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Three Notes on Nash, Cournot and Wald Equilibria

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WHEN ARE NASH EQUILIBRIA INDEPENDENT OF THE DISTRIBUTION OF AGENTS' CHARACTERISTICS?

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Abstract. We present some examples of Nash equilibria that are independent of the distribution of some parameter across the economic agents and describe a general theorem that characterizes this phenomenon.

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In several Nash games to be described below the 'outcome' of the game turns out to be independent of the distribution of some characteristic across the agents. The easiest way to describe what we mean is through some examples.

Example 1. Cournot Equilibrium

Let y_i be the output of firm i, c_i its constant marginal cost, Y industry output, and $P(Y)$ the industry price. Then the first order conditions for a Cournot–Nash equilibrium can be written as:

$$P(Y) + P'(Y)y_i - c_i = 0$$

Summing these equations across the n firms we have:

$$nP(Y) + P'(Y)Y = \sum_{i=1}^{n} c_i$$

Making the weak assumption that the left hand side of this equation is downward sloping, there will be a unique Y that solves this equation, which depends only on the *sum* of the marginal costs, not on its distribution across the firms. The observation that output and price in a Cournot industry are independent of the distribution of marginal costs has undoubtably been noted and used several times in the literature. See for example, Dixit and Stern (1982) or Katz (1984).

We can show the precise way that individual behavior changes when the distribution of costs changes. Simply note that when costs change by (Δc_i), then if each firm changes its output by:

$$\Delta y_i = -\frac{\Delta c_i}{P'(Y)}$$

it will continue to satisfy the appropriate first order conditions, assuming of course, that $y_i + \Delta y_i > 0$, i.e., that we maintain an interior solution.

Note that the same independence result holds in any conjectural variations model, as long as all firms have the *same* conjectural variation and we consider only interior equilibria.

Example 2. 'Oil'igopoly

Loury (1983) has presented an interesting model of oligopoly involving exhaustible resources. Suppose that we have n firms that each own some amount R_i of an exhaustible resource. Each firm acts as a Nash competitor. It turns out that in this model, if each firm supplies the resource in all of the periods, then the entire time path of the market price is independent of the distribution of R_i

across the firms. Loury (1983) establishes this result in a continuous time model and investigates the conditions under which this 'interior' equilibrium will occur. Here we provide a somewhat different proof of his result in a discrete time model, and extend Loury's results by showing show precisely which sorts of redistribution will result in an interior equilibrium and how each individual firm's behavior will change.

Assuming zero extraction costs and an interior solution, the first order conditions for firm i are:

$$\alpha^t[P(Y^t) + P'(Y^t)y_i^t] = P(Y^0) + P'(Y^0)(R_i - \sum_{t=1}^{T} y_i^t) \quad \text{for } t = 1, ..., T.$$

These conditions state that the marginal revenue in each time period must equal the marginal revenue in period 0. Summing these equations over all firms for each time period t we have:

$$\alpha^t[nP(Y^t) + P'(Y^t)Y^t] = nP(Y^0) + P'(Y^0)[\sum_{i=1}^{n} R_i - \sum_{t=1}^{T} Y^t].$$

Under appropriate monotonicity assumptions, there will be a unique Y^t that solves this equation for each t, and it depends only on the total endowment of resources.

We can also show how the output of each firm changes when the distribution of resources changes but the analysis is not quite as straightforward as in the previous example. Let (ΔR_i) be a change in resource ownership that keeps the total amount of the resource fixed, and let (Δy_i^t) be the associated change in firm i's output in period t. Then in order to satisfy the first order conditions given above we must have:

$$\alpha^t[P(Y^t) + P'(Y^t)(y_i^t + \Delta y_i^t)] = P(Y^0) + P'(Y^0)(y_i^0 + \Delta y_i^0) \quad \text{for } t = 1, ..., T.$$

Solving for Δy_i^t as a function of Δy_0^t gives us:

$$\Delta y_i^t = P'(Y^0)\Delta y_i^0/\alpha^t P'(Y^t) \qquad \text{for } t = 1, \ldots, T.$$

Now sum these expressions for Δy_i^t over $t = 0, \ldots, T$ and set the result equal to ΔR_i. Solving for Δy_i^0 gives us:

$$\Delta y_i^0 = \Delta R_i/[1 + P'(Y^0)]\sum_{t=1}^{T}[1/\alpha^t P'(Y^t)]$$

Note that these equations are linear in ΔR_i. Thus we can easily choose values of ΔR_i such that $y_i^t + \Delta y_i^t > 0$ for $t = 1, \ldots, T$ and $i = 1, \ldots, n$ and $\sum_1^n \Delta R_i = 0$.

Example 3. Private Provision of a Public Good

Warr (1983) has recently presented an interesting neutrality result which has been extended in various directions by Bergstrom, Blume, and Varian (1983). Here we describe Warr's original result, using the setup of the latter paper. Suppose that each agent's utility depends on his private consumption x_i and some public good, G. The amount of the public good is determined by private donations; each agent has some initial wealth level w_i which he uses to finance his gift and his private consumption. Letting g_i be the gift of agent i, G be the sum of the gifts, and G_{-i} be the sum of the gifts of all agents except for agent i we can pose agent $i's$ maximization problem as:

$$\max u_i(x_i, g_i + G_{-i})$$
$$\text{s.t. } x_i + g_i = w_i$$
$$g_i \geq 0$$

It follows from the structure of this maximization problem that an individual's optimal gift will take the form:

$$g_i = f_i(w_i + G_{-i}) - G_{-i}$$

where $f_i(\cdot)$ is the individual's unconstrained demand for the public good as a function of wealth.

Warr (1983) showed that in this sort of model redistributions of wealth among agents who actually contributed to the public good would not change the equilibrium level of the public good. It turns out that each dollar given to or taken from an individual results in a one–for–one increase or reduction in his or her contribution to the public good in equilibrium. Since the total dollars taken equal the total dollars given in a wealth redistribution, the aggregate amount of the public good will remain unchanged.

Kemp (1983) extended Warr's argument to the case of multiple public goods. Bergstrom, Blume, and Varian (1983) provide a somewhat different proof and examine other sorts of comparative statics in this class of models.

2. A General Result

The examples given above are all members of a general class of Nash equilibria which we characterize in the following theorem.

THEOREM 1. Let x_i be agent $i's$ choice, X the sum of the choices, c_i some characteristic of agent i, and C the sum of the characteristics. Assume that the Nash equilibrium values of x_i can be expressed as the solution to n equations of the form:

$$x_i = f_i(c_i, X) \quad \text{for} \quad i = 1, \ldots, n$$

where each f_i is a continuous function. Then a sufficient condition for the equilibrium value of X to be independent of the distribution of the agents' characteristics is that each f_i has the form:

$$f_i(c_i, X) = a_i(X) + b(X)c_i.$$

If $n \geq 3$ this is also necessary. Thus a change (Δc_i) that keeps C constant will result in an equilibrium response of $\Delta x_i = -b(X)\Delta c_i$.

Proof. Sufficiency is obvious. For notational convenience we will prove necessity in the case of $n = 3$; the extension to larger n follows trivially.

Fix C and sum the equations to get:

$$X = f_1(c_1, X) + f_2(c_2, X) + f_3(C - c_1 - c_2, X)$$

As c_1 and c_2 change, c_3 adjusts to keep C constant. By hypothesis, since C is fixed, X must be fixed. Hence this can be viewed as an equation in c_1 and c_2 alone. An equation of this form is known as a Pexider functional equation (Aczel (1966), p. 141). The only continuous solution to such an equation is of the form:

$$f_i(c_i, X) = a_i(X) + b(X)c_i$$

which is what we wanted to show. []

It is not hard to see that the Cournot examples are of the required form. The public good example is a bit more obscure. Take the expression for g_i and add G_{-i} to each side to get:

$$G = f_i(w_i + G_{-i}).$$

Let $\phi_i(\cdot)$ be the inverse of $f_i(\cdot)$. Applying this inverse to each side of the above express and subtracting G from each side of the result gives us:

$$g_i = w_i + G - \phi_i(G)$$

which clearly has the required form. This example shows that some rearrangement of the "natural" equilibrium conditions may be necessary in order to apply our result.

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TWO REMARKS ON COURNOT EQUILIBRIA

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Abstract. We describe two simple results in the theory of Cournot equilibria. The first has to do with the effect of taxation in a Cournot industry and the second describes under what conditions a Cournot equilibrum will implicitly maximize an objective function.

TWO REMARKS ON COURNOT EQUILIBRIA

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In this note we describe a few simple results having to do with Cournot equilibria. These are probably known, but are not *well* known, and they seem useful enough to be worth spelling out.

1. Taxation in a Cournot Industry

Let y_i be the output of firm i, c_i its constant marginal cost, Y industry output, and $P(Y)$ the industry price. Then the first order conditions for a Cournot–Nash equilibrium can be written as:

$$P(Y) + P'(Y)y_i - c_i = 0 \tag{1}$$

Summing these equations across the n firms we have:

$$nP(Y) + P'(Y)Y = \sum_{i=1}^{n} c_i \tag{2}$$

Making the weak assumption that the left hand side of this equation is downward sloping, there will be a unique Y that solves this equation, which depends only on the *sum* of the marginal costs, not on their distribution across the firms. The observation that output and price in a Cournot industry is independent of the distribution of marginal costs has undoubtably been noted and used several times in the literature. See for example, Dixit and Stern (1982), Katz (1984), Loury (1983) or Bergstrom and Varian (1985).

Note that the same independence result holds in any conjectural variations model, as long as all firms have the *same* conjectural variation and we consider only *interior* equilibria. The latter condition becomes increasingly important as the market equilibrium approaches a competitive structure, since in that case only the low cost producer will produce a positive amount.

Here we consider an application of this result to a taxation problem. Suppose that each of the n firms in the industry faces a quantity tax t_i. Then this tax is just the same as a marginal cost, so we can apply the above result to show that the equilibrium output and price is independent of the distribution of taxes across the firms. Furthermore, if we make a tax change (Δt_i) that preserves the sum of the taxes, we must satisfy the first order condition:

$$P(Y) + P'(Y)(y_i + \Delta y_i) = c_i + t_i + \Delta t_i$$

which implies that

$$\Delta y_i = -\Delta t_i / P'(Y) \tag{3}$$

Thus we have an exact expression for the impact of a tax change on the equilibrium output of each firm.

It is of interest to consider how tax revenue is affected by this sort of operation. It is most convenient to analyze this in the case of identical costs. Making this assumption, we take equation (1), multiply through by y_i, and sum to get:

$$P(Y)Y - cY + P'(Y)\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} t_i y_i \tag{4}$$

The first two terms on the left hand side are industry profits, which we denote by Π. The term on the right hand side is total tax revenue. Using the standard variance identity, we can rewrite this equation as:

$$R = \Pi + P'(Y)[n\sigma_y^2 + Y^2/n^2] \tag{5}$$

where σ_y^2 is the variance of output across firms. Using (1), we can solve for y_i:

$$y_i = \frac{c + t_i - P(Y)}{P'(Y)} \tag{6}$$

which implies that $\sigma_y^2 = \sigma_t^2/P'(Y)^2$. Plugging this back into (5) we have the final result:

$$R = \Pi + n\frac{\sigma_t^2}{P'(Y)} + \frac{P'(Y)Y^2}{n^2} \tag{7}$$

As we change the distribution of taxes across firms, the industry price, output and profits remain constant but tax revenue will change. The above formula shows that tax revenue is a *decreasing* function of the variance of the taxes across the firms in the industry. This shows for example that tax revenue is maximized when all firms face the same tax rates, as economic intuition would suggest.

2. What Does a Cournot Equilibrium Maximize?

This question was first raised and answered by Spence (1976) in the context of his monopolistic competition model. It was later examined by Loury (1983) in a model of intertemporal Cournot equilibria. As far as we know this question has not been addressed for the standard Cournot model, although it may well be part of the folklore.

Let us first consider the case of identical cost functions. In what follows we will assume that $P'(Y) < 0$, $P''(Y) < 0$, and $c''(y) > 0$, although weaker assumptions will work for some of the results. In this case a symmetric Cournot equilibrium satisfies the first and second order conditions:

$$P(Y) + P'(Y)y - c'(y) = 0 \tag{8}$$

$$2P'(Y) + P''(Y)y - c''(y) < 0 \tag{9}$$

where $y = Y/n$. Under our assumptions the second order condition is satisfied so there is at most one symmetric Cournot equilibrium.

Now consider the function:

$$F(Y) = (n-1)[U(Y) - nc(Y/n)] + P(Y)Y - nc(Y/n)$$

where

$$U(Y) = \int_0^Y P(t)dt.$$

That is, $U(Y)$ is just the area beneath the market demand curve — also known as consumers' surplus.

Then a local maximum of $F(Y)$ is characterized by the first and second order conditions:

$$(n-1)[P(Y) - c'(Y/n)] + P(Y) + P'(Y)Y - c'(Y/n) = 0 \tag{10}$$

$$(n+1)P'(Y) + P''(Y)Y - c''(Y/n) < 0 \tag{11}$$

Again our assumptions imply that there will be a unique maximum that satisfies these conditions. But by inspection, equations (8) and (10) are the same — the industry output in a symmetric Cournot equilibrium maximizes the function $F(Y)$.

It is not necessary to give the $U(Y)$ term a welfare interpretation in this context; all that matters is that its derivative is the inverse demand curve. However, the welfare interpretation is suggestive. The first term is simply the net consumers' surplus — what society should be maximizing — and the second term is profits. If there is only one firm in the market — a monopoly — industry output is determined entirely by profit maximization. As the number of firms increases, more and more weight is given to the welfare term as compared to the profit term, and as n approaches infinity, industry structure approaches pure competition and thus the socially optimal level of output.

How far can we generalize this result? One interesting direction of extension would be to nonsymmetric equilibria. Suppose there were some function $f(y_1, \ldots, y_n)$ such that the singular points of this function were the first order conditions that characterize a Nash equilibrium; i.e.:

$$\frac{\partial f(y_1, \ldots, y_n)}{\partial y_i} = P(Y) + P'(Y)y_i - c_i'(y_i) = 0.$$

Then applying the standard integrability conditions, we would have to have:

$$\frac{\partial^2 f}{\partial y_i \partial y_j} = P'(Y) + P''(Y)y_i = P'(Y) + P''(Y)y_j = \frac{\partial^2 f}{\partial y_j \partial y_i}.$$

It follows that we must have $y_i = y_j = Y/n$. This observation implies that we must restrict ourselves to symmetric equilibria.

However we can relax the assumption of zero conjectural variation. If we restrict ourselves to symmetric equilibria and let $dY/dy = 1 + \gamma$ be the conjectural variation, then it is straightforward to show that industry output maximizes the following expression:

$$(n-1)[U(Y) - nc(Y/n)] + P(Y)Y - nc(Y/n) + \gamma[P(Y)Y - U(Y)]$$

This interpretation of this expression is similar to that given above, but not as natural, at least to us.

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ADDITIVE UTILITY AND GROSS SUBSTITUTES

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Abstract. Abraham Wald has shown that if a consumer has an additively separable utility function of the form $\sum_{i=1}^{k} u_i(x_i)$ and this utility function satisfies the condition that $u_i''(x)x/u_i'(x) < 1$ for all i and x then the consumer's demand function will exhibit the gross substitutes property. If all consumers satisfy this condition it follows that the general equilibrium is unique. I provide a more accessible proof of this proposition and discuss its relevance to certain general equilibrium models.

ADDITIVE UTILITY AND GROSS SUBSTITUTES

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We will say that utility function for k goods is *additively separable* if it can be written in the form:

$$U(x_1,\ldots,x_k)=\sum_{i=1}^{k}u_i(x_i).$$

Gorman (1968), Debreu (1959) and others have described conditions under which utility has this additive structure; Blackorby, Primont and Russell (1978) give a complete survey.

Two important examples of this kind of utility are vonNeuman–Morgenstern expected utility functions and time–additive intertemporal utility functions. In these cases, utility is often assumed to be state or time independent so that $u_i(x_i)=\pi_i u(x_i)$.

Given an additive utility function we will define parameters $\rho_i(x)$ by

$$\rho_i(x)=-\frac{u_i''(x)x}{u_i'(x)}.$$

In the vonNeuman–Morgenstern case, $\rho_i=\rho$ is known as the Arrow–Pratt measure of relative risk aversion. This parameter often arises in comparative statics calculations; see Arrow (1965) and the exercises in Diamond and Rothschild (1978), section 7. It turns out that the value of this parameter determines whether the demand functions for this kind of separable utility are gross substitutes or gross complements. This in turn has interesting implications for uniqueness of general equilibrium.

The original proof of this is due to Abraham Wald (1936). However, although most economic theorists recognize that Wald showed that a gross substitutes condition implied uniqueness of equilibrium, few realize that he also provided a condition for gross substitutes to occur. Perhaps this is due to the abbreviated nature of Wald's argument. In any event, it seems worthwhile to present a modern proof of Wald's theorem, as it seems to be quite useful in certain classes of general equilibrium models.

1. Individual Comparative Statics

Consider the utility maximization problem:

$$\max\sum_{i=1}^{k}u_i(x_i)$$

$$\text{s.t.}\sum_{i=1}^{k}p_ix_i=m.$$

and the resulting demand functions $x_i(p,m)$ for $i=1,\ldots,k$.

Theorem. *If utility is additively separable and $u_i'(x)>0$ and $u_i''(x)<0$ for all $i=1,\ldots k$ then:*

1. All goods are normal goods.

2. All own price effects are negative.

3. All goods are Hicksian substitutes.

4. If $\rho_i(x) < 1$ for all x then all the other goods are gross substitutes for good i. If $\rho_i(x) > 1$ for all x then all the other goods are gross complements for good i.

Proof. The first three properties are well known; indeed they are often used as exercises in standard textbooks. See Silberberg (19??), page ?? or Varian (1984), Exercise 3.22. Rader (1972) also devotes considerable discussion to properties of additively separable utlity and provides proofs of some of these results. However, we include a proof for completeness.

In order to prove statement 1, write the first order conditions as:

$$u_i'(x_i) = \lambda p_i.$$

Then increasing m will change λ so all the x_i terms must change in the same direction. In order to satisfy the budget constraint, the x_i terms must therefore all increase.

Statement 2 follows immediately from Slutsky's equation. In order to prove statement 3 we write the first order conditions as:

$$\frac{u_1'(x_1)}{p_1} = \frac{u_2'(x_2)}{p_2} = \ldots = \frac{u_k'(x_k)}{p_k}.$$

Now increasing p_1 may increase or decrease the first fraction, but all of the other fractions must change in the same direction. Thus the compensated cross price effect must be the same for all goods. Since the substitution matrix is necessarily singular, each row must have at least one positive term, and therefore all the compensated cross price effects are positive.

In order to prove statement 4 we first note that all the gross cross price effects have the same sign: for if we increase p_1, then all of the x_j terms, for $j \neq i$ must move in the same direction to satisfy the first order conditions. So it suffices to examine the sign of $\partial x_2 / \partial p_1$ for example.

To this end, it is convenient to define the matrices:

$$A_2 = \begin{pmatrix} u_1'' & \lambda & -p_1 \\ 0 & 0 & -p_2 \\ -p_1 & x_1 & 0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} u_1'' & \lambda & 0 & -p_1 \\ 0 & 0 & 0 & -p_2 \\ 0 & 0 & u_3'' & -p_3 \\ -p_1 & x_1 & -p_3 & 0 \end{pmatrix}$$

and so on. It follows from a standard comparative statics exercise that:

$$\text{sign } \frac{\partial x_2}{\partial p_1} = \text{sign } (-1)^k |A_k|.$$

I claim the the sign of the determinant $|A_k|$ is determined by ρ_1. We prove this by induction, starting with the case of $k = 2$. Expanding the determinant on the second row, we have:

$$|A_2| = p_2(u_1'' x_1 + \lambda p_1).$$

Substituting from the first order condition, we see that the sign of the right hand side depends on the sign of $u_1''x_1 + u_1'$ which gives us the desired result.

Now assume that the statement is true for k and examine the matrix for A_{k+1}. Expanding by principal minors on row $k+1$ we have:

$$|A_{k+1}| = u_k''|A_k| + |B_k|$$

where B_k has row of zeroes in it. Thus $|B_k| = 0$ and the result is proved. []

Remark 1. In the case of vonNeuman–Morgenstern utility, it is commonly assumed that ρ is greater than one. It follows from the above that in this case all contingent consumption goods are gross complements, rather than gross substitutes as one might have thought.

Remark 2. Blomquvist (1985) has generalized statement 3 to the case where the x_i terms are vectors.

Remark 3. Fisher (1972) examined restrictions on the utility function that would give rise to gross substitutes. However, he did not explicitly consider the case of additive utility. Rader (1972) gives an alternative form of the Wald condition along with a detailed discussion of the comparative statics in an additive utility model.

2. Uniqueness of General Equilibrium

In a general equilibrium model, wealth is given by the value of the endowment vector; i.e., $m = \sum_{i=1}^{k} p_i\omega_i$. Thus

$$\frac{\partial x_i(p, p\cdot\omega)}{\partial p_j} = \frac{\partial x_i(p,m)}{\partial p_j} + \omega_j \frac{\partial x_i(p,m)}{\partial m}.$$

We have already shown that all goods are normal goods, so the second term on the right hand side is necessarily positive. If all of the off diagonal terms in the substitution matrix are positive, then adding positive terms will not change their sign. What about the diagonal terms? Since demand functions are homogeneous of degree 0 in prices, Euler's law implies that for each i

$$\sum_{j=1}^{k} \frac{\partial x_i}{\partial p_j} p_j = 0$$

Thus each row must contain at least one negative sign, which can only be the diagonal term.

We have shown that if each consumer has $\rho_i < 1$, each consumer's demand function will exhibit the gross substitutes property. It follows that the aggregate excess demand function exhibits the same property, and it is well-known that this implies equilibrium is unique. (The original proof was by Wald (1936); see Arrow and Hahn (19??) for further discussion and a proof.)

Kehoe (1984) has shown that the gross substitutes property need only hold at equilibrium in order to ensure uniqueness; thus the restriction on ρ_i need only hold in equilibrium as well.

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