3) and (5) as:

$$(6) \qquad C_H \sum_{j=1}^{n} \frac{\partial U_j}{\partial C} \quad \frac{\partial U_j}{\partial M_j} = p_H$$

and

$$(7) \qquad \frac{\partial U_i}{\partial D_i} \div \frac{\partial U_i}{\partial M_i} = p_F - C_D \sum_{j \neq 1} \frac{\partial U_j}{\partial C} \div \frac{\partial U_j}{\partial M_j}$$

for all i. Define

$$(8) \qquad \lambda_i = \frac{\sum_{j \neq 1} \frac{\partial U_j}{\partial C} \div \frac{\partial U_j}{\partial M_j}}{\sum_{j=1}^{n} \frac{\partial U_j}{\partial C} \div \frac{\partial U_j}{\partial M_j}}$$

Then substituting from (6) and (8), we can rewrite (7) as:

$$(9) \qquad \frac{\partial U_i}{\partial D_i} \div \frac{\partial U_i}{\partial M_i} = p_F - \lambda_i \frac{p_H C_D}{C_H} \qquad \text{for all } i.$$

For large economies, λ_i will be close to one. Thus (9) implies the following approximation

$$(9') \qquad \frac{\partial U_i}{\partial D_i} \div \frac{\partial U_i}{\partial M_i} \cong p_F - p_H \frac{C_D}{C_H}$$

Since $C_D > 0$ and $C_H < 0$, $\frac{C_D}{C_H} < 0$.

Condition (9') asserts that i's marginal rate of substitution between driving and Big Macs should be approximately equal to the fuel cost of driving plus the cost of adding enough extra highway to eliminate the extra congestion caused by the marginal bit of driving.

If all Hot Rodders were allowed to drive as much as they liked so long as they paid for their own fuel, they would drive to the point where

$$(10) \qquad \frac{\partial U_i}{\partial D_i} \div \frac{\partial U_i}{\partial M_i} = p_F.$$

Comparing (10) with (8) and noticing that $\lambda_i p_H \frac{C_D}{C_H} < 0$, we see that there would then be "too much" driving. (At least if there are diminishing marginal rates of substitution).

It is reasonable to guess that a system of tolls for driving could be imposed to induce Pareto efficient behavior. In case Hot Rodders had identical preferences and wealths, it turns out that an equalitarian Pareto optimum could be enforced by a uniform system of tolls. Suppose that a toll of T is charged by the government per unit of driving. Suppose also that the total revenue from tolls is returned in equal shares to all citizens of Hot Rod. Suppose likewise that the cost of highways are paid for by lump sum taxes collected in equal amounts by all citizens of Hot Rod. Then the budget constraint for citizen i of Hot Rod is:

(11) $M_i + (p_F + T)\, D_i = W_i - \frac{p_H H}{n} + \frac{T \sum_{j=1}^{n} D_j}{n}$

or rearranging terms slightly:

(12) $M_i + (p_F + (1 - \frac{1}{n})\; TD_i = W_i - \frac{p_H H}{n} + \frac{T \sum_{j \neq i} D_j}{n}$

The right side of (12) consists of variables that i does not control. From the left side of (12) we see that he is choosing M_i and D_i where the "prices" are respectively 1 and $p_F + (1 - \frac{1}{n})T$. Therefore he will choose M_i and D_i so that:

(13) $\frac{\partial U_i}{\partial D_i} \div \frac{\partial U_i}{\partial M_i} = p_F + (1 - \frac{1}{n})T.$

Recalling (9) we see that the toll T would lead to efficient amounts of driving only if:

(14) $(1 - \frac{1}{n})T = -\lambda_i p_H \frac{C_D}{C_H}$.

Since we are looking at a symmetric solution, we see that $\lambda_i = 1 - \frac{1}{n}$ so that (14) is equivalent to

(15) $T = -p_H \frac{C_D}{C_H}$.

Thus condition (15) is a solution for the toll that induces an efficient amount of driving.

The government in this model has two tasks. Collecting (and returning) tolls and providing (and taxing for) highways. An interesting question is whether the revenue collected from efficient tolls would be sufficient to pay for highway construction.

Let us suppose that the congestion function is homogeneous of some degree, k. If $k = 0$, there are "constant returns to scale" in the sense that doubling both the amount of driving and the amount of highways leaves the level of congestion unchanged. If $k > 0$, there are "decreasing return Doubling driving and highways makes congestion worse. If $k < 0$, there are increasing returns. Doubling driving and highways leads to less congestion. Using Euler's theorem we note that

(16) $DC_D + HC_H = kC.$

From (16) we see that

(17) $DC_D \gtreqless -HC_H$

if $k \lesseqgtr 0.$

From (15) we see that if an optimal toll is charged, total government revenue from tolls will be:

(18) $TD = -p_H \frac{C_D}{C_H} D$

If $k = 0$, we see from (17) and (18) that

(19) $TD = p_H H.$

Thus in the case of constant returns to scale, the revenue from tolls just covers construction costs. Also from (17) and (18) we see that

(20) $TD \gtrless p_H H$ if $k \lessgtr 0.$

Thus if there are decreasing (increasing) returns, efficient toll revenue will more than (less than) cover total highway construction costs.

Let there by one private good and one public good. Consumer i has the utility function.

(1) $$U_i(X_i, Y) = X_i + F_i(Y)$$

where X_i is his private good consumption and Y is the amount of public good. Each i has an initial endowment of W_i units of private good. Public good must be produced using private goods as an input. The total amount of private goods needed to produce Y units of public good is a function C(Y). Assume that F_i is a strictly concave function and C a convex function. If we consider only allocations in which everyone receives at least some private good, then for this economy there is a unique Pareto optimal amount of public good. This amount is the amount that maximizes

(2) $$\sum_i F_i(Y) - C(Y)$$

Consumers are asked to reveal their functions F_i to the government. Let M_i (possibly different from F_i) be the function that consumer i claims. Let $M = (M_1, \ldots, M_n)$ be the vector of functions claimed by the population. If the reported vector is M, the government chooses an amount of public goods, Y(M), such that

(3) $$\sum_i M_i(Y(M)) - C(Y(M)) \geqq \sum_i M_i(Y) - C(Y)$$

for all $Y \geqq 0$.

Taxes, $T_i(M)$, are then assigned to each consumer i where

(3) $$T_i(M) = C(Y(M)) - \sum_{j \neq i} M_j(Y(M)) - R_i(M)$$

and where $R_i(M)$ is some function that may depend on the functions, M_j, reported by consumers other than i but is constant with respect to M_i.

If the vector of functions reported to the government is $M = (M_1, \ldots, M_n)$, then Consumer i's private consumption is

(4) $$X_i(M) = W_i - T_i(M)$$

and his utility is

(5) $$X_i(M) + F_i(Y(M)) = W_i + \sum_{j \neq i} M_j(Y(M)) + F_i(Y(M)) - C(Y(M)) + R_i(M)$$

Since $W_i + R_i(M)$ is independent of M_i, we notice that the only way in which i's stated function M_i affects his utility is through the dependence of $Y(M)$ on M_i.

We see, therefore from (5) that given any choice of strategies by the other players, the best choice of M_i for i is the one that leads the government to choose $Y(M)$ so as to maximize

(6) $$\sum_{j \neq i} M_j(Y) + F_i(Y) - C(Y).$$

But recall that the government attempts to maximise

(7) $$\sum_{j=1}^{n} M_j(Y) - C(Y).$$

Therefore if consumer i reports his true function, so that, i.e. $M_i = F_i$, then the government in maximizing (7) will maximize (6). It follows that the consumer can not do better and could do worse than to report the truth. Honest revelation is therefore a dominant strategy.

Since everyone chooses his dominant strategy, true preferences are revealed and the government's choice of $Y(M)$ is the value of Y that maximizes

(8) $$\sum_{j=1}^{n} F_j(Y) - C(Y)$$

This leads to the correct amount of public goods. Of course for the device to be feasible, it must be that total taxes collected are at least as large as the total cost of the public goods. If the outcome is to be Pareto optimal, the total amount of taxes collected must be no greater than the total cost of public goods. Otherwise private goods are wasted. We are left, therefore, with the task of trying to rig the functions $R_i(M)$ in such a way to establish this balance. In general, it turns out to be impossible to find functions $R_i(M)$ that are independent of M_i for each i and such that

(9) $$\sum_i T_i(M) = C(Y(M)).$$

However, Clarke and also Groves and Loeb found functions $R_i(M)$ that guarantee that tax revenues at least cover total costs.

Their idea can be explained as follows. Suppose that for each i, the government sets a "target share" $\theta_i \geqq 0$ where $\sum_i \theta_i = 1$. The government then tries to fix $R_i(M)$ so that $T_i(M) \geqq \theta_i C(Y(M))$ for each i. Then, of course, $\sum_i T_i(M) \geqq C(Y(M))$. From equation (3), it follows that

(10) $$T_i(M) - \theta_i C(Y(M)) = [(1-\theta_i)C(Y(M)) - \sum_{j \neq i} M_j(Y(M))] - R_i(M).$$

Therefore the government could set $T_i(M) = \theta_i C(Y(M))$ if and only if it could set

$$R_i(M) = (1-\theta_i)\ C(Y(M)) - \sum_{j \neq i} M_j(Y(M)).$$

But in general such a choice of $R_i(M)$ would be inadmissable for our purpose because $R_i(M)$ depends on M_i, since $Y(M)$ depends on M_i.

Suppose that the government sets

(11) $$R_i(M) = \min_Y [(1-\theta_i)\ C(Y) - \sum_{j \neq i} M_j(Y)]$$

Then $R_i(M)$ depends on the M_j's for $j \neq i$ but is independent of M_i. From (10) it follows that with this choice of $R_i(M)$ we have:

(12) $$T_i(M) - \theta_i\ C(Y(M)) \geqq 0 \text{ for all } i.$$

Therefore

(13) $$\sum_i T_i(M) \geqq C(Y(M)).$$

This establishes the claim we made for the Clarke tax.

THE GROVES-LEDYARD DEMAND REVEALING MECHANISM

LECTURE 6

T. BERGSTROM

Groves and Ledyard propose a demand revealing mechanism which they call "An Optimal Government". The mechanism formulates rules of a "game" in which the amount of public goods and the distribution of taxes is determined by the government as a result of messages which the citizens choose to communicate. Although the government has no independent knowledge of preferences, and citizens are aware that sending deceptive signals might possibly be beneficial, it turns out that Nash equilibrium for this game is Pareto optimal. The Groves-Ledyard mechanism is defined for general equilibrium and applies to arbitrary smooth convex preferences.

In contrast, the Clarke tax (discovered independently by Clarke [1971] and Groves and Loeb [1975]) is well defined only for economies in which relative prices are exogenously determined and where utility of all consumers takes the quasi-linear form: $U_i(X_i,Y) = X_i + F_i(Y)$.

The Clarke tax has the advantage that for each consumer, equilibrium is a dominant strategy equilibrium rather than just a Nash equilibrium. Thus there are no complications related to stability or multiple equilibria. On the other hand, the Clarke tax has the disadvantages that although it leads to a Pareto efficient amount of public goods it generally will lead to some waste of private goods.

It is of some interest to examine the nature of the Groves-Ledyard mechanism as applied to the case of transferable utility.

This helps us to develop some "feel" for the device by seeing how it performs in a manageable environment. It also is of some interest to compare the merits of this system with the Clarke tax when both are operating on the Clarke tax's home turf. We are able to show quite generally that when there is quasi-linear utility, the Groves-Ledyard mechanism has exactly one Nash equilibrium. Furthermore, this equilibrium is quite easily computed and described. This is of some interest because, in general, little is known about the uniqueness of Groves-Ledyard's equilibrium and the question of the existence of equilibrium is also less than satisfactorily resolved.

Suppose that there are n consumers, and one public good and one private good. Each consumer has an initial endowment of W_i units of private good. Public good is produced at a constant unit cost of q. Utility of each consumer takes the form $U_i(X_i,Y) = X_i + F_i(Y)$.

The government asks each consumer i to submit a number, (positive or negative) m_i. The government will supply an amount of public goods $Y = \sum_i m_i$. It will charge a tax to consumer i equal to

(1) $$C^i(m_1,\ldots,m_n) \equiv \alpha_i q \sum_k m_k$$

$$+ \frac{\gamma}{2}\left[\frac{n-1}{n}\left(m_i - \frac{1}{n-1}\sum_{k\neq i} m_k\right)^2 - \frac{1}{2(n-1)(n-2)}\sum_{k\neq i}\sum_{k'\neq i}(m^k - m^{k'})\right]$$

where the α_i's and γ are arbitrarily chosen positive parameters and $\sum_k \alpha_k = 1$. If the vector of messages is $(m_1,\ldots,m_n)$ then consumer i's utility will be

(2) $$W_i - C^i(m_1,\ldots,m_n) + F_i\left(\sum_k m_k\right)$$

where $F_i' > 0$ and $F_i'' < 0$.

Therefore in a Nash equilibrium each consumer i would be choosing m_i to maximize (2).

The first order condition for maximizing (2) reduces to

(3) $$F_i'(\sum_k m_k) = \gamma[m_i - \frac{1}{n}\sum_k m_k] + \alpha_i q$$

Summing the equations in (3) and recalling that $\sum_k \alpha_k = 1$, we see that

(4) $$\sum_k F_k'(\sum_k m_k) = q.$$

This is the Samuelson condition for efficient provision of public goods.

Since $F_k'' < 0$, (4) has a unique solution for $\sum_k m_k$. Let $\bar{Y}$ denote this solution. Now define

(5) $$\beta_i = \frac{1}{q} F_i'(\bar{Y}).$$

Then (3) can be rewritten as

(6) $$\beta_i = \gamma[m_i - \frac{1}{n}\bar{Y}] + \alpha_i q.$$

Now α_i, q and γ are parameters and β_i is uniquely solved for by (4) and (5). Thus we solve uniquely for m_i as follows.

(7) $$m_i = \frac{1}{\gamma}(\beta_i - \alpha_i q) + \frac{\bar{Y}}{n}.$$

This establishes our claim that in case of quasi-linear utility. Nash equilibrium exists, is unique and is easily computed.

OUR TOWN - AN EXPOSÉ

T.C. Bergstrom
Christchurch 15 July 1981

Life in Our Town is simple. Folks here are interested in only three things. One of these we are not allowed to discuss. So we will assume they are interested in only two things. These are hot dogs and the Circus. There are really only two kinds of people in town - the toads and the blades. Toads don't care at all about the circus but always prefer more hot dogs to less. Blades like both hot dogs and circus. As it happens, preferences of blades can be represented by the utility function $U_B(X_i,Y)= X_i+2\sqrt{Y}$ while toads' utility functions are simply $U_T(X_i,Y)=X_i$, where X_i is the amount of hot dogs that person i consumes and Y denotes the size of the circus. Each citizen, i, of Our Town has an intial endowment of wealth W_i which can be used either to buy hot dogs or to pay taxes. Tax revenue is used to pay for the circus. The bigger the circus, the more it costs. In fact, let us choose units of measurement for the size of the circus so that the cost of a circus of size Y is just $\$Y$. Let us also suppose that hot dogs cost $\$1$ each. There are N people in Our Town. Let us define an <u>allocation</u> to be a vector $(X_1,\ldots,X_N,Y)$ where X_i is the number of hot dogs consumed by person i and Y is the size of the circus. An allocation is <u>feasible</u> for the town if the total cost of hot dogs consumed plus the cost of the circus just equals total wealth of its citizens. The set of feasible allocations can therefore be denoted by $S=\{(X_1,\ldots,X_N,Y)\mid \sum_{i=1}^{N} X_i+Y=\sum_{i=1}^{N} W_i\}$

A feasible allocation is said to be <u>Pareto optimal</u> if there is no other feasible allocation that is as good for everyone and better for someone. A classic result of Samuelson is that a necessary condition for Pareto optimality in a place like <u>Our Town</u> is that the sum of everyone's marginal rate of substitution of public for private goods must equal the marginal cost of public goods in terms of private goods. In our town the marginal cost of public goods is always one. Therefore the Samuelson condition takes the special form:

$$(1) \qquad \sum_{i=1}^{N} \frac{\partial U}{\partial Y} \div \frac{\partial U}{\partial Y} = 1.$$

Recalling the special form of utility functions assumed we see that for a toad $\frac{\partial U_i}{\partial Y} \div \frac{\partial U_i}{\partial X_i}$ is always zero. For a blade, we calculate $\frac{\partial U_i}{\partial Y} \div \frac{\partial U}{\partial X_i} = \frac{1}{\sqrt{Y}}$. Therefore in Our Town equation (1) takes the special form

$$(2) \qquad N_B \frac{1}{\sqrt{Y}} = 1.$$

From (2) we see that the Pareto optimal amount of public goods for Our Town is

$$(3) \qquad Y = N_B^2.$$

Our Town is a democracy. Everybody pays the same tax rate. We decide by majority vote how much circus to have. Of course toads always vote for no public goods, since they have to pay taxes but don't enjoy the circus. As it turns out, toads are in the minority in Our Town. Therefore blades always out-vote the toads and get a positive amount of circus. (You might want to know why the toads haven't all moved to a town that has a majority of toads and no circus. The answer is that some of the necessary jobs in town can only be done by toads. For example, we need a banker, a mortician, some accountants, an estate agent, a lawyer, a school principal and some mothers-in-law.)

How much circus would a blade like to have? Where Y is the amount of circus, his tax bill will be just $\frac{Y}{N}$. Therefore his after-tax wealth is just $W_i - \frac{Y}{N}$. Therefore he will be able to consume $X_i = W_i - \frac{Y}{N}$ hot dogs when the amount of circus is Y. His utility would then be

$$(4) \qquad U_B(W_i - \frac{Y}{N}, Y) = W_i - \frac{Y}{N} + 2\sqrt{Y}$$

From (4) we see that

$$(5) \qquad \frac{dU_B(W_i - \frac{Y}{N}, Y)}{dY} = \frac{1}{\sqrt{Y}} - \frac{1}{N}$$

Therefore a blade's utility is an increasing function of Y for $Y<N^2$, a decreasing function of Y for $Y>N^2$ and is maximized at $Y=N^2$. Since there are more blades than toads, it is clear that the only amount of circus that "wins" in majority voting is

$$(6) \qquad Y = N^2.$$

For a long time the toads in Our Town have been grousing about high taxes and too much circus. Blades never paid much attention. The other day an economist visited us. (Claimed he wasn't a toad). He said the toads were right. He showed us equation (3) and pointed out that we have more than the Pareto efficient amount of public goods. He said he had just come from Their Town in the next county, where the problem was just the opposite. A majority of the people in Their Town (but not everyone) are toads. They have no circus at all.

This economist suggested that we try a different political system where we require unanimity instead of majority rule. But, since we have people with different tastes, we would have to set different tax rates for different people so as to get unanimity about quantities. He called this idea Lindahl equilibrium. In Our Town, the only way we could get the toads to agree to any positive amount of circus is if we don't tax them for the circus. Then blades would have to pay all the taxes. Suppose that all blades are taxed at the same rate. Then each blade would have a tax bill of $\frac{Y}{N_N}$. He could therefore consume $X_i = W_i - \frac{Y}{N_B}$ hot dogs and would have a utility of

$$(7) \quad U(W_i - \frac{Y}{N_B}, Y) = W_i - \frac{Y}{N_B} + 2\sqrt{Y}.$$

This is maximized when

$$(8) \quad Y = N_B^2.$$

Therefore all blades would choose the amount N_B^2 as their most preferred quantity of circus. Since toads pay no taxes and have no interest in the circus, this amount is as good as any other amount for them. Therefore the amount, N_B^2, receives unanimous approval. The allocation in which $Y=N_B^2$, $X_i=W_i$ if i is a toad and $X_i=W_i-\frac{Y}{N_B}=W_i-N_B$ if i is a blade is therefore a Lindahl equilibrium.

The economist said that Lindahl equilibrium was both more equitable and more efficient than our old ways. The toads said he was right. The blades were not so sure. A blade made the following calculations. Under the current system a blade has the utility:

$$(9) \quad W_i - \frac{N^2}{N} + 2\sqrt{N^2} = W_i + N.$$

Under the Lindahl system a blade has the utility

$$(10) \quad W_i - \frac{N_B^2}{N_B} + 2\sqrt{N_B^2} = W_i + N_B$$

Since $N > N_B$, moving to the Lindahl system is bad for blades. The economist said that the blade had a point (though he was being a bit piggish). But the economist said that since we know that the current system is not Pareto optimal, it should be possible for the toads to bribe the blades to move to Lindahl equilibrium. The economist pointed out that under the current system each toad has a utility of

$$(11) \quad X_i = W_i - \frac{N^2}{N} = W_i - N$$

while under the Lindahl system he would have no taxes so his utility would be

(12) $X_i = W_i$.

We can see from expressions (9) and (10) that a blade could be bribed to accept the Lindahl system if he was given $N-N_B=N_T$ hot dogs. Since there are N_B blades, it would take N_BN_T hot dogs to bribe all of the blades to accept the Lindahl system. Therefore if each toad gave up N_B hot dogs to bribe the blades, there would be just enough hot dogs to do so. If this is done, each toad would have a utility of

(13) $X_i = W_i - N_B$.

Equation (10) expresses the utility of each blade in the Lindahl system without bribes. With bribes of N_T for each blade, the utility of each blade would be

(14) $W_i + N_B + N_T = W_i + N$

which is the same as his utility under the current system. Since (13) is greater than (11) and (14) equals (9), we see that moving to Lindahl equilibrium with this system of bribes benefits all toads and leaves all blades as well off as before. If we made the bribes slightly larger, everyone would be better off than in the current system.

The blades and the toads were all impressed by this argument. The bribes were paid, and the entire community agreed to switch to the Lindahl system. There was one small hitch. You can't always tell by looking, whether a person is a toad or a blade. To solve this problem, the mayor asked everyone to come down to the town hall and answer the simple question:

"Are you a toad?"

To his amazement almost everyone in town squatted down and croaked:

Each blade made the following calculation. If all the other blades are telling the truth, then if I confess to being a blade, the Lindahl equilibrium amount of circus will be N_B^2, and my tax bill will be N_B so that my utility will be

$$(15) \qquad W_i - N_B + 2\sqrt{N_B^2} = W_i + N_B.$$

If I claim to be a toad, the amount of public good will be only $(N_B-1)^2$ but I won't have any taxes, so my utility will be

$$(16) \qquad W_i + 2\sqrt{(N_B-1)^2} = W_i + 2(N_B-1).$$

But (16) is bigger than (15) so long as $N_B>2$. Since in Our Town I know that $N_B>2$, it is therefore worthwhile for me to pretend to be a toad. As a result of this experience, blades in Our Town are inclined to look at economists (and at each other) with suspicion. True toads, of course, are pleased and amused with the outcome.

It is time, I think, to draw the curtain on the sordid situation in Our Town, while we seek aid from some more general analysis. So far, we have learned the following lessons which apply not only in Our Town but quite generally.

(1) For an arbitrary distribution of taxes, majority voting will not in general lead to a Pareto optimal supply of public goods.

(2) Lindahl equilibrium is Pareto optimal. However imposition of a Lindahl equilibrium requires the central authority to know individual preferences.

(3) If people are asked to state their preferences, knowing that their statements will be used to calculate a Lindahl equilibrium that is then imposed, the situation where everyone tells the truth is <u>not</u> a best

response (Nash) equilibrium.

The difficulty alluded to in (3) is often called the "free-rider problem". It is representative of a fascinating class of problems of the firm. "How do you get someone else to tell you the truth about something that only he knows?" A related question is "When can you design a system of rewards and punishments such that when selfish people who are willing to lie will act in such a way as to yield a Pareto optimal outcome?"

A philosopher who dabbles in economics, Alan Gibbard, and an economist who dabbles in philosophy, Mark Satterthwaite, independently showed that in general it is not possible to design such mechanisms. There are, however, some interesting special cases where the truth can be elicited even though the answers are used to choose a Pareto optimal policy.

The first example is William Vickerey's suggestion for a sealed bid auction. Suppose that there are n people and one object to the allocated among them. Let V_i be the maximum amount that person i would be willing to pay for the object. Pareto efficient allocations would have the object go to the person with the greatest willingness to pay. Why? If a sealed-bid auction were held, with the object going to the highest bidder at his bid price, it would not be wise for anyone to bid his true valuation. Why? Vickerey suggested that the object be given to the highest bidder at the second highest bid price. With this system, it turns out that bidding ones true valuation is the best thing to do no matter what other people bid. A strategy that is best no matter what others do is known as a dominant strategy. A social outcome where everyone is using a dominant strategy the outcome where everyone bids his true valuation and the object goes to the person with the highest valuation at a price equal to the second highest valuation is a dominant strategy equilibrium. Lets see why this is so. Suppose that you bid more than your true evaluation. If your bid is not the highest bid, you are no better (or worse) off than if you had told the truth. If your bid is the highest bid, then there are two possible cases. If your true valuation would also have been the highest bid, then you are no better (or worse) off than if you had bid the truth. If your true valuation is lower than the second highest bid, then you get the object but you must pay more than it is worth to you. You would have been better off bidding the truth and not getting the object. Thus we see that you can not gain but you can lose by overbidding. You should be able to construct a similar argument to show that you can not gain and may lose by underbidding. Therefore, bidding the truth is a dominant strategy.

The idea of Vickerey's auction can be extended to other kinds of discrete choices. Of particular interest are all-or-nothing choices on public issues, such as whether to allow public nudity or public rugby playing, or the sale of handguns. I have promised not to discuss certain interests of the people in Our Town, so we will consider the Springbok issue in New Zealand. Define V_i to be person i's willingness to pay to have the Springboks allowed into New Zealand. More formally, let $U_i(X_i,0)$ and $U_i(X_i,1)$ denote respectively person i's utility when his wealth is X_i and the Springboks are allowed or not allowed to tour, and let $\overline{X}_i$ denote i's current wealth. Then V_i is the solution to the equation

$$U_i(\overline{X}_i+V_i,0) = U_i(\overline{X}_i,1).$$

Thus V_i is positive for people who want them to come and negative for people who don't want them.

One possible decision mechanism is to decide the issue by majority vote. The weakness of this mechanism is that it may not be Pareto optimal. The minority may be intensely concerned, while members of the majority each care very little. In this case it might be possible to find a Pareto superior outcome which reverses the result since the minority cares enough to buy off the majority.

To make an efficient decision, we need to compute the sum over the entire population, $\sum_i V_i$. If $\sum_i V_i>0$, then with the current allocation of wealth, allowing the Springboks to come is Pareto optimal and not allowing them to come is not. If $\sum_i V_i<0$, the story is reversed.

If we just asked people to state V_i, and then decided on the Springbok issue by the sign of ΣV_i, they would have an incentive to overstate the intensity of their preferences. We need a more subtle device. Here is one that works. Each i is asked to state V_i. The state then calculates $\sum_i V_i$ and allows the Springboks to come if and only if $\sum_i V_i>0$. In addition some taxes are assessed in the following way. If person j's answer does not affect the outcome, that is if the sign of $\sum_{j\neq i} V_j$ is the same as the sign of $\sum_i V_i$, then he pays no tax. If person j's answer does make a difference, then he pays the amount $\sum_{j\neq 1} V_j$. Any revenue from this scheme is thrown away. Using exactly the same kind of reasoning that we did in the case of the Vickerey auction, we can show that the equilibrium. Furthermore, the resulting decision is "Pareto optimal". There is, however, some waste in the process, since the tax revenue is thrown away. In large economies, it can be shown that under reasonable assumptions the amount of waste of this type will be small.