VOLUME XXIV, NUMBER 2; December, 2013

Articles and Notes

Click on the cloud button to see a word cloud as an "abstract" of the adjacent article!

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A New Topographic Functional
Edward Earl and David Metzler

Klein 4 Group: Beth Olem Cemetery Application
Sandra Lach Arlinghaus

GeoMusic: Linked Selections
Sandra Lach Arlinghaus

Links of Interest

From Joseph Kerski: Why Geography Education Matters
From Rafael Pereira: If Christaller Had Google Earth...


Solstice: An Electronic Journal of Geography and Mathematics, Institute of Mathematical Geography

1. ARCHIVE
2. Editorial Board, Advice to Authors, Mission Statement
3. Awards

RECENT NEWS, 2013
1. Chene Street History Project.
3. The work above is the first volume in a series of books to be published by CRC Press in its series "Cartography, GIS, and Spatial Science: Theory and Practice." If you have an idea for a book to include, or wish to participate in some other way, please contact the series Editor, Sandra L. Arlinghaus.
4. Virtual Cemetery with William E. Arlinghaus; an ongoing project that continues in development run in the virtual world in parallel with the trust-funded model of a real-world cemetery.


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Solstice was a Pirelli INTERNETional Award Semi-Finalist, 2001 (top 80 out of over 1000 entries worldwide)

One article in Solstice was a Pirelli INTERNETional Award Semi-Finalist, 2003 (Spatial Synthesis Sampler).

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Solstice is listed in Geoscience e-Journals

IMaGe is listed on the website of the Numerical Cartography Lab of The Ohio State University: http://ncl.sbs.ohio-state.edu/4_homes.html

Congratulations to all Solstice contributors.

Remembering those who are gone now but who contributed in various ways to Solstice or to IMaGe projects, directly or indirectly, during the first 28 years of IMaGe:
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1 Introduction

The simplest and most familiar number associated to a mountain peak is the elevation of its summit above sea level. However, absolute elevation often does not correlate well with the visual impressiveness of a peak, which has more to do with the amount of local relief and the steepness of the flanks of the peak. For example, the summit of Mount Elbert, the highest point in the Rocky Mountains, is 4401 meters above sea level\[1\], while Devils Thumb\[2\] a striking rock spire on the border between Alaska and British Columbia, rises only to 2767 meters\[2\]. Based on pure elevation, Elbert far surpasses Devils Thumb. However, Mount Elbert rises from a high base in central Colorado, so its local relief is not nearly as great as its elevation would indicate; nor is it a particularly steep peak. For example, Elbert rises about 1600 meters (one mile) over a horizontal distance of 6.5 kilometers on its southeast flank\[2\] which is not unimpressive. However, the northwest face of Devils Thumb soars an amazing 2000 meters in 1.6 km, and it is similarly steep in other directions. To get 2000 meters of vertical relief from the summit of Mount Elbert, one has to go about 30 km away, to the town of Aspen; if one goes 30 km from Devils Thumb, one gets to tidewater, yielding 2767 meters of relief. See Table 1 for representative profiles of the two peaks; also see topographical maps for Mount Elbert and for Devils Thumb\[3\].

In this article we introduce a functional that takes into account the relief and steepness of a peak in a mathematically elegant way, and which has substantial correlation with the visual impressiveness of the peak. In fact, our functional can be applied to any point on a landscape (not necessarily a summit—for example, see the discussion below of the famous granite cliff of El Capitan in Yosemite), or indeed, any point on the graph of a function. We will also briefly introduce two concepts derived from the main functional; one takes into account how independent a particular feature is from nearby “better” features, and the other calculates a kind of “ruggedness” for a domain.

A pedagogical note: using the basic definitions provides good exercises in multivariable calculus, suitable for strong students in an introductory course. Proving theorems about these measures involve good workouts with elementary real and functional analysis.

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\[1\]There is no apostrophe in the official spelling of the name of this peak.

\[2\]One can verify these numbers using the public-domain mapping website map-per.acme.com, among others.

\[3\]Google Earth produces a good virtual tour of Mount Elbert. However it has very inaccurate (and misleadingly smoothed-out) elevation data for Devils Thumb. Getting accurate elevation data for steep features in obscure locations, such as Devils Thumb, is one of the challenges of this research.
2 Omnidirectional Relief and Steepness (ORS)

Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a bounded, Lebesgue measurable function, thought of as the height function of a landscape. (We do not require $h$ to be continuous, to permit the presence of vertical cliffs.) Consider a fixed base point $p \in \mathbb{R}^2$, and a corresponding reference point $(p, h_0)$. (It is theoretically useful, and no more complicated, to let the height $h_0$ of the reference point vary independently, so $h_0$ need not equal $h(p)$.) Physically, one can imagine, for example, $h_0 > h(p)$ to be the height of the top of a flagpole placed atop a peak. However we will primarily be interested in the case where $h_0 = h(p)$. We will define a functional of this data, which we call omnidirectional relief and steepness (ORS), which will capture a kind of average of the relief and steepness of the terrain as viewed from the reference point.

More precisely, let $h \in L^\infty (\mathbb{R}^2)$, and let $(p, h_0) \in \mathbb{R}^2 \times \mathbb{R}$. We will presently define ORS of the reference point $(p, h_0)$ relative to the landscape $h$, yielding a functional

$$\text{ORS} : \mathbb{R}^2 \times \mathbb{R} \times L^\infty (\mathbb{R}^2) \to \mathbb{R}$$

$(p, h_0; h) \mapsto \text{ORS} (p, h_0; h)$

(In fact we will define a whole family of possible functionals, but we will immediately specialize to one particularly appealing case.)

We first consider a simple landscape, both to fix ideas and to define an important normalization for the general case. It is a radially symmetrical conical peak, rising from a flat plain. Given a point $x \in \mathbb{R}^2$, denote the distance from

\footnote{We could use $S^2$ as the domain, to take into account the spherical nature of the Earth, but we will see that all of the calculations localize strongly, making the difference minuscule. Generalizing everything in this paper to $\mathbb{R}^n$ is straightforward, but we use $\mathbb{R}^2$ throughout for simplicity and because of the application to physical landscapes. However, we do not take into account overhanging cliffs, since that would vastly complicate the mathematical model.}
the origin to \( x \) by \( r(x) \), or just \( r \) for short (i.e. it is the usual \( r \) of polar coordinates).

**Definition 1** Let \( h_0, b > 0 \), let \( s = h_0/b \), and let \( \phi = \arctan s \). Then the cone function \( c : \mathbb{R}^2 \to \mathbb{R} \) associated to \( h_0, b \) is given by

\[
c(x) = \begin{cases} 
h_0 - sr, & r < b \\ 0, & r > b
\end{cases}
\]

(We suppress the dependence on \( h_0, b \) for tidiness.) Note that \( s \) is the slope of the cone, and \( \phi \) is the angle its sides make with the \( xy \)-plane.

See Figure 1 for the cross-section of the cone. We wish to define the ORS of the summit of this cone, i.e. \( \text{ORS}(0, h_0; c) \). It should take into account its height, and also its steepness. The combination \( h_0 s \) does not work, since it is unbounded for large \( s \), even if \( h_0 \) is small. The combination \( h_0\phi \) is just as natural, and is bounded. We will actually choose \( \frac{2}{\pi} h_0 \phi \) ("height times angle over 90°"), so that the limiting case \( \phi \to \pi/2 \), which we will call a flagpole, yields simply \( h_0 \). Hence we have the following.

**Definition 2** We say that \( \text{ORS} \) is angle-normalized if it yields \( \frac{2}{\pi} h_0 \phi \) when applied to the vertex of the cone with height \( h \) and angle \( \phi \).

\[
\text{ORS}(0, h_0; c) = \frac{2}{\pi} h_0 \phi
\]

Note two further important features of the conical case: first, if two cone functions \( c_1, c_2 \) share the same angle but have different heights \( h_2 = Ah_1 \), then the ORS of \( c_2 \) will be \( A \) times the ORS of \( c_1 \). In other words, scaling up every dimension (heights and horizontal distances) by a factor of \( A \) results in scaling.

---

\[^{5}\text{We discuss other possible normalizations just after Theorem 5.}\]
up ORS by the same factor. We will see below that this homogeneity, or scale-
_covariance, property is true of ORS in general; in particular, it means that ORS
has meaningful units, namely, units of length. (In the topographic examples
below, ORS is given in meters.)

Second, if we take a low-slope cone \( c \) with height \( h \) and base \( b \gg h \) and
scale up \( h \) by a factor of \( A \), leaving \( b \) unchanged, then the ORS will increase by
approximately \( A^2 \). This low-slope quadratic behavior is also a general feature
of ORS.

Now we turn to the general case of a non-conical peak or other topographic
feature. We imagine standing at the reference point—say the summit of a
mountain—and looking down in all directions, gauging the impressiveness of
the view. We want to take some sort of average of the impressiveness information
obtained by looking in all directions. One can also think of stationing a host of
tourists (mathematically, these will be called sample points) everywhere around
the mountain, all looking up at the summit, and surveying them for their idea
of the impressiveness of the summit\(^6\) Hence ORS will involve an integral over
the set of all sample points; we will denote a typical sample point by \( x \), and we
will set \( r = \|p - x\| \), the distance from the reference point to a sample point.

For every sample point \( x \), we calculate the slope \( u(x) = (h_0 - h(x))/r \). If
we integrated \( u \) itself, the integral over all \( x \in \mathbb{R}^2 \) would clearly diverge for most
landscapes. Instead, we use an appropriate function to turn \( u \) into a sensible
integrand. We first present a general definition, using an arbitrary such function,
and then use the cone normalization to determine what function we desire.

**Definition 3** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function with \( f(u) = 0 \) for \( u \leq 0 \)\(^7\). Let \( h \in L^{\infty}(\mathbb{R}^2) \), and let \((p, h_0) \in \mathbb{R}^2 \times \mathbb{R} \). Let \( r = \|x - p\| \) be the radial
coordinate based at \( p \), and let \( u(x) = (h_0 - h(x))/r \). The omnidirectional
relief and steepness (ORS) of the reference point \((p, h_0) \) relative to the
landscape \( h \), using \( f \), is

\[
ORS_f(p, h_0; h) = \|f \circ u\|_2 = \left[ \iint_{\mathbb{R}^2} f^2 \left( \frac{h_0 - h(x)}{r} \right) dA(x) \right]^{1/2}
\]

Before examining the general properties of ORS, we first derive the correct
function \( f \) based on our normalization.

---

\(^6\)Note that ORS ignores line-of-sight issues: we make no distinction between points that are actually in view from the reference point and points that are obscured by intervening terrain. Hence phrases such as "looking up at the mountain" should not be taken too literally.

\(^7\)It is not absolutely necessary to require that \( f \) vanish for negative \( u \). It has the effect of ignoring surrounding higher terrain in evaluating the reference point. This usually has a negligible effect when the reference point is a summit, which is our main application. Dropping this requirement turns out to make the reduced version of ORS, discussed at the end of this paper, difficult to define.
Proposition 4 Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable and assume that $f(u) = a(u^{1+\varepsilon})$ for some $\varepsilon > 0$, as $u \to 0$. Let $h_0, b > 0$ and let $c$ be the associated cone function, with slope $s = h_0/b$. Then $\text{ORS}_f(0, h_0; c) = h_0 F(s)$, where $F$ satisfies the initial value problem

$$\frac{1}{\pi} (F^2(s))' = \frac{1}{s^2} (f^2(s))', \quad F(0) = 0.$$ 

Proof. Defining $F$ as in the theorem, we have

$$F'(s) = \left( \frac{h_0 F(s)}{s^2} \right) \frac{1}{b}$$

where we have set $u = h_0/r$ and hence $du = -(h_0/r^2) dr$ or $r dr = -(h_0/r) du = - (h_0^2/u^2) du$. Note that the order of vanishing assumed for $f$ makes all the integrals converge. Hence

$$\frac{1}{2\pi} F^2(s) = \frac{1}{2} f^2(s) \frac{h_0^2}{s^2} + \int_0^s f^2(u) \frac{h_0^2}{u^3} du$$

Integration by parts yields

$$\frac{1}{2\pi} F^2(s) = \frac{1}{2} f^2(s) \frac{h_0^2}{s^2} + \lim_{u \to 0} \frac{f^2(u)}{2u^2} + \frac{1}{2} \int_0^s (f^2(u))' \frac{du}{u^2}$$

or, taking the derivative of both sides,

$$\frac{1}{\pi} (F^2(s))' = \frac{1}{s^2} (f^2(s))', \quad F(0) = f(0) = 0.$$

Proposition 5 Let

$$f(u) = \left[ \frac{4}{\pi^3} (2u \arctan u - \ln (u^2 + 1) - \arctan^2 u) \right]^{1/2}$$

(1)
for $u \geq 0$ and $f(u) = 0$ for $u < 0$. Then the function $F$ associated to $f$ by Proposition 4 is $F(s) = \frac{2}{\pi} \arctan s$, and hence the resulting ORS$_f$ is angle-normalized:

$$\text{ORS}_f(0, h_0; c) = \frac{2}{\pi} h_0 \phi.$$  

Proof.

$$f^2(s) = \frac{1}{\pi} \int_0^s u^2 (F^2(u))' \, du$$

$$= \frac{4}{\pi^3} \int_0^s u^2 (\arctan^2 u)' \, du$$

$$= \frac{8}{\pi^3} \int_0^s \frac{u^2}{u^2 + 1} \arctan u \, du$$

$$= \frac{8}{\pi^3} \left[ 2 \int_0^s \arctan u \, du - \int_0^s (\arctan^2 u)' \, du \right]$$

$$= \frac{4}{\pi^3} \left[ 2u \arctan u \big|_0^s - 2 \int_0^s \frac{u}{u^2 + 1} \, du - \arctan^2 s \right]$$

$$= \frac{4}{\pi^3} \left[ 2s \arctan s - \ln (s^2 + 1) - \arctan^2 s \right].$$

We exclusively use this angle-normalized $f$, shown in Figure 2, in our calculations of ORS in this paper. However we can say a word about what happens when one chooses different functions for $f$. Since ORS combines information about local relief with information about steepness, there is an issue of how much to weight relief versus steepness: should we assign a greater value to a very steep, but only moderately high peak, or to a moderately steep, but very high peak? At the risk of making apples-to-oranges comparisons, we boldly proceed to assign one number that makes a certain tradeoff between relief and steepness. Different
choices for $f$ will result in somewhat different tradeoffs, either more “heightist” (favoring relief over steepness) or more “slopist” (the opposite). In past work we have also tried other normalizations, notably $F(s) = s/(s+1)$, which is more heigthist than the angle normalization. We work with angle normalization for reasons of simplicity, elegance, and a good fit with visual impressiveness.

For the remainder of this paper, we will use the modified slope integrand $f$ given in Theorem 5 and we will suppress $f$ from the notation; that is, we define

$$\text{ORS}(p, h_0; h) = \text{ORS}_f(p, h_0; h)$$

With this definition, ORS has many good properties, including strong versions of continuity, which are essential for dealing with the discretized data encountered in practice. Note that the square root in the definition is an order-preserving function; hence for the purposes of comparing peaks (one of our main uses for ORS), it is enough to use $\text{ORS}^2$, which will be simpler to analyze. One can think of the square root serving mainly to make normalization easier (in particular, it produces a quantity with units of length). The root does make it tricky to analyze the behavior of ORS for landscapes where ORS is very small. This is not a major concern for our purposes, since we focus primarily on reference points for which ORS is relatively large. Also, taking the square root halves relative error, so any relative error result for $\text{ORS}^2$ yields a corresponding, and stronger, relative error result for ORS.

**Proposition 6** The functional $\text{ORS} : \mathbb{R}^2 \times \mathbb{R} \times L^\infty(\mathbb{R}^2) \to \mathbb{R}$ has the following properties:

1. ORS is weakly increasing as a function of $h_0$ and weakly decreasing as a function of $h$: for every $p \in \mathbb{R}^2$, $h_0, k_0 \in \mathbb{R}$, and $h, k \in L^\infty$, if $h_0 \leq k_0$ and $h \geq k$, then $\text{ORS}(p, h_0; h) \leq \text{ORS}(p, k_0; k)$.

2. ORS is bounded by the maximum height of the landscape: for every $p \in \mathbb{R}^2$, $h_0 \in \mathbb{R}$, and $h \in L^\infty$, $\text{ORS}(p, h_0; h) \leq \|h_0 - h\|_\infty$. In particular, it is finite for any bounded landscape and any reference point.

3. ORS is invariant under vertical translation and horizontal translation: for every $p \in \mathbb{R}^2$, $h_0 \in \mathbb{R}$, $h \in L^\infty$, $a \in \mathbb{R}$, and $q \in \mathbb{R}^2$,

$$\begin{align*}
\text{ORS}(p, h_0 + a; h + a) &= \text{ORS}(p, h_0; h) \\
\text{ORS}(p + q, h_0; h(x - q)) &= \text{ORS}(p, h_0; h)
\end{align*}$$

4. ORS is invariant under reflections and rotations about the reference point: let $A$ be a 2 by 2 orthogonal matrix and define $h_A(x) = h(A(x - p) + p)$. Then

$$\text{ORS}(p, h_0; h_A) = \text{ORS}(p, h_0; h)$$

---

8In particular, the Lipschitz continuity in Corollary 13 would not hold if we used a 1-norm instead of a 2-norm, which might otherwise seem simpler.
5. ORS$_f$ is scale-covariant (if we scale the landscape both horizontally and vertically), and in particular it has units of length. That is, if $h_M$ is obtained from $h$ by dilating horizontally about the point $p$ by $M > 0$ and scaling vertically by $M$ (i.e. $h_M(x) = M \cdot h((x - p)/M + p)$) then

$$\text{ORS}(p, Mh_0; h_M) = M \cdot \text{ORS}(p, h_0; h)$$

**Proof.** Monotonicity [1] and vertical and horizontal translation invariance [3] are clear from the definition. Invariance under reflections and rotations follows from the corresponding invariance of the integral. Scale-covariance follows from the change of variables indicated in item [5]. The bound given in item [2] follows from monotonicity and the flagpole case of the cone normalization; hence we will refer to this bound as the flagpole bound.  

**Remark 7** By using vertical and horizontal invariance, we can always reduce to the case where the reference point is the origin and the reference height is zero. We do this below for simplicity, denoting the result by $\text{ORS}(h) = \text{ORS}(0, 0; h)$. Note that in any statement involving a variation of the landscape $h$, we can recover a more general version, with variation in $h_0$ as well: for example, simply replace any quantity of the form $\|h - k\|_\infty$ by $\|(h_0 - h) - (k_0 - k)\|$.

Before turning to results about the continuity and robustness of ORS, we need a lemma about the function $f$ which appears in the definition. This lemma summarizes all of the features of $f$ that are necessary for the results about ORS that follow.

**Lemma 8** The function $f$ defined in Proposition [3] is $C^1$ on $\mathbb{R}$ and has the following properties for $u > 0$. (Recall that $f$ is identically zero for $u \leq 0$.)

1. $f$ is strictly increasing.
2. $f^2(u) = \frac{2}{\pi^3} u^4 + O(u^6)$.
3. $f(u) = \sqrt{\frac{2}{\pi}} \cdot u^2 + O(u^3)$ as $u \to 0^+$ and $f(u) \leq \min \left\{ \sqrt{\frac{2}{\pi}} \cdot u^2, \frac{2}{\pi} \sqrt{u} \right\}$.
4. $f^2$ is strictly convex.
5. $0 \leq (f^2)'(u) < \frac{4}{\pi^2}$ and $(f^2)'(u) \leq \frac{4}{\pi^2} u^3$.

**Proof.** Let $u > 0$. The function

$$f^2(u) = \frac{4}{\pi^3} \left( 2u \left( u - \frac{u^3}{3} \right) - \left( u^2 - \frac{u^4}{2} \right) - \left( u - \frac{u^3}{3} \right)^2 \right) + O(u^6)$$

is clearly $C^\infty$. Its Taylor expansion at $u = 0$ is

$$f^2(u) = \frac{2}{\pi^3} u^4 + O(u^6)$$
Hence
\[
\begin{align*}
f(u) &= \sqrt{\frac{2}{\pi^3} u^4 (1 + O(u^2))} \\
&= \sqrt{\frac{2}{\pi^3} u^2 (1 + O(u^2))}
\end{align*}
\]
This shows that, even with the proviso that \( f(u) = 0 \) for \( u < 0 \), \( f \) is \( C^1 \) for all \( u \in \mathbb{R} \).

Next we calculate the derivative of the squared function:
\[
(f^2)'(u) = \frac{8}{\pi^3} u^2 \frac{\arctan u}{u^2 + 1}
\]
(recall from Prop.\[4\] that it is not accidental that this is relatively simple). This is clearly positive for \( u > 0 \); hence \( f^2 \) and \( f \) are both increasing (in fact, strictly increasing as long as \( u > 0 \)). (Again, this follows also from Prop. \[4\].) Since \( \arctan u < \min \left\{ \frac{\pi}{2}, u \right\} \) for \( u > 0 \), we see also that \( (f^2)'(u) < \min \left\{ \frac{4}{\pi^2}, \frac{8}{\pi^3} u^3 \right\} \) for \( u > 0 \). We take the second derivative and obtain
\[
0 < (f^2)''(u) = \frac{8}{\pi^3} \frac{u(u+2\arctan u)}{(u^2+1)^2} < \frac{24}{\pi^3} u^2 = \frac{d^2}{du^2} \left( \frac{2}{\pi^3} u^4 \right) \quad (u > 0)
\]
which shows that \( (f^2)'' \) is convex, and also, since \( f^2(0) = (f^2)'(0) = 0 \), that
\[
f^2(u) < \frac{2}{\pi^3} u^4
\]
and hence that
\[
f(u) < \sqrt{\frac{2}{\pi^3} u^2}
\]
as desired. \( \blacksquare \)

We now consider the sensitivity of ORS and ORS\(^2\) to the landscape data \( h \) (and hence also to the height \( h_0 \) of the reference point, as in Remark \[7\]). We certainly want continuity, but we actually want a bit more; continuous functions can have unpleasantly large derivatives. This is important when dealing with discrete, and often somewhat inaccurate, digital data. In fact, a previous attempt at defining such a function using a 1-norm instead of a 2-norm led to poor behavior in this regard.

To quantify the sensitivity of ORS\(^2\)(\( h \)) to variations in \( h \), we recall the following standard notion from functional analysis.\[3\]

**Definition 9** Given a function \( F : V \to W \) between two topological vector spaces, the Gâteaux differential of \( F \) is the function \( dF \) given by
\[
dF(h,v) = \frac{d}{dt} \bigg|_{t=0} F(h + tv)
\]
\( F \) is said to be Gâteaux differentiable at \( h \in V \) if \( dF \) exists for all \( v \in V \).

9
In general, $dF$ need not be continuous or linear. In our case, we are most interested in the following. Suppose that $V$ and $W$ are Banach spaces, and that $F$ is Gâteaux differentiable at $h$. Then define

$$mF(h) = \sup_{\|v\|=1} \|dF(h,v)\|$$

(which may be infinite). If $F$ is actually (Fréchet) differentiable at $h$, this is clearly just the norm of the derivative, as a linear operator between Banach spaces. It measures the worst-case sensitivity of $F$ at $h$. We are interested in a simpler quantity, namely

$$MF(H) = \sup_{\|h\|=H} mF(h) = \sup_{\|h\|=H} \sup_{\|v\|=1} \|dF(h,v)\|$$

which gives the worst-case sensitivity of $F$ over all inputs of given norm $H$. We are interested in the case where $V = L^\infty$ and $W = \mathbb{R}$. For example, if $F(h) = \|h\|_\infty^2$, a simple calculation yields $MF(H) = 2H$. With this notation, we can state our main result about the sensitivity of ORS.

**Theorem 10** The worst-case sensitivity of $\text{ORS}^2$ satisfies

$$\text{MORS}^2(H) = 2H$$

That is, it is exactly as sensitive, in the worst case, as the function $H^2$. For ORS itself, we have

$$m\text{ORS}(h) \leq \frac{\|h\|_\infty}{\text{ORS}(h)}$$

Before proving the theorem, we first note the following easy consequence of monotonicity, whose proof we omit.

**Lemma 11** Let $H > 0$ be fixed and consider all pairs of landscapes $h, k \in L^\infty$, with $\|h - k\|_\infty = H$. Then $|\text{ORS}^2(h) - \text{ORS}^2(k)|$ is maximized when $h - k$ is a constant function (a.e.).

**Proof of the Theorem.** Let $h \in L^\infty$. By the lemma, to calculate $m\text{ORS}(h)$, we need only consider the case where $v$ is constant function; let’s say $v = z$ everywhere. So

$$m\text{ORS}(h) = \left. \frac{d}{dz} \right|_{z=0} \text{ORS}^2_f(h + z)$$

$$= \left. \frac{d}{dz} \right|_{z=0} \int_{\mathbb{R}^2} f^2 \left( \frac{h(x) + z}{r} \right) dA$$

$$= \int_{\mathbb{R}^2} \left. \frac{\partial}{\partial z} \right|_{z=0} \left( f^2 \left( \frac{h(x) + z}{r} \right) \right) dA$$

$$= \int_{\mathbb{R}^2} (f^2)' \left( \frac{h(x)}{r} \right) \frac{1}{r} dA$$
where we can pass the derivative inside the integral since \((f^2)' \left( \frac{h(x)}{r} \right) \frac{1}{r} \) is integrable. \(\text{[It is integrable near the origin since } (f^2)' \text{ is bounded, and at infinity since } (f^2)'(u) \leq \frac{8}{r^6} u^3, \text{ both by Lemma } [8]\) Now let \(H > 0\) and consider all functions \(h\) with \(\|h\|_\infty = H\). Since \((f^2)\) is increasing by Lemma [8], \(\text{mORS}(h)\) will be maximized when \(h\) is a constant function, with value \(H\). But this reduces us to the case where \(h = H\) and \(v\) are both constant, that is, the flagpole case, and this is normalized to give

\[
\text{ORS}^2(H) = H^2
\]

for which we have already noted that

\[
\text{MORS}^2(H) = 2H
\]

The result about \text{ORS} itself follows by the chain rule:

\[
\begin{align*}
\text{mORS}(h) &= \left| \frac{d}{dz} \right|_{z=0} \sqrt{\text{ORS}^2(h + z)} \\
&= \left| \frac{d}{dz} \right|_{z=0} \frac{\text{ORS}^2(h + z)}{2 \sqrt{\text{ORS}^2(h)}} \\
&\leq \frac{\text{MORS}^2(\|h\|_\infty)}{2 \sqrt{\text{ORS}^2(h)}} \\
&= \frac{\|h\|_\infty}{\sqrt{\text{ORS}^2(h)}}
\end{align*}
\]

\[\blacksquare\]

**Corollary 12** The function \text{ORS}^2 is locally Lipschitz continuous, and the Lipschitz bound depends only on \(\|h\|_\infty\). More precisely, on any set \(S\) with \(\|h\|_\infty \leq H\) for all \(h \in S\),

\[
|\text{ORS}^2(h_0) - \text{ORS}^2(h_1)| \leq 2H
\]

for all \(h_0, h_1 \in S\).

**Proof.** For \(h_0, h_1 \in S\), let \(h_t = th_1 + (1 - t) h_0\). The corollary follows from the mean value theorem applied to the function \(t \mapsto \text{ORS}^2(h_t)\), since the preceding theorem implies that the derivative of this function is bounded by \(2H\). \[\blacksquare\]

**Corollary 13** The function \text{ORS} is continuous, and it is locally Lipschitz continuous away from the zero (a.e.) landscape. Further, on a set \(S\) on which \text{ORS} is bounded away from zero, \text{ORS} is uniformly Lipschitz continuous.

**Proof.** \text{ORS} is continuous since \text{ORS}^2 is. If \(h\) is not the zero landscape, then \(\text{ORS}(h) \neq 0\), and by continuity, there is a neighborhood around \(h\) where \text{ORS} is
bounded away from zero. Hence a mean value theorem argument as in the last corollary, using the bound on $m\text{ORS} (h)$ in the theorem, yields local Lipschitz continuity. If ORS is bounded away from zero a priori, then the same argument gives a uniform Lipschitz constant. 

**Remark 14** Even for landscapes with small ORS values, ORS tends to be better-behaved than this corollary would indicate, but the result given is satisfactory for our purposes.

While the previous theorem and its corollaries address the sensitivity of ORS to an arbitrary bounded change in the landscape, we get a sharper result if the change in the landscape occurs only far away from the reference point. This is important to the interpretation of ORS as a measure of local impressiveness, without regard to absolute elevation above the level of a distant ocean. As before, it is simpler to discuss $\text{ORS}^2$.

**Theorem 15 (Locality)** $\text{ORS}^2$ is local: the contribution $I$ to $\text{ORS}^2(h)$ from points $x$ with $\|x\| > R$ satisfies

$$I \leq \frac{2 \|h\|_\infty^4}{\pi^2 R^2}.$$ 

Hence for every $h, k \in L^\infty$, if $h(x) = k(x)$ for all $x$ with $\|x\| \leq R$, and $\|h\|_\infty, \|k\|_\infty \leq H$, then

$$|\text{ORS}^2(h) - \text{ORS}^2(k)| \leq \frac{2H^4}{\pi^2 R^2}.$$

**Proof.** Let $E = \{x \in \mathbb{R}^2: \|x\| \geq R\}$. Then

$$I = \int_E f^2(h(x)/r) \, dA$$

$$\leq \frac{2}{\pi^3} \int_E \left( \frac{h(x)}{r} \right)^4 \, dA$$

$$= \frac{2}{\pi^3} \int_0^{2\pi} \int_R^\infty \left( \frac{h(x)}{r} \right)^4 \, r \, dr \, d\theta$$

$$\leq \frac{4}{\pi^2} \int_0^\infty \int_R^\infty \frac{|h|_\infty^4}{r^3} \, dr$$

$$\leq \frac{2 \|h\|_\infty^4}{\pi^2 R^2}.$$ 

We noted above that in the case of a low-slope cone, ORS is approximately quadratic in the height (for a fixed base radius). This is true in general as long as the slopes near the reference point are bounded.
Theorem 16  For terrain that has bounded slope near the origin, $\text{ORS}(h)$ approximately scales quadratically in the height (with no horizontal scaling). More precisely, assume that $h(x)/r$ is bounded and let $M > 0$. Then

$$\text{ORS}(Mh) = CM^2 + O(M^4)$$

as $M \to 0$, for some $C$ depending on $h$.

Proof. Let $u(x) = -h(x)/r$ and let $H = \|h\|_{\infty}$. Then the corresponding slope function for the (vertically) scaled landscape is $Mu$, and

$$\text{ORS}^2(Mh) = \int_{\mathbb{R}^2} f^2(Mu(x)) \, dA$$

$$= \int_{\mathbb{R}^2} \left[ \frac{2}{\pi^3} M^4 u(x)^4 + g(Mu(x)) \right] \, dA$$

$$= \frac{2}{\pi^3} M^4 \int_{\mathbb{R}^2} u(x)^4 \, dA + \int_{\mathbb{R}^2} g(Mu(x)) \, dA$$

where $|g(u)| \leq C_1 u^6$ for all $u$. Hence

$$\left| \text{ORS}^2(Mh) - \frac{2}{\pi^3} M^4 \int_{\mathbb{R}^2} u(x)^4 \, dA \right| \leq \left| \int_{\mathbb{R}^2} g(Mu(x)) \, dA \right|$$

$$\leq \int_{\mathbb{R}^2} |g(Mu(x))| \, dA$$

$$\leq \int_{\mathbb{R}^2} C_1 M^6 u(x)^6 \, dA$$

Since $h$ is bounded, $u$ decays at least as $1/r$ at infinity, and it is assumed to be bounded at the origin. Hence

$$\int_{\mathbb{R}^2} u(x)^n \, dA < \infty \quad \text{for } n \geq 3$$

We can apply this for $n = 4$ to the expression above to see that

$$C_2 = \frac{2}{\pi^3} \int_{\mathbb{R}^2} u(x)^4 \, dA$$

is finite. Applying the case $n = 6$ gives

$$|\text{ORS}^2(Mh) - C_2 M^4| \leq C_3 M^6$$

where $C_3 = C_1 \int_{\mathbb{R}^2} u(x)^6 \, dA$. Therefore

$$\text{ORS}^2(Mh) = C_2 M^4 + O(M^6)$$

and

$$\text{ORS}(Mh) = CM^2 + O(M^4)$$
as desired, where $C = \sqrt{C_2}$. ■

To state the next result, we return to considering ORS as a function of $p$, $h_0$, and $h$. We look at how ORS depends on the horizontal location of the reference point, if we do not change its height. (This is a little strange physically, as the reference point is usually at ground level; we will address this immediately after the theorem.)

**Theorem 17** Let $H > 0$ be fixed. Then $\text{ORS}^2$ and ORS are continuous in $p$, uniformly in $p$, $h_0$, and $h$, provided that $\|h_0 - h\|_\infty \leq H$.

We first need a lemma regarding $f^2(h/r)$.

**Lemma 18** Given $r_1, r_2$ with $0 < r_1 < r_2$, $f^2(h/r_1) - f^2(h/r_2)$ is an increasing function of $h$ for $h \geq 0$.

**Proof.** We have

$$\frac{d}{dh} \left( f^2 \left( \frac{h}{r_1} \right) - f^2 \left( \frac{h}{r_2} \right) \right) = (f^2)' \left( \frac{h}{r_1} \right) \frac{1}{r_1} - (f^2)' \left( \frac{h}{r_2} \right) \frac{1}{r_2}$$

$$> \frac{1}{r_1} \left( (f^2)' \left( \frac{h}{r_1} \right) - (f^2)' \left( \frac{h}{r_2} \right) \right)$$

$$> 0$$

since $f^2$ is convex. ■

**Proof of the Theorem.** We wish to bound $|\text{ORS}^2(q, h_0; h) - \text{ORS}^2(p, h_0; h)|$ for $q$ near $p$. Without loss of generality, we can let $p$ be the origin, $h_0 = 0$, and $q = (\delta, 0)$, for some $\delta > 0$, and we can look at the case where $\text{ORS}^2(q, h_0; h) \geq \text{ORS}^2(p, h_0; h)$. We have

$$\text{ORS}^2(q, 0; h) - \text{ORS}^2(0, 0; h) \leq \int_{\mathbb{R}^2} \left( f^2 \left( \frac{h(x)}{||x - q||} \right) - f^2 \left( \frac{h(x)}{r} \right) \right) dA$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( f^2 \left( \frac{h(x)}{||x - q||} \right) - f^2 \left( \frac{h(x)}{r} \right) \right) dy \, dx + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( f^2 \left( \frac{h(x)}{||x - q||} \right) - f^2 \left( \frac{h(x)}{r} \right) \right) dy \, dx$$

where the second integral drops out because $||x - q|| > r$ on that region and $f^2(h/r)$ is a decreasing function of $r$. Also, by the previous lemma, the difference between the $f^2$ values at a particular $x$ will be maximized when $h(x)$ is as large
as possible, so we have

\[
\text{ORS}^2(\delta, 0, 0; h) - \text{ORS}^2(0, 0, 0; h) \leq \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\infty}^{\infty} \left( f^2 \left( \frac{H}{r} \right) - f^2 \left( \frac{H}{r} \right) \right) \, dy \, dx
\]

\[
= \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\infty}^{\infty} f^2 \left( \frac{H}{r} \right) \, dy \, dx - \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\infty}^{\infty} f^2 \left( \frac{H}{r} \right) \, dy \, dx
\]

\[
= \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\infty}^{\infty} f^2 \left( \frac{H}{r} \right) \, dy \, dx
\]

(where the first equality follows from the change of variables \( u = x - \delta \), which is exactly \( \text{ORS}^2 \) applied to an infinitely long, thin “mesa” of constant height \( H \).

This in turn can be estimated as follows, using Lemma 8:

\[
\int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\infty}^{\infty} f^2 \left( \frac{H}{r} \right) \, dy \, dx = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} f^2 \left( \frac{H}{r} \right) \, dy \, dx
\]

\[
+ 2 \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} f^2 \left( \frac{H}{r} \right) \, dy \, dx
\]

\[
+ 2 \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} f^2 \left( \frac{H}{r} \right) \, dy \, dx
\]

\[
\leq 2\pi \int_{0}^{\frac{a}{2}} f^2 \left( \frac{H}{r} \right) \, r \, dr
\]

\[
+ 2\delta \int_{\frac{a}{2}}^{H} f^2 \left( \frac{H}{y} \right) \, dy
\]

\[
+ 2\delta \int_{H}^{\infty} f^2 \left( \frac{H}{y} \right) \, dy
\]

\[
\leq 2\pi \cdot \frac{4}{\pi^2} H \cdot \sqrt{\frac{2}{\delta}}
\]

\[
+ 2\delta \cdot \frac{4}{\pi} \cdot H \cdot \ln \left( \frac{2H}{\delta} \right)
\]

\[
+ 2\delta \cdot \frac{2}{\pi^3} \cdot \frac{H}{3}
\]

Hence we have

\[
\text{ORS}^2(\delta, 0, 0; h) - \text{ORS}^2(0, 0, 0; h) \to 0 \quad \text{as} \quad \delta \to 0 \quad \text{(with \( H \) fixed)}
\]

so \( \text{ORS}^2 \) and \( \text{ORS} \) are continuous in \( p \). Since the bound we derived only depends on \( H \), and not on \( p \), \( h \), or \( h_0 \), the continuity is uniform as desired. \[ \blacksquare \]

(Note: if we put a bound on the slope of \( h \) near \( p \), this can be sharpened to yield Lipschitz continuity.)

We are usually interested in the case where \( h_0 = h(p) \), yielding the function (with \( h \) fixed and \( p \) variable) \( \text{ORS}(p, h(p); h) \). Note that in general (when \( h \) is
not continuous) we do not expect this function of \( p \) to be continuous, since the reference height follows the discontinuous function \( h \). However, wherever \( h \) is continuous, Theorem \( 17 \) and Corollary \( 13 \) together imply that \( \text{ORS}(p, h(p); h) \) will also be continuous.

## 3 Examples

To get a feel for the meaning of ORS, it is most instructive to look at explicit examples, preferably with pictures. Below we display a sample cross-section for a few representative peaks. In addition, the Peaklist website \[5\] and viewing packages such as Google Earth\[9\] are very useful. All of the ORS values for the examples were generated by computer, using gridded digital elevation models (DEMs)\[10\].

First let us dispatch our introductory contrasting examples, Mount Elbert and Devils Thumb. Mount Elbert has an ORS of 237 meters, while Devils Thumb’s is 828 meters, corresponding to their dramatically different profiles as shown in Table \[1\]. These values show that a comparison between these two peaks based on ORS gives the opposite result from the comparison suggested by their absolute elevations.

Another illustrative contrast is provided by Yosemite National Park. The highest point in the park is Mount Lyell (Google Earth Tour) at 3999 meters. It has a respectable ORS value of 200 meters. See Table \[2\]. However, far more famous is the huge granite cliff on the side of Yosemite Valley known as El Capitan (GE). It is hardly a mountain at all (there is higher terrain quite nearby), and its “summit” (a minor knoll some distance back from the brow of the cliff) has an elevation of only 2307 meters. El Capitan is a good example of a feature whose maximum ORS value is not obtained at the “summit” (local maximum of height). Rather, it is obtained by placing the reference point just atop the steepest portion of the cliff. The resulting ORS value is 575 meters. See Table \[2\]. (The similarly famous and impressive Half Dome (GE) nearby gets an ORS of 580 meters; these are easily the two best ORS values in the park, and in the whole Sierra Nevada.) Here ORS clearly correlates much better with the notability of the features than does absolute elevation.

Table \[3\] lists the six U.S. states with the highest maximum ORS value. Not surprisingly, Alaska tops the list, although Mount McKinley (GE) (ORS = 1243 m, Elev = 6194 m) is not the best point in Alaska. The lower Mount Saint Elias (GE) is very close to tidewater (about 10 km away), and is comparably steep, so it gets a higher ORS value. Most of the other peaks are well-known, except perhaps Mount Cleveland (GE), the high point of Glacier National Park. (The glaciers there are fast disappearing, but they have carved a number of

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\[9\] However note that, as of 2013, in some regions (typically non-U.S. regions with high relief), the dataset that underlies Google Earth is still of varying, and sometimes strikingly low, quality.

\[10\] The typical accuracy of the ORS values presented in this section is a few percent. More details on the calculations can be found on the peaklist website.[5]
Table 2: El Capitan (left) and Mount Lyell (right) profiles

<table>
<thead>
<tr>
<th>Peak</th>
<th>ORS</th>
<th>Elev</th>
<th>State</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mount Saint Elias</td>
<td>1334</td>
<td>5489</td>
<td>Alaska</td>
</tr>
<tr>
<td>Mount Rainier</td>
<td>827</td>
<td>4392</td>
<td>Washington</td>
</tr>
<tr>
<td>Grand Teton</td>
<td>683</td>
<td>4197</td>
<td>Wyoming</td>
</tr>
<tr>
<td>Mount Shasta</td>
<td>675</td>
<td>4317</td>
<td>California</td>
</tr>
<tr>
<td>Mount Cleveland</td>
<td>672</td>
<td>3190</td>
<td>Montana</td>
</tr>
<tr>
<td>Mount Hood</td>
<td>649</td>
<td>3452</td>
<td>Oregon</td>
</tr>
</tbody>
</table>

Table 3: State best points by ORS

- Mount Rainier
- Grand Teton
- Mount Shasta
- Mount Cleveland
- Mount Hood

exceptionally steep peaks.) It is interesting to also compare [Mount Whitney (GE), the high point of the contiguous U.S. (ORS = 418 m, Elev = 4421 m); note that it is bested within California not only by the huge stratovolcano [Mount Shasta (GE), but also by El Capitan and Half Dome (among others).

Worldwide, we have Table 4, which lists the top five independent peaks in the world. Four are in the Himalaya, while Rakaposhi is in the nearby Karakoram range. While three of these peaks are in the famed group of fourteen "eight-thousanders" (with elevation over 8000 meters), two are not; in fact [Machhapuchhre] is not even in the top 300 peaks in the world by elevation. (It is a tremendously steep peak, near low terrain, in the Annapurna region of Nepal; it is highly sacred and is off-limits to climbing.) For comparison, Mount Everest, elevation 8848 m, gets a very respectable ORS value of 1302 m. Also note the dramatic difference in scale between these peaks and peaks in the contiguous U.S. (Mount Saint Elias does, however, come close to the top five, and actually beats Everest.)

---

11Since uniform topographic mapping is not available for these peaks, the links are to Google Earth tours. They give the general impression, but be aware that they are not always highly accurate.

12This list was actually generated by taking the five highest points as ranked by reduced ORS, as in Section 4, to ensure five truly independent peaks.
Table 4: World’s top five independent peaks by ORS

<table>
<thead>
<tr>
<th>Peak</th>
<th>ORS</th>
<th>Elev</th>
<th>State</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nanga Parbat</td>
<td>1740</td>
<td>8125</td>
<td>Pakistan</td>
</tr>
<tr>
<td>Dhaulagiri</td>
<td>1680</td>
<td>8167</td>
<td>Nepal</td>
</tr>
<tr>
<td>Rakaposhi</td>
<td>1628</td>
<td>7788</td>
<td>Pakistan</td>
</tr>
<tr>
<td>Machhapuchhare</td>
<td>1596</td>
<td>6993</td>
<td>Nepal</td>
</tr>
<tr>
<td>Manaslu</td>
<td>1550</td>
<td>8163</td>
<td>Nepal</td>
</tr>
</tbody>
</table>

4 Derived concepts

We have created two main concepts derived from ORS: reduced ORS (RORS) and domain relief and steepness (DRS). We will discuss both briefly, without proofs.

RORS is used for building a list of the “best” peaks (as judged in terms of relief and steepness) in a region. Since, for a fixed, continuous landscape function $h$, $\text{ORS}(p, h(p); h)$ is a continuous function of $p$, it is nonsensical to compile a list of points with the highest possible ORS values in a given region. This is true of height, as well; lists of the “highest $N$ peaks” in a given region usually use some cutoff criterion to eliminate trivial subpeaks. Instead of pursuing this strategy, we created RORS, which is a variant of ORS which takes into account the degree of independence of a given peak from nearby “better” peaks. Hence it measures a combination of relief, steepness, and independence. For details, we refer the reader to [5], but we can briefly note the most important feature of RORS. It is automatically discrete: for any $\varepsilon > 0$, the set of points $p$ with $\text{RORS}(p) > \varepsilon$ is discrete (and hence finite, in a bounded domain). This makes it a valid list-making criterion; the list of the top $N$ points in a given region, as ranked by RORS, is meaningful. Various such lists are presented on the website [5].

The second concept derived from ORS is more straightforward to define. It is a measure of the ruggedness of a given domain, taking into account both relief and steepness. It is easy to create such a measure using ORS: roughly, we (RMS) average the ORS value for every point in the domain, yielding what we call domain relief and steepness, DRS. However there are two additional issues. First, given a bounded domain $K \subset \mathbb{R}^2$, and a landscape function $h$, we redefine ORS to use sample points only within the given domain. Second, instead of declaring our modified slope integrand $f$ to have $f(u) = 0$ for $u < 0$, we extend it as an even function [14].

With notation as in Section 2, we define the new version of ORS, appropriate

\footnote{Part of the inspiration for this strategy was topographic prominence, a popular alternate mountain measure. See for example [6].}

\footnote{This change is not essential, but it does make the resulting formula more symmetric. It is easy to verify that using the original convention for $f$ instead results in a definition of DRS that is $1/\sqrt{2}$ times that given here.}
to this setting, as

\[ \text{ORS}(p, h_0; h, K) = \| f \circ u \|_{2,K} \]

\[ = \left[ \int \int_{K} f^2 \left( \frac{h_0 - h(x)}{\|p - x\|} \right) \, dA(x) \right]^{1/2} \]

Then we define

\[ \text{DRS}(h, K) = \left[ \frac{1}{A(K)} \int \int_{K} \text{ORS}^2(p, h(p); h, K) \, dA(p) \right]^{1/2} \]

where \( A(K) \) is the area of \( K \). This can be expressed directly in terms of the (new) modified slope integrand \( f \) as follows. Abusing notation slightly, let \( u(p, x) = (h(p) - h(x)) / \|p - x\| \). Then

\[ \text{DRS}(h, K) = \frac{1}{\sqrt{A(K)}} \| f \circ u \|_2 \]

\[ = \left[ \frac{1}{A(K)} \int_{K \times K} f^2 \left( \frac{h(p) - h(x)}{\|p - x\|} \right) \, dA(p) \, dA(x) \right]^{1/2} \]

Note that this (quadruple) integral is symmetric in the variables \( p \) and \( x \), and that it has units of length, just as ORS does (recall that \( f \) is dimensionless).

We will not go into detail regarding DRS here; see [5] for more. However we will make two notes about it.

First, DRS is sensitive to the overall slope of the terrain, but it is continuous in the \( L^\infty \) norm, unlike a functional based on derivatives. Hence it will not give an unreasonably high value to a landscape with low relief, no matter how rugged, nor will its value depend (absurdly) on a particular microscale model of matter. (Think of applying the derivative to the surface of a “flat”, “level” table, but taking into account the atomic-scale bumpiness of the surface—one will not obtain the expected value of zero.)

Second, empirical investigations indicate that the following problem is well-defined (with perhaps some mild regularity assumptions): within a given domain \( K_0 \), what is the domain \( K \subset K_0 \) with maximal ruggedness? Doing this is a tricky problem in calculus of variations, one which we have not investigated completely. However a coarse-gridded numerical approximation to this problem yields stable results. For example, our calculations indicate that the most rugged region in the contiguous 48 states is the Picket Range (GE) of the North Cascades, in Washington State.

5 Future work

There are several directions which we expect to pursue to extend this work. First is the continued measurement and tabulation of the world’s mountain
topography, aided by the progressive improvement in availability of digital data for regions outside the United States. Second are theoretical issues such as dealing with overhanging terrain for ORS and the problem of finding optimally rugged regions for DRS. Third is the possible application of these ideas to image analysis. A grayscale image, for example, is usually modeled as a real-valued function of two variables, to which our functionals could apply. It would be interesting to see if ORS, RORS, and DRS could be used to accomplish some of the standard tasks (or novel ones) in image analysis.

6 Acknowledgements

The seed of the idea of ORS came from Bob Bolton. He and other members of the Prominence electronic discussion group contributed a great deal of feedback to the early work on ORS (then known as “spire measure”). Data sets for the computer calculations of spire measure (done in MATLAB) primarily came from the National Elevation Dataset (U.S.), Canadian Digital Elevation Data (Canada), and the Shuttle Radar Topography Mission (worldwide)—the last with major, invaluable improvements due to Jonathan de Ferranti.

References


Introduction  A summary made for A Guide to the Beth Plan  A quarter of a century ago, in the mid-1980s, General Motors Corporation built the Detroit/Hamtramck Assembly Plant near the intersections of major Detroit freeways and railroad tracks. A number of years ago, in the mid-1980s, General Motors Corporation built the Detroit/Hamtramck Assembly Plant near the intersections of major Detroit freeways and railroad tracks. Proximity to transportation links made sense from a variety of viewpoints. To acquire the land for the large new plant (eventually to cover 362 acres), a combination of deals were employed (eminent domain, purchase, and so forth); some met with more favor than did others (Wikipedia).

The Detroit/Hamtramck Assembly Plant, has extensive security surrounding it. The compound is walled off with a high security gate that requires specific access. The gate is not open to the public. The compound is not known to the majority of people. The compound includes an extensive parking lot and a complex of buildings. This is a typical scene in an urban area; the visual effect is not satisfactory. One has a sense that the single tile might be a challenge with modeling the walls is getting the surface to look correct so that the created visualization is realistic.

A First Step in Creating the Virtual Beth Olem:  The Walls  The walls around it delineate it clearly and make it quickly recognizable. In terms of distance issues for loved ones to visit 24/7, but it will also serve as a basic study in the systematic use (by blog associations) of the CSHS archive, added to the present 'GEOMAT' methodology.

Figure 5. Beth Olem cemetery. Small white circles may be golf balls. Cemetery maintenance crews collect golf balls from the grounds that executives apparently hit at lunchtime into the cemetery from nearby parking. Link from the tombstone to a blog of associated materials. The process of building a virtual marker and get taken to materials from the archive (insofar as privacy concerns permit).

Records in the Chene Street History Study (CSHS) and elsewhere show that this cemetery is the oldest in Detroit. The archives of the Chene Street History Study have many photos taken from inside Beth Olem. The image in Figure 5 is one example that shows clearly the proximity of the cemetery to the Detroit/Hamtramck Assembly Plant. General Motors was not able to acquire that small adjacent to the giant parking lot. Figure 3 shows a small patch of trees that appear more mature than the others on the plant site. The trees appear walled into a rectangular area.

The cemetery is no longer taking new 'residents.' In that regard, it offers to researchers an urban Chene Street scene. Take a closer look; the area to the north end of the plant contains quite a bit of grass. It is a simple matter to capture a swatch of the pattern on the walls from a photograph. It is not possible to use that swatch, only, to create the full wall--at least not in a realistic manner. In Figure 6a, a single swatch of an arbitrary pattern is used to tile a broad area; the visual effect is not satisfactory. One has a sense that the single tile might be a challenge with modeling the walls is getting the surface to look correct so that the created visualization is realistic.

Surface Pattern  There are a number of approaches in the use of digital swatches for modeling pattern. To model a two-dimensional pattern, one must consider the graphical rendering of that pattern. The visual effect is satisfied if the total image is considered. Two-dimensional patterns are typically tiles. The process of creating a two-dimensional pattern is reasonably straightforward, but it is not as simple as it seems. There is a lack of change in the image, and the process is difficult to automate. In Figure 6a, a single pattern is used to tile a broad area; the visual effect is not satisfactory. One has a sense that the single tile might be a challenge with modeling the walls is getting the surface to look correct so that the created visualization is realistic.

A Klein 4 Group:  Beth Olem Cemetery Application  The Klein 4 Group is a group of four elements. It has an order of 4, which means that the group contains four elements. It is a mathematical object that is used in the study of symmetry. In the field of algebra, a Klein 4 group is a group that is defined by four elements. These elements are denoted by e, a, b, and c. The group is defined by the following properties:

- The identity element e.
- The element a satisfies the relation a^2 = e.
- The element b satisfies the relation b^2 = e.
- The element c satisfies the relation c^2 = e.
- The operation is defined by the following table:

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>e</td>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>a</td>
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<tr>
<td>b</td>
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</tr>
<tr>
<td>c</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>e</td>
</tr>
</tbody>
</table>

The Klein 4 group is a simple example of a non-abelian group. It is a group in which the order of the elements matters. The Klein 4 group is also an example of a cyclic group. It is a group that can be generated by a single element. In the case of the Klein 4 group, it can be generated by the element a.

The Klein 4 group is a group that is used in the study of symmetry. It is a fundamental concept in the study of group theory. The Klein 4 group is also a group that is used in the study of geometry. It is a group in which the elements are geometric objects.

The Klein 4 group is a group that is used in the study of symmetry. It is a fundamental concept in the study of group theory. The Klein 4 group is also a group that is used in the study of geometry. It is a group in which the elements are geometric objects.
Institute of Mathematical Geography

walls is that images from the interior side of the walls were used as textures on the outsides; disentangled from the tree limbs. One might further refine the detail of images; that action, flipping it, was employed with the sign for the cemetery, again so that it too might be different times of the year. A similar strategy of using a view from the inside, and then grass stains on the outsides of the walls. The reason there are grass stains on the modeled 'second' permutations 'spatial mathematics' (Arlinghaus and Kerski). What other transformations of the base tile new larger tile by appending the flipped tile to one side of the base tile; it was an exercise in. The case above employed a vertical flip of a rectangular (non-square) base tile to create a. The Klein 4 Group: Pattern Alignment Issues

To introduce appropriate notation, replace the visual pattern in the non-square rectangles with numerical pattern, labelling the vertices of the rectangles as 1, 2, 3, 4. Thus the answer using a structure from a branch of mathematics called group theory. Clearly, one might flip the base tile about a horizontal axis. Further, one might rotate the base tile through 180 degrees and still maintain tile orientation. The 'landscape' tile will rotate to a 'portrait' tile under such a transformation. Are these possibilities. Verify that no new elements were created: all are displayed in the table. Verify that the element 'goes' to the first one, so here '2 goes to 1'. Similarly, represent the horizontal flip '1 goes to 2' then once the end of a parenthetical notational phrase is reached, the last go to bottom. To illustrate how to use the numbers, represent the base tile as the identity permutation on. To improve alignment and consequent appearance and visual impression, one might flip the applied pattern has alignment issues resolved in horizontal strips but not across the entire wall. Look at the grass stains on the bottom to see where the vertical flip was made. Figure 10 confronts the model with the reality of a photograph. There are no alignment if the tile covers the wall from top to bottom. The edges along the tops of the walls, as well as the dots in the walls, align. In Figure 8b, the applied pattern has alignment issues resolved in horizontal strips but not from side to side. To figure out how to use the numbers, represent the base tile as the identity permutation on. Figure 7. Flip of tile about a vertical axis.

The one selected to be exhibited is one that works well for modeling the Beth

Figure 8a shows the flipped tile appended to the base tile; Figures 14a-c suggest one such strategy: apply a vertical flip to the base tile, append that to the base tile (Figure 14a) then apply a 180 rotational flip as (13)(24). Finally, read the table to see that each element is its own inverse:

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>(12)(34)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(13)(24)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(14)(23)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1)(2)(3)(4)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For the Klein 4 Group, element (12)(34) will permute 1 to 2 and 3 to 4, so the first two elements of the product (12)(34)(13)(24) are (12)(34). In the second permutation, 1 goes to 2; in the second permutation, 2 goes to 4. Thus, in the resulting (12)(34)(13)(24) is executed as starting on the left (for example) with 1--in the first permutation, 1 goes to 2; in the second permutation, 2 goes to 4. Thus, in the resulting product, 1 goes to 4 or (14. Now, where does 4 go? Start in the first permutation--4 goes to 3 and in the second permutation, 3 goes to 1. Thus, in the resulting product, 4 goes to 2 and it is correct to close the parentheses: (23). Parenthetically enclosed permutations. It is straightforward from the table that (1)(2)(3)(4) is the identity permutation; (12)(34) is a flip around the vertical axis, (13)(24) is a horizontal flip, and (14)(23) is a 180 degree rotation. Does this set of permutations represent the Klein 4 Group? To determine this, verify that each element has an inverse. Finally, represent the 180 rotational flip as (13)(24). Does this set of permutations represent the Klein 4 Group? To determine this, verify that each element has an inverse. Clearly, there is an identity element: (1)(2)(3)(4), if there is an inverse for each element, if there is an associative law for permutation composition: a(bc)=(ab)c, so (12)(34)*(13)(24) is a valid expression in the Klein 4 Group. Is it so? Is it the case that (1)(2)(3)(4)* (12)(34) = (12)(34)*(1)(2)(3)(4) = (13)(24)? If so, then the set of permutations represents the Klein 4 Group. The group is defined to have four elements that are closed under composition, has an identity, and each element has an inverse. A Klein 4 Group has the following properties:

- Closure: The composition of any two elements is another element in the group.
- Associativity: For all a, b, and c in the group, (a * b) * c = a * (b * c).
- Identity: There exists an element e in the group such that for every element a in the group, e * a = a * e = a.
- Inverse: For every element a in the group, there exists an element b in the group such that a * b = b * a = e.

In summary, the set of permutations represent the Klein 4 Group. To verify this, check that each element has an inverse:

<table>
<thead>
<tr>
<th>Element</th>
<th>Inverse</th>
<th>Verification</th>
</tr>
</thead>
</table>
scales. The process is similar and employs the same general style of reasoning. One may carve out shaped Escher-like fish that fit together perfectly in two different directions and at different pieces of a rectangular tile and glue them on top or bottom or sides and create oddly-tilings. Here, only a simple non-square rectangular tile was considered. One might carve out this document.

Figure 14a. Vertical flip of base tile appended to base tile alignment of the vertical and horizontal flips. The tile is new, the alignment pattern of wall of any tiling can be properly new, a proper way to extend the Klein 4 Group: \[ \text{http://en.wikipedia.org/wiki/Group%28mathematics%29} \]

Klein Four-group, \[ \text{http://en.wikipedia.org/wiki/Klein_four-group} \]

Group Theory, \[ \text{http://en.wikipedia.org/wiki/Group_theory} \]

References for further reading are suggested at the end of this document.

Wikipedia, ‘Wallpaper group’). References for further reading are suggested at the end of this document.


Please contact an appropriate party concerning citation of this article:

All rights reserved worldwide, by IMaGe and by the authors.

Remembering those who are gone now but who contributed in various ways to Solstice.

One article in Solstice was a Pirelli INTERNETional Award Semi-Finalist, 2001 (top 80 out of over 1000 entries worldwide) was a Pirelli INTERNETional Award Semi-Finalist, 2003 (top 80 out of over 1000 entries worldwide).

Build a GEOMAT Web Architecture for any Investigation or Case Study.

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Arlinghaus, Sandra L. 2010.

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In their own special ways to the boldness and grandeur of a mighty river. Our setting beyond the spherical horizon with quiet admiration while others respond trying her hand with portions of the Indonesian Gamelan interactive display!

The Earth inspires us in different ways; some respond to the beauty of the sun, others to the transition of the seasons, to the changing landscape, or even to the musical notes of a song. Music often accompanies us, or sets the scene for a film or performance. Music can touch the soul, stir emotions, and transport us to faraway places. For example, the Indonesian Gamelan is a traditional music of Indonesia, consisting of a variety of orchestras which are usually assembled into a single large ensemble. The music is rich in rhythms and melodic patterns, each of which represents a different region or historical period. The Gamelan instruments include gongs, drums, and xylophones, played both individually and in unison to create a harmonic and complex sound. They are often accompanied by vocal parts that tell stories or convey emotions.

Music, whether traditional or contemporary, can be a powerful tool for cultural expression and education. It can help us understand and connect with other cultures and their histories. It can also serve as a catalyst for creativity and innovation. We often refer to geography and cultural studies as the study of the relationships between people and their environment, and music is one of the most significant cultural expressions that connects us to the world around us. Music can be a window into the past, allowing us to experience the voices and stories of the past, or it can be a tool for shaping the future, allowing us to express our hopes and dreams for the world to come.

In the context of this volume of Solstice, we are exploring the role of music in our daily lives, its relationship to geography, and its cultural significance. We invite you to join us in this exploration, to reflect on the ways in which music shapes our experiences and our understanding of the world.

Music

- Frankie Valli,  "Can't Stop Loving You"
- Donna Summer,  "Last Dance"
- Barry Manilow,  "Mandy"
- Bruce Springsteen,  "Streets of Philadelphia"
- Whitney Houston,  "I Will Always Love You"
- Elton John,  "Candle in the Wind"
- Luciano Pavarotti,  "Nessun Dorma"
- Andrea Bocelli,  "Time to Say Goodbye"
- Pavarotti & Friends,  "The Prayer"

Music with Place Name in Title

- "Geography of the World"
- "Cultural Geography"
- "Geography of Music"
- "Geography and Music"
Institute of Mathematical Geography

http://www.imagenet.org

http://deepblue.lib.umich.edu/handle/2027.42/58219

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One article in Solstice was a Pirelli INTERNETional Award Semi-Finalist, 2003 (Spatial Synthesis Sampler).

Solstice is listed in the Directory of Open Access Journals maintained by the University of Lund where it is maintained as a "searchable" journal.

Solstice is listed on the journals section of the website of the American Mathematical Society, http://www.ams.org/

Solstice is listed in Geoscience e-Journals

IMaGe is listed on the website of the Numerical Cartography Lab of The Ohio State University: http://ncl.sbs.ohio-state.edu/4_homes.html

Congratulations to all Solstice contributors.

Remembering those who are gone now but who contributed in various ways to Solstice or to IMaGe projects, directly or indirectly, during the first 28 years of IMaGe:

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