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WHEN IS THERE A REPRESENTATIVE CONSUMER OF PUBLIC GOODS? - ON SEPARATING THE ALLOCATION AND DISTRIBUTION BRANCHES OF GOVERNMENT

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If all the seas were one sea,
What a *great* sea that would be!
If all the trees were one tree,
What a *great* tree that would be!
And if all the axes were one axe,
What a *great* axe that would be!
And if all the men were one man,
What a *great* man that would be!
And if the *great* man took the *great* axe,
And cut down the *great* tree,
And let it fall into the *great* sea,
What a splish-splash that would be!

Normally aggregate demand for private goods can not be treated as if it were the demand of a single gigantic rational consumer. This is possible only if "income distribution doesn't affect aggregate demand". Gorman (1953) discovered restrictions on the form of indirect utility functions that are necessary and sufficient to allow such aggregation. In early partial equilibrium treatments of public goods theory by Lindahl (1910) and Bowen (1943) the efficient amount of public goods appears to be determined independently of income distribution. Samuelson (1955), (1966) observes that generally an efficient amount of public goods cannot be determined independently of the distribution of private goods. He points out that such separation is possible in the special case where preferences of all consumers are *quasi-linear*, that is, representable by utility functions that are linear in private goods. Musgrave (1966) responds that although independence of allocation from distribution is not legitimate in a strict logical sense, separation of allocational decisions from distributional decisions is a useful simplification of reality that may in practical situations lead to better decision making than attempts to simultaneously determine allocation and distribution.

If quasi-linear preferences were necessary for separation of allocation from distribution, then Musgrave's case for separation, even as an approximation, would not be good. Quasi-linearity has strong and rather easily re-

futable implications. For example, it implies a zero income elasticity of demand for public goods. Several recent studies of the demand for public goods strongly reject the hypothesis that the income elasticity of demand for local public goods is close to zero.[1] As it turns out, however, separation of allocation from distribution is possible for a much broader class of preferences. This class is essentially dual to the class of preferences found by Gorman to admit construction of a "representative consumer" in the theory of demand for private goods.[2]

In the next section we develop a rigorous theory of when allocation can be separated from income distribution, or equivalently of when there is a representative consumer of public goods. It remains an open question whether empirical data can be found that refute the hypothesis that preferences for public goods allow such aggregation. This being the case, it also remains to be decided whether the assumption that preferences satisfy this hypothesis yields misleading guidance for public policy.

In the final section of this paper we show that the assumption that preferences belong to the class that allows a representative consumer has interesting implications for the theory of social welfare functions, for Lindahl's allocation theory, and for Bowen's majority voting theory. We also demonstrate that demand revealing mechanisms of the kind introduced by Clarke (1971) and Groves and Loeb (1975) for the case of quasi-linear utility can be extended in a simple way to this much broader class of preferences.

SECTION 1

A. Two heuristic claims and their implications

Let there be n public goods and one private good.[3] Let Y denote the vector of public goods supplied and let X_i denote the consumption of private goods by consumer i. Subject to certain qualifications, we establish the following claims.

Claim S - If all consumers have utility functions of the functional form $A(Y)X_i + B_i(Y)$, then the Pareto efficient quantity of public goods is independent of the distribution of private goods.

Claim N - If the Pareto efficient quantity of public goods is independent of the distribution of private goods, then all consumers have utility functions of the form $A(Y)X_i + B_i(Y)$.

In these claims there is some good news and some bad news. The good news is Claim S. The class of preferences that allow separation of allocation from distribution is much larger than the quasi-linear family. The special case where $A(Y) = 1$ is the quasi-linear case. If $A(Y) = Y^{\alpha}$ for $\alpha > 0$ and $B_i(Y) = 0$, preferences are identical and Cobb-Douglas. But $A(Y)$ and $B_i(Y)$ could be chosen so that preferences are neither homothetic, separable, or identical. The bad news is found in Claim N. This result states restrictions on the class of preferences which admit separation between allocation and distribution. Some of the implications of membership in this class are stated in Theorem 1.

Theorem 1

Let preferences of consumer i be represented by a differentiable utility function $A(Y)X_i + B_i(Y)$ where $A(Y) > 0$ for all $Y \geqq 0$.

(a) Preferences of consumer i are homothetic if and only $A(Y)$ is homogeneous of some degree α and $B_i(Y)$ is either constant or homogeneous of degree $\alpha + 1$. If there is only one public good, $A(Y)X_i + B_i(Y) = Y^{\alpha}X_i + k_i Y^{\alpha+1}$ for some k_i.

(b) Preferences of consumer i are additively separable between public and private goods if and only if they are representable by a utility function of the form $A(Y)(X_i + k_i)$ for some k_i.

(c) Preferences of each consumer i are additively separable and homothetic if and only if all consumers have identical preferences representable by utility functions of the form $A(Y)X_i$ where $A(Y)$ is a homogeneous function. If there is only one public good, this implies that utility functions all have the Cobb-Douglas form $Y^{\alpha}X_i$ for some $\alpha > 0$.

The assumptions that $A(Y)X_i + B_i(Y)$ and $A(Y)X_i + \sum_i B_i(Y)$ are quasi-concave will play an important part in the development of our theory. It is therefore useful to identify some necessary and some sufficient conditions for quasi-concavity of these functions.

<u>Theorem 2</u>

Define the function $\beta_i(Y) = \frac{B_i(Y)}{A(Y)}$. Then $A(Y)X_i + B_i(Y) \equiv A(Y)(X_i + \beta_i(Y))$. The following conditions are each sufficient for $A(Y)X_i + B_i(Y)$ to be quasi-concave.

(i) $A(Y) > 0$ and $\beta_i(Y) \geqq 0$ for all $Y \geqq 0$ and the functions $A(Y)$ and $\beta_i(Y)$ are concave.

(ii) There is only one public good, $A(Y) > 0$ and $B_i(Y) \geqq 0$ for all $Y > 0$ and the functions $A(Y)$ and $B_i(Y)$ have non-negative first derivatives and negative second derivatives.

(iii) The two quadratic forms, $\nabla^2 A(Y) - \frac{2}{A(Y)} \nabla A(Y) \nabla A(Y)^T$ and $\beta_i(Y)[\nabla^2 A(Y) - \frac{2}{A(Y)} \nabla A(Y) \nabla A(Y)^T] + A(Y)\nabla^2 \beta_i(Y)$ are both negative semi-definite and one of them is negative definite for all $Y \geqq 0$.

The following conditions are each necessary for $A(Y)X_i + B_i(Y)$ to be quasi-concave.

(i) $A(Y)$ and $B_i(Y)$ are both quasi-concave functions.

(ii) The quadratic forms in sufficient condition (iii) are both negative semi-definite for all $Y \geqq 0$.

The condition on quadratic forms is seen to be (essentially) a necessary and sufficient condition for quasi-concavity. However, in general this condition is rather difficult to verify or to interpret. Sufficient conditions (i) and (ii) are easier to interpret but are not necessary conditions. In our applications, however, they will serve adequately.

An easy consequence of theorem 2 is the following.

Corollary 1 - If all individual utility functions satisfy necessary condition (i) or (ii), of theorem 2 then the function $A(Y)X + \sum_i B_i(Y)$ is quasi-concave.

B. Establishing Claim S - Sufficiency

Define an outcome to be a vector $(X_1,\ldots,X_m,Y)$ where X_i is the amount of private good for i and Y is the (n-dimensional) vector of public goods. Let the set of feasible outcomes be $\{(X_1,\ldots,X_m,Y) \mid (\sum_i X_i,Y) \in F\}$ for some set $F \subset R^{n+1}$. An interior outcome is an outcome such that $X_i > 0$ for all i. An interior Pareto optimum is a Pareto optimal interior outcome.

If all consumers have utility functions of the form $A(Y)X_i + B_i(Y)$, a possible mandate for the allocation branch is: "Choose (X,Y) to maximize $A(Y)X + \sum_i B_i(Y)$ on the set F". We show that an allocation branch that follows this instruction will choose an aggregate output level that yields a Pareto optimal outcome no matter how the private good is divided. Furthermore, given convexity, all of the "interesting" Pareto optima are found in this way.

Theorem 3

Let all consumers have utility functions of the form $A(Y)X_i + B_i(Y)$. If $(\overline{X},\overline{Y})$ maximizes $A(Y)X + \sum_i B_i(Y)$ on F, then every outcome $(\overline{X}_1,\ldots,\overline{X}_m,\overline{Y})$ such that $\sum_i \overline{X}_i = \overline{X}$ is Pareto optimal. If $A(Y)X + \sum_i B_i(Y)$ is quasi-concave, $A(Y) > 0$, and F is a convex set, then $(\overline{X}_1,\ldots,\overline{X}_m,\overline{Y})$ is an interior Pareto optimum only if $(\sum_i \overline{X}_i,\overline{Y})$ maximizes $A(Y)X + \sum_i B_i(Y)$ on F.

It is instructive to consider the application of Theorem 3 to a special case.

Example 1. There are two consumers with identical quasi-linear utility functions $X_i + \sqrt{Y}$. The set of feasible aggregate outputs is $F = \{(X,Y) \geqq 0 | X + Y = 3\}$.

In this example $A(Y)X + \sum_i B_i(Y)$ is maximized on F at $\overline{X} = 2$, $\overline{Y} = 1$. According to Theorem 3, all outcomes that have one unit of public good and some distribution of two units of private good between the consumers are Pareto optimal. Since the convexity conditions of theorem 3 apply, it must also be that all interior Pareto optima have exactly one unit of public good and two units of private goods. There are also some non-interior Pareto optima for which $Y \neq 1$. In fact, an outcome is Pareto optimal if $\frac{1}{4} \leqq Y \leqq 1$ and one of the consumers gets no private goods while the other gets $3 - Y$. The utility possibility frontier for example 1 is shown in figure 1. Points on the line between (1,3) and (3,1) correspond to outcomes where $Y = 1$. Points on the curved lines from (3,1) to $(3\frac{1}{4}, \frac{1}{2})$ and from (1,3) to $(\frac{1}{2}, 3\frac{1}{4})$ represent non-interior Pareto optima.

In theorem 3, although we assumed preferences to be monotone increasing in private goods, no assumption was made about monotonicity in public goods. This is fortunate, since to model externalities in a natural way we need some public goods that are desirable to some consumers and undesirable, at least in certain quantities, to others. Theorem 2 and its corollary specify an interesting class of functions for which our assumption that utility is quasi-concave applies. Nevertheless, the assumption of convex preferences over public goods is not entirely appealing. In fact, as Starrett (1972) pointed out, in the case of public "bads" convexity is very implausible. It is therefore interesting to look closely at the use we have made of the convexity assumptions and to find the best results available without convexity.

We notice that the first statement of Theorem 3 is true independent of convexity. Example 2 helps us to see why convexity is needed for the second result of theorem 3.

> Example 2. Preferences are as in example 1. There are only two possible outputs of public good, $Y = 1/4$ and $Y = 1$. The set F is therefore $\{(2\frac{3}{4}, \frac{1}{4}), (2,1)\}$.

Possible distributions of utility are represented by the two diagonal lines in Figure 2. The upper line represents utility distributions possible if $(X,Y) = (2,1)$. The lower line represents utility distributions possible if $(X,Y) = (2\frac{3}{4}, \frac{1}{4})$. Clearly $A(Y)X + \sum_i B_i(Y)$ is greater at $(2,1)$ than at $(2\frac{3}{4}, \frac{1}{4})$. But there are some interior Pareto optima for which aggregate output levels are $(2\frac{3}{4}, \frac{1}{4})$. One such outcome is $(X_1, X_2, Y) = (2\frac{5}{8}, \frac{1}{8}, \frac{1}{4})$. This outcome corresponds to the utility distribution $(3\frac{1}{8}, \frac{5}{8})$ marked P in figure 2. From the figure it is apparent that $(2\frac{5}{8}, \frac{1}{8}, \frac{1}{4})$ is an interior Pareto optimum. There are other outcomes obtained by distributions from the aggregate outputs $(2\frac{3}{4}, \frac{1}{4})$ which are not Pareto optimal. An example is $(1\frac{3}{8}, 1\frac{3}{8}, \frac{1}{4})$. This yields the utility distribution marked Q in figure 1. This distribution is clearly dominated by feasible utility distributions such as R in the figure. We see from this example that without convexity, it is not true that every outcome obtained by redistribution from an interior Pareto optimum is Pareto optimal.

Define an aggregate output $(\overline{X}, \overline{Y})$ to be <u>certainly efficient</u> if every outcome $(\overline{X}_1, \ldots, \overline{X}_n, \overline{Y})$ such that $\sum_i \overline{X}_i = \overline{X}$ is Pareto optimal. Let $E \subset F$ be the set of all certainly efficient aggregate outputs. In general, the set E may be empty. However if utility is of the form $A(Y)X_i + B_i(Y)$ for all consumers i, it follows from Theorem 1 that even without convexity, E will be non-empty given some weak technical assumptions. The following is a direct consequence of Theorem 2 and the Weierstrass theorem on the existence of maximal elements for continuous functions on compact sets.

Corollary 2 - If preferences of all consumers are represented by continuous utility functions of the form $A(Y)X_i + B_i(Y)$ and if F is closed and bounded, then the set E of certainly efficient aggregate outputs is non-empty.

According to theorem 3, every aggregate output (X,Y) that maximizes $A(Y)X + \sum_i B_i(Y)$ on F belongs to the set E of certainly efficient aggregate outputs. In general, however, E is larger than the set of maximizers of $A(Y)X + \sum_i B_i(Y)$ on F. To see this, consider the following example.

<u>Example 3</u>. There are two consumers. Consumer i has a utility function of the form $A(Y)X_i + B_i(Y)$ where $A(Y) = 1$ for all Y and where $B_1(0) = B_2(0) = 2$, $B_1(1) = 6$ and $B_2(1) = 0$. Let $F = \{(1,1),(2,0)\}$.

The utility possibility frontier is described in figure 3. Utility distributions obtainable when $Y = 0$ are represented on the line AB. Utility distributions obtainable when $Y = 1$ are represented on the line CD. All of these distributions can be seen to be Pareto optimal. Therefore the set E contains both of the points (1,1) and (2,0) from F. However if $(X,Y) = (2,0)$, then $A(Y)X + \sum_i B_i(Y) = 6$ and if $(X,Y) = (1,1)$, $A(Y)X + \sum_i B_i(Y) = 7$. Therefore the set E is larger than the set of maximizers of $A(Y)X + \sum_i B_i(Y)$.

It is of interest to find a binary relation whose maximal elements on F constitute precisely the set E. As it turns out, characterizing this result is also the key to our converse result, claim N. Let us define a binary relation $\circledS$ over aggregate outputs so that $(X,Y) \circledS (X',Y')$ if some outcome obtained by distributing X is Pareto superior to some outcome obtained by distributing X'. More formally, $(X,Y) \circledS (X',Y')$ if and only if there exist outcomes $(X_1,\dots,X_m,Y)$ and $(X_1',\dots,X_m',Y')$ such that $\sum_i X_i = X$, $\sum_i X_i' = X'$ and $(X_1,\dots,X_m,Y)$ is Pareto superior to $(X_1',\dots,X_m',Y')$.[4] From the definitions, it is easy to verify the following.

<u>Remark 1</u> - For arbitrary preferences, the set E of certainly efficient aggregate outcomes is equal to the set of $\circledS$ maximal elements of F.

In general, the binary relation ⊗ may have cycles and therefore may not have a maximal element even on a finite set F. One such case is illustrated in example 4.

Example 4. There are two consumers. Let $U_1(X_1,Y) = X_1$ for all Y and $U_2(X_2,0) = X_2$ and $U_2(X_2,1) = 2X_2$. Let $F = \{(2,1),(3,0)\}$.

The functions $U_1(\cdot)$ and $U_2(\cdot)$ are not of the special form $A(Y)X_i + B_i(Y)$. Utility distributions possible when aggregate output is (3,0) are represented by the line AB in figure 4. Distributions possible from (2,1) are represented by CD. From figure 4 and the definition of ⊗ it is clear that (1,1) ⊗ (2,0) and also (2,0) > (1,1). Therefore ⊗ has a cycle (of length 2). Clearly ⊗ has no maximal elements on F and (equivalently) the set E is empty.

Suppose preferences of all consumers are representable by utility functions of the form $A(Y)X_i + B_i(Y)$. Then if $(X,Y) > (X',Y')$ it must be that there exist outcomes $(X_1,\ldots,X_m,Y)$ and $(X_1',\ldots,X_m',Y')$ such that $\sum_i X_i = X$, $\sum_i X_i' = X'$ and $A(Y)X_i + B_i(Y) \geqq A(Y')X_i' + B_i(Y')$ for all i with strict inequality for some i. Therefore if $(X,Y) > (X',Y')$ it must be that $A(Y)X + \sum_i B_i(Y) > A(Y')X' + \sum_i B_i(Y')$. From this fact, the following is obvious.

Remark 2 - If preferences of all consumers are of the form $A(Y)X + \sum_i B_i(Y)$, then the relation ⊗ has no cycles.

A standard theorem (see Bergstrom (1975)) is that if a continuous binary relation has no cycles, then it takes maximal elements on compact sets. This fact, with remark 2 provides an alternative derivation of corollary 2.

It is interesting to compare the relation ⊗ with the more familiar Kaldor-Hicks-Samuelson partial order which was central to discussions of the "new welfare economics" (see Chipman and Moore (1978)). The K.H.S. relation is defined over aggregate outputs as follows: (X,Y)K.H.S.(X',Y') if and only if for every outcome $(X_1',\ldots,X_m',Y')$ such that $\sum_i X_i' = X'$ there exists some outcome

$(X_1,\ldots,X_m,Y)$ such that $\sum_i X_i = X$ and such that $(X_1,\ldots,X_n,Y)$ is Pareto superior to $(X_1',\ldots,X_m',Y')$.

From the definition of the K.H.S. relation and from the fact that a continuous binary relation with no cycles takes minimal elements on compact sets, it is easily shown that:

Remark 3 - If individual preferences are symmetric and transitive, then the relation K.H.S. has no cycles. If individual preferences are also continuous and F is compact, then the set of K.H.S. maximal elements of F is non-empty.

Define an aggregate output $(\overline{X},\overline{Y})$ to be potentially efficient if there is some Pareto optimal outcome $(\overline{X}_1,\ldots,\overline{X}_m,\overline{Y})$ such that $\sum_i \overline{X}_i = \overline{X}$. Let E^* denote the set of all potentially efficient aggregate outputs in F. From the definitions it is easy to verify the following.

Remark 4 - The set of K.H.S. maximal elements of F is equal to the set E^* of potentially efficient aggregate outputs.

In general the set E^* of potentially efficient aggregate outputs is larger than the set E of certainly efficient aggregate outputs. Equivalently the set of K.H.S. maximal elements on F contains the set of $>$ maximal elements of F. For instance, in example 1 above, $E = \{(2,1)\}$ while $E^* = \{(X,Y) \mid X+Y = 3$ and $\frac{1}{4} \leq Y \leq 1\}$. In example 4, E is empty, while $E^* = F = \{(2,0),(1,1)\}$.

Although, as we see from remark 3, the set of K.H.S. maximal elements is never empty for "well-behaved" preferences and compact F, this set is in general too large to serve the purpose of decentralizing allocation from distribution, since the distribution branch can not be assured that every outcome obtained from distributing a K.H.S. maximal outcome will be Pareto optimal.

C. Establishing Claim N - Necessity

A minimal requirement for the allocation branch to be able to choose efficient aggregate output independently of income distribution, is that the set E of certainly efficient aggregate outputs must be non-empty. As remarked in the previous section, E is equal to the set of $\circledS$ maximal elements on F. Suppose that the set of $\circledS$ maximal elements of F is non-empty for all closed bounded sets F. Then, in particular, $\circledS$ must have at least one maximal element on all finite sets F. But this is possible if and only if $\circledS$ has no cycles. Therefore a necessary condition for independence of allocation from distribution on all closed bounded feasible sets F is that the relation $\circledS$ has no cycles. Theorem 4 states conditions under which the absence of cycles in $\circledS$ implies that utility functions take the form $A(Y)X_i + B_i(Y)$. This establishes our claim N. Since we regard the proof of theorem 4 to be novel and interesting we include it in the text rather than in the appendix.

Theorem 4

Let preferences of each consumer be representable by a continuous utility function $U_i(X_i,Y)$ that is monotone increasing in X_i. Let there be some Y^o such that $U_i(X_i,Y) \geqq U_i(X_i,Y^o)$ for all $Y \geqq 0$ and assume that for all X_i, Y, and Y' there exists X_i' such that $U_i(X_i',Y') > U_i(X_i,Y)$. If the binary relation > has no cycles, then for each i, $U_i(X_i,Y)$ is a monotone transformation of a utility function of the form: $A(Y)X_i + B_i(Y)$.

Proof of Theorem 4

From our assumptions it follows that for every (X_i,Y), there exists a unique X_i' such that $U_i(X_i',Y^o) = U_i(X_i,Y)$. Therefore we can define $U_i{*}(X_i,Y)$ so that $U_i(U_i{*}(X_i,Y),Y^o) = U_i(X_i,Y)$ represents i's preferences. Furthermore, we see from the definition that $U_i{*}(X_i,Y^o) = X_i$ for all X_i.

The assumption that ⓢ has no cycles implies that if $\Sigma X_i = \Sigma X_i' = X$, then $\sum_i U_i^*(X_i,Y) = \sum_i U_i^*(X_i',Y)$. For suppose not. Then without loss of generality, let $\sum_i U_i^*(X_i,Y) - \sum_i U_i^*(X_i',Y) = \delta > 0$. Let $Z_i = U_i^*(X_i,Y) - \frac{\delta}{2n}$ and let $Z_i' = U_i^*(X_i',Y) + \frac{\delta}{2n}$. Then $\sum_i Z_i = \Sigma Z_i' \equiv Z$. Now $U_i^*(Z_i,Y^o) = Z_i < U_i^*(X_i,Y)$ for all i and $U_i^*(Z_i',Y^o) = Z_i' > U_i^*(X_i',Y)$ for all i. Therefore (X,Y) ⓢ (Z,Y^o) and $(Z,Y^o) > (X,Y)$. But this contradicts the assumption that ⓢ has no cycles.

From the result of the previous paragraph it follows that $\sum_i U_i^*(X_i,Y) = U(\sum_i X_i,Y)$ for some function U. An equation of this functional form is known as Pexider's functional equation. It is well known (see Aczel (1966)) that Pexider's functional equation implies that $U_i^*(X_i,Y) = A(Y)X_i + B_i(Y)$ for all i. Since $U_i^*(X_i,Y)$ represents i's preferences, Theorem 4 is established.

Q.E.D.

SECTION 2 Applications

If preferences are representable by utility functions of our special form, then several of the standard problems in the theory of public finance and welfare economics have interesting special solutions.

A. Cardinal Utility and Social Welfare Functions

Classical utilitarians (Edgeworth (1881)) proposed that the goal of society should be to maximize the sum of human happiness. Since Pareto, it has been well-known that to maximize the sum of utility functions representing individual preference is not a well-defined prescription, since such functions are unique only up to monotone transformations. Bergson (1938) and Samuelson (1947) have suggested that ethical value systems which respect the Pareto ordering can generally be represented by a social welfare function which depends (not necessarily in a linear way) on the level of "utility" of each consumer according to an arbitrary, but prespecified, utility representation of each consumer's preferences.

Where utility functions are of the form: $U_i(X_i,Y) = A(Y)X_i + B_i(Y)$, consider the function $W(X_1,\ldots,X_m,Y) = \sum_i U_i(X_i,Y) = A(Y)\sum_i X_i + \sum_i B_i(Y)$.

If the technical conditions of Theorem 3 are satisfied, then finding an interior Pareto optimum is equivalent to finding an allocation $(X_1,\ldots,X_n,Y)$ that maximizes $W(\cdot)$ subject to the feasibility constraint, $(\sum_i X_i, Y) \in F$. The function, $W(\cdot)$, is therefore very useful as a guide to solving for efficient aggregate outputs. On the other hand, it provides absolutely no guidance about distribution of private goods. In fact all redistributions of the same amount of private goods yield the same value of W. This is as it should be since we sought conditions under which efficient levels of aggregate output could be chosen independently of income distribution.

It would be misleading to think of $W(\cdot)$ as a "social welfare function" in the sense employed by Bergson and Samuelson. Notice that $W(\cdot)$ ascribes a higher "score" to any Pareto optimal allocation, no matter how asymmetrically private goods are distributed, than it does to any less efficient but much more symmetric distribution. Clearly then such a function could not represent the distributional preferences of a person with continuous preferences, who although he wishes all individuals to be happier, has some taste for equality.

In fact, the function $W(\cdot)$ defined above is not the only function that will always pick a Pareto optimum when maximized on the feasible set. Consider the function $W^*(X_1,\ldots,X_m,Y) = W(X_1,\ldots,X_m,Y) - \sum_i (X_i - \overline{X})^2$ where $\overline{X} = \frac{1}{n}\sum_i X_i$. Clearly W^* gives a higher value to some non-optimal equalitarian allocations than it does to some asymmetric Pareto optimal allocations. On the other hand, it is easily seen that maximizing W^* on the feasible set will always yield a Pareto optimal allocation. In fact this procedure will yield an allocation that maximizes W on the feasible set and has the additional property that $X_1 = X_2 = \ldots = X_m = \overline{X}$. Of course W^* is not the only "social welfare function" that reflects concern for the distribution of private income. An infinity of other functions could be constructed, each of which recommended a Pareto optimum with a different distribution of private goods and excluded all other distributions. It should be clear, from this discussion, that the function $W(\cdot)$ does not typically represent a full schedule of belief about the ethics of distribution. In fact the statement "Maximize W on the feasible set" has no more content than we built into it. Under appropriate conditions this condition is equivalent to "Find aggregate output levels that can be distributed to yield interior Pareto optima". There is no implicit recommendation about distribution.

B. Lindahl Equilibrium

A Lindahl equilibrium occurs when individual "tax prices" are adjusted in such a way that, given their tax prices, consumers agree unanimously on the amount of public goods to be provided. Lindahl equilibrium is known to be Pareto optimal and to belong to the "core" when public goods are desirable. (Foley (1970)). In case utility functions are of the special form

$$(1) \qquad A(Y)X_i + B_i(Y)$$

Lindahl equilibrium has a very special structure. As it turns out, in this case Lindahl tax schedules will be affine in wealth. This means that such taxes could be collected by means of a proportional wealth tax (at the same rate for everyone) augmented by a "head tax" that may be positive or negative for an individual depending on the private functions, $B_i(Y)$.

We conduct this discussion with a simplified formal model which could be extended in a straightforward way to more general environments. Let there be one private good and one public good. Each i has an initial endowment W_i of private good. Public goods can be made from private goods at constant unit cost c. The set of feasible allocations, then, is

$$\{(X_1,\ldots,X_m,Y) \mid \sum_i X_i + cY = \sum_i W_i\}.$$

A Lindahl equilibrium consists of tax shares t_i for each i where $\sum_i t_i = 1$ and a feasible allocation $(\overline{X}_1,\ldots,\overline{X}_m,\overline{Y})$ such that for all i, $(\overline{X}_i,\overline{Y})$ maximizes $U_i(X_i,Y)$

subject to the budget constraint $X_i + t_i cY = W_i$. In Lindahl equilibrium, therefore, each consumer's marginal rate of substitution between public and private goods equals his tax price $t_i c$.

If utility functions are of the form (1), then marginal rates of substitution take the form $\alpha(Y)X_i + \gamma_i(Y)$ where $\alpha(Y) = \frac{A'(Y)}{A(Y)}$ and $\gamma_i(Y) = \frac{B_i'(Y)}{A(Y)}$. Therefore, in Lindahl equilibrium,

$$(2) \qquad t_i c = \alpha(\bar{Y})\bar{X}_i + \gamma_i(\bar{Y}) = \alpha(\bar{Y})[W_i - t_i c\bar{Y}] + \gamma_i(\bar{Y}).$$

Rearranging equation (2), we have:

$$(3) \qquad t_i c = \frac{\alpha(\bar{Y})}{1 + \alpha(\bar{Y})\bar{Y}} W_i + \frac{\gamma_i(\bar{Y})}{1 + \alpha(\bar{Y})\bar{Y}}$$

From (2) we see that, as promised, each consumer's tax share is an affine function of his wealth. Summing equation (2) over the i's, recalling that $\sum_i t_i = 1$, and rearranging terms, we find that:

$$(4) \qquad c = \alpha(\bar{Y})[\sum_i W_i - c\bar{Y}] + \sum_i \gamma_i(\bar{Y})$$

which is just the Samuelson first order condition for efficiency applied to this case. Thus if the government knows the utility functions, it could compute Lindahl equilibrium simply by solving equation (4) for $\bar{Y}$ and then assessing taxes $t_i c\bar{Y}$ where $t_i c$ is found from equation (3). These taxes will just pay for $\bar{Y}$ and all consumers, given their tax rates, will agree that $\bar{Y}$ is the "right" amount of public goods.

C. Majority Voting Equilibrium

A serious disadvantage of the Lindahl allocation method is that it requires the government to know details of individual preferences which are private information and which individuals may have an incentive to conceal. A less stringent requirement would be that the government knows $A(Y)$ and has a good estimate of $\bar{B}(Y) = \frac{1}{n}\sum_i B_i(Y)$. Then the government would know an "average utility function"

(1) $A(Y)X_i + \overline{B}(Y)$

although it would not know detailed individual preferences. The government would know enough to find an efficient amount of public goods since it needs only to choose Y to maximize:

(2) $A(Y)\sum_i X_i + \sum_i B_i(Y) = A(Y)\sum_i X_i + n\overline{B}(Y)$

subject to the feasibility constraint.

If taxes are assessed according to an "average" Lindahl schedule, we have:

(3) $$t_i c = \frac{\alpha(\overline{Y})}{1 + \alpha(\overline{Y})\overline{Y}} W_i + \frac{\overline{\gamma}(\overline{Y})}{1 + \alpha(\overline{Y})\overline{Y}}$$

where $$\alpha(\overline{Y}) = \frac{A'(\overline{Y})}{A(\overline{Y})}$$

and $$\overline{\gamma}(\overline{Y}) = \frac{\overline{B}'(\overline{Y})}{A(\overline{Y})} = \frac{1}{n}\sum_i \gamma_i(\overline{Y}).$$

Suppose tax shares are set by the schedule (3) and consumers are allowed to vote on the amount of public goods. If the amount of public goods were Y, then the utility of consumer i after paying his taxes would be

(4) $\tilde{U}_i(Y) = A(Y)[W_i - t_i cY] + B_i(Y).$

If consumer i has convex preferences, then the function $\tilde{U}_i(Y)$ will be quasi-concave in Y and hence single-peaked. Consumer i's "peak" is Y_i^* where Y_i^* maximizes $\tilde{U}_i(Y)$. Let $\hat{Y}^*$ be the median of the Y_i^*'s. Since preferences are single peaked, $\hat{Y}^*$ would be the only stable outcome of a pairwise majority voting process. By straightforward calculation we see that:

(5) $U_i'(\overline{Y}) \gtrless 0$ as $\alpha(\overline{Y})[W_i - t_i c\overline{Y}] + \gamma_i(\overline{Y}) \gtrless t_i c.$

Substituting from (3) into (5) and rearranging terms we have:

(6) $U_i'(\overline{Y}) \gtrless 0$ as $\gamma_i(\overline{Y}) \gtrless \gamma(\overline{Y})$.

Therefore $Y_i^* \gtrless \overline{Y}$ as $\gamma_i(\overline{Y}) \gtrless \overline{\gamma}(\overline{Y})$.

Suppose, now, that the functions $\gamma_i(\cdot)$ are symmetrically distributed over the population. Then the mean, $\overline{\gamma}(\overline{Y})$, of the terms, $\gamma_i(\overline{Y})$, will equal their median. This fact, together, with (6) implies that just as many people will want more as will want less public good. Therefore $\overline{Y} = \hat{Y}^*$, the median of the favorite amounts. It follows that if taxes are assessed according to (3) and if preferences are symmetrically distributed in this sense, then majority vote will select the Pareto efficient quantity, $\overline{Y}$. This generalizes a result of Bowen (1943) who showed that if preferences are symmetrically distributed and quasi-linear and if taxes are the same for everyone, then the majority rule outcome is Pareto optimal.

D. Demand-Revealing Mechanisms[5]

Clarke (1971) and Groves and Loeb (1975) have demonstrated that if utility is quasi-linear, then there exists an "incentive compatible" mechanism that determines the supply of public goods and individual tax rates. This mechanism uses information supplied by consumers about their own preferences and has the property that honest revelation of preferences is a dominant strategy for each consumer. The amount of public goods selected will satisfy the Samuelson marginal rate of substitution conditions. Groves and Ledyard (1976) suggest that in more realistic cases where the demand for public goods is income responsive, it may be necessary to settle for a preference revelation mechanism in which honest revelation is a Nash equilibrium but not a dominant strategy. They concede however, that Nash equilibrium in this context is a less persuasive game theoretic "solution" than dominant strategy. We show here that the Clarke, Groves-Loeb results generalize to the case where utility functions are of the form:

(1) $U_i(X_i,Y) = A(Y)X_i + B_i(Y)$

where U_i is strictly quasi-concave.[6]

The procedure in its simplest form assumes that the function $A(Y)$ is public information.[7] The mechanism induces consumers to honestly reveal the "private information" $B_i(Y)$, in their utility functions. Let the technology be as follows. Each consumer i has an initial endowment of private goods, W_i. Public good is produced from private goods at a total cost $C(Y)$ where $C'(Y) > 0$ and $C''(Y) \geqq 0$. The set of feasible allocations is then the convex set: $\{(X_1,\ldots,X_m,Y)\,|\,\sum_i X_i + C(Y) = W\}$. All consumers are asked to reveal their functions $B_i(\cdot)$. Each i then reports a function $M_i(\cdot)$ (possibly different from $B_i(\cdot)$). Let $M = (M_1(\cdot),\ldots,M_n(\cdot))$ be the vector of functions reported.

The government chooses an amount of public goods $Y(M)$ so as to maximize:

(2) $$A(Y)(\sum_j W_j - C(Y)) + \sum_j M_j(Y).$$

Consumer i is assessed a tax bill equal to

(3) $$T_i(M) = W_i - \sum_j W_j + C(Y(M)) - \left(\frac{\sum_{j\neq i} M_j(Y) + R_i(M)}{A(Y(M))} \right)$$

where $R_i(M)$ is a function that may depend on the information sent by all consumers other than i but must be entirely independent of i's own message. Since for each i,

(4) $$X_i(M) = W_i - T_i(M),$$

it follows from (1), (3) and (4) that if the vector of functions reported is M, then i's utility is

(5) $$A(Y(M))[\sum_j W_j - C(Y(M))] + \sum_{j\neq i} M_j(Y(M)) + B_i(Y(M)) + R_i(M).$$

Since $R_i(M)$ is independent of $M_i(\cdot)$, we notice from (5) that the only way in which i's stated function $M_i(\cdot)$ affects his final utility is through the dependence of (5) on $Y(M)$. Therefore, given any choose of messages by the other players, the best choice of M_i for i is the one that leads the government to choose Y(M) to maximize

$$(6) \qquad A(Y)[\sum_j W_j - C(Y)] + \sum_{j\neq i} M_j(Y) + B_i(Y).$$

But recall that the government seeks to maximize (3). Therefore if i reports his true function, so that $M_i(Y) \equiv B_i(Y)$, then the government in maximizing (3) will also maximize (6). It follows that regardless of the message sent by others, consumer i can do no better than to report the truth. Honest revelation is therefore a dominant strategy.

Let $(\overline{T}_1,\ldots,\overline{T}_n,\overline{Y})$ be an equilibrium for this process. That is, $\overline{Y}$ maximizes (2) where $M_i(\cdot) = B_i(\cdot)$ for all i and $\overline{T}_i = T_i(B_1(\cdot),\ldots,B_n(\cdot))$. If it happened that $\sum_i \overline{T}_i = C(\overline{Y})$, then the allocation $(\overline{X}_1,\ldots,\overline{X}_n,\overline{Y})$ where $\overline{X}_i = W_i - \overline{T}_i$ would be Pareto optimal. This is a consequence of Theorem 3 and the fact that $\overline{Y}$ maximizes $A(Y)[\sum_j W_j - C(Y)] + \sum_j B_j(Y)$.

Here, as in the case of quasi-linear utility, it is in general impossible to find functions, $R_i(\cdot)$, that guarantee that $\sum_i \overline{T}_i = C(\overline{Y})$. For the quasi-linear case, Clarke and Groves-Loeb were able to find functions $R_i(M)$ that guarantee feasibility in the sense that tax revenues at least cover costs.[8] We can extend their idea to our broader class of preferences. Suppose that for each i, the government sets a "target share", $\theta_i \geqq 0$ where $\sum_i \theta_i = 1$. The government tries to fix $R_i(M)$ so that $T_i(M) \geqq \theta_i C(Y(M))$ for each i. From (3) we see that

$$(7) \qquad A(Y(M))[T_i(M) - \theta_i C(Y(M))] = A(Y(M))[(1-\theta_i)C(Y(M)) - \sum_{j\neq i} W_j] - \sum_{j\neq i} M_j(Y(M)) - R_i(M).$$

From equation (7) and the assumption that $A(Y(M)) > 0$, we see that the government could guarantee that $T_i(M) \geqq \theta_i C(Y(M))$ if it could set:

$$(8) \qquad R_i(M) \leqq A(Y(M))[(1-\theta_i)C(Y(M)) - \sum_{j\neq i} W_j] - \sum_{j\neq i} M_j(Y(M)).$$

To this end, the government may choose

$$(9) \qquad R_i(M) = \underset{Y}{\text{Min}}\{A(Y)[(1-\theta_i)C(Y) - \sum_{j\neq i} W_j] - \sum_{j\neq i} M_j(Y).$$

It can be seen that $R_i(M)$ as defined in (9) does not depend in any way on i's stated function $M_i(\cdot)$. Furthermore, it is clear that $R_i(M)$ defined in this way satisfies the inequality (8). From (7) it follows that $T_i(M) \geqq \theta_i C(Y(M))$. Since $\sum_i \theta_i = 1$, it must be that $\sum_i T_i(M) \geqq C(Y(M))$.

The fact that a simple extension of the Clarke tax performs equally satisfactorily on a much larger class of preferences than the quasi-linear reduces the sting of one of the list of criticisms of this mechanism found in Groves and Ledyard (1976). Whether in this environment, the Clarke tax is likely to perform as well as alternative mechanisms in which honest preference revelation is a Nash equilibrium rather than a dominant strategy equilibrium remains an open question.

Appendix

Proof of Theorem 1

(a) To verify sufficiency is straightforward. To prove necessity, let $\Pi_i(X_i,Y)$ be the vector of marginal rates of substitution between public and private goods. If preferences are homothetic then $\Pi_i(X_i,Y) = \frac{\nabla A(Y)X_i + \nabla B_i(Y)}{A(Y)}$ must be homogeneous of degree zero in X_i and Y. Therefore $\frac{\nabla B_i(kY)}{A(kY)}$ and $\frac{\nabla A(kY)kX_i}{A(kY)}$ must both be constant as functions of k. If $\frac{\nabla A(k,Y)kX_i}{A(kY)}$ is independent of k, it must be that A(Y) is homogeneous of some degree α. If $\frac{\nabla B_i(kY)}{A(ky)}$ is constant, it must be that either $\nabla B_i(Y) = 0$ for all Y or $B_i(Y)$ is homogeneous of degree $\alpha + 1$.

(b) If preferences are additively separable between public and private goods, then there exist functions $F(\cdot)$, $V_1(\cdot)$ and $V_2(\cdot)$ for consumer i such that $A(Y)X_i + B_i(Y) = F(V_2(X_i) + V_2(Y))$. Therefore the vector of marginal rates of substitution of public for private goods can be written $\Pi(X_i,Y) = \frac{\nabla V_2(Y)}{V_1'(X_i)}$. Hence the effect of a change in X_i is to change all the components of the vector $\Pi(X_i,Y)$ proportionately. But we also have $\Pi(X_i,Y) = \frac{\nabla A(Y)X_i + \nabla B_i(Y)}{A(Y)}$. If changing X_i changes all components of $\Pi(X_i,Y)$ proportionately it must be that the gradient vectors $\nabla B_i(Y)$ and $\nabla A(Y)$ are proportionate. Therefore for some constants, k_i and c_i, $B_i(Y) = k_iA(Y) + c_i$. Ignoring the inessential constant c_i we have $V_i(X_i,Y) = A(Y)(X_i + k_i)$. Conversely, $A(Y)(X_i + k_i) = F(V_1(X_i) + V_2(Y))$ where $V_1(X_i) = \ln(X_i + k_i)$, $V_2(X_i) = \ln A(Y)$ and $F(V_1 + V_2) = e^{V_1+V_2}$.

(c) Result c is immediate from (a) and (b).

Proof of Theorem 2

Sufficient condition (i)

Suppose that $A(Y) > 0$ and $B_i(Y) \geqq 0$ for all $Y \geqq 0$. Then the function

$\ln A(Y)(X_i + \beta_i(Y)) = \ln A(Y) + \ln(X_i + \beta_i(Y))$ is a well defined monotone increasing function of $A(Y)(X_i + \beta_i(Y))$. Therefore $A(Y)(X_i + \beta_i(Y))$ will be quasi-concave if $\ln A(Y) + \ln(X_i + \beta_i(Y))$ is quasi-concave. If $\beta_i(Y)$ is a concave function and $A(Y)$ is a concave function, then the functions $A(Y)$ and $X_i + \beta_i(Y)$ are concave functions. Since an increasing concave function of a concave function is concave, $\ln A(Y)$ and $\ln(X_i + \beta_i(Y))$ are both concave functions. Since the sum of concave functions is concave, $\ln A(Y) + \ln(X_i + \beta_i(Y))$ is concave and therefore quasi-concave. It follows that $A(Y)(X_i + \beta_i(Y)) = A(Y)X_i + B_i(Y)$ is quasi-concave.

Sufficient condition (ii)

This condition is most easily demonstrated by examining the bordered Hessian of $U_i(X_i,Y)$. The principle minors of the bordered Hessian are $-A(Y)^2$ and $A(Y)A'(Y)(A'(Y)X_i + B_i'(Y)) - [A''(Y)X_i + B_i''(Y)]A(Y)^2$. A sufficient condition for quasi-concavity of $U(X_i,Y)$ is that the first principle minor be negative and the second be positive for all X_i and Y. If $A(Y) > 0$, then $-A(Y)^2 < 0$. If $A(Y) > 0$, $A'(Y) \geqq 0$, $B_i'(Y) \geqq 0$, $A''(Y) \leqq 0$ and $B_i''(Y) \leqq 0$ for all $Y \geqq 0$, then the second principle minor is positive for all $X_i > 0$ and $Y \geqq 0$. This establishes quasi-concavity.

Sufficient condition (iii)

A standard result in the theory of quasi-concave functions is that a function is quasi-concave if its Hessian matrix is negative definite on the null space of its gradient. In the case of $U(X_i,Y) = A(Y)X_i + B_i(Y)$, we have $\nabla U = (A(Y), (X_i + \beta_i(Y))\nabla A(Y) + A(Y)\nabla\beta_i(Y))$ and

$$\nabla^2 U = \begin{pmatrix} 0 & \nabla A(Y)^T \\ \nabla A(Y) & (X_i + \beta_i(Y))\nabla^2 A(Y) + 2\nabla A(Y)\nabla\beta_i(Y) + A(Y)\nabla^2\beta_i(Y) \end{pmatrix}$$

The condition for quasi-concavity states that for all $\gamma \varepsilon R$ and $c \varepsilon R^n$, such that $(\gamma, C^T)\nabla U = 0$, and $(\gamma, c) \neq 0$, $(\gamma, c^T)\nabla^2 U\begin{pmatrix}\gamma \\ c\end{pmatrix} < 0$. But $(\gamma, c^T)\nabla U = 0$ if and only if $\gamma = -\frac{1}{A(Y)}[(X_i + \beta_i(Y))\nabla A(Y)^T + A(Y)\nabla\beta_i(Y)^T]c$. Substituting this expression for γ into the quadratic expression, we see that U will be quasi-concave if $c^T[(X_i + \beta_i(Y))(\nabla^2 A(Y) - \frac{2}{A(Y)}\nabla A(Y)\nabla A(Y)^T) + A(Y)\nabla^2\beta_i(Y))]c < 0$ for all $c \varepsilon R^n$, $c \neq 0$ and for all (X_i, Y) such that $X_i > 0$, $Y \geq 0$. This expression will be negative for all such X_i, Y and c if the two quadratic forms $\nabla^2 A(Y) - \frac{2}{A(Y)}\nabla A(Y)\nabla A(Y)^T$ and $\beta_i(Y)[\nabla^2 A(Y) - \frac{2}{A(Y)}\nabla A(Y)\nabla A(Y)^T] + A(Y)\nabla^2\beta_i(Y)$ are negative semi-definite while at least one of them is negative definite.

Necessary condition (i) - If $A(Y)X_i + B_i(Y)$ is a quasi-concave function, then it must be quasi-concave in Y holding X_i constant at zero. This implies that $B_i(Y)$ is quasi-concave. Furthermore $A(Y)X_i + B_i(Y)$ must be quasi-concave in Y holding X_i constant for arbitrarily large values of X_i. Therefore A(Y) must also be quasi-concave.

Necessary condition (ii) - A necessary condition for a function to be quasi-concave is that its Hessian be negative semi-definite on the null space of its gradient. Reasoning as in the proof of sufficient condition (iii) we show that necessary condition (ii) is implied.

Proof of Corollary 1

If individual utility functions satisfy sufficient condition (i) or (ii) then the utility function $A(Y)X + B(Y)$ also satisfies condition (i) or (ii) respectively where we define $B(Y) = \sum_i B_i(Y)$. This is true since the sum of concave functions is concave. But if $A(Y)X + B(Y)$ satisfies either of these conditions it must, according to theorem 2, be concave.

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Proof of Theorem 3

Suppose that $(\overline{X},\overline{Y})$ maximizes $A(Y)X + \sum_i B_i(Y)$ on F and let $(\overline{X}_1,\ldots,\overline{X}_m,\overline{Y})$ be an outcome such that $\sum_i \overline{X}_i = \overline{X}$. If $(\overline{\overline{X}}_1,\ldots,\overline{\overline{X}}_m,\overline{\overline{Y}})$ is Pareto superior to $(\overline{X}_1,\ldots,\overline{X}_m,\overline{Y})$, then $A(\overline{\overline{Y}})\overline{\overline{X}}_i + B_i(\overline{\overline{Y}}) \geqq A(\overline{Y})\overline{X}_i + B_i(\overline{Y})$ for all i with strict inequality for some i. Therefore $A(\overline{\overline{Y}})\sum_i \overline{\overline{X}}_i + \sum_i B_i(\overline{\overline{Y}}) > A(\overline{Y})\overline{X} + \sum_i B_i(\overline{Y})$. Since $(\overline{X},\overline{Y})$ maximizes $A(Y)X + \sum_i B_i(Y)$ on F, it must be that $(\overline{\overline{X}},\overline{\overline{Y}}) \notin F$. Therefore $(\overline{\overline{X}}_1,\ldots,\overline{\overline{X}}_n,\overline{\overline{Y}})$ is not feasible. It follows that $(\overline{X}_1,\ldots,\overline{X}_n,\overline{Y})$ is Pareto optimal. This proves the first statement of the theorem.

Suppose that $(\overline{X}_1,\ldots,\overline{X}_n,\overline{Y})$ is an interior Pareto optimum and that $A(\overline{\overline{Y}})\overline{\overline{X}} + \sum_i B_i(\overline{\overline{Y}}) > A(\overline{Y})\sum_i \overline{X}_i + \sum_i B_i(\overline{Y})$ for some $(\overline{\overline{X}},\overline{\overline{Y}}) \in F$. Define $\overline{X} = \sum_i \overline{X}_i$ and for $0 \leqq \lambda \leqq 1$, define $X(\lambda) = \lambda\overline{X} + (1-\lambda)\overline{\overline{X}}$ and $Y(\lambda) = \lambda\overline{Y} + (1-\lambda)\overline{\overline{Y}}$. Since F is a convex set, $(X(\lambda),Y(\lambda)) \in F$ for all $\lambda \in [0,1]$. Since $A(Y)X + \sum_i B_i(Y)$ is assumed to be quasi-concave, $A(Y(\lambda))X(\lambda) + \sum_i B_i(Y(\lambda)) > A(\overline{Y})\overline{X} + \sum_i B_i(\overline{Y})$ for $0 \leqq \lambda < 1$. Let $\delta(\lambda) = A(Y(\lambda))X(\lambda) + \sum_i B_i(Y(\lambda)) - [A(\overline{Y})\overline{X} + \sum_i B_i(\overline{Y})]$. Then $\delta(\lambda) > 0$ for $0 \leqq \lambda < 1$. Define $\tilde{X}_i(\lambda) = \dfrac{A(\overline{Y})\overline{X}_i + B_i(\overline{Y}) - B_i(Y(\lambda)) + \frac{1}{n}\delta(\lambda)}{A(Y(\lambda))}$. Since $\overline{X}_i > 0$, $\delta(\lambda) > 0$, $B_i(\cdot)$ is a continuous function, and $A(Y) > 0$ for all Y_j it follows that for some $\lambda^* < 1$ but sufficiently close to one, $\tilde{X}_i(\lambda^*) > 0$ for all i. But $A(Y(\lambda^*))\tilde{X}_i(\lambda^*) + B_i(Y(\lambda^*)) = A(\overline{Y})\overline{X}_i + B_i(\overline{Y}) + \frac{1}{n}\delta(\lambda) > A(\overline{Y})\overline{X}_i + B_i(\overline{Y})$ for $0 \leqq \lambda < 1$. Therefore $(\tilde{X}_1(\lambda^*),\ldots,\tilde{X}_n(\lambda^*), Y(\lambda^*))$ is Pareto superior to $(\overline{X}_1,\ldots,\overline{X}_n,\overline{Y})$. Furthermore, $\sum_i \tilde{X}_i(\lambda^*) = X(\lambda^*)$. Since $(X(\lambda^*),Y(\lambda^*)) \in F$, the outcome $(\tilde{X}_1(\lambda^*),\ldots,\tilde{X}_n(\lambda^*),Y(\lambda^*))$ is feasible as well as Pareto superior to $(\overline{X}_1,\ldots,\overline{X}_n,\overline{Y})$. This contradicts the assumption that $(\overline{X}_1,\ldots,\overline{X}_n,\overline{Y})$ is Pareto optimal. Therefore if $(\overline{X}_1,\ldots,\overline{X}_n,\overline{Y})$ is an interior Pareto optimum, there can not be an $(\overline{\overline{X}},\overline{\overline{Y}}) \in F$ such that $A(\overline{\overline{Y}})\overline{\overline{X}} + \sum_i B_i(\overline{\overline{Y}}) > A(\overline{Y})\overline{X} + \sum_i B_i(\overline{Y})$. This proves the second assertion of the theorem.

Q.E.D.

FOOTNOTES

1. Examples of such studies are Borcherding and Deacon (1972) and Bergstrom and Goodman (1973). Other similar studies are reviewed by Inman (1979).

2. This duality is discussed explicitly in Bergstrom and Cornes (1981).

3. Throughout this paper we confine our attention to this case. Possibly the private good is an aggregate. Aggregation would be possible either if relative prices of private goods were constant throughout the analysis or if homothetic separability permitted aggregation under varying prices.

4. The natural extension of this relation to the case where there is more than one private good would require that $(X_1',\ldots,X_n',Y)$ entail a Pareto efficient distribution of the total vector $\Sigma X_i'$ of private goods.

5. After this paper was written, we discovered a recent paper by Joseph Sicilian which reports results very similar to the results of this section.

6. Conn (1980) has shown a different way in which the Clarke-Groves-Loeb results can be extended beyond the quasi-linear case.

7. This does not seem unreasonable since if A(Y) is common to everyone's utility function, anyone could discover A(Y) by introspection. If one wished, however, it would not be difficult to devise a mechanism in which honest revelation of A(Y) is a Nash equilibrium and honest revelation of $B_i(Y)$ is dominant strategy.

8. For expository simplicity, we extend the Clarke tax. The Groves-Loeb tax includes the Clarke tax as a special case. Loeb (1976). The generalization of the Groves-Loeb tax is a straightforward extension of the argument used here.

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Figure 1

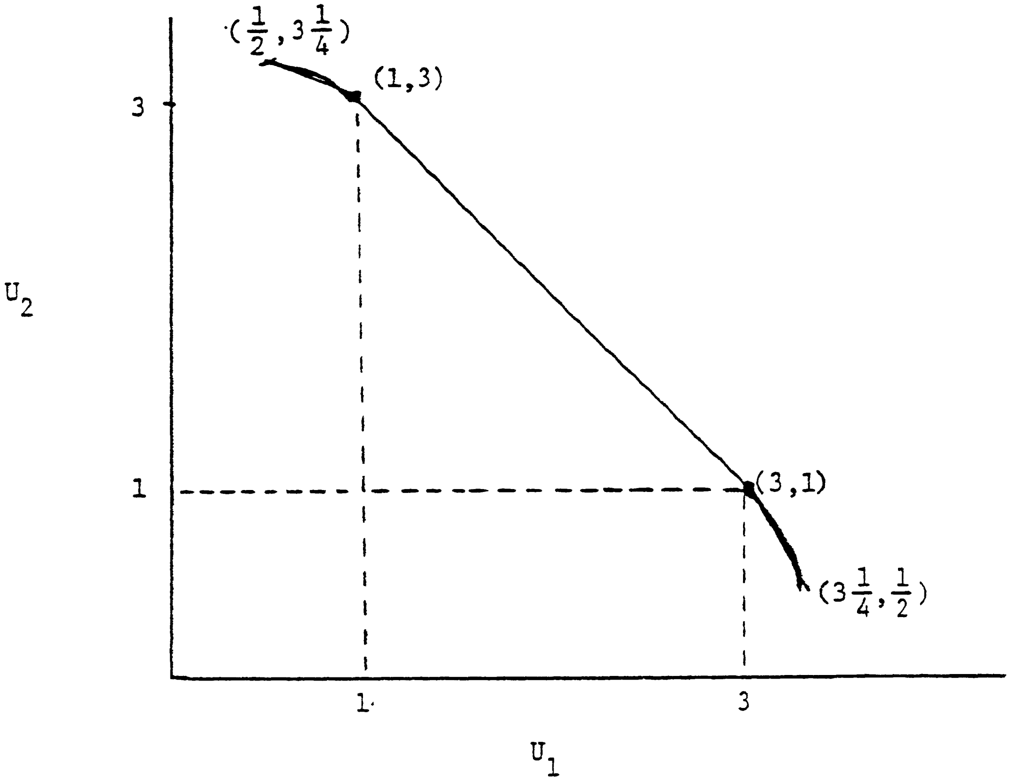

Figure 2

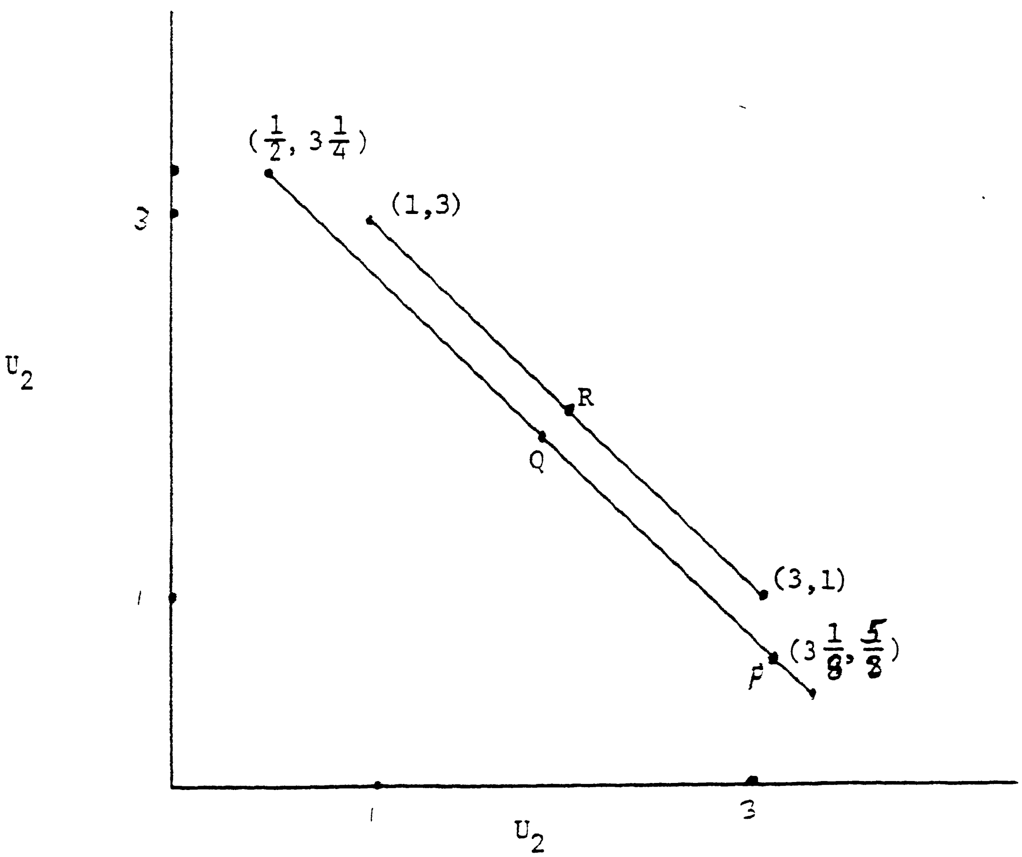

Figure 3

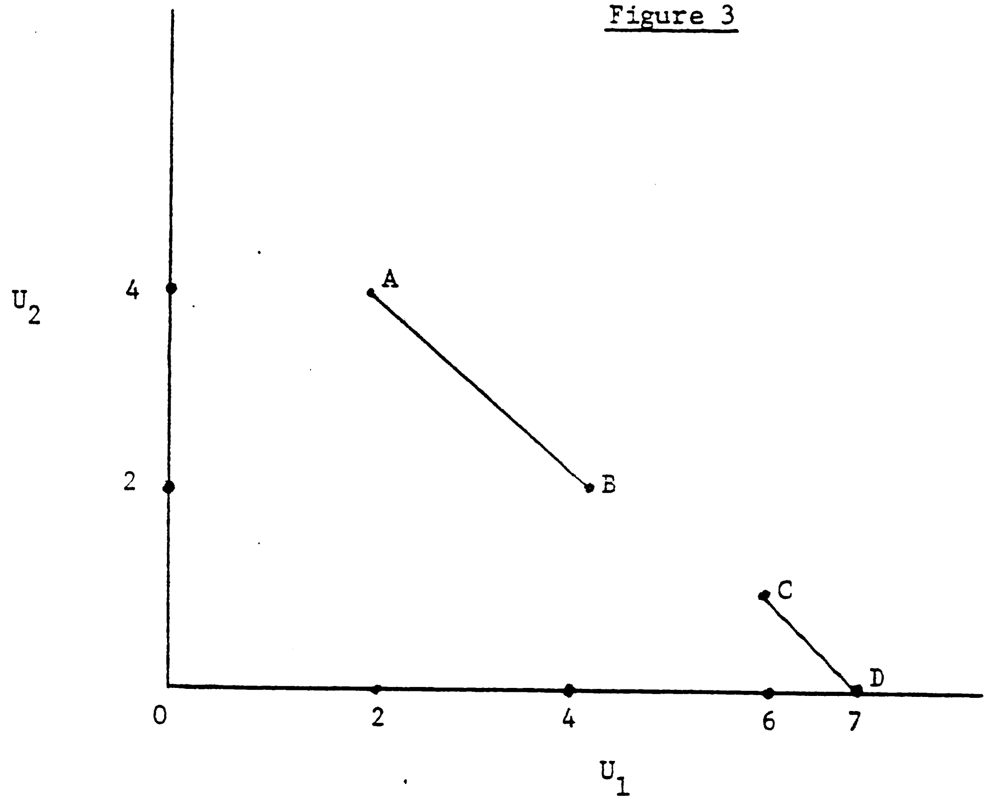

Figure 4