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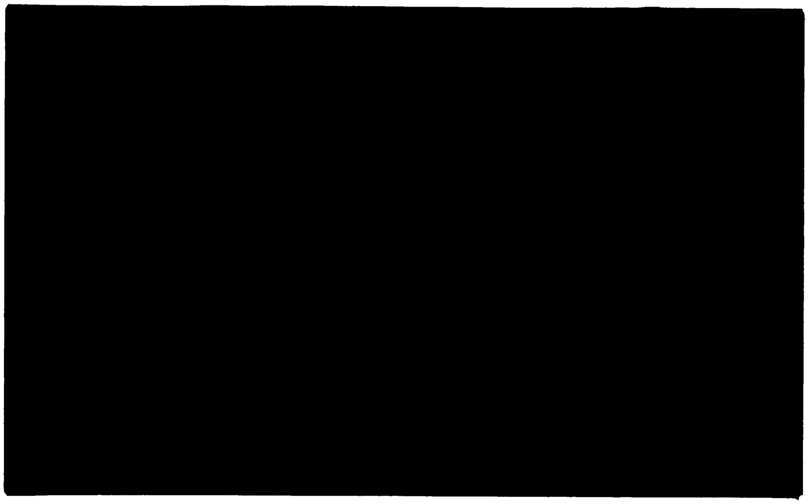

# WHEN DO MARKET GAMES HAVE TRANSFERABLE UTILITY?

by

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Game theorists like to work with transferable utility. In fact, several familiar notions of game theoretic equilibrium are either undefined or generalize in unconvincing ways when utility is not transferable.[1] Economists are suspicious of transferable utility because it is not in general possible to model a well-behaved exchange economy as a transferable utility game. It is well-known that in an exchange economy if preferences are of the quasi-linear form, $u_i(x_1,...,x_m) = x_1 + f_i(x_2,...,x_m)$, for all i, then there is transferable utility over a range of utility distributions.[2] However, quasi-linear utility implies that individual demands for all goods except one are independent of income. For many economic problems, this is not an attractive assumption.

The question of whether there are other, more

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[1]The kernel and the nucleolus of a game have not been defined for games without transferable utility. (Owen (1982)). Shapley (1969) proposed an extension of the Shapley value concept to games without transferable utility. However, Roth (1980) and Shafer (1980) have demonstrated that this extension is not in general a satisfactory solution concept.

[2]Kaneko (1976), shows if there is quasilinear utility, then the utility possibility frontier is linear. He <u>defines</u> there to be "transferable utility" whenever utility functions are of the form that we call "quasilinear". In view of the results of our paper, it seems clear that this nomenclature is inappropriate, since quasilinear utility is sufficient but not necessary for "transferable utility" in the sense of game theory.

generates a game with transferable utility whenever aggregate demand is determined by prices and aggregate income, independently of income distribution. As is well-known in consumer theory, this is precisely when indirect utiity is of the Gorman polar form over an appropriate domain of prices and incomes. We also prove a converse result. If there is transferable utility in a neighborhood of a Pareto efficient allocation then it must be that in this neighborhood, at competitive prices, aggregate demand will depend only on aggregate income and not on income distribution. This implies that at least locally, utility must be representable in the Gorman polar form.

## 1. PRELIMINARIES

We will deal with an exchange economy in which there is a set T of n consumers and where there are k commodities. In this paper we adopt the following notational conventions. Let $x \in R_+^{nk}$ be an allocation. Then $x = (x_1,...,x_n)$ where $x_i \in R_+^k$ is consumer i's commodity bundle and $X = \Sigma x_i$ is the aggregate consumption bundle. We let $u(x) = (u_1(x_1),...,u_n(x_n)) \in R^n$ denote the utility distribution resulting from allocation x. A similar convention applies to allocations denoted y and z.

Consumer preferences will be assumed to satisfy the following condition:

Condition P Preferences of all consumers are reflexive, is global transferable utility is where preferences are homothetic and identical. Since there are interesting cases where there is not global transferable utility but where there is transferable utility over a substantial range of utility distributions, we find it useful to develop the weaker notion of "local transferable utility".

Let x be a Pareto efficient allocation. There is <u>local transferable utility</u> at x if there is a neighborhood $N_u$ of $u(x)$ in $R^n$ and a neighborhood $N_X$ of X in $R^k_+$ such that for any allocation y, if $Y \in N_X$, then $N_u \cap W(T,Y) = \{u \in N_u | \Sigma u_i = \Sigma u_i(y_i)\}$.

Indirect utility functions are of the Gorman polar form if they can be written as $v_i(p,m_i) = \alpha(p)m_i + \beta_i(p)$ for all i. When indirect utility is of the Gorman form, all consumers have parallel linear Engle curves. Therefore aggregate demand is determined by aggregate income and the price vector, independently of income distribution. Preferences for which indirect utility is of the Gorman form over some domain include quasi-linear utility, but they also include interesting cases where income elasticities are non-zero.

In the next section we assume that indirect utility is of the Gorman polar form for all price-income configurations that allow every consumer to purchase some bundle that he likes as well as his initial endowment, $\omega_i$. Formally this condition is stated as follows.

$\varepsilon$ **M**, the allocation $x(m) = (h_1(p^*,m_1),\ldots,h_n(p^*,m_n))$ is a competitive equilibrium and thus a Pareto optimum. Therefore if $m \varepsilon$ **M**, the utility allocation $u(x(m))$ is on the utility possibility frontier. But $\Sigma u_i(x_i(m_i)) = \Sigma v_i(p^*,m_i) = \alpha(p^*)\Sigma m_i + \Sigma\beta_i(p^*)$. Therefore the portion of the utility possibility frontier consisting of allocations that are Pareto superior to $\omega$ must be the set $\{(u_1,\ldots,u_n)|\Sigma u_i = \alpha(p^*)(p^*\Sigma\omega_i) + \Sigma\beta_i(p^*)$ and $u_i \geq u_i(\omega_i)$ for all $i\}$. This set is equivalent to a simplex under appropriate linear scaling of utility. QED

Theorem 1 can be applied to the subeconomy consisting only of members of S where $S \subset T$. This enables us to claim the following:

<u>Corollary 1</u> If Conditions P and G hold, then there exist utility functions such that for every $S \subset T$, the utility possibility frontier for S generated by $\sum_{i\varepsilon S} \omega_i$ is a simplex over the range of utility distributions corresponding to allocations that are Pareto superior to $\omega$.

## 3. TRANSFERABLE UTILITY IMPLIES GORMAN POLAR FORM (LOCALLY)

In general if there are convex preferences and if an allocation x is Pareto optimal, then there exists a price vector p* such that if x is a competitive equilibrium allocation at prices p* and if y is Pareto superior to x,

(ii) If $\Sigma u_i(y_i) > \Sigma u_i(x_i^*)$ then $p^*Y > p^*X^*$

Proof.

Since there is local transferable utility at $x^*$, there are neighborhoods $N_X^*$ of $X^*$ and $N_u$ of $u(x^*)$ such that for any allocation y, if $Y \varepsilon N_X^*$ then $N_u \cap W(T,Y) = \{u \varepsilon N_u | \Sigma u_i = \Sigma u_i(y_i)\}$. Consider an allocation y where $Y \varepsilon N_X^*$, $u(y) \varepsilon N_u$ and $\Sigma u_i(y_i) \geq \Sigma u_i(x_i^*)$. Let $\lambda = (\Sigma u_i(y_i))/(\Sigma u_i(x_i^*)) \geq 1$ and let $u' = \lambda u(x^*)$. Then $\Sigma u_i' = \Sigma u_i(x_i^*)$ and $u' \varepsilon N_u$. Therefore, since there is local transferable utility at $x^*$, there is an allocation z such that $Z = X^*$ and $u_i(z_i) = u_i' \geq u_i(x_i^*)$ for all i. But $x^*$ is a competitive equilibrium at prices $p^*$. Therefore $p^*z_i \geq p^*x_i^*$ for all i and hence $p^*Z \geq p^*X^*$. This proves conclusion (i) of the theorem.

Since the functions $u_i(\S)$ are strictly monotonic it must be that $p^* >> 0$. A simple continuity argument then establishes conclusion (ii) as a consequence of conclusion (i). QED

Lemma 2 Let preferences satisfy Condition P and let $x^*$ be a competitive equilibrium at prices $p^*$. If there is local transferable utility at $x^*$, then there exists a neighborhood $N_u$ of $u(x^*)$ such that if y is a Pareto optimal allocation with $Y=X^*$ and $u(y) \varepsilon N_u$, then y is a competitive equilibrium at the same prices $p^*$.

Proof.

According to Lemma 1, there is a neighborhood $N_u$ of $u(x^*)$ such that if z is an allocation for which $u(z) \varepsilon N_u$, Z the allocation $(h_1(p*,m_1),...,h_n(p*,m_n))$. Since demand is a continuous function of income and since utility functions are continuous, there exists a neighborhood $N_m$ of m* such that for all $m \varepsilon N_m$, $u(x*(m)) \varepsilon N_u$. Suppose that $m \varepsilon N_m$ and $\Sigma m_i = \Sigma m_i^*$. Let $u' = \lambda u(x(m))$ where $\lambda = (\Sigma u_i(x_i^*))/(\Sigma u_i(x(m)))$. Then $\Sigma u_i' = \Sigma u_i(x_i^*)$ and since $u(x(m)) \varepsilon N_u$, $u' \varepsilon N_u$. Since there is transferable utility at x*, it must be that there exists a Pareto optimal allocation y such that Y = X* and $u_i(y_i) = u_i'$ for all i. According to Lemma 2, y must be a competitive equilibrium at prices p*. Therefore $y_i = h_i(p*,p*y_i)$ for all i. But $u_i(y_i) = \lambda u_i(x_i(m))$ and $x_i(m) = h_i(p*,m_i)$. Either $\lambda \leq 1$ in which case $p*y_i \leq p*x_i(m) = m_i$ for all i, or $\lambda \geq 1$ in which case $p*y_i \geq p*x_i(m) = m_i$ for all i. Since $\Sigma p*y_i = p*Y = p*X* = \Sigma m_i^* = \Sigma m_i$, it must be that $p*y_i = m_i$ for all i. Therefore $y_i = h_i(p*,m_i)$ for all i. This implies that $\Sigma h_i(p*,m_i) = Y = X* = \Sigma h_i(p*,m_i*)$. QED

<u>Theorem 2</u> Let preferences satisfy Condition P, let x* be a Pareto optimal allocation and suppose that there is a neighborhood $N_{x*}$ of x* such that there is local transferable utility at every Pareto optimal allocation in $N_{x*}$. Then there is a price vector p* such that x* is a competitive equilibrium at prices p* with income distribution $m* = (p*x_1^*,...,p*x_n^*)$ and for each consumer i there is a neighborhood $N_i$ of $(p*,m_i^*)$ in $R_+^{k+1}$ such that on the domain $N_i$, consumer i's demand function and indirect utility

literature on functional equations as a Pexider functional equation (Aczel (1966)). Where Pexider's functional equation holds for all m in an open rectangular region, according to a standard result of functional equations it must be that $h_i(p,m_i) = a(p)m_i + b_i(p)$ for all $m_i$ in the interval $\{m_i | \underline{m}_i < m_i < \bar{m}_i\}$ and for some functions a(p) and $b_i(p)$.

As we demonstrated in the previous paragraph, if $p \varepsilon N_p$, there is an aggregate demand function $h(p,\Sigma m_i) = \Sigma h_i(p,m_i)$ for all $m \varepsilon N_m$. It follows that if $m \varepsilon N_m$ and $m' \varepsilon N_m$ and $\Sigma m_i = \Sigma m_i'$, then x(p,m) and x(p,m') are Pareto optimal allocations with the same aggregate endowment. Since there is local transferable utility, it follows that $\Sigma u_i(h_i(p,m_i)) = \Sigma u_i(h_i(p,m_i'))$. But $u_i(h_i(p,m_i))$ is by definition equal to $v_i(p,m_i)$. Therefore, if m and m' are in $N_m$ and if $\Sigma m_i = \Sigma m_i'$, then $\Sigma v_i(p,m_i) = \Sigma v_i(p,m_i'))$. As before, we have a Pexider functional equation. It follows that there are functions $\alpha(p)$ and $\beta_i(p)$ for each i such that for all i and for all $p \varepsilon N_p$ and $m \varepsilon N_m$, $v_i(p,m) = \alpha(p)m_i + \beta_i(p)$. QED

## 4. TWO EXAMPLES

In the previous section we showed that if there is local transferable utility then in some neighborhood, demand functions and indirect utility functions are linear in income with identical slopes for all individuals. As we will demonstrate, there is global transferable utility only good 1. For example, the allocation in which consumer 1 receives all of the good 1 and 1.5 units of good 2 is Pareto optimal. The efficient way to make a small transfer of utility from person 1 to person 2 would now be for 1 to give 2 some good 2. In fact the utility possibility frontier in the neighborhood of this utility allocation can be shown to be described by the (nonlinear) equation $u_1 = 2 + 2\sqrt{2-u_2^2}$

If $m_i > p_2^{-2}$ then $h_i(p,m_i) = a(p)m_i + b_i(p)$ where $a(p) = (1,0)$ for all p and $b_i(p) = (-p_2^{-2})/p_1, p_2^{-2})$. If $m_i \geq p_2^{-2}$ then $h(p,m_i) = (0,m_i/p_2)$. For p and $m_i$ such that $m_i > p_2^{-2}$ we find by substituting demand into the utility function that $v_i(p,m_i) = \alpha(p)m_i + \beta_i(p)$ where $\alpha(p) = (1/p_1)-(1/p_1p_2)$ and $\beta_i(p) = 2/p_2$. This function is of the Gorman form. On the other hand, if $p_2^{-2} > m_i$, then $v_i(p,m_i) = \sqrt{m_i/p_2}$ which is not of the Gorman form.

<u>EXAMPLE 2</u>(Translated homothetic demand)

Suppose that there are two consumers and two commodities. Each consumer has a bundle of "minimum requirements" such that for any bundle that exceeds minimum requirements, he has homothetic preferences on the excess over minimum requirements. Any commodity bundle that does not meet the minimum requirements is regarded as worse than any bundle that does. While the two consumers may have different minimal requirements, their preferences on the excess over minimal requirements are identical. For expository purposes, let us look at a particular case of

2 to give us a theorem about the global implications of global transferable utility.

Lemma 4 Let preferences satisfy Condition P. Let p be a price vector and suppose that for some income distribution $\underline{m}$, if $m >> \underline{m}$, then there is local transferable utility at the allocation $x(p,m) = (h_1(p,m_1),...,h_n(p,m_n))$. Under these assumptions, there exists a vector a(p) and vectors $b_i(p)$ for every consumer i such that for all i if $m_i > \underline{m}_i$ then:

$$\text{(i)} \quad h_i(p,m_i) = a(p)m_i + b_i(p)$$

Proof

From Theorem 2 it follows that for any consumer i, if $m_i' > \underline{m}_i$ then there is an open interval $I_{mi'}$ containing $m_i'$ such that the demand function of consumer i is of the form (i) at price p and income $m_i \in I_{mi'}$. In order to prove Lemma 4, we must show that these local demand functions "fit together" to yield functions of the same form over the entire half line $\{m_i | m_i > \underline{m}_i\}$. This would not be possible if there were some price p and two incomes, $m_a > \underline{m}_i$ and $m_b > \underline{m}_i$ such that $h_i(p,m_i) = a(p)m_i + b_i(p)$ for $m_i$ in an open interval $I_a$ containing $m_a$, $h_i(p,m_i) = a'(p)m_i + b_i'$ in a neighborhood $I_b$ containing $m_b$ and such that $a'(p) \neq a(p)$ or $b_i(p) \neq b_i'$.

In this paragraph, we show that that if the open interval $I_a$ intersects the open interval $I_b$ and if $h_i(p,m_i) = a(p)m_i + b_i(p)$ for all $m_i$ in $I_a$, and $h_i(p,m_i) = a'(p)m_i + b_i'(p)$ for $m_i$ in the open interval $I_b$, then it must be that allocation, $\underline{x}$, there is local transferable utility at all Pareto optimal allocations, x, such that $x >> \underline{x}$. Then for all (p,m) such that $h_i(p,m_i) >> \underline{x}_i$, it must be that:

(i) $h_i(p,m_i) = a(p)m_i + b_i(p)$

(ii) $v_i(p,m_i) = \alpha(p)m_i + \beta_i(p)$

Proof

Let p be any price vector. Since all goods are assumed to be normal, the set of income distributions, m, such that $h_i(p,m_i) >> \underline{x}_i$ for all i is a set of the form $\{m \mid m >> \underline{m}\}$ for some $\underline{m}$. From Lemma 4 it follows that for any p, there exist vectors a(p) and $b_i(p)$ for all i such that if $h_i(p,m_i) >> \underline{x}_i$, then $h_i(p,m_i) = a(p)m_i + b_i(p)$. The same line of reasoning that establishes Condition (ii) of Theorem 2 as a consequence of Condition (i) of that theorem will establish Condition (ii) of Theorem 3 as a consequence of Condition (i) of this theorem. QED

In the case of quasilinear utility, the conditions of Theorem 3 are met where $\underline{x} = 0$. In the case of homothetic identical preferences on bundles in excess of individual "minimal requirements bundles" $r_i$, the conditions of Theorem 3 are met where $\underline{x} = r = (r_1,\ldots,r_n)$.

An interesting consequence of Theorem 3 is the following.

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assumption, however, the proof is rather intricate and seems barely worth the effort. It can be shown that under the other assumptions of the theorem, demand for any good must be either monotone increasing, monotone decreasing, or constant as a function of income so long as $h_i(p,m_i) >> \underline{x}_i$.

## APPENDIX

It is interesting to compare our results with those of Aumann (1960). Aumann assumes that there is a single transferable private good and a set P of possible public outcomes. Each consumer, i, has a utility function of the form $u_i(x_i,p)$ where $x_i$ is the amount of private good he receives and p is the public outcome. He considers the case where the set of utility distributions attainable by redistributing the private good <u>while holding the public outcome fixed</u> is contained in a simplex. Stated formally, Aumann's condition is:

<u>Condition A</u> For all $p \varepsilon P$, if $\Sigma x_i = \Sigma y_i$, then $\Sigma_i u_i(x_i,p) = \Sigma_i u_i(y_i,p)$.

Aumann proves the following:

<u>Aumann's Theorem</u> If $n > 3$ and if $u_i(x_i,p)$ is monotonic in $x_i$ for all p, then Condition A implies that for all i and all $p \varepsilon P$, $u_i(x_i,p) = c(p)x_i + k_i(p)$ for some functions, c(p) and $k_i(p)$.

While Aumann's theorem appears to be similar to ours, it is not really a theorem about transferable utility. Given Aumann's other conditions, Condition A is in general neither necessary nor sufficient for the utility possibility frontier to be contained in a simplex. The reason is that

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