Thin components made from advanced brittle glasses or ceramics are becoming increasingly important due to the widespread adoption of portable consumer products as well as other modern electronic and medical devices. The strength of these brittle materials is traditionally estimated from empirical relationships relating the stress at failure to characteristic lengths derived from the fracture surface’s topography. One example is Orr’s relationship, \( \sigma_r R_m^{1/2} = A_m \), which correlates the material strength, \( \sigma_r \), to the radius of the “mirror-mist boundary region,” \( R_m \), through the empirical constant, \( A_m \).

Although various studies have shown that, for flexural fractures (failed in bending), \( A_m \) depends on the specimen’s geometry, this effect has been generally neglected by arguing that the magnitude of \( A_m \) is almost constant for thicker specimens. However, we show that this argument cannot be applied to thin geometries, and that by not accounting for the thickness of the sample, the flexural strength will be grossly underestimated. In this work, we introduce an expression based on an iterative fracture mechanics algorithm which yields more accurate estimates of flexural strength for thin brittle components in bending. The accuracy of the model is validated both through flexural strength tests on glass and by comparing our predictions to an extensive literature survey of experimental results.

**I. Introduction**

The increasing demand for smaller and lighter devices with a wide range of applications generates growing interest in the manufacturing of thin brittle components, such as glass screens, silicon chips, and ceramic resistors. For example, boro-silicate glasses (BSGs) as thin as 0.3 mm (Corning’s Willow™; Corning, One RiverFront Plaza, Corning, NY) for liquid crystal displays (LCD) and aluminosilicate glasses (ASGs) as thin as 0.5 mm (Gorilla™; Corning) as cover glasses are becoming the norm in portable consumer products. It is also expected that the first consumer devices to use Corning’s Willow™ Glass, a type of glass so thin (0.1 mm) and flexible that it can be rolled onto spools, will appear in 2013.

The importance of thin brittle components highlights the necessity for an adequate means to estimate fracture strength. Unfortunately, most of the current tools are based on outdated models obtained from testing relatively thick bulk samples. In this manuscript, we propose and validate a fracture mechanics method that accounts for the sample thickness to estimate the fracture strength. Although the proposed model is consistent with Orr’s equation for thick samples, it additionally provides an accurate prediction for thin geometries.

Our findings were validated by comparing predictions with extensive experimental flexural strength data from BSG, soda-lime glass (SLG) and ASG. These glasses are widely employed in electronic consumer products including television sets, laptop computers, smart phones, and tablets. The proposed model also applies to other brittle materials such as ceramics, single crystals, semicrystalline polymers, and glass-ceramics, as the form of the expressions used in the model’s derivation are mechanism independent.

**II. The Mirror Constant**

The mirror-mist constant refers to the proportionality constant, \( A_m \), first described by Orr’s equation [Eq. (1)]. The value of the mirror constant is directly related to the crack tip’s surface morphology during catastrophic crack growth. As cracks propagate during fracture, the surface roughness of the tip increases, as a progressively larger surface area is necessary to dissipate the potential strain energy stored in the material. When the crack tip’s surface roughness approaches the wavelength of visible light, light is scattered by the fracture surface rather than being reflected. This optical interference effect causes the characteristic “hazy” or “misty” appearance at the boundary of the mirror-mist region. The distance from the fracture origin to this hazy region is referred as the mirror radius, \( R_m \).

Figure 1 shows a schematic illustration of the mirror-mist transition on the surface of a glass fractured in flexure or bending. In Figs. (a) and (c) indicate the depth and the half-width of the crack at any given instant, respectively. The general convention for samples fractured in flexure or bending is to measure the mirror radius along the free surface of the plate on the tension side.

The strength of brittle materials is often estimated using well-established empirical relationships such as Orr’s equation. Orr showed that the magnitude of stress at failure, \( \sigma_r \), is linearly correlated with the inverse of the square root of the mirror radius:

\[
\sigma_r = A_m \frac{1}{\sqrt{R_m}} \tag{1}
\]

Figure 2 shows a plot of the average mirror constant, \( A_m \), versus the thickness, \( H \), of glass specimens as obtained by flexural strength tests conducted by various authors. Although the thickness of the sample is not accounted for in Eq. (1), the plot strongly suggests that a correlation might exist between \( A_m \) and the thickness \( H \).

For the last half century, ceramists have tested and correlated the flexural strength of brittle materials with the corresponding lengths of the mirror radii, \( R_m \), resulting in a rich literature. Generally, the mirror constant, \( A_m \), is computed by fitting a linear function between the strength of the material and the inverse of the square root of the mirror radius as...
The addition of the term \( \Delta \sigma_o \) has often been justified by attributing it to “residual stresses” at the surface of the tested sample. Interestingly, \( \Delta \sigma_o \) is almost always reported as a positive term, indicating “tension.” Nonetheless, the presence of tension stresses at the glass surface is inconsistent with the glass-forming process which predicts compressive stresses at the surface prior to annealing. We propose an alternative explanation where the mirror constant, \( A_{m} \), is a function of the sample thickness and the term \( \Delta \sigma_o \) corresponds to the residual surface stress only in the case of thick specimens.

(1) The Mirror Constant’s Magnitude

In the past, various studies showed that the magnitude of the mirror constant, \( A_{m} \), is related to material properties as well as the dimensions of the sample. For instance, Mecholsky et al.\(^6\) gives an equation for the mirror constant relating its value to the critical stress intensity of the material, \( K_{IC} \), the ratio of the initial flaw radius to the mirror radius, \( c_{0}/R_{m} \), and the shape factor for the initial flaw, \( Y_0 \). Mecholsky et al.\(^7\) also showed that for thick samples the ratio \( c_{0}/R_{m} \) is almost equal to the fractal number, \( D^* \). Hence, based on these two works:

\[
A_{m} = \frac{Y_0}{\sqrt{2}} \frac{K_{IC}}{\sqrt{Y_{0m}}^{1/2}} = \frac{Y_0}{\sqrt{2}} \left( \frac{Y_0}{Y_{0m}} \right)^{1/4} \frac{K_{IC}}{D^{*1/2}} \quad (3)
\]

The fractal number, \( D^* \), in Eq. (3) describes the degree of tortuosity of the fracture surface and its value is relatively constant. The term \( Y_0/Y_{0m} \) is the ratio of the shape factor of the initial flaw over the crack’s shape factor at the onset of the mist region. As shape factors are related to the geometry of the sample, it follows that the mirror constant, \( A_{m} \), must also depend on the glass thickness, \( H \).

III. Stress Intensity and Mist Formation

The quasi-static stress intensity factor (SIF), \( K_{I} \), is generally defined as the product of the far-field stress multiplied by the square root of the crack depth, \( a_{m} \), and the shape factor \( Y_{0m} \). Note that although the mist appears at \( R_{m} \), no mist is necessarily present at \( a_{m} \) as the stress intensity is not constant along the crack front. Combining Eqs. (4) and (1), we obtain a relationship describing \( A_{m} \) as a function of \( K_{Im} \) and \( \Delta \sigma_o \) to Eq. (1), an improved fit to the experimental flexural strength data is obtained:

\[
s_{f} = \frac{A_{m}}{\sqrt{R_{m}}} + \Delta \sigma_o \quad (2)
\]

As it will be shown in later in this study, Eq. (5) provides the means to estimate the value of the SIF at the onset of the mist region, \( K_{Im} \).

IV. Crack Evolution Model

The value of the mirror constant, \( A_{m} \), is a function of the crack shape at the onset of the mist region as shown in Eq. (3). To calculate the mirror constant, \( A_{m} \), the crack shape during fast-growth first needs to be calculated and subsequently used to evaluate the shape factor, \( Y_{0m} \). We employ a numerical algorithm based on fracture mechanics principles to estimate the evolution of the crack shape as it propagates into a sample of constant thickness, \( H \). A schematic block diagram describing the steps involved in this iterative crack evolution model is shown in Fig. 3. The algorithm used in this work combines both the works of Dwivedi and Green\(^10\) for subcritical crack growth with the work of Sharon and Fineberg\(^11\) for fast propagating cracks.

The evolution of the crack shape can be divided into two distinct regimes, depending on the factors driving the growth: the subcritical range is driven by environmental factors such as moisture, whereas the fast crack range is driven by the strain energy release rate. The evolution of the crack shape in both regimes can be determined from a marching time numerical algorithm provided that the stress profile, the initial geometry of the specimen, and the crack tip velocity are...
known. In the proposed model the tested samples are monotonically loaded in bending. The original flaw therefore initially propagates in the subcritical growth range until \( K_1 \) reaches the critical value for the material, \( K_{ic} \). As soon as \( K_1 = K_{ic} \), the crack is assumed to be driven by the strain energy release rate.

The crack growth model for subcritical crack growth has been developed and validated by Dwivedi and Green.\textsuperscript{10} The crack velocity for the subcritical crack growth region can be described by the equation:

\[ v = v_c \left( \frac{K_1}{K_{ic}} \right)^n \quad \text{for} \quad K_1 < K_{ic} \]  

\( (6) \)

In this study, \( n = 20 \) and \( v_c = 0.26 \text{ mm/s} \), based on typical SLG values reported by Dwivedi \textit{et al.}\textsuperscript{10}

For fast propagating cracks, the crack tip velocity, \( v_c \), correlates with the square of the critical SIF over the local SIF\textsuperscript{10,11} [Eq. (7)], where \( c_R \) represents the Raleigh speed. Observations by numerous authors indicate that the speed of the crack tip practically only reaches a maximum speed, \( v_{\text{max}} \), of less than half its theoretical \( c_R \) value. Hence, the velocity of the crack tip can be approximated by the equation:

\[ v = c_R \left( 1 - \frac{K_1^2}{K_{ic}^2} \right) \quad \text{for} \quad v < v_{\text{max}} \left( \approx \frac{c_R}{2} \right) \]  

\( (7) \)

Equation 7 is a variation of Freund\textsuperscript{12} equation of motion where the SIF instead of the energy release rate is used in the computation of the crack speed. The term \( K_{ld} \) in Eq. (7) stands for the dynamic SIF. \( K_{ld} \) is related to the static SIF using test data for SLG reported by Sharon and Fineberg.\textsuperscript{11} The values of \( c_R \) and \( v_{\text{max}} \) used in this work are also based on the experimental data reported by Sharon \textit{et al.},\textsuperscript{11} i.e., \( c_R = 3300 \text{ m/s} \), and \( v_{\text{max}} = 1550 \text{ m/s} \). Quinn\textsuperscript{7} and Swartz\textsuperscript{13} also report similar values.

For the fast crack regime, the Newman and Raju\textsuperscript{8} equations are used to determine the quasi-static shape factor at the crack’s tip along the principal axis (i.e., \( Y_0 \) and \( Y_{90} \)). These values are then used to calculate the corresponding dynamic SIF. Based on this SIF calculation, the extensions of the crack along the principal axes of the elliptical crack are computed as \( \Delta a = v_{90} \Delta T \) and \( \Delta a = v_{90} \Delta T \), respectively, where the crack speed is obtained based on either Eqs. (6) or (7) depending on whether the crack is propagating in the subcritical range \( (K_1 < K_{ic}) \) or in the fast crack range \( (K_1 \geq K_{ic}) \). For the fracture of brittle plates in bending, the crack shape is assumed semieliptical until the crack front reaches the depth \( a = 0.8 \cdot H \). Experiments conducted by Sherman and Be’ery,\textsuperscript{14} on SLG plates indicate that, for flexural tests, the crack does not grow into the depth direction as \( a = 0.8 \cdot H \), but rather propagates longitudinally while maintaining a constant crack front shape.

A final remark should be made about how the formation of mist and corresponding length of the mirror radius, \( R_{\text{mir}} \), was determined in our numerical algorithm. Various authors suggested that the mist forms at the crack tip as the SIF reaches \( K_{\text{mir}} \). For convenience in the numerical code, we calculated the onset of the mist region based on the velocity of the crack tip rather than \( K_{\text{mir}} \). Based on Eq. (7), the value of \( K_{\text{mir}} \) uniquely defines the velocity of the crack tip, \( v_{\text{m}} \), at the onset of the mist, and hence the two criteria are equivalent. In particular, for a value of the mist SIF, \( K_{\text{mir}} = 2.3 \text{ MPa-m}^{1/2} \), and \( K_{ic} = 0.75 \text{ MPa-m}^{1/2} \), the velocity of the crack as mist forms is predicted by Eq. (7) as \( v_{\text{m}} \approx 0.5 c_R \), which is consistent with the reported velocity at the onset of the mist region from previous studies.\textsuperscript{15}

\[ \sigma_c = \frac{3 L d}{b H^2} \]  

\( (8) \)

where \( L \) describes the breaking load, \( d \) is the moment arm or distance between adjacent supports and loading edges, \( b \) is the width of the specimen, and \( H \) is the thickness of the specimen. All samples that broke at the rollers were rejected. Equation (8) is valid for small displacements only. For large displacement corrections are generally applied to predict the correct stress at failure. For instance, ASTM Standard D790-02\textsuperscript{16} considers what correction should be applied to the stress equation if the beam experiences large deflections (greater than 10% of the support span). In this work, the fracture strength was computed by finite element analysis software to include the nonlinear effects due to the large deflections.

\section*{VI. Results}

In an effort to better understand the relationship between the mirror radius and the flexural strength of a material fractured in bending, a numerical fracture mechanics model that incorporates sample thickness dependence to predict the mist formation was developed. In this section, we describe the results obtained from testing thin BSG and ASG by 4PTB tests, and augment our experimental data with results.
collated from the literature. Overall, the data analyzed include mirror radii and flexural strength measurements from glass specimens with thicknesses ranging from 0.3 to 38 mm. Table I compares the fitting constants computed using Orr’s equation to our proposed fit for the results obtained from various experimental studies.

To understand the effects of sample thickness, simulations based on the crack evolution algorithm are carried out. With various values of $H$ and initial flaw sizes. For each simulation, a pair of flexural strength and mirror radius is obtained. Repeating this process for a broad set of flaw sizes conditions allows us to obtain trends relating the strength of the glass to the inverse of the square root of the mirror radius $R_d$. Finally, the residual stress based on Orr’s model, $\Delta \sigma_d$, increases as $H$ decreases, whereas no correlation is found between $\Delta \sigma_d$ and $H$. The “residual stresses” predicted by Orr’s relationship therefore appear, for the most part, to be a mismatch artifact caused by the simplified fitting equation.

Figure 4 shows a plot for $A_{ind}/K_{im}$ versus $R_m/H$ for over 770 fractured specimens obtained both from the literature (7, 12, 19, 26, 27) and tests conducted by the authors. It should be noted that Zaccaria and Overend’s data for flint glass flexural strength was omitted in the plot as well as in the calculations of $K_{im}$. As Zaccaria’s manuscript did not report the fixture span used in their measurements, it was not possible to independently verify whether a corrections for large deflection in the fracture strength calculations should have been applied to this set of data. In this analysis, mirror radii larger than 20% of the sample’s width are omitted as the crack-evolution model described in the previous section assumes an infinite sample width and finite sample thickness. SLG, ASG, and BSG are all shown on the same plot as the values of $K_{im}$ are similar for the three cases. For reference, Fig. 4 also shows the trend from the numerical algorithm

### Table I. Summary of Fitting Constants and Test Conditions for Flexural Strength Data on Boro-Silicate (BSG), Soda-Lime (SLG), Alumino-Silicate (ASG), and Flint Glass

<table>
<thead>
<tr>
<th>Author</th>
<th>Glass</th>
<th>$H$ (mm)</th>
<th>$A_{im}$ (Pa·m$^{-1/2}$)</th>
<th>$\Delta \sigma_d$ (MPa)</th>
<th>$K_{im}$ (MPa·m$^{1/2}$)</th>
<th>$\Delta \sigma_o$ (MPa)</th>
<th>Samples tested</th>
<th>Test conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dugnani</td>
<td>BSG</td>
<td>0.3</td>
<td>1.19</td>
<td>88.0</td>
<td>2.18 ± 0.16</td>
<td>−0.7</td>
<td>35</td>
<td>4PTB</td>
</tr>
<tr>
<td>Dugnani</td>
<td>BSG</td>
<td>0.7</td>
<td>1.37</td>
<td>31.6</td>
<td>2.26 ± 0.12</td>
<td>−0.02</td>
<td>12</td>
<td>4PTB</td>
</tr>
<tr>
<td>Dugnani</td>
<td>ASG</td>
<td>0.7</td>
<td>1.14</td>
<td>35.0</td>
<td>1.88 ± 0.05</td>
<td>−12.1</td>
<td>28</td>
<td>4PTB</td>
</tr>
<tr>
<td>Gulati</td>
<td>BSG</td>
<td>0.9</td>
<td>1.94</td>
<td>11.9</td>
<td>2.63 ± 0.08</td>
<td>−25.5</td>
<td>32</td>
<td>RoR</td>
</tr>
<tr>
<td>Dugnani</td>
<td>BSG</td>
<td>1.0</td>
<td>1.48</td>
<td>34.1</td>
<td>2.37 ± 0.15</td>
<td>−15.6</td>
<td>140</td>
<td>4PTB</td>
</tr>
<tr>
<td>Ruggero</td>
<td>SLG</td>
<td>1.0</td>
<td>1.41</td>
<td>15.0</td>
<td>2.06 ± 0.12</td>
<td>−22.7</td>
<td>41</td>
<td>4PTB, annealed</td>
</tr>
<tr>
<td>Choi and Gyekenyi</td>
<td>SLG</td>
<td>1.5</td>
<td>1.52</td>
<td>13.4</td>
<td>1.77 ± 0.27</td>
<td>0.0</td>
<td>12</td>
<td>RoR, annealed</td>
</tr>
<tr>
<td>Mecholsky et al.</td>
<td>SLG</td>
<td>2.0</td>
<td>1.31</td>
<td>18.4</td>
<td>2.34 ± 0.16</td>
<td>−19.5</td>
<td>22</td>
<td>4PTB</td>
</tr>
<tr>
<td>Gaume and Pelletier</td>
<td>SLG</td>
<td>2.2</td>
<td>1.49</td>
<td>19.5</td>
<td>1.90 ± 0.17</td>
<td>0.0</td>
<td>24</td>
<td>4PTB</td>
</tr>
<tr>
<td>Mecholsky and Rice</td>
<td>SLG/Silicate</td>
<td>2.5</td>
<td>2.13</td>
<td>8.7</td>
<td>2.62 ± 0.43</td>
<td>−6.9</td>
<td>21</td>
<td>RoR</td>
</tr>
<tr>
<td>Zacaria and Overend</td>
<td>SLG/Silicate</td>
<td>3.0</td>
<td>1.37</td>
<td>11.0</td>
<td>n/a</td>
<td>n/a</td>
<td>33</td>
<td>4PTB</td>
</tr>
<tr>
<td>Kerper and Scuderí</td>
<td>BSG</td>
<td>4.1</td>
<td>1.85</td>
<td>16.6</td>
<td>2.48 ± 0.16</td>
<td>−2.4</td>
<td>22</td>
<td>Flexure, rods</td>
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<tr>
<td>Schwartz</td>
<td>SLG</td>
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<td>1.84</td>
<td>10.3</td>
<td>2.53 ± 0.19</td>
<td>−9.3</td>
<td>25</td>
<td>4PTB, annealed</td>
</tr>
<tr>
<td>Kirchner and Conway</td>
<td>SLG</td>
<td>4.8</td>
<td>1.88</td>
<td>10.1</td>
<td>1.88</td>
<td>2.7</td>
<td>2</td>
<td>Flexure, rods</td>
</tr>
<tr>
<td>Kirchner and Kirchner</td>
<td>FLint</td>
<td>5.0</td>
<td>1.88</td>
<td>25.0</td>
<td>2.09 ± 0.16</td>
<td>26.3</td>
<td>25</td>
<td>Flexure, rods</td>
</tr>
<tr>
<td>Quinn</td>
<td>BSG</td>
<td>5.3</td>
<td>1.98</td>
<td>9.6</td>
<td>2.60 ± 0.12</td>
<td>−5.0</td>
<td>45</td>
<td>RoR, Annealed</td>
</tr>
<tr>
<td>Kirchner and Kirchner</td>
<td>FLint</td>
<td>6.0</td>
<td>1.68</td>
<td>24.7</td>
<td>2.16 ± 0.10</td>
<td>11.2</td>
<td>13</td>
<td>Flexure, rods</td>
</tr>
<tr>
<td>Kerper and Scuderí</td>
<td>BSG</td>
<td>6.1</td>
<td>2.00</td>
<td>7.0</td>
<td>2.51 ± 0.16</td>
<td>0.3</td>
<td>20</td>
<td>Flexure, rods</td>
</tr>
<tr>
<td>Orr</td>
<td>SLG</td>
<td>6.4</td>
<td>2.01</td>
<td>4.1</td>
<td>2.81 ± 0.05</td>
<td>−12.8</td>
<td>46</td>
<td>RoR</td>
</tr>
<tr>
<td>Gaume and Pelletier</td>
<td>BSG</td>
<td>7.9</td>
<td>1.92</td>
<td>6.9</td>
<td>2.22 ± 0.29</td>
<td>0.0</td>
<td>25</td>
<td>4PTB</td>
</tr>
<tr>
<td>Kerper and Scuderí</td>
<td>BSG</td>
<td>9.9</td>
<td>1.78</td>
<td>8.1</td>
<td>2.17 ± 0.14</td>
<td>4.9</td>
<td>25</td>
<td>Flexure, rods</td>
</tr>
<tr>
<td>Shand</td>
<td>BSG</td>
<td>11.6</td>
<td>2.23</td>
<td>35.1</td>
<td>2.59 ± 0.19</td>
<td>0.0</td>
<td>16</td>
<td>4PTB, annealed</td>
</tr>
<tr>
<td>Ball et al.</td>
<td>SLG</td>
<td>12.5</td>
<td>1.80</td>
<td>1.4</td>
<td>2.11 ± 0.11</td>
<td>0.3</td>
<td>16</td>
<td>3PTB, annealed</td>
</tr>
<tr>
<td>Kerper and Scuderí</td>
<td>BSG</td>
<td>19.1</td>
<td>1.87</td>
<td>7.1</td>
<td>2.13 ± 0.27</td>
<td>8.4</td>
<td>63</td>
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<tr>
<td>Kerper and Scuderí</td>
<td>BSG</td>
<td>25.4</td>
<td>1.98</td>
<td>6.9</td>
<td>2.53 ± 0.17</td>
<td>2.2</td>
<td>39</td>
<td>Flexure, rods</td>
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<tr>
<td>Kerper and Scuderí</td>
<td>BSG</td>
<td>38.1</td>
<td>1.77</td>
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<td>2.28 ± 0.25</td>
<td>4.8</td>
<td>39</td>
<td>Flexure, rods</td>
</tr>
</tbody>
</table>
available in the literature. Nonetheless, Eq. (5) provides a relatively simple way to estimate $K_{im}$ for brittle, isotropic materials.

For a thick plate (i.e., $a_m \ll H$) in bending, or for any sample tested in tension, the most likely shape of the initial flaw after subcritical crack growth is $a_m/c_0 \approx 0.85$ as shown by Dwivedi et al.\textsuperscript{16} A survey conducted on numerous fractured ASG samples\textsuperscript{31} also confirmed that $a_m/c_0 \approx 0.8$. For tensile tests on large specimens (or flexural tests on thick samples), the shape of the flaw will not change substantially during growth; hence, $K_{im}$ can be estimated from Eq. (5), assuming $a_m/R_m = 0.85$ and $Y_{im} \approx 1.3$. This yields a simple expression for estimating $K_{im}$:

$$K_{im} \approx 1.2 \cdot A_{m,T}$$

$A_{m,T}$ in Eq. (11) refers to the mirror constant obtained from tension tests, which are more common in the literature. For instance, using the value $A_{m,T} = 1.72 \pm 0.28$ MPa$\cdot$m$^{1/2}$ for SLG\textsuperscript{32} and $A_{m,T} = 1.9$ MPa$\cdot$m$^{1/2}$ for BSG,\textsuperscript{33} $K_{im}$ can be estimated as 2.1 MPa$\cdot$m$^{1/2}$ (SLG) and 2.3 MPa$\cdot$m$^{1/2}$ (BSG).

(2) The Mirror Behavior

Figure 4 shows a plot of the “dynamic” mirror coefficient, $A_{md}$, normalized by $K_{im}$ versus the ratio $R_m/H$. $A_{md}$ has two asymptotic values corresponding to “infinitely small” and “infinitely large” values of $H$. The first part of our discussion focuses on the expected asymptotic behavior of $A_{md}$ for the two cases of very thin and very thick specimens. Two different approaches can be used to calculate the behavior of the mirror coefficient in these limiting cases.

The first approach used to investigate the behavior for the dynamic mirror constant, $A_{md}$, employs Eq. (5). For the infinitely thick sample case, we expect the crack aspect ratio at the onset of the mirror region, $R_m/a_0 \approx 1.25$ and hence $Y_{im} \approx 1.3$. Substituting into Eq. (5), and assuming an average value of $K_{im} = 2.4$ MPa$\cdot$m$^{1/2}$ for SLG and BSG, the dynamic mirror coefficient $A_{md} \approx 2.1$ as $H \rightarrow \infty$. This value is in good agreement with reported values for the mirror constant found in the literature for tests in tension indicating $A_{m,T} = 1.72$ MPa$\cdot$m$^{1/2}$ (SLG)\textsuperscript{32} and 1.9 MPa$\cdot$m$^{1/2}$ (BSG).\textsuperscript{33} For thin samples, the effective crack’s aspect ratio $R_m/a_0 \approx 3.125$ based on observations made by Sherman et al.\textsuperscript{16} and hence $Y_{im} \approx 1.1$. Substituting into Eq. (5) yields the dynamic mirror coefficient $A_{md} \approx 3.3$ as $H \rightarrow 0$. These analytical asymptotic values based on Eq. (5) are in reasonable agreement with the values expected from the crack evolution model.

A second approach to estimate the value of the dynamic mirror coefficient, $A_{md}$, for the case of a thick sample is using Eq. (3). As for thick samples Eq. (10) tends to Orr’s equation because $A_{m} = A_{md}$ as $H \rightarrow \infty$, it follows that Eq. (3) can be used to study the asymptotic behavior of the mirror coefficient. Taking the limit of Eq. (3) for an infinitely thick sample, the mirror constant, $A_{md}$, can be expressed as follows:

$$\lim_{H \rightarrow \infty} A_{md} = \lim_{H \rightarrow \infty} A_{md} = K_{ic} \frac{Y_0}{\sqrt{2}(c_0/R_m)^{1/2}} \approx \frac{K_{ic}Y_0}{\sqrt{2D^*}}$$

For an infinitely thick sample, the most likely shape for the initial flaw after subcritical crack growth is $a/c \approx 0.85$, as shown by Dwivedi et al.\textsuperscript{16} Corresponding to $Y_0 \approx 1.3$. Various studies estimate the fractal exponent $D^*$ for SLG $^{7}$ ($D^* = 0.08$), BSG $^{34}$ ($D^* = 0.07-0.10$) and ASG $^{7}$ ($D^* = 0.08$). In all cases, only thick samples were considered in the estimation of $D^*$. Assuming a value for the critical strength intensity factor $K_{ic} = 0.7-0.75$ MPa$\cdot$m$^{1/2}$ for BSG, SLG, and ASG, it follows based on Eq. (11) that $A_{md}/H \rightarrow \infty = 2.0-2.6$ MPa$\cdot$m$^{1/2}$. This range is in agreement with the prediction.
of Eq. (5). More accurate measurements for $D^*$ would be required to improve the estimate. Since $D^*$ are similar for the types of glass considered, based on Eqs. (10) and (12), we should also expect $K_{lm}$ to have similar magnitudes. This observation is consistent with the results from the previous section, which indicate almost identical values of $K_{lm}$ for the types of glass considered.

Figure 5 shows that Orr’s equation can accurately describe the behavior of the system for thick glasses in bending (i.e., $H > 2.5$ mm). Nonetheless, Orr’s equation becomes more and more inaccurate for $H < 2.5$ mm, especially for larger values of $R_m$. Equations (13) and (14) give the first and second derivatives of Eq. (10) with respect to $(1/\sqrt{R_m})$:

$$\frac{\partial \sigma_f}{\partial (1/\sqrt{R_m})} = K_{lm} \left\{ 2.02 - 1.2 \exp\left( -0.459 \frac{R_m}{H} \right) \cdot \left[ 1 + 0.918 \frac{R_m}{H} \right] \right\}$$

(13)

$$\frac{\partial^2 \sigma_f}{\partial (1/\sqrt{R_m})^2} = -1.016 \frac{R_m}{H^2} \sqrt{R_m} \exp\left(-0.459 \frac{R_m}{H}\right) \cdot$$

$$\left[ 0.918 \frac{R_m}{H} - 1 \right]$$

(14)

The first derivative [Eq. (13)] can be thought of as the local value of the mirror constant for a given glass thickness and mirror radius. Near the origin of the axes (i.e., for $1/\sqrt{R_m} \to 0$) the slope of the curve is the steepest and its magnitude has a value of 2.02$K_{lm}$. The slope decreases as $1/\sqrt{R_m}$ increases up until the inflection point (i.e., second derivative equals zero) is reached. Past the inflection point, the slope of the curve slowly increases and its value tends to the asymptotic value of 0.81$K_{lm}$.

As seen in Fig. 5, most of the strength data available from the literature fall in the region past the inflection point where the slope of the strength versus $1/\sqrt{R_m}$ is almost constant. In this region, a linear fit as the one proposed by Orr’s equation generally describes the behavior of the curve but it inevitably results in a positive intercept with the ordinate axis. This effect is more obvious for thinner samples. As expected, thicker samples require a higher positive intercept with the ordinate axis when fitted using Orr’s equation. All data analyzed are in the range $R_m < 7$, as the literature does not report experimental results for very thin, low strength glasses.

To obtain the value of the inflection point, Eq. (14) can be set equal to zero, which leads to the equation, $R_m = 1.089H$. For values of $1/\sqrt{R_m}$ to the right of the inflection point (i.e., $R_m < 1.089H$) the slope of the strength versus $1/\sqrt{R_m}$ curve changes gradually and Orr’s equation might be used accurately as long as the intercept $\Delta \sigma_l$ is either kept small ($H < 2.5$ mm) or is known from testing. For approximate values of the mirror radius such that $R_m > 5$ : (1.089H), the slope of the curve changes dramatically as shown in Fig. 5. It follows that flexural strength data in the approximate range 1.089H < $R_m$ < 5 : (1.089H) are likely to fit Orr’s linear equation rather poorly. A typical flexural strength of 70 MPa corresponds to a mirror radius of ~0.75 mm, and falls in the nonlinear range of the strength versus $1/\sqrt{R_m}$ curve for values of $H$ between 0.15 and 0.75 mm.

**VIII. Conclusions**

In this study, we show that the radius of the mirror region obtained from brittle bending fractures is a function of the sample thickness, $H$. A new relationship to estimate the flexural strength of samples using fractographic measurements that include the effect of $H$ is proposed. This has important implications for modern devices in a wide range of industries, where advanced brittle materials in thin geometries are becoming increasingly common.

Although previous models based on Orr’s equation are reasonably accurate for $H > 2.5$ mm, it is found that for $H < 2.5$ mm the linear fit suggested by Orr needs to be offset by a phenomenological stress fit constant, $\Delta \sigma_l$, which is not compatible with stress states typically observed in thin components; although our approach does not necessarily require such adjustments, an analogous $\Delta \sigma_l$ term that better represents the underlying residual stress can be included if desired. In this work, we also describe a simple method for estimating $K_{lm}$ in brittle isotropic materials based on traditional tension tests that are more readily available in the literature.

Our approach is based on a general fracture mechanics crack evolution framework that is applicable to any brittle system. As an example to verify the validity of our model, we applied it to data available from flexural strength tests on SLG, ASG, and BSG. Flexural data from SLG, ASG, and BSG indicate that a single fitting parameter, i.e., the static SIF at the onset of the mist region, $K_{lm}$, can be used to reasonably predict the strength of glass based on the mirror radius measurements.

**References**