C-47

Center for Research on Economic and Social Theory
Research Seminar in Quantitative Economics

Discussion Paper

DEPARTMENT OF ECONOMICS

University of Michigan

Ann Arbor, Michigan 48109

[redacted]

Soldiers of Fortune?

Ted Bergstrom
Department of Economics
University of Michigan

C-47

Revised, May 1982

I am grateful for helpful comments and encouragement from Sherwin Rosen of the University of Chicago, P. Srinagesh of the University of Illinois, Chicago Circle, and my colleagues Paul Courant, George Johnson, Richard Porter, David Sappington, Harold Shapiro and Hal Varian of Michigan. None of them are to be implicated for any unpopular opinions or false statements found herein.

If you asked ten economists "Which is more efficient, a volunteer army or a draft lottery?", I think that at least nine would choose the volunteer army. This paper reports that "man bites dog." It is shown that if tastes and abilities are identical, then a draft lottery is Pareto superior to a volunteer army. Where people differ in their tastes and abilities, a volunteer army is likely to be superior to a draft lottery because of its ability to sort the "right" people into the army. However, even in this case, it is generally possible to improve on an all-volunteer army by a draft lottery if probabilities of being drafted can be bought and sold.[1/]

This analysis applies as well to the choice of garbagemen or C.P.A.'s as to selection of an army. It suggests that lotteries may have a more important role to play in resource allocation than is generally recognized. There is also a striking implication for welfare economics. In a neoclassical economy, even with risk-averse consumers, Pareto efficient allocation which is equalitarian <u>ex ante</u> may have to be non-equalitarian <u>ex post</u>.[2/] Stated in another way, if an allocation is to be Pareto efficient and if no one is to envy the opportunities of others then it may be necessary that some envy the luck of others.

I. The Fortunes of Soldiers in a Homogeneous Community

Imagine a country where all N citizens have the same tastes and abilities. There is a single consumption good, bread. To defend itself, the country must raise an army of A soldiers. The other N-A citizens are farmers who produce a total of B units of bread. Farmers are taxed to pay the soldiers. Let w_A be the amount of bread paid to each soldier and let w_F be the amount of bread left to each farmer after taxes. The "wages" w_A and w_F must satisfy the feasibility constraint

$$(1) \qquad Aw_A + (N - A)w_F = B.$$

Equivalently, if we define $\bar{\pi} = \frac{A}{N}$ and $\bar{B} = \frac{B}{N}$,

$$(2) \qquad \bar{\pi}w_A + (1 - \bar{\pi})w_F = \bar{B}.$$

We will call (w_A, w_F) a _feasible wage structure_ if (2) is satisfied.

Being a soldier is unpleasant and dangerous. If soldiers and farmers were paid the same wage, everyone would want to be a farmer. The country might offer soldiers high enough wages to attract a volunteer army. Alternatively it could choose its army by lottery. We will call a lottery a _fair lottery_ if everyone has the same probability $\bar{\pi} = \frac{A}{N}$ of being drafted. The country could select its army by a fair lottery with any feasible wage structure (w_A, w_F). In general, with a fair lottery, citizens will not be indifferent about being drafted. But since everyone faces the same prospects before the lottery makes its selection, everyone will have the same _ex ante_ expected utility. Let us assume that each citizen is an expected utility maximizer with a utility function of the form

$$(3) \qquad \bar{\pi}u_A(w_A) + (1 - \bar{\pi})u_F(w_F)$$

where $u_A(\cdot)$ and $u_F(\cdot)$ are smooth, increasing, strictly concave functions. As we see from (3), the expected utility of each citizen from a fair lottery depends on the wage rates w_A and w_F. We define the _best fair lottery_ as a fair lottery where wages (w_A, w_F) are chosen from among the feasible wage structures to maximize the expected utility of a representative participant in the lottery. The best fair lottery then has wages (w_A^*, w_F^*) which maximize (3) subject to (2). Simple calculus informs us that there is a unique solution for (w_A^*, w_F^*) and at this solution,

$$(4) \qquad u_A'(w_A^*) = u_F'(w_F^*).$$

If there is to be a volunteer army, then the prospect of being a soldier must be just as attractive as that of being a farmer. For the time being, let us suppose that for any wage of farmers, there is some wage for soldiers that would make the army as attractive as the farm. Then there will be exactly one feasible wage structure $(\hat{w}_A, \hat{w}_F)$ that will just attract a volunteer army. The wage structure $(\hat{w}_A, \hat{w}_F)$ will satisfy equation (2) and

$$(5) \qquad u_A(\hat{w}_A) = u_F(\hat{w}_F).$$

With the wage structure $(\hat{w}_A, \hat{w}_F)$ no one cares whether he is a farmer or a soldier. Therefore everyone would be indifferent between having a volunteer army and selecting an army by a fair lottery with wage structure $(\hat{w}_A, \hat{w}_F)$. That is to say

$$(6) \qquad \bar{\pi} u_A(\hat{w}_A) + (1 - \bar{\pi}) u_F(\hat{w}_F) = u_A(\hat{w}_A) = u_F(\hat{w}_F)$$

Since the expected utility achievable with a volunteer army is also achievable with a lottery, a volunteer army will give citizens as high an expected utility as the best fair lottery if and only if the best fair lottery has wage structure $(\hat{w}_A, \hat{w}_F)$. But in general this is not the case. The volunteer army wage structure $(\hat{w}_A, \hat{w}_F)$ equalizes total utilities $u_A(\hat{w}_A)$ and $u_F(\hat{w}_F)$, while the best lottery equalizes marginal utilities $u_A'(w_A^*)$ and $u_F'(w_F^*)$. Only under special circumstances does the former condition imply the latter.

Looking at an example will help us to understand the reason for this result. Suppose the country has just two citizens. One must be a soldier, the other a farmer. They have identical von Neuman-Morgenstern utility functions such that $u_A(w) = u_F(w) - \alpha$ for all w where $\alpha > 0$ and where $u_F'(w) > 0$, $u''(w) < 0$. There are two units of bread to be distributed between them. The best fair lottery requires a wage structure (w_A^*, w_F^*) such that $u_A'(w_A^*) = u_F'(w_F^*)$ which in this instance implies $w_A^* = w_F^* = 1$. But if $w_A^* = w_F^*$, then $u_A(w_A^*) = u_F(w_F^*) - \alpha < u_F(w_F^*)$. Therefore the best fair lottery leaves farmers better off than soldiers.

In Figure 1 we draw the utility possibility frontier representing those distributions of von Neuman-Morgenstern utility possible if one of the citizens is assigned to the army with certainty and the other to the farm. This frontier is the outer envelope of the two curves AB and CD. Curve AB represents utility distributions possible if person 1 is certain to be in the army and CD the utility distributions possible if person 2 is certain to be in the army. Although each of these curves encloses a convex region, their outer envelope does not. The point E represents the distribution of utility from a volunteer army. Consider a point X on AB and a point X' on CD. Any distribution of von Neuman-Morgenstern utility on the line XX' can be achieved by a lottery which with some probability λ puts person 1 in the army and pays wages that distribute utility as at point X and with probability $1 - \lambda$ puts person 2 in the army and pays wages that distribute utility as at point X'. We see, therefore, that the equalitarian volunteer army is Pareto dominated by a draft lottery. Because utility is state dependent and because one must be discretely either in one state or the other, the set of von-Neuman-Morgenstern utilities attainable as sure things is not convex even though utility is concave in income and there are no economies of scale. When this set is non-convex,[3/] it is always possible to reach additional points in the convex hull of the set by means of a lottery.

Where citizens are identical, a strong case can be made that normally the best fair lottery is one in which soldiers are paid less than would be required to recruit a volunteer army. Assuming that it is always possible to pay soldiers enough to be as well off as farmers, define $c(w)$ to be the wage premium necessary to make a soldier as well off as a farmer with wage w. Then $c(w)$ is the unique solution to the equation

$$(7) \qquad u_A(w + c(w)) = u_F(w).$$

If the amenity of being a farmer rather than a soldier is a "normal good", we would expect that $c'(w) > 0$ for all w. Proposition 1 shows that if being a farmer rather than a soldier is a normal good in this sense, then the best fair lottery leaves soldiers worse off than farmers.

Proposition 1. If all citizens have identical expected utility functions of the type discussed above then the best fair lottery has a wage structure (w_A^*, w_F^*) such that

(1) $u_A(w_A^*) < u_F(w_F^*)$ if $c'(\hat{w}_F) > 0$

(2) $u_A(w_A^*) = u_F(w_F^*)$ if $c'(\hat{w}_F) = 0$

(3) $u_A(w_A^*) > u_F(w_F^*)$ if $c'(\hat{w}_F) < 0$

where $c(w)$ is the wage premium function defined above and $\hat{w}_F$ is the wage received by farmers if there is a volunteer army.

Informal proof:

A formal proof of this proposition is found in the appendix. We can, however, give the idea for the proof (glossing subtleties that are treated in the formal proof) by a simple diagram. For any $u = \bar{u}$, the horizontal distance in Figure 1 between the curves $u_A(\cdot)$ and $u_F(\cdot)$ at $\bar{u}$ is just $c(\bar{w})$ where $u_F(\bar{w}) = \bar{u}$. If $c'(w) > 0$ for all w, then the horizontal distance between these curves increases as u increases. Therefore the curve $u_A(w)$ must be flatter than the curve $u_F(w)$. Hence, at the volunteer army solution where $u_A(\hat{w}_A) = u_F(\hat{w}_F)$, it must be that $u_A'(\hat{w}_A) < u_F'(\hat{w}_F)$. The best fair lottery has $u_A'(w_A^*) = u_F'(w_F^*)$. Since we have assumed that u_A and u_F are concave functions, it must be that $w_A^* < \hat{w}_A$ and $w_F^* > \hat{w}_F$. Therefore if $c'(w) > 0$ for all w, then $u_A(w_A^*) < u_F(w_F^*)$. Using this basic idea and reasoning a bit more closely we can establish Proposition 1.

If the country had a draft lottery but set a feasible wage structure (w_A, w_F) different from (w_A^*, w_F^*), then there would be reason for private "draft insurance" markets to develop. These would enable people to arrange contingent consumption plans different from their contingent wages. If he can buy actuarially fair insurance and if the wage structure is (w_A, w_F) then a citizen can afford a consumption c_A contingent on being drafted and c_F contingent on not being drafted so long as the budget constraint

$$(8) \qquad \bar{\pi}(c_A - w_A) = (1 - \bar{\pi})(w_F - c_F)$$

is satisfied. But (8) is equivalent to

$$(9) \qquad \bar{\pi}c_A + (1 - \bar{\pi})c_F = \bar{\pi}w_A + (1 - \bar{\pi})w_F = \bar{B}$$

where the second equality follows from the fact that (w_A, w_F) is a feasible wage structure. Each consumer will therefore choose c_A and c_F to maximize

$$(10) \qquad \bar{\pi}u_A(c_A) + (1 - \bar{\pi})u_F(c_F)$$

subject to (9). But this is just the same problem that we solved to find the best fair lottery (w_A^*, w_F^*). This proves the following proposition.

<u>Proposition 2</u>. Under the assumptions of this section, if the army is selected by a fair lottery with any feasible wage structure, then if there are actuarially fair "draft insurance" markets, citizens will buy insurance in such a way that $(c_A, c_F) = (w_A^*, w_F^*)$ where c_A and c_F are consumption contingent on being drafted or not and (w_A^*, w_F^*) is the wage structure of the best fair lottery.

From proposition 2, we see that if it can be assumed that perfect insurance markets will arise, then a draft lottery is better than a volunteer army regardless of what feasible wage structure is set.

It would not be so surprising to find that a lottery is "better" than a volunteer army if the usefulness of income were somehow diminished for those in the army. This could be the case if, say, soldiers must spend much of their time in combat areas and have little access to consumer goods or if soldiers have a high probability of early death. It is true that these effects would make a lottery more attractive, but as our example shows they are not necessary for it to be the case that a lottery that leaves soldiers worse off than farmers is "better" than a volunteer army.

The result that a lottery is better than a volunteer army is our model depends in no way on illusion or misapprehension of probabilities. Every citizen understands the system perfectly and everyone is a rational decision-maker under uncertainty. This result may trouble readers with strong equalitarian instincts. Although all citizens have equally favorable prospects <u>ex ante</u>, the optimal lottery could produce <u>ex post</u> utilities that are very different. This might lead some to question the values implicit in defining the "best fair lottery" to be the fair lottery that maximizes <u>ex</u>

ante expected utility. Before one rejects *ex ante* utility, however, he should consider carefully the alternative. If in the example studied above, citizens were asked beforehand whether they would rather have a volunteer army with wage structure $(\hat{w}_A, \hat{w}_F)$ or the fair lottery with wage structure (w_A^*, w_F^*), all would prefer to have the lottery. Rejecting *ex ante* expected utility in favor of *ex post* equalitarianism would require imposing a policy against the unanimous wishes of an informed and rational population.

In my opinion an equalitarian case against choosing an army by lottery must be made not by disputing the appropriateness of *ex ante* expected utility, but must be argued from a richer model of individual preferences. For example it might be argued that the model we have studied is misleading because it assumes everyone to be selfish. If people envy the good fortune of risk-takers who win and pity those who take risks and lose, then they may wish their neighbors to avoid risks which the neighbors regard as good bets. If this is the case, people would prefer more equalitarian outcomes than would be predicted by the selfish model. If, however, citizens are benevolent in the sense of preferring better *ex ante* prospects for others, then the solution for the best fair lottery would be the same as in the selfish case.

II. Selective Service in a Heterogeneous Country

If some people are more averse to serving in the army than others, or if people differ in their comparative advantage in the two occupations, then it is important to select the right people for each occupation. Clearly a volunteer army is better able to select its members according to comparative advantage than is a lottery in which probabilities of being drafted are the same for everyone. But even in a heterogeneous country, the reasons we advanced for preferring a lottery to a volunteer army retain some force. It is probably true that for a diverse modern economy, at least in peacetime, the advantages of a volunteer army outweigh those of a lottery with equal probabilities. Both of these systems, however, can be Pareto dominated by an allocation mechanism in which the army is selected by a lottery where probability of being drafted can be bought and sold in fractional units.

Consider a country just like that of the previous section except that tastes differ. Consumer i has an expected utility function

$$(11) \quad \pi^i u_A^i(c_A^i) + (1 - \pi^i) u_F^i(c_F^i)$$

where π^i is the probability that he is assigned to the army and c_A^i and c_F^i are his consumption levels contingent on being in the army or not. If the wage structure is (w_A, w_F) then a citizen i will willingly join the army if $u_A^i(w_A) \geqq u_F^i(w_F)$. The higher is w_A and the lower is w_F, the greater the number of citizens who would willingly join the army. A volunteer army can be recruited with a wage structure $(\hat{w}_A, \hat{w}_F)$ such that the number of people for whom $u_A^i(w_A) \geqq u_F^i(w_F)$ is at least A and the number of people for whom $u_A^i(w_A) > u_F^i(w_F)$ is no larger than A. Since preferences differ, some citizens will typically be inframarginal. Thus there will be soldiers who strictly prefer their lot to that of farmers and vice versa. On the other hand no one will envy anyone else in the sense of wanting to exchange jobs and wage rates with him.

Now let us consider a draft lottery. If there are actuarially fair markets for consumption contingent on whether one is drafted, then a citizen with wealth B_i and probability π_i of being drafted could afford any contingent consumption plan (c_A^i, c_F^i) such that $\pi_i c_A^i + (1-\pi_i) c_F^i \leqq B_i$. Therefore an indirect utility function for such a citizen can be defined as

$$(12) \quad V_i(\pi_i, B_i) \equiv \text{maximum } \pi_i u_A(c_A^i) + (1-\pi_i) u_F(c_F^i)$$

$$\text{s.t. } \pi_i c_A^i + (1-\pi_i) c_F^i \leqq B_i.$$

If everyone has initially the same wealth, $\bar{B}$, and the same probability of being drafted, $\bar{\pi}$, and if there is no possibility of trading draft probabilities, then the utility of any consumer i will be $V_i(\bar{\pi},\bar{B})$. In general, it will not be the case that either the volunteer army or a draft lottery with no exchange of probabilities is Pareto superior to the other. Nor can it be determined in general whether changing from a draft lottery to a volunteer army or vice versa can benefit everyone given appropriate redistribution of bread. As we will show in examples, depending on the specific situation, the outcome of such a comparison could be either way.

Suppose that a competitive market is allowed to develop for probability of being drafted. Then there will be three commodities of interest to each consumer. These are consumption contingent on being drafted, consumption contingent on not being drafted, and probability of being drafted. Models of general competitive equilibrium routinely incorporate contingent commodities (Debreu (1959)), but it is not usual for there also to be markets for probabilities. In fact, the standard theory does not extend automatically to this case because preferences will typically not be convex jointly in the three variables π_i, c_A^i and c_F^i.[4/] However there is a trick that allows us to extend the usual results to this case.

Instead of dealing with direct utility functions in the three variables, π_i, c_A^i and c_B^i, we treat the indirect utility functions, $V_i(\pi_i,B_i)$. If, as we have assumed, $u_i^A(\cdot)$ and $u_i^B(\cdot)$ are continuous (strictly) concave functions then $V_i(\pi_i,B_i)$ is continuous and strictly concave.[5/] Consider, now the formal two-commodity pure exchange model in which the commodities are π_i and B_i, where there are initial endowments (π_i^0,B_i^0) for each i and where $\sum_i \pi_i^0 = \pi N$ and $\sum_i B_i^0 = N\bar{B}$. In the case of equal initial endowments this would mean for each i, $(\pi_i^0,B_i^0) = (\bar{\pi},\bar{B})$.

Let bread be the numeraire and let p be the price of draft probability in terms of bread. (Typically p will be negative in equilibrium, which is to say that one is paid a positive amount for accepting some of another person's initial probability of being drafted.) Citizen i's competitive budget allows him to choose any combination (π_i, B_i) such that

$$(13) \quad B_i + p\pi_i \leqq B_i^0 + p\pi_i^0. \text{ 6/}$$

Since $V_i(\pi_i, p)$ is continuous and concave, standard theorems 7/ of general equilibrium theory inform us that a competitive equilibrium exists for the formal exchange economy just described.

Where there are many commodities, competitive equilibrium for this two-commodity economy corresponds in a simple way to a full competitive equilibrium with contingent commodity markets. In particular, suppose that p^* is the competitive equilibrium price of draft probability and (π_i^*, B_i^*) is i's competitive allocation. Let c_A^{i*} and c_F^{i*} be contingent consumptions that maximize $\pi_i^* u_A^i(c_A^i) + (1-\pi_i^*) u_F^i(c_F^i)$ subject to

$$(14) \quad \pi_i^* c_A^i + (1-\pi_i^*) c_F^i \leqq B_i^*.$$

Summing the inequalities (14) over i, we find that the expected amount of bread needed to fulfill all of the contingent consumption contracts does not exceed $\sum_i B_i = N\bar{B}$. For a large economy, we can treat expected aggregate consumption as if it were certain without serious distortion. 8/ Therefore the allocation that assigns consumer i a draft probability π_i^* and contingent consumption (c_A^{i*}, c_F^{i*}) is feasible and can reasonably be called a competitive allocation with marketable draft probability. Simple extensions of the usual methods of proof allow us to extend the first and second theorems of welfare economics to this notion of equilibrium. In summary, we have:

Proposition 3.

(1) For the country modelled in section 2, there exists a competitive equilibrium with marketable draft probability. (2) This equilibrium is Pareto optimal. (3) Every Pareto optimal allocation of contingent commodities and draft probabilities is a competitive equilibrium with marketable draft probability given some initial distribution of ownership of bread and draft probability.

A volunteer army, on the other hand, will not in general result in Pareto optimal allocation. The intuitive reason is that in a volunteer army, everyone must have a value of π_i equal either to zero or one. Soldiers will be paid a discrete amount more than farmers. If there is a near continuum of tastes in the economy, then there will be a marginal soldier whose preferences are very similar to those of the marginal farmer. If the marginal soldier is paid a discrete amount more than the marginal farmer and their preferences are very similar, then since the function $V(\pi,B)$ is strictly concave, the marginal farmer will have a higher marginal rate of substitution between B and π than the marginal soldier. They could both gain if the farmer accepted some of the soldier's π in return for some bread. We will show explicitly how this works in the examples of the next section.

III. Two Examples

To fully understand the general principles of Section II it is useful to look at some computable special cases. These special cases also serve as counterexamples to some plausible but central conjectures.

Example 1

Let each citizen i be classified by an index $T_i > 0$ of his "tolerance" for the army. Citizens with higher tolerance are less averse to the army. Specifically, let citizen i's expected utility function be such that

$$(14) \quad u^i_A(c^i_A) = \ln c^i_A - \frac{1}{T_i} \quad \text{and} \quad u^i_F(c^i_F) = \ln c^i_F$$

Then to maximize $\pi_i u^i_A(c^i_A) + (1-\pi_i)u^i_F(c^i_F)$ subject to $\pi_i c^i_A + (1-\pi_i)c^i_F = B_i$, we must have $c^i_A = c^i_F = B_i$. Therefore

$$(15) \quad V_i(\pi_i, B_i) \equiv \ln B_i - \pi_i \frac{1}{T_i}$$

Suppose that there is a draft lottery with marketable probability of being drafted and full contingent commodity markets. We solve for equilibrium quantities and prices as follows. Where p is the price of draft probability, citizen i chooses (B_i, π_i) to maximize (15) subject to (13). If this maximum is interior, then his marginal rate of substitution between π_i and B_i must equal the price ratio, p. Therefore his demand equation would satisfy

$$(16) \quad B_i = -pT_i$$

Let us provisionally assume that all citizens have interior equilibria. Then market equilibrium requires

$$(17) \quad \sum_{i=1}^{N} B_i = -p \sum_{i=1}^{n} T_i = N\overline{B}$$

From (17) we see that the equilibrium price p* must satisfy

$$(18) \quad p^* = -\frac{\overline{B}}{\overline{T}}$$

where $\overline{T} \equiv (\frac{1}{N}) \sum_{i=1}^{N} T_i$ is the mean tolerance level in the country. Given the equilibrium price p* in (18) we can solve for the equilibrium demands of each citizen from (16) and (13). These are

$$(19) \quad B_i = \left(\frac{T_i}{\overline{T}} \right)\overline{B}$$

$$(20) \qquad \pi_i = \overline{\pi} + (T_i - \overline{T})$$

These solutions were provisional on the assumption that each consumer found an interior solution for B_i and π_i. This assumption is justified if and only if the solution for π_i in (20) lies in the interval from zero to one. Otherwise there would be some consumers with corner solutions who either joined the army with certainty or avoided the army with certainty.

Whether there are corner solutions for some consumers will in general depend on the distribution of tolerance levels in the community and on the fraction $\overline{\pi}$. In this example we choose a distribution and a $\overline{\pi}$ so that everyone does have an interior solution. Let $\overline{\pi} = \frac{1}{6}$ and let the T_i's be uniformly distributed on the interval, $(1, \frac{4}{3})$. Then $\overline{T} = \frac{7}{6}$ and $0 \leqq \pi_i = \frac{1}{6} + T_i - \overline{T} \leqq \frac{1}{3}$ for each i. Therefore everyone has an interior solution. From (18), (19) and (20) we find that the equilibrium quantities for citizen i are

$$(21) \qquad \pi_i = T_i - 1 \text{ and } B_i = \overline{B}(\frac{T_i}{\overline{T}}) = (\frac{6}{7})\overline{B}T_i.$$

Therefore expected utility of i is

$$(22) \qquad V(T_i - 1, \frac{6}{7}\overline{B}T_i) = \ln(\frac{6}{7}\overline{B}T_i) - (\frac{T_i - 1}{T_i}).$$

If there is a draft lottery without a market in draft probabilities but with contingent commodity markets for bread, then citizen i has expected utility

$$(23) \qquad V(\frac{1}{6}, \overline{B}) = \ln\overline{B} - \frac{1}{6T_i}.$$

If there is a volunteer army, then the wage structure $(\hat{w}_A, \hat{w}_B)$ must be such that the $\frac{1}{6}$ of the population with highest tolerances is willing to join the army while no one else wants to join. Since tolerance is uniformly distributed on $(1, \frac{4}{3})$ the marginal soldier must have tolerance $T = 1 + \frac{5}{6} \cdot \frac{1}{3} = \frac{23}{18}$. Therefore it must be that

$$(24) \quad \ln \hat{w}_F = \ln \hat{w}_A - \frac{1}{\hat{T}} = \ln \hat{w}_A - \frac{18}{23}.$$

Equations (2) and (24) determine

$$(25) \quad \hat{w}_A = 1.83\overline{B} \quad \text{and} \quad \hat{w}_F = .84\overline{B}$$

Therefore citizen i will have expected utility

$$(26) \quad \ln(.84\overline{B}) \text{ if } 1 \leqq T_i \leqq \frac{23}{18} \text{ and } \ln(1.83\overline{B}) - \frac{1}{T_i} \text{ if } \frac{23}{18} \leqq T_i \leqq \frac{4}{3}.$$

From expressions (22), (23) and (26) we can calculate the expected utility of a citizen of any tolerance level T_i under each of the possible arrangements -- draft lottery with marketable probability, draft lottery without marketable probability, volunteer army. These solutions are tabulated for selected values of T_i in Table 1 on the assumption that $\overline{B} = 1$

Table 1

Expected Utility of Consumer with Tolerance T_i when $\overline{B} = 1$ and where T_i's are Uniformly Distributed on $(1, \frac{4}{3})$

Selection mechanism \ T_i	1	$\frac{25}{24}$	$\frac{26}{24}$	$\frac{27}{24}$	$\frac{28}{24}$	$\frac{29}{24}$	$\frac{30}{24}$	$\frac{31}{24}$	$\frac{32}{24}$
Lottery with marketable probability	-.154	-.153	-.151	-.147	-.143	-.137	-.131	-.124	-.116
Lottery without marketable probability	-.166	-.160	-.154	-.148	-.143	-.138	-.133	-.129	-.125
Volunteer army	-.182	-.182	-.182	-.182	-.182	-.182	-.182	-.175	-.151

In this example, the draft lottery with marketable probability is Pareto superior to the draft lottery without marketable probability and both lottery solutions are Pareto superior to a volunteer army. In general, however, the equalitarian lottery solution need not be Pareto superior to an equalitarian volunteer army. This is illustrated in the next example.

Example 2

Let preferences be as in Example 1 and let $\bar{\pi} = \frac{1}{6}$ but suppose that tolerance levels are uniformly distributed on the interval (1,5). In this case, some consumers will have corner solutions. Let T^* be the lowest tolerance level at which one joins the army with certainty and let T_* be the highest tolerance level at which one avoids the army with certainty. From equation (20), we see that $1 = \bar{\pi} + (T^* - \bar{T})$ and $0 = \bar{\pi} + (T_* - \bar{T})$. Subtracting the second equation from the first, we find that $T_* = T^* - 1$. From (16) we see that if $T_* \leqq T_i \leqq T^*$ then $B_i = -T_i p$. Therefore it follows from (13) and the fact that $\bar{\pi} = \frac{1}{6}$ that

$$(27) \qquad \pi_i = \frac{\bar{B}}{p} + \frac{1}{6} + T_i$$

If $T_i = T^*$, then $\pi_i = 1$ so that (27) implies

$$(28) \qquad T^* = \frac{5}{6} - \frac{\bar{B}}{p} ,$$

In equilibrium there must be A soldiers. The number of soldiers will be the number of citizens with $T_i \geqq T^*$ plus those citizens with $T_* < T_i < T^*$ who happen to be drafted. This means that

$$(29) \qquad A = \left(\frac{5-T^*}{4}\right) N + \sum_{T_* \leqq T_i \leqq T^*} \pi_i$$

Since the T_i's are uniformly distributed on (1,5) and since the interval (T_*, T^*) is of length one, there are $\frac{N}{4}$ citizens for whom $T_* \leqq T \leqq T^*$. Substituting from (27) and (28) into (29), noticing that $\sum_{T_* \leqq T_i \leqq T^*} T_i = \frac{N}{4}(T^* - \frac{1}{2})$, we find that (29) reduces to

$$(30) \quad \frac{A}{N} = \frac{9}{8} + \frac{\overline{B}}{4p} + \frac{\overline{\pi}}{4}.$$

But $\frac{A}{N} = \overline{\pi} = \frac{1}{6}$ so that (27) implies that the equilibrium price p* satisfies

$$(31) \quad p^* = \frac{-\overline{B}}{4}$$

From (31) and (28) we find $T^* = 4\frac{5}{6}$ and $T_* = 3\frac{5}{6}$. Citizens with tolerance T_i between $4\frac{5}{6}$ and 5 all serve in the army with certainty. Citizens with tolerance $T_i < 3\frac{5}{6}$ choose to avoid the army with certainy. People with tolerance T_i between $3\frac{5}{6}$ and $4\frac{5}{6}$ choose positions which lead them to be drafted with probability $T_i - 3\frac{5}{6}$. One fourth of the army ($\frac{1}{24}$ of the total population) joins the army with certainy. The other three fourths of the army is chosen by lottery from among those citizens for whom $2\frac{5}{6} < T_i < 3\frac{5}{6}$.

As in Example 1, we can solve for the expected utility of citizens at each tolerance level if the army is chosen by lottery with or without marketable probability and if there is a volunteer army. Results of such a tabulation are listed in Table 2.

Table 2

Expected Utility of Consumer with Tolerance T_i when $\overline{B}$=1 and where T_i's are Uniformly Distributed on (1,5).

Selection Mechanism \ T_i	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5
Lottery with Marketable Probability	-.0425	-.0425	-.0425	-.0425	-.0425	-.0425	-.0417	-.033	-.011
Lottery without Marketable Probability	-.167	-.111	-.083	-.067	-.055	-.048	-.0417	-.037	-.033
Volunteer Army	-.0429	-.0429	-.0429	-.0429	-.0429	-.0429	-.0429	+.008	+.030

As we see from Table 2, an equalitarian lottery with marketable probability is not Pareto superior to an equalitarian volunteer army nor vice versa. "Civilians" are worse off with the volunteer army than with a lottery while "professional soldiers" are better off. A volunteer army is better for all but a small fraction of the population than a draft lottery without marketable probability. In fact it is easy to show in this case that with appropriate side-payments, everyone can be made better off with a volunteer army than with a draft lottery if probability is not marketed.

Part II - On the concavity of $V(\pi, B_i)$

This result is of some interest as a general method for treating marketable probabilities. The proof is stated for two events but extends readily to many events.

Theorem - Let $U(\pi, c_A, c_F) \equiv \pi u_A(c_A) + (1-\pi)u_F(c_F)$ and let $V(\pi, B) =$ maximum $U(\pi, c_A, c_F)$ subject to $\pi c_A + (1-\pi)c_F \leqq B$. If $u_A(c_A)$ and $u_F(c_F)$ are continuous (strictly) concave functions, than $V(\pi, B)$ is continuous and (strictly) concave.

Proof:

The argument for continuity is well-known. We prove concavity here. Let (c'_A, c'_F) maximize $U(c_A, c_F)$ subject to

(1) $\pi' c'_A + (1-\pi')c'_F \leqq B'$

and let (c''_A, c''_F) maximize

(2) $\pi'' c''_A + (1-\pi'')c''_F \leqq B''$

Then from the definition of V it follows that

(3) $V(\pi', B') = U(c'_A, c'_F)$ and

(4) $V(\pi'', B'') = U(c''_A, c''_F)$.

For every λ between zero and one define

(5) $\pi_A(\lambda) = \lambda\pi' + (1-\lambda)\pi''$

(6) $\pi_F(\lambda) = \lambda(1-\pi') + (1-\lambda)(1-\pi'')$

(7) $B(\lambda) = \lambda B' + (1-\lambda)B''$.

In order to prove that V is a concave function we must demonstrate that

(8) $V(\pi_A(\lambda), B(\lambda)) \geqq \lambda V(\pi', B') + (1-\lambda)V(\pi'', B'')$

for all λ between zero and one. For arbitrary λ between zero and one define

(9) $\theta_A = \frac{\lambda\pi'}{\lambda\pi'+(1-\lambda)\pi''}$

(10) $\theta_F = \frac{\lambda(1-\pi')}{\lambda(1-\pi')+(1-\lambda)(1-\pi'')}$

From (5), (6), (9) and (10) it follows that:

(11) $\pi_A(\lambda)\theta_A = \lambda\pi'$

(12) $\pi_A(\lambda)(1-\theta_A) = (1-\lambda)\pi''$

(13) $\pi_F(\lambda)\theta_F = \lambda(1-\pi')$

(14) $\pi_F(\lambda)(1-\theta_F) = (1-\lambda)(1-\pi'')$.

From (5) and (6) it follows that

(15) $\pi_A(\lambda) + \pi_F(\lambda) = 1$.

Multiplying inequalities (1) and (2) by λ and $1-\lambda$ respectively yields

(16) $\lambda\pi' c_A' + (1-\lambda)\pi'' c_A'' + \lambda(1-\pi')c_F' + (1-\lambda)(1-\pi'')c_F'' \leqq B(\lambda)$

From (11), (12), (13), (14) and (16) it follows that

(17) $\pi_A(\lambda)[\theta_A c_A' + (1-\theta_A)c_A''] + \pi_F(\lambda)[\theta_F c_F' + (1-\theta_F)c_F''] \leqq B(\lambda)$

Since $\pi_F(\lambda) = 1-\pi_A(\lambda)$, it follows from (17) and the definition of V that

(18) $V(\pi_A(\lambda),B(\lambda)) \geqq \pi_A(\lambda)u_A(\theta_A c_A' + (1-\theta_A)c_A'') + \pi_F(\lambda)u_F(\theta_F c_F' + (1-\theta_F)c_F'')$

From (18) and the assumption that u_A and u_F are concave functions it follows that

(19) $V(\pi_A(\lambda),B(\lambda)) \geqq \pi_A(\lambda)\theta_A u_A(c_A') + \pi_A(\lambda)(1-\theta_A)u_A(c_A'') + \pi_F(\lambda)\theta_F u_F(c_F')$

$+ \theta_F(\lambda)(1-\theta_F)u_F(c_F'')$.

Substituting into (19) from (11), (12), (13) and (14) and rearranging terms we have

(20) $V(\pi_A(\lambda),B(\lambda)) \geqq \lambda[\pi' u_A(c_A') + (1-\pi')u_F(c_F')] +$

$(1-\lambda)[\pi'' u_A(c_A'') + (1-\pi'')u_F(c_F'')]$.

Then recalling (3) and (4) we see that

$$(21) \quad V(\pi_A(\lambda), B(\lambda)) \geqq \lambda V(\pi', B') + (1-\lambda)V(\pi'', B'').$$

Therefore V is a concave function.

Figure 1

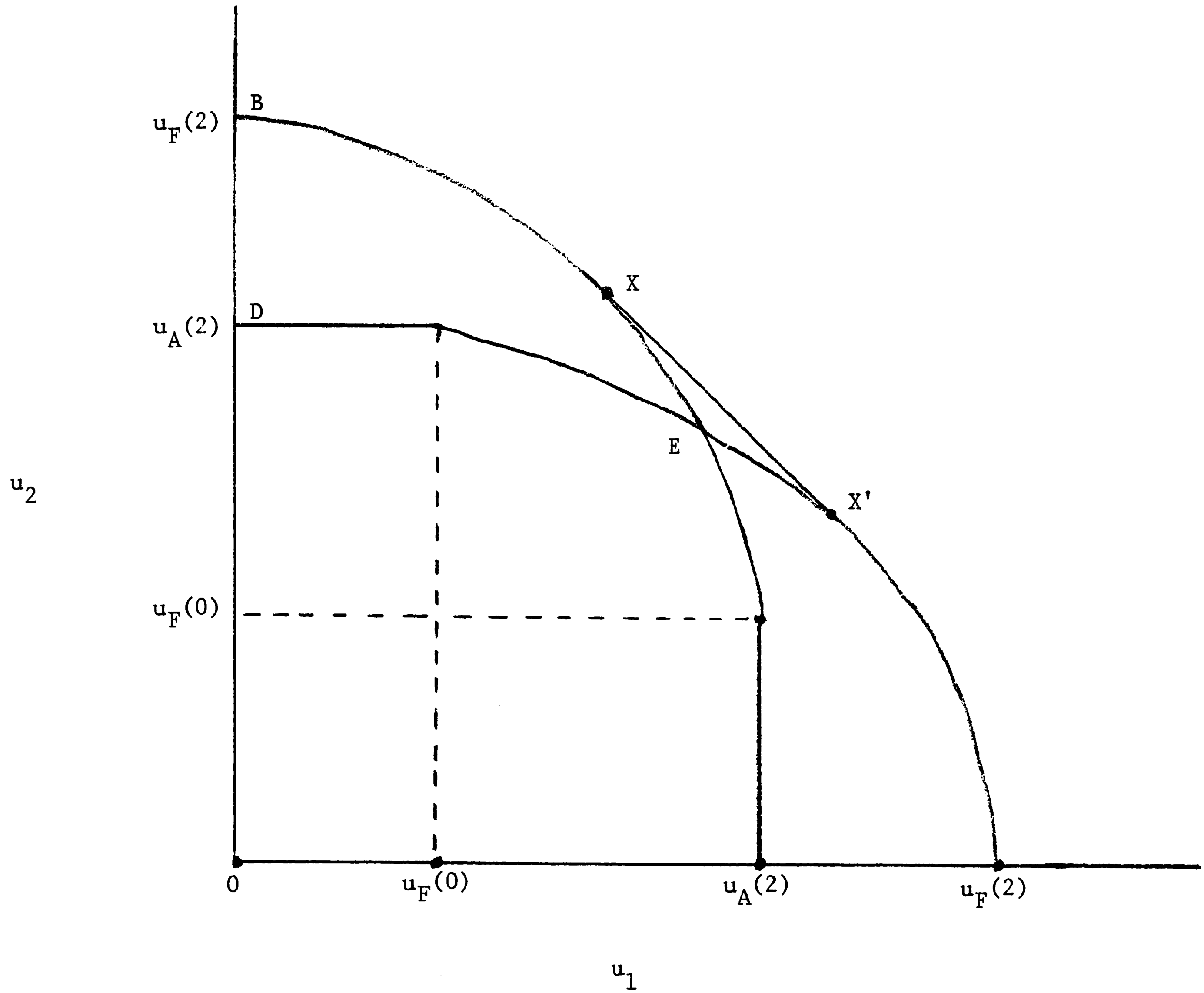
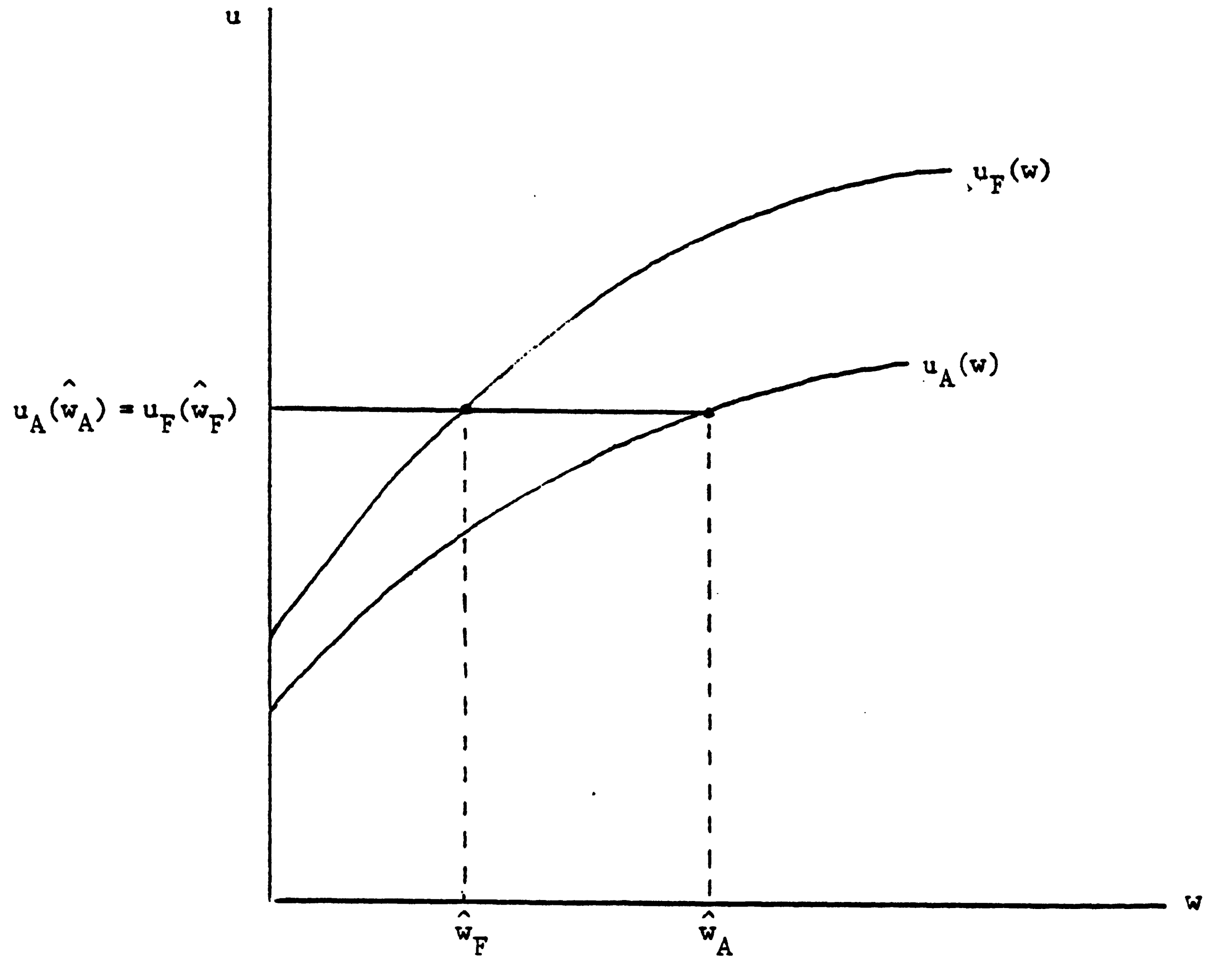


Figure 2



Appendix

Part I - Proof of Proposition 1

It is convenient to rewrite the constrained maximization problem of finding the best fair lottery as an unconstrained maximum problem in w_A by substituting for w_F from the feasibility equation (2). Thus the best fair lottery has soldiers paid w_A^* where w_A^* maximizes

$$\text{(A.1)} \qquad U^*(w_A) \equiv \bar{\pi} u_A(w_A) + (1 - \pi)u_F - \left(\frac{\bar{\pi}}{1-\bar{\pi}}\right) w_A).$$

By straightforward calculation we see that

$$\text{(A.2)} \qquad U^{*\prime}(w_A) = \bar{\pi}[u_A'(w_A) - u_F'(w_F)]$$

$$\text{(A.3)} \qquad U^{*\prime\prime}(w_A) = \bar{\pi}[u_A''(w_A) + \left(\frac{\bar{\pi}}{1-\bar{\pi}}\right) u_F''(w_F)].$$

where $w_F = \frac{\bar{B}}{1-\bar{\pi}} - \left(\frac{\bar{\pi}}{1-\bar{\pi}}\right) w_A$. From (A.2) we see that $U^{*\prime}(w_A) \gtreqless 0$ accordingly as $u_A'(w_A) \gtreqless u_F'(w_F)$ where $w_F = \frac{\bar{B}}{1-\bar{\pi}} - \left(\frac{\bar{\pi}}{1-\bar{\pi}}\right) w_A$. From (A.3) and the assumption that $u_A(\cdot)$ and $u_F(\cdot)$ are strictly concave, it follows that $U^{*\prime\prime}(w_A) < 0$ for all $w_A \geqq 0$. Therefore $w_A \lesseqgtr w_A^*$ accordingly as $u_A'(w_A) \gtreqless u_F'(w_F)$ where $w_F = \frac{\bar{B}}{(1-\bar{\pi})} - \left(\frac{\bar{\pi}}{1-\bar{\pi}}\right) w_A$.

Since $u_A(w + c(w)) = u_F(w)$ for $w \geqq 0$, it follows that

$$\text{(A.4)} \qquad (1 + c'(w))u_A'(w + c(w)) = u_F'(w) \quad \text{for all} \quad w \geqq 0.$$

From the definitions of $c(\cdot)$, $\hat{w}_A$ and $\hat{w}_F$ it must be that $\hat{w}_A = \hat{w}_F + c(\hat{w}_F)$. Therefore from (A.4) it follows that

$$\text{(A.5)} \qquad (1 + c'(\hat{w}_F))u_A'(\hat{w}_A) = u_F'(\hat{w}_F)$$

From (A.5) we see that $u_A'(\hat{w}_A) \lesseqgtr u_F'(\hat{w}_F)$ accordingly as $c'(\hat{w}_F) \gtreqless 0$. Therefore from the argument of the previous paragraph we see that $\hat{w}_A \gtreqless w_A^*$ accordingly as $c'(\hat{w}_F) \gtreqless 0$. The feasibility constraint implies therefore that $\hat{w}_F \lesseqgtr w_F^*$ accordingly as $c'(\hat{w}_F) \gtreqless 0$. From the assumption that $u_A(\cdot)$ and $u_F(\cdot)$ are monotone increasing functions and the fact that $u_A(\hat{w}_A) = u_F(\hat{w}_F)$, Proposition 1 is now immediate.

Q.E.D.

Footnotes

[1] Bradford (1968) presents a useful analysis of individual decision making in the presence of a draft lottery. He does not, however study the efficiency issues raised here.

[2] Stiglitz (1976) makes a similar observation in a different context. In particular, he demonstrates that random taxation may be a Pareto improvement on non-random taxation in a standard model with endogenous labor supply.

[3] Since von Neuman-Morgenstern utility functions are unique up to affine transformations and since convexity of a set is preserved under affine transformations of the axes, we see that the convexity of the set of von Neuman-Morgenstern utility distributions attainable as sure things is a well-defined property.

[4] To see this consider the following example. A possibly biased coin with probability π_H of turning up heads is tossed. Contingent commodities are consumption if heads, c_H, and consumption with tails c_T. A consumer has a von Neuman-Morgenstern utility function $U(\pi_H, c_H, c_T) = \pi_H u(c_H) + (1-\pi_H)u(c_T)$ where $u(1) > u(\frac{1}{2})$. Then $U(1,1,0) = U(0,0,1) = U(1)$. But $U(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = u(\frac{1}{2}) < u(1)$. This means that U is not a quasi-concave function.

[5] A proof is found in the Appendix.

[6] His choice is also constrained by the "boundary" constraints $0 \leqq \pi_i \leqq 1$ and $B_i > 0$. These constraints present no theoretical difficulties to standard general equilibrium theory since they can be incorporated in the description of the (convex) consumption possibility set.

[7] The one apparently unorthodox element of the theory is that one of the "goods" is a bad. General existence theorems have been extended to cover this case (see e.g., Bergstrom (1976)). Alternatively the model could be made to appear entirely orthodox by treating probability of being a farmer as the commodity instead of probability of being a soldier.

[8] If all contingent claims were met, then total bread consumption would have to be a random variable $\beta = \sum_i c^i_F + \sum_i I_i(c^i_A - c^i_F)$ where $I_i = 1$ if citizen i is drafted and $I_i = 0$ otherwise. Standard statistical results for sampling without replacement inform us that the per capita variance of β approaches zero as the country gets large. Therefore in a large economy the "risks" that we ignore by treating as equal to its expected value are negligible. If we wished an exact model we could have large numbers of individual "stockholders" in "insurance company" who each absorb a tiny share of the difference between β and its expected value.

References

Bergstrom, T. "How to Dispose of Free Disposability -- At No Cost," *Journal of Economic Theory*, 1976.

Bradford, D. "The Enlistment Decision Under Draft Uncertainty," *Quarterly Journal of Economics*, November 1968, pp. 621-638.

Debreu, G. *Theory of Value*, New York, Wiley, 1959.

Stiglitz, J. "Utilitarianism and Horizontal Equity: The Case for Random Taxation," mimeo, 1976.

CREST Working Papers

C-1 The Ergodic Behavior of Stochastic Processes of Economic Equilibrium, by Lawrence E. Blume.

C-2 The Token Economy, Reinforcement and the Consumer Model, by John G. Cross.

C-3 Notes on a Theory of Learning to Search, by John G. Cross.

C-4 Redistributive Taxation as Social Insurance, by Hal R. Varian.

C-5 Catastrophe Theory and the Business Cycle, by Hal R. Varian.

C-6 Household Bequests, Perfect Expectations, and the National Distribution of Wealth, by John P. Laitner.

C-7 Inflationary Disequilibrium, by John G. Cross.

C-8 Ellet's Transportation Model of an Economy with Differentiated Commodities and Consumers, I: Generic Cumulative Demand Functions, by Carl P. Simon.

C-9 Cournot Equilibrium in Factor Markets, by Theodore Bergstrom.

C-10 Is a Man's Life Worth More Than His Human Capital, by Theodore Bergstrom.

C-11 A Model of Sales, by Hal R. Varian.

C-12 New Techniques for the Study of Stochastic Equilibrium Processes, by Lawrence E. Blume.

C-13 Consistent Expectations, by Lawrence E. Blume.

C-14 On Baubles and Bubbles, by John G. Cross.

C-15 When Does Majority Rule Supply Public Goods Efficiently?, by Theodore Bergstrom.

C-16 "Rational" Duopoly Equilibria, by John P. Laitner.

C-17 The Natural Rate of Unemployment, by John P. Laitner.

C-18 Learning to be Rational, by Lawrence E. Blume and David Easley.

C-19 Notes on Cost-Benefit Analysis, by Hal R. Varian.

C-20 On Capturing Oil Rents with a National Excise Tax, by Theodore Bergstrom.

C-21 Cournot, Novshek and the Many Firms, by Theodore Bergstrom.

C-22 The Nonparametric Approach to Demand Analysis, by Hal R. Varian.

C-23 National Debt, Social Security, and Bequests*, by John P. Laitner.

C-24 The Taking of Land: When Should Compensation be Paid?, by Lawrence Blume.

C-25 Efficiency-Inducing Taxation for a Monopolistically Supplied Depletable Resource, by Theodore C. Bergstrom, John G. Cross and Richard C. Porter.

C-26 Monopoly and Long-Run Capital Accumulation*, by John Laitner.

C-27 Information Asymmetry and Optimal Inefficiency Between Principal and Agent, by David Sappington.

C-28 Micro-Based Estimates of Demand Functions for Local School Expenditures, by Theodore C. Bergstrom, Daniel L. Rubinfeld and Perry Shapiro.

C-29 A Quantity-Constrained Macroeconomic Model with Price Flexibility, by Allan Drazen.

C-30 Limited Liability Contracts Between Principal and Agent, by David Sappington.

C-31 Nonparametric Tests of Consumer Behavior, by Hal R. Varian.

C-32 The Nonparametric Approach to Production Analysis, by Hal R. Varian.

C-33 Nonparametric Tests of Models of Investor Behavior, by Hal R. Varian.

C-34 Indirect Utility and Nonparametric Demand Analysis, by Hal R. Varian.

C-35 Social Indifference Curves and Aggregate Demand, by Hal R. Varian.

C-36 Some Aspects of Risk Sharing in Nonclassical Environments, by Hal R. Varian.

C-37 A Statistical Test of the Revealed Preference Axioms, by Hal R. Varian.

C-38 When is There a Representative Consumer of Public Goods? - On Separating the Allocation and Distribution Branches of Government, by Theodore Bergstrom and Richard Cornes.

C-39 Gorman and Musgrave are Dual - An Antipodal Theorem, by Theodore Bergstrom and Richard Cornes.

C-40 Michigan State Expenditures and The Provision on Public Services, by John G. Cross.

C-41 Rational Expectations Equilibrium: An Alternative Approach, by Lawrence E. Blume and David Easley.

C-42 Estimating the Dispersion of Tastes and Willingness to Pay, by Philip E. Howrey and Hal R. Varian.

C-43 Lectures on Public Economics, by Ted Bergstrom.

C-44

C-45 Optimal Regulation of Research and Development Under Imperfect Information, by David Sappington.

C-46 Oligopoly Behavior When Conjectural Variations are Rational, by John Laitner.

C-47 Soldiers of Fortune? by Ted Bergstrom.

C-48 Scalar and Vector Maximization: Calculus Techniques with Economics Applications, by Carl P. Simon.