Endoscopy for Nilpotent Orbits of G_2

by

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To my parents,
Dan Altschul and Sherry Wallingford

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CHAPTER I

Introduction

In this paper we compute the image of stable, nilpotent combinations of orbital integrals under the transfer map for the Lie algebra of the p-adic group G_2.

1.1 Motivation

Let k be a p-adic field with Weil group W_k. Let G^L be the Langlands dual group of G. The Local Langlands program, stated roughly, conjectures a correspondence between continuous homomorphisms (called L-parameters) $v : W_k \to G^L$ and collections of complex, irreducible, admissible representations of (pure inner forms of) G. The Langlands correspondence is supposed to carry certain number theoretic data associated to the L-parameters to certain representation theoretic information of the members of the L-packets. Furthermore, for nice morphisms between Langlands dual groups, Langlands conjectured a relation amongst the L-packets of the original groups.

The simplest, interesting case of a map between Langlands dual groups was explored by Kottwitz, Langlands, and Shelstad [Kot3, LanShe]; in which one takes the root system Φ_G of a fixed group G and looks at all inclusions of sub root systems

Φ_H that come about in a nice way. There is a reductive k-group $\mathbf{H}$ with the same character lattice as G but with Φ_H as its root system. Such inclusions of root systems give a map from geometric, semisimple conjugacy classes in $\mathbf{H}$ to those of $\mathbf{G}$ which, with a little work, becomes a map of 'stable' invariant distributions. If a distribution D^H maps to a distribution D under this process we call D a transfer of D^H. Conjecturally, to each L-packet Π_v of H we should be able to find a canonical stable combination of the characters of the members of Π_v, call it $S\Theta_v^H$. As an application, one expects that for a fixed G, as one varies H over all endoscopic groups and v over all Langlands parameters of the various H, the $\mathbb{C}$-span of the transfers of all $S\Theta_v^H$ should contain the characters of all admissible, irreducible representations of G. This is very satisfying for a p-adic group theorist because it moves the Langlands Correspondence from a classification of L-packets to a classification of representations.

Work of Kottwitz and Waldspurger has reduced to the Lie algebra many interesting questions in the Langlands program. Particularly, we have a theory of endoscopy on the Lie algebra that is analogous to that of the group with the added benefit that it is more computationally tractable. Where before we had L-packets split up by endoscopic character identities, now we have stable combinations of orbital integrals that are split up by the image of stable combinations on Lie algebras of endoscopic groups.

Of particular interest is the cone of nilpotent elements within the Lie algebra. These are the singular limits of semi-simple classes. A theorem of Harish-Chandra gives that the character of a representation can be written as a sum of Fourier transforms of nilpotent orbital integrals on a neighborhood of the identity. Furthermore, the nilpotent elements have a rather refined, interesting structure theory, lending hope that additional structure can come into play. Hence, it is a natural question to work out the theory of endoscopy for the nilpotent cone in a Lie algebra in the hopes that such a theory provide insight into the theory for representations of groups.

1.2 Prior Results

The problem of computing the stable, nilpotent distributions and their image under the endoscopic transfer maps is undertaken and solved for classical groups in the book of Waldspurger [W3]. We map out Waldspurger's approach:

1. Waldspurger proves a homogeneity theorem which roughly says that the restriction of the nilpotent invariant distributions to a specified subspace of the smooth functions is the restriction of the invariant distributions supported on compact elements to that same subspace.

2. One then constructs a basis for the G-orbits for the left translation action in the subspace of the smooth functions in question utilizing lifts of Lusztig's generalized Green functions. One writes distributions dual to each of these generalized Green functions that are constructed explicitly from orbital integrals coming from regular, semisimple elements.

3. Galois cohomology results for tori of Kottwitz and a little further effort provide enough tools to understand the stability of the tori constructed in step 2. These Galois cohomology computations determine exactly which combinations

of the distributions dual to lifts of generalized Green functions are stable. There is an additional, technical trick here that we shall refer to in the body of the text.

4. Using the homogeneity theorem from part 1 and our stable combinations from part 3, we expand our stable combinations from part 3 in terms of nilpotent orbital integrals. These expansions give us a basis for the stable nilpotent invariant distributions. This step is somewhat elaborate and requires a series of technical tools, including formulas of Kawanaka and Lusztig on Gelfand-Graev characters. Waldspurger uses this stable basis to write down all stable combinations of nilpotent orbital integrals for all elliptic, unramified endoscopic groups of G.

5. Waldspurger then writes explicit transfers for all generalized Green functions, which in turn provides transfers of our stable distributions in part 3, which, via homogeniety arguments, provides the image under the transfer map of the stable, nilpotent combinations from part 4.

Waldspurger's program for attacking such problems has proven fairly amenable to generalization to all reductive, p-adic groups. DeBacker generalized the homogeneity theorem to general groups in [D1]. Major (and inspiring to the author of this work) progress was made in generalizing Waldspurger's program in the paper of DeBacker and Kazhdan [DKaz1], which worked out step 2 in detail for Green functions, and, in the case of G_2, for Lusztig functions. DeBacker and Kazhdan then go on to execute steps 3 and 4 for the group G_2, producing the stable, nilpotent orbital integrals on the Lie algebra of G_2. A follow-up paper worked out step 2 for all Lusztig functions [DKaz2]. The work of Kazhdan and Varshanksy [KazV] found 'nice transfers of the Green functions (but not, to the author's knowledge, the Lusztig functions) in part 2 in complete generality. Particularly, they found that the transfer of a Green function was an explicit combination of related Green functions with some explicitly determined signs defined originally by Weil.

1.3 New Material, a Road Map to this Paper

Our goal was to finish Waldspurger's program for G_2. Specifically, we have found the nilpotent, stable distributions for the Lie algebras of the endoscopic groups of G_2 and found methods to transfer them to $\mathfrak{g}_2$, the Lie algebra of G_2. As G_2 is an exceptional group, we must use a different parameterization of the structure theory than that used by Waldspurger. For this, we go to results of DeBacker [D2]. [D3]. We must then construct various generalized Lusztig functions and distributions dual to them. Most of these constructions and Lemmas are close to the results of DeBacker and Kazhdan, but some (slightly) new arguments are required to deal with SO_4, which occurs as an endoscopic group of G_2. Homogeneity arguments suffice to compute the image of transfer for all groups but SO_4, and all stable, nilpotent combinations in $\mathfrak{so}_4$, the Lie algebra of SO_4, except the two composed of subregular, nilpotent orbits. Finally, we generalize a trick we learned from DeBacker and Kazhdan by transferring twists of the stable combinations so that we can work out the image in $\mathfrak{g}_2$ of the subregular orbits in $\mathfrak{so}_4$.

In Chapters 2, 3, and 4 we lay out the basics of Bruhat-Tits theory, harmonic analysis, and Galois cohomology that we shall require. Chapter 5 is devoted to the definition of endoscopic groups and transfer factors following Kottwitz [Kot3] and Waldspurger [W1]. Chapter 6 constructs the needed generalized Green functions, focusing on concrete definitions for the cases of interest. Chapter 7 constructs the various distributions we need. Up to this point everything is general.

Chapter 8 starts the computations particular to G_2. We describe the structure theory of all endoscopic groups and write down all the distributions from part 6 that we need. Chapter 8 contains all the various cohomology computations we need. We also build a basis for the stable part of the distributions dual to the generalized Green functions in Chapter 8. Chapter 9 contains our Gelfand-Graev character computations. The computations in Chapter 9 allow us to relate the distributions that are dual to generalized Green functions to nilpotent orbital integrals. Here we make some slight improvements on the work of DeBacker and Kazhdan that we expect will be useful for higher rank groups. The work on SO_4 is new.

Chapter 11 then computes all stable, nilpotent distributions on all endoscopic groups of G_2. Chapter 12 proves some small (but delightful!) lemmas that we require to execute the transfer, and chapter 13 computes the image of the transfer map. For the majority of the transfer map, homogeneity arguments combined with some easy non-vanishing results suffice to solve the problem, but to compute the image of the subregular orbits in $\mathfrak{so}_4$, some creativity is required. It is fairly likely the trick used can be combined with a more general formula of Lusztig for writing Gelfand-Graev characters in terms of Deligne-Lusztig representations to get interesting, general results.

In the appendix we include our computations of Gelfand-Graev characters and their pairings with Green functions on G_2 and SO_4. These computations are first done for SL_2, then bootstrapped to SO_4. Here, Kawanaka's formula fails, but Lusztig has a general formula that applies to this case [Lus1]; however, as SO_4 is quite easy to work with directly, we simply work everything out explicitly. The rest of the appendix works out the case of G_2. I am in rough agreement with DeBacker-Kazhdan, but skip one adjustment on the p-adic level and hence compute slightly different factors that arise in Kawanaka's formula.

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CHAPTER II

Basics and Bruhat Tits Theory

In this section we recall the structure theory for a reductive p-adic group. Our notation will defer to that of DeBacker and Kazhdan in [DKaz1]. Before we begin I would like to point out that we shall eventually consider only split groups and tori over fields of large residual characteristic (relative to the rank of the groups). These assumptions significantly reduce the level of caution needed in our definitions. The theory of the building developed here was done first in [BT1, BT2, and MP].

2.1 Basic Structure Theory

Let k be any field. Let $\bar{k}$ be a fixed algebraic closure of k. Let $\mathrm{Gal}(\bar{k}/k) = \Gamma$ be the absolute Galois group of k. Let $\mathbf{V}$ be any k-variety. If A is a k-algebra, let $\mathbf{V}(A)$ be the group of A-points of $\mathbf{V}$ and $V = \mathbf{V}(k)$. We denote by ${}^{\sigma}v$ the action of $\sigma \in \Gamma$ on $v \in \mathbf{V}$. Let $\mathbf{G}$ be a k-group, by which we mean $\mathbf{G}$ is both a k-variety and a group with both the multiplication map and the inversion map being k-morphisms. Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$, again thought of as a k-variety. For any $\mathbf{G}$-space $\mathbf{X}$ with action $(g,x) \to g.x$ and $S \subset \mathbf{X}$, let $Z_{\mathbf{G}}(S) = \{g \in \mathbf{G} | g.x = x \forall x \in S\}$ be the fixator of S and $N_{\mathbf{G}}(S) = \{g \in G | g.S = S\}$ be the stabilizer of S. We write $N_G(S) = \{g \in G | g.S = S\}$ and $Z_G(S) = \{g \in G | g.x = x \forall x \in S\}$ as well. Note that when $\mathbf{X}$ is a k-variety with k-action by $\mathbf{G}$ and $\mathbf{S} \subset \mathbf{X}$ is closed in the Zariski toplogy, both $Z_{\mathbf{G}}(\mathbf{S})$ and $N_{\mathbf{G}}(\mathbf{S})$ are naturally k-varieties. For $X \in \mathfrak{g}(\bar{k})$ we shall sometimes write $\mathbf{G}_X$ for $Z_{\mathbf{G}}(X)$. For $g \in \mathbf{G}(\bar{k})$ and $X \in \mathfrak{g}(\bar{k})$ we will write ${}^gX$ for X acted on by g via the adjoint action.

We now add the hypothesis that $\mathbf{G}$ be a k-quasi split reductive k-group. Let $\mathbf{T_G}$ be a maximal k-torus contained in $\mathbf{B_G}$, a Borel k-subgroup of $\mathbf{G}$. To such a pair $(\mathbf{T_G}, \mathbf{B_G})$ let the quadruple $(X^*(\mathbf{G}, \mathbf{T_G}), \Phi, X_*(\mathbf{G}, \mathbf{T_G}), \check{\Phi})$ be the root datum of $\mathbf{G}$ with choice of positive roots $\Delta_{\mathbf{G}}$ corresponding to $\mathbf{B_G}$. As usual, the reflections in $X^*(\mathbf{G}, \mathbf{T_G}) \otimes \mathbb{R}$ over the hyperplanes on which the roots vanish generate the Weyl group of $\mathbf{T_G}$ in $\mathbf{G}$, $W_{(\mathbf{T_G}, \mathbf{G})} \cong N_{\mathbf{G}}(\mathbf{T_G})/Z_{\mathbf{G}}(\mathbf{T_G})$. Given two maximal tori $\mathbf{T}'$ and $\mathbf{T}''$ we shall mean by $W(\mathbf{T}', \mathbf{T}'')$ the set of cosets $\mathbf{T}'g$ with ${}^g\mathbf{T}' = \mathbf{T}''$. We shall write W_T for $N_G(T)/Z_G(T)$, which may not be the k-points of $W_{\mathbf{T}}$. Note that not all the roots need be defined over k, however all of these objects are acted on by Γ. If one changes to a different maximal k-torus of $\mathbf{G}$, the datum stays the same, however the Γ-action and the fields of definition of the various roots may be altered. We will occasionally drop arguments and subscripts when no confusion is possible.

Let $\mathfrak{g}_\alpha$ be the eigenspace of $\mathfrak{g}$ where $\mathbf{T_G}$ acts by the root α. Then $\mathfrak{g} = \mathfrak{t}_{\mathbf{G}} \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \oplus_{\alpha \in \Delta} \mathfrak{g}_{-\alpha}$. Note that $\mathfrak{b}_{\mathbf{G}} = \mathfrak{t}_{\mathbf{G}} \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ and we get another Borel subgroup called $\mathbf{B}_{\mathbf{G}}^{op}$ that has corresponding Lie algebra $\mathfrak{t} \oplus_{\alpha \in \Delta} \mathfrak{g}_{-\alpha}$.

For any field $\mathbb{F}$, a splitting of any reductive, $\mathbb{F}$ split, $\mathbb{F}$-group is a collection $(\mathbf{T}, \mathbf{B}, \{X_\alpha\}_{\alpha \in \Delta})$ where $\mathbf{T} < \mathbf{B}$ is a $\mathbb{F}$-split, maximal $\mathbb{F}$-torus contained in a Borel $\mathbb{F}$-subgroup, Δ the corresponding set of positive roots and each $X_\alpha \in \mathfrak{g}_\alpha$ is non-zero with the relations ${}^{\sigma}X_{\alpha} = X_{{}^{\sigma}\alpha}$ for all $\sigma \in \mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F})$ and $\alpha \in \Phi$.

We now introduce notions special to the case of a p-adic, reductive group. Let k be a p-adic field with finite residue field $\mathfrak{f}$. Let R be the ring of integers of k. Let $\mathcal{P}$ be the maximal ideal of R. Let K be the maximal unramified extension of k in the fixed algebraic closure $\bar{k}$. Let $\mathfrak{F}$ be the residue field of K. Then $\mathfrak{F}$ may be identified with $\bar{\mathfrak{f}}$. For any extension E/K let R_E be the ring of integers of E and let $\mathcal{P}_E$ be the maximal ideal of R_E. Let Γ_{un} =Gal(K/k). Let Frob$_{\mathfrak{f}}$ be a topological generator for Gal$(\mathfrak{F}/\mathfrak{f})$, and let Frob$_k$ be a topological generator for Γ_{un} lifting Frob$_{\mathfrak{f}}$. Let $q = |\mathfrak{f}|$. Let ν be a discrete, non-trivial $\mathbb{Z}$-valued valuation of k. This valuation extends to K uniquely. This valuation provides a norm and hence a p-adic toplogy on the field k that passes to the k-points of any k-variety. In order to make our work more explicit, we fix a uniformizer ϖ of R, so $\mathcal{P} = \varpi R$.

Let $\mathbf{H}$ be a reductive, complex group on which Γ acts. We call this action an L-action if Γ fixes some splitting of $\mathbf{H}$. A dual group for $\mathbf{G}$, denoted $\hat{\mathbf{G}}$ is a reductive, complex group with root datum $(X_*(G), \check{\Phi}, X^*(G), \Phi)$ and any L-action of Γ. Any two dual groups for G are isomorphic, and any two Γ-fixed splittings are $\hat{\mathbf{G}}^{\Gamma}$-conjugate (See [Kot1, section 1.5), so we can safely consider $\hat{\mathbf{G}}$ unique. For any topological group G_1, we shall also use the Pontryagin dual of a group G_1, which is the set of characters of irreducible, complex representations of G_1. We will only use the Pontryagin dual for abelian or finite groups, so there are no topological or finite dimensionality difficulties to consider. We shall denote the Pontryagin dual of G_1 by G_1^D to distinguish it from the dual group we just defined.

If $\mathbf{G}$ is k-split, after fixing a Borel $\mathbf{B_G}$ containing a maximal k-split k-torus $\mathbf{S_G}$ with set of positive roots $\Delta_{\mathbf{G}}$, one may compute the root datum with respect to this maximal, k-split torus $\mathbf{S_G}$ and the resulting Γ action on the root datum is trivial. Furthermore, one gets a Chevalley basis subordinate to $(\mathbf{S_G}, \mathbf{B_G})$ of $\mathfrak{g}$, $\{X_\phi, H_\delta | \phi \in \Phi, \delta \in \Delta_{\mathbf{G}}\}$ with $H_\delta \in \mathfrak{s}_{\mathbf{G}}(k)$, $X_\delta \in \mathfrak{g}_\delta(k) \subset \mathfrak{b}_{\mathbf{G}}(k)$, and $X_{-\delta}. \in \mathfrak{b}_{\mathbf{G}}^{op}(k)$ with all commutators in $\mathrm{Span}_{\mathbb{Z}}\{X_\phi, H_\delta\}$. As is, there are many options for Chevalley bases. We shall pick a particular Chevalley basis in the near future.

2.2 Bruhat Tits Theory

Let E be a Galois extension of k. Let $\mathcal{B}(\mathbf{G}, E)$ be the Bruhat-Tits building of $\mathbf{G}(E)$. Let $\mathcal{B}(G)$ be $\mathcal{B}(\mathbf{G}, k)$. We may identify $\mathcal{B}(G)$ with $\mathcal{B}(\mathbf{G}, K)^{\Gamma_{un}}$. To each maximal k-split torus $\mathbf{S}$ in $\mathbf{G}$ we may associate an apartment $\mathcal{A}(\mathbf{S}) = \mathcal{A}(\mathbf{S}, k) \cong X_*(\mathbf{S}) \otimes \mathbb{R}$ in $\mathcal{B}(\mathbf{G}, k)$. We identify $\mathcal{B}(\mathbf{S}, k)$ with $\mathcal{A}(\mathbf{S}, k)$. Recall that ν is our discrete valuation. Let $\Psi = \{\nu \circ \phi + n : \mathbf{S} \to \mathbb{Z} | \phi \in \Phi, n \in \mathbb{Z}\}$ be the set of affine roots of $\mathbf{G}$ with respect to $\mathbf{S}$ and ν. There is a natural pairing $\mathbb{Z}\Phi \times \mathbb{Z}\check{\Phi} \to \mathbb{Z}$ that can be extended to $\Psi \times \check{\Phi} \to \mathbb{Z}$. This lets us think of the affine roots as functions on $\mathcal{A}(\mathbf{S})$ with their zero sets giving a simplicial decomposition of $\mathcal{A}(\mathbf{S})$. We call the simplicies in this decomposition facets. We have a partial order on facets given by $F < F'$ if $\bar{F} \subset \bar{F}'$. A maximal facet is called an alcove, while a minimal facet is called a vertex. The reflection group generated by all zero sets of all affine roots of $\mathbf{G}$ with respect to $\mathbf{S}$ is called the affine Weyl group $W_{(G,S)}^{aff}$. The normalizer $N_G(T)$ acts on $\mathcal{A}(T)$ while G acts on $\mathcal{B}(G)$ by simplicial isometries. The stabilizer of an alcove C is called Ω_C or 'the omega group of C.' For any two alcoves C, C' in $\mathcal{B}(G)$ there is some $g \in G$ such that $gC = C'$.

To each point $x \in \mathcal{B}(\mathbf{G}, K)$, Bruhat-Tits theory associates a smooth, R_K-structure on $\mathbf{G}$. When $x \in \mathcal{B}(G)$ the resulting R_K-structure is even a R-structure. In the latter case let $\mathbf{G}_x$ be the resulting R scheme while in the former we are forced to let $\mathbf{G}_x$ be only a R_K group-scheme. In either case, denote the R_K points of $\mathbf{G}_x$ by $G(K)_x$. The result is a parahoric subgroup of $G(K)$. Of course, in the special case $x \in \mathcal{B}(G)$ we may look at the R_k points $\mathbf{G}(k)_x$ to get a parahoric of G, and $\mathbf{G}(K)_x^{\Gamma_{un}} = \mathbf{G}(k)_x$. Let $G(K)_x^+$ be the pro-unipotent radical of $G(K)_x$. There is a connected, $\mathfrak{F}$-group such that $\mathsf{G}_x(\mathfrak{F}) = G(K)_x/G(K)_x^+$. The groups that arise depend only on the facet F to which x belongs, so we will frequently say $G(K)_F$, $G(K)_F^+$, and G_F rather than worrying about choosing a point within F. For a point $x \in \mathcal{B}(G)$ the root system attached to x is the set of all vector parts of the affine roots that vanish at x. We call a point x special if ϕ_x contains a multiple of every root in ϕ. We call a vertex x hyperspecial if it is special in $\mathcal{B}(\mathbf{G}, K)$ and $\mathbf{G}$ is k-quasi split.

Following Moy-Prasad we can define R_K structures on $\mathfrak{g}(K)$ yielding $\mathfrak{g}(K)_x$, $\mathfrak{g}(K)_x^+$, $\mathfrak{g}(K)_F$, and $\mathfrak{g}(K)_F^+$. Let L_F be the Lie algebra of G_F. Then $\mathsf{L}_F(\mathfrak{F}) \cong \mathfrak{g}(K)_F/\mathfrak{g}(K)_F^+$. Let $\dot{\psi}$ be the vector part of the affine root ψ. In the case that x lies in the zero set of some affine root ψ we will also look at $\mathfrak{g}(K)_\psi = \mathfrak{g}_{\dot{\psi}} \cap \mathfrak{g}(K)_x$. This does not depend on which x we pick in the zero set of ψ, as any two points will differ by an element of the zero set of ψ and hence not effect the valuations allowed for the $\dot{\psi}$ root subspace. For $X \in \mathfrak{g}_F$, we shall sometimes let $\bar{X}$ denote the image in $\mathsf{L}_F(\mathfrak{F})$ of X and likewise $\bar{g}$ for the image of $g \in G(k)_x$ in $\mathsf{G}_f(\mathfrak{f})$.

In Chapter 12 we will need to do some depth r analysis. Again from Moy-Prasad, for any $r \in \mathbb{R}$ we also get depth r filtrations, $\mathfrak{g}_{x,r}$, on the Lie algebra. We let $\mathfrak{g}_r = \cup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r}$.

Let $G_0 = \cup_{x \in \mathcal{B}(G)} G_x$ and $G_{0^+} = \cup_{x \in \mathcal{B}(G)} G_x^+$. These are the compact elements and the topologically unipotent elements of the group G respectively. Likewise, $\mathfrak{g}_0 = \cup_{x \in \mathcal{B}(G)} \mathfrak{g}_x$ and $\mathfrak{g}_{0^+} = \cup_{x \in \mathcal{B}(G)} \mathfrak{g}_x^+$ are the compact and topologically nilpotent elements in the Lie Algebra $\mathfrak{g}$.

We now specialize to $\mathbf{G}_2$, the k-split k-group of type G_2. Fix a preferred maximal, k-split, k-torus of $\mathbf{G}_2$ called $\mathbf{S}_{G_2}$ and a preferred Borel k-subgroup $\mathbf{B}_{G_2}$ containing $\mathbf{S}_{G_2}$. Let $\mathfrak{s}_{G_2}$ and $\mathfrak{b}_{G_2}$ be their respective Lie algebras. These choices specify an apartment $\mathcal{A}_{G_2} \subset \mathcal{B}(G_2)$ and a system of positive roots as before. Fix a preferred hyperspecial vertex x_0. Then there is exactly one alcove A_{G_2} contained in $\mathcal{A}_{G_2}$ and containing x_0 in its closure such that all the affine roots that vanish at x_0 and who have vector part equal to one of the simple roots corresponding to $\mathbf{B}_{G_2}$ are positive on A_{G_2}. Fix that A_{G_2}.

Fix a Chevalley basis $\{X_\phi^{\mathbf{G}_2}, H_\delta^{\mathbf{G}_2}\}$ contained in $\mathfrak{g}_{x_0}$. For every facet $F \subset \bar{A}_{G_2}$, we can construct a Chevalley basis, $\{\bar{X}_\phi^{\mathbf{G}_2,F}, \bar{H}_\delta^{\mathbf{G}_2,F}\}$, of L_F contained in $\mathsf{L}_F(\mathfrak{f})$, consisting of images of elements of $(\mathfrak{g}_2)_F(k)$ of the form $\{\varpi^{i_{\phi,F}} X_\phi^{\mathbf{G}_2}, H_\delta^{\mathbf{G}_2}\}$ where $i_{\phi,F} \in \{0, \pm 1\}$. We will use all these Chevalley bases to explicitly describe the various elements and orbits in question. Subscripts and superscripts are surreptitiously absent when no confusion is possible, and by convention elements of a Chevalley basis for a $\mathfrak{f}$-Lie algebra appear with a bar overhead while that of a k-Lie algebra shall appear without a bar.

We shall need some method to determine facets in the buildings of the various groups we consider. In DeBacker and Kazhdan [DKaz1] there is a beautiful parameterization of the facets of G_2 relating the facets to the residue groups that occur and some information about the length of the root. The facets that occur within the confines of this paper occur with a great degree of degeneracy, in particular 2 pairs of facets that cannot be easily distinguished occur in SO_4. Furthermore, similar degeneracies should arise with increasing frequency for higher rank groups, so we adapt a somewhat more cumbersome notation that will fit our goals.

For a k-group $\mathbf{G}$ with alcove A_G, and torus $\mathbf{S_G}$ with $A_G \subset \mathcal{A}(\mathbf{S_G})$, let Ψ_A be the collection of all affine roots with respect to $\mathbf{S_G}$ with zero sets intersecting $\bar{A}_G$ and that are strictly positive on A_G. For $S \subset \Psi_{A_G}$, $F_S = \cap_{\psi \in S}\{x \in \bar{A}_G | \psi(x) = 0\}$. For example, for G_2 with the alcove chosen by DeBacker and Kazhdan, $\Psi_A = \{\alpha, \beta, 1 - 3\alpha - 2\beta\}$. The vertex of type G_2 is $F_{\{\alpha,\beta\}}$, the vertex of type SO_4 is $F_{\{\alpha,1-3\alpha-2\beta\}}$, the facet of type SL_3 is $F_{\{\beta,1-3\alpha-2\beta\}}$, the longest 1 dimensional facet in $\bar{A}_{G_2}$ is $F_{\{\beta\}}$, the second longest 1 dimensional facet is $F_{\{\alpha\}}$, the shortest 1 dimensional facet is $F_{\{1-3\alpha-2\beta\}}$, and $A_{G_2} = F_{\emptyset}$.

2.3 Jacobson-Morosov Theorem and Corresponding Filtrations

Let $\mathbf{G}$ be a reductive, k-quasi split k-group. Let $\mathcal{N}(\mathfrak{g})$ be the nilpotent variety of $\mathfrak{g}$ and $\mathcal{U}(\mathfrak{g})$ be the unipotent variety of G. Let $\mathfrak{g}^{s.s.}$ be the semisimple elements of $\mathfrak{g}$. Let $\mathfrak{g}^{reg}$ be the elements of $\mathfrak{g}$ with $\dim \mathbf{Z_G}(X)$ minimal. Let $\mathfrak{g}^{reg.s.s.}$ be the regular, semisimple elements, that is the elements X s.t. $Z_{\mathbf{G}}(X)^0$ is a torus, and $\mathfrak{g}^{s.r.s.s.}$ be the strongly regular semi simple elements of $\mathfrak{g}$, that is $X \in \mathfrak{g}$ such that $Z_{\mathbf{G}}(X)$ is a torus.

Fix $X \in \mathcal{N}(\mathfrak{g})(k)$. After making certain assumptions on p ($p > h + 2$ where h is the Coxeter number of the group suffices), the Jacobson-Morosov theorem assures the existence of a triple (X, H, Y) with the following properties:

- $[X, Y] = H$, $[X, H] = 2X$, $[H, Y] = 2Y$, and
- The k-linear span is closed under the Lie bracket of $\mathfrak{g}$ and $\mathrm{Span}_k(\{X, Y, H\}) \cong \mathfrak{sl}_2$.
- We have a co-character $\mu \in X_*(G)$ with $d\mu(1) = H$.

Let $\mathfrak{g}(i)$ be the i eigenspace of $\mathrm{Ad}(H)$. Let $\mathfrak{g}(\geq j) = \oplus_{i \geq j}\mathfrak{g}(i)$. All these definitions also make sense over $\mathfrak{f}$. Fix a facet $F \subset \mathcal{B}(G)$ and $e \in \mathcal{N}(\mathsf{L}_F)(\mathfrak{f})$ lifting to $X \in \mathcal{N}(\mathfrak{g})(k)$ then we can choose a compatible triple (e, f, g) of elements of $\mathsf{L}_F(\mathfrak{f})$ such that (e, f, g) lifts to (X, Y, H).

We will require that the residual characteristic of our p-adic field be sufficiently large so that the following conditions are met:

- Every torus of every group we will encounter will be at worst tamely ramified.
- The exponential map of every reductive group over a finite field we encounter will be a k-isomorphism between that group's nilpotent and unipotent varieties.
- Jacobson-Morosov triples will exist for every group and Lie algebra encountered.
- All Killing forms we encounter will be non-degenerate.

- We assume that every torus contains a strongly regular semi-simple element.

These restrictions eliminate finitely many characteristics when working with a fixed group. While some of these assumptions are not particularly demanding, removing the first would certainly require wildly different techniques than we have available in this paper. This would be the most interesting assumption to remove, and the most likely to be of interest for (far) future applications.

CHAPTER III

Some Harmonic Analysis

In this chapter we recall some definitions and theorems on harmonic analysis for reductive, p-adic groups.

3.1 Various Spaces of Functions

Let $\mathbf{V}$ be a variety defined over $\mathfrak{f}$ and let $\mathbf{W}$ be a variety defined over k. Let $\mathfrak{e}$ be a finite field extension of $\mathfrak{f}$ and E be a finite, unramified field extension of k. For $U \subset \mathbf{W}(E)$ (or $U \subset \mathbf{V}(\mathfrak{e})$) let $[U]$ be the characteristic function of U. That is, $[U] : \mathbf{W}(E) \to \mathbb{C}$ is defined by $[U](x) = 1$ if $x \in U$ and $[U](x) = 0$ otherwise. In the special case where $U = \{x\}$ is a one element set we shall write $[x]$ for $[\{x\}]$. Define the space $C_c^\infty(\mathbf{V}(\mathfrak{e}))$ of all $\mathbb{C}$-valued functions on $\mathbf{V}(\mathfrak{e})$. Any extension of the discrete valuation on k to E gives a topology on E called the p-adic topology. This topology extends to a topology of $\mathbf{W}(E)$, again called the p-adic topology on $\mathbf{W}(E)$. Let $C_c^\infty(\mathbf{W}(E))$ denote the space of locally constant, compactly supported $\mathbb{C}$-valued functions on $\mathbf{W}(E)$, with the topology on $\mathbf{W}(E)$ given by the p-adic topology rather than the Zariski topology. For $X \subset \mathbf{W}(E)$ we can restrict the p-adic topology on $\mathbf{W}(E)$ to X and write $C_c^\infty(X)$ for the locally constant, compactly supported $\mathbb{C}$-valued functions on X.

3.2 Haar Measure and the Fourier Transform on the Lie algebra

Let μ_G be the Haar measure on G with $meas_{\mu_G}(G(k)_F^+) = |\mathsf{L}_F(\mathfrak{f})|^{-\frac{1}{2}}$ for all facets F, and $\mu_{\mathfrak{g}}$ be the Haar measure on $\mathfrak{g}(k)$ with respect to the additive group of the underlying vector space with $meas_{\mu_{\mathfrak{g}}}(\mathfrak{g}(k)_F^+) = |\mathsf{L}_F(\mathfrak{f})|^{-\frac{1}{2}}$. This normalization is independent of the facet chosen. We shall occasionally suppress the measure when writing integrals, writing $\int f(g)dg$ rather than $\int f(g)d_\mu(g)$. Let $B(X,Y) = tr(\mathrm{ad}(X)\,\mathrm{ad}(Y))$ be the Killing form, which, by our assumptions on p, is nondegenerate. Fix a non-trivial character on $R/\mathcal{P}$ and lift and extend it to Λ, a character of k. Note that by construction, Λ also gives an unique character of $\mathfrak{f} \cong R/\mathcal{P}$ which we shall also call Λ. For $f \in C_c^\infty(\mathfrak{g})$ define the Fourier transform of f to be:

$$\mathcal{F}(f)(X) = \int_{\mathfrak{g}} \Lambda(B(X,Y))f(Y)dY$$

For $F \subset \mathcal{B}(G)$ a facet, L_F is a variety over $\mathfrak{f}$, so we have $C_c^\infty(\mathsf{L}_F(\mathfrak{f}))$. We shall frequently compare $C_c^\infty(\mathfrak{g}(k)_F)$ with $C_c^\infty(\mathsf{L}_F(\mathfrak{f}))$. For $f \in C_c^\infty(\mathsf{L}_F(\mathfrak{f}))$, define its Fourier transform $\hat{f} \in C_c^\infty(\mathsf{L}_F(\mathfrak{f}))$ to be:

$$\hat{f}(\bar{X}) = |\mathsf{L}_F(\mathfrak{f})|^{-\frac{1}{2}} \Sigma_{\bar{Y} \in \mathsf{L}_F(\mathfrak{f})} \Lambda(B(\bar{X},\bar{Y}))f(\bar{Y})$$

If F is a facet in $\mathcal{B}(G)$ and f is a function on $\mathsf{L}_F(\mathfrak{f})$ then for all $X \in \mathfrak{g}(k)_F$ let $f_F(X) = f(\bar{X})$ and extend by zero to all of $\mathfrak{g}(k)$. We call f_F the 'lift of f.' Note $f_F \in C_c^\infty(\mathfrak{g}(k)_F) \subset C_c^\infty(\mathfrak{g}(k))$. We have chosen our normalizations for Haar measures and the Fourier transform on $C_c^\infty(\mathsf{L}_F(\mathfrak{f}))$ so that $(\hat{f})_F(X) = \mathcal{F}(f_F)(X)$.

Our Haar measure gives us an inner product on $C_c^\infty(\mathfrak{g}(k))$ defined by the equation $< f, g >= \int_{\mathfrak{g}} \bar{f}(x)g(x)dx$, where $\bar{z}$ denotes complex conjugation for $z \in \mathbb{C}$. The fact that both f and g are compactly supported eliminates any convergence questions. The fact that G is reductive and hence unimodular implies this measure is conjugation invariant. While $C_c^\infty(\mathfrak{g}(k))$ is not complete with respect to the metric induced by this inner product, it at least allows us to discuss orthogonality in a sensible way.

3.3 Basics on Induction and Restriction

For any reductive $\mathfrak{f}$-group G with Lie algebra L we have defined $C_c^\infty(\mathsf{L}(\mathfrak{f}))$. Let $C^{\mathsf{G}}(\mathsf{L}(\mathfrak{f})) \subset C_c^\infty(\mathsf{L}(\mathfrak{f}))$ be the subset of $\mathsf{G}(\mathfrak{f})$-invariant functions on $\mathsf{L}(\mathfrak{f})$. There is a pairing $(f, g)_{\mathsf{L}}$ on $C_c^\infty(\mathsf{L}(\mathfrak{f}))$ invariant under the adjoint action of $\mathsf{G}(\mathfrak{f})$ defined by:

$$(f, g)_{\mathsf{L}} = \Sigma_{\bar{X} \in \mathsf{L}(\mathfrak{f})} \bar{f}(\bar{X}) g(\bar{X})$$

Where, for $z \in \mathbb{C}$, we use $\bar{z}$ to denote complex conjugation.

Given a parabolic $\mathfrak{f}$-subgroup $\mathsf{P} < \mathsf{G}$ with Levi decomposition into $\mathfrak{f}$-subgroups $\mathsf{P} = \mathsf{MU}$, with Lie algebras $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}$, we can define a map $\mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}} : C^{\mathsf{M}}(\mathfrak{m}) \to C^{\mathsf{G}}(\mathfrak{g})$ by:

$$\mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}} f(\bar{X}) = \frac{1}{|\mathsf{P}(\mathfrak{f})|} \Sigma_{(\bar{x}, \bar{Z}, \bar{Y}) \in \mathsf{G}(\mathfrak{f}) \times \mathfrak{u}(\mathfrak{f}) \times \mathfrak{m}(\mathfrak{f})} f(\bar{Y})[\bar{X}]({}^{\bar{x}}(\bar{Y} + \bar{Z}))$$

We also get the restriction map $\mathfrak{r}_{\mathsf{P}}^{\mathsf{G}} : C^{\mathsf{G}}(\mathsf{L}(\mathfrak{f})) \to C^{\mathsf{M}}(\mathfrak{m}(\mathfrak{f}))$ defined by:

$$\mathfrak{r}_{\mathsf{P}}^{\mathsf{G}} f(\bar{Y}) = \frac{1}{|\mathsf{G}(\mathfrak{f})||\mathsf{U}(\mathfrak{f})|} \Sigma_{(\bar{z},\bar{Z}) \in \mathsf{G}(\mathfrak{f}) \times \mathfrak{u}(\mathfrak{f})} f({}^{\bar{z}}(\bar{Y} + \bar{Z}))$$

As usual, we have $(\mathfrak{r}_{\mathsf{P}}^{\mathsf{G}} h, f)_{\mathfrak{m}} = (h, \mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}} f)_{\mathfrak{g}}$. If $\mathfrak{r}_{\mathsf{P}}^{\mathsf{G}} f = 0$ for all proper parabolic $\mathfrak{f}$-subgroups $\mathsf{P} < \mathsf{G}$, we call f cuspidal.

We shall need a similar definition of cuspidality over k. For a k-split reductive k-group $\mathbf{G}$ fix a hyperspecial vertex x. Let $K = G_x$. Fix a parabolic k-subgroup $\mathbf{P}$ with Levi decomposition $\mathbf{P} = \mathbf{MN}$ and k-points $P = MN$ and with Lie algebras $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$. Then as N is a closed subgroup of G, we can restrict the Haar measure of G to provide a left-invariant measure on N. We can define a map $f \to f_P : C_c^\infty(\mathfrak{g}) \to C_c^\infty(\mathfrak{m})$ given by the formula:

$$f_P(Y) = \int_K \int_{\mathfrak{n}} f({}^k(Y + Z)) dZ d_K k$$

Now let $\mathbf{G}$ be a p-adic, reductive group. Say $H \subset \bar{F} \subset \mathcal{B}(G)$ are two facets. Then $G_H^+ \subset G_F^+ \subset G_F \subset G_H$ and G_F/G_H^+ can be identified with the $\mathsf{P}(\mathfrak{F})$ for some parabolic $\mathfrak{f}$ subgroup of G_H. Additionally, $\mathcal{U}(\mathsf{P})(\mathfrak{f}) \cong G_F^+/G_H^+$ and any Levi $\mathfrak{f}$-subgroup part of P is isomorphic to G_F. Similar results hold for $\mathfrak{g}_F$, $\mathfrak{g}_H$, etc. We shall let $\mathrm{Ind}_F^H = \mathrm{Ind}_{\mathsf{P}}^{\mathsf{G}_H}$ and $\mathfrak{r}_F^H = \mathfrak{r}_{\mathsf{P}}^{\mathsf{G}_H}$.

3.4 Orbital Integrals and Invariant Distributions

Fix $Y \in \mathfrak{g}$. Let $\mathcal{O}_Y$ be the G-orbit of Y. Note $\mathcal{O}_Y \cong G/Z_G(Y)$. By work of Ranga Rao and Deligne [R] there is an unique up to scaling G-invariant measure on $\mathcal{O}_Y$ which induces an invariant distribution on $C_c^\infty(\mathfrak{g})$ we shall denote by μ_Y. We call μ_Y

the 'orbital integral associated to Y.' In this case we normalize the measure on $\mathcal{O}_Y$ to be the quotient measure derived from G.

Say $X \subset \mathfrak{g}$. Let $\mathcal{N}$ be the nilpotent variety of $\mathfrak{g}$. For any (possibly infinite) dimensional $\mathbb{C}$-vector space V let V^* be the linear dual of V. Let $D(X) = (C_c^\infty(X))^*$, called the distributions supported on X. If $\mathcal{S} \subset C_c^\infty(X)$, let $D(X,\mathcal{S}) = \mathcal{S}^*$ and we get a restriction map $\mathrm{res}_{\mathcal{S}} : D(X) \to D(X,\mathcal{S})$. If $Y \subset X$ then we can identify $D(Y)$ with the subspace $\{T \in D(X) | \forall f \text{ with } \mathrm{supp}(f) \cap Y = \emptyset, T(f) = 0\}$ contained in $D(X)$ via the map $E : D(Y) \to D(X)$, $ET(f) = T(f|_Y)$. For any smooth function f on X and any $g \in G$ let ${}^g f(x) = f({}^g x)$. We call a distribution on X G-invariant if $T({}^x f) = T(f)\ \forall f$ smooth, supported on X. Let $J(X)$ be the space of G-invariant distributions supported on X. We will be particularly interested in $J(\mathcal{N}(k))$, the invariant distributions supported on the nilpotent set. The following results are known: [H-C]

Theorem III.1. *(Harish-Chandra):Let $\mathcal{K} = \{f \in C_c^\infty(\mathfrak{g}) | \mu_X(f) = 0$ for all $X \in \mathfrak{g}^{reg,s.s.}\}$. Say $T \in D(\mathfrak{g})$. Then $T \in J(\mathfrak{g})$ if and only if res $_{\mathcal{K}}T = 0$.*

Theorem III.2. *(Harish-Chandra):$Span_{\mathbb{C}}\{\mu_N | N \in \mathcal{N}(k)\} = J(\mathcal{N}(k))$. In particular, for p-adic reductive groups $dim_{\mathbb{C}} J(\mathcal{N}(k)) < \infty$.*

Less well known is the following definition: for $X \in \mathfrak{g}^{reg,s.s.}$, let $\{X_\sigma\}$ be a choice of representatives of the rational conjugacy classes in the geometric conjugacy class of X. Then $\mu_X^{stab} = \Sigma_\sigma \mu_{X_\sigma}$ is an invariant distribution that does not depend on the choice of the X_σs. Let $\mathcal{K}^{stab} = \{f \in C_c^\infty(\mathfrak{g}) | \mu_X^{stab}(f) = 0$ for all $X \in \mathfrak{g}\}$. We say that a distribution $T \in D(\mathfrak{g})$ is stable if $\mathrm{res}_{\mathcal{K}^{stab}} T = 0$. Let $J^{st}(X)$ be the stable distributions supported on the set $X \subset \mathfrak{g}$.

Recall that we have fixed an alcove $A \subset \mathcal{B}(G)$. Define the space of functions $\mathcal{D}_0 = \Sigma_{A' \subset \mathcal{B}(G)} C_c^\infty(\mathfrak{g}/\mathfrak{g}_{A'})$, where the sum is over all A' alcoves in $\mathcal{B}(G)$. Let $\mathcal{D}_0^0 = \Sigma_{F \subset \bar{A}} C_c^\infty(\mathfrak{g}_F/\mathfrak{g}_A)$. Let $\mathcal{D}_{0+} = \Sigma_{x \in \mathcal{B}(G)} C_c^\infty(\mathfrak{g}/\mathfrak{g}_x^+)$. Notice that the Fourier transform maps $C_c^\infty(\mathfrak{g}_\circ)$ to $\mathcal{D}_{0^+}$.

Theorem III.3. *(DeBacker, Waldspurger)[D1, W3]:*$res_{\mathcal{D}_0} J(\mathfrak{g}_0) = res_{\mathcal{D}_0} J(\mathcal{N})$ *and for* $T \in J(\mathfrak{g}_0)$, $res_{\mathcal{D}_0} T = 0$ *if and only if* $res_{\mathcal{D}_0^0} T = 0$.

The Fourier transform of a distribution is given by the formula $\hat{T}(f) = T(\hat{f})$.

We shall also need the following theorem of Harish-Chandra:

Theorem III.4. *(Harish-Chandra):Let h be a cuspidal function on $\mathfrak{g}$. The function $G \to \mathbb{C}$ given by $g \to \int_{\mathfrak{g}} f({}^g Z) h(Z) dZ$ is smooth and compactly supported.*

Theorem III.5. *(Waldspurger)[W3]:Say $T \in J^{st}(\mathfrak{g}_0)$. By Theorem 2.4.3:*

$\mathrm{res}_{\mathcal{D}_0} T = \Sigma_{\mathcal{O} \in \mathcal{N}(k)/G(k)} c_{\mathcal{O}}(T) \mu_{\mathcal{O}}$

Then: $\Sigma_{\mathcal{O} \in \mathcal{N}(k)/G(k)} c_{\mathcal{O}}(T) \mu_{\mathcal{O}}$ is itself stable.

CHAPTER IV

Galois Cohomology

This chapter covers basics on Galois cohomology that we shall use in our computations. While some results in this section could be stated for mildly more general fields (namely those of cohomological dimension ≤ 1), we shall restrict our attention to the case where k is p-adic. We will frequently use that the Galois group of an unramified extension of k and of a tamely, totally ramified extension of k are both pro-cyclic. In this chapter, we shall attempt to explicitly write the Galois group we are computing cohomology of to assist the reader in following the arguments. Reader be warned: we shall occasionally identify a cocycle representative with its cohomology class.

4.1 Rational Orbits in a Geometric Class

Suppose $\mathbf{V}$ is a k-variety. Let $\mathbf{G}$ act on $\mathbf{V}$ via k-automorphisms. Say $x \in V$. We will often be interested in ${}^{\mathbf{G}(\bar{k})}x \cap V$. Note ${}^{\mathbf{G}(\bar{k})}x \cong [\mathbf{G}(\bar{k})/\mathbf{Z}_{\mathbf{G}}(x)(\bar{k})]$ as k-varieties in all cases we shall study. Consider the exact sequence:

$$0 \to \mathbf{Z}_{\mathbf{G}}(x) \to \mathbf{G} \to \mathbf{G}/\mathbf{Z}_{\mathbf{G}}(x) \to 0$$

Taking k points we get the long exact sequence:

$$0 \to Z_G(x) \to G \to [\mathbf{G}/\mathbf{Z_G}(x)](k) \to H^1(\mathrm{Gal}(\bar{k}/k), \mathbf{Z_G}(x)) \to H^1(\mathrm{Gal}(\bar{k}/k), \mathbf{G})$$

We are interested in G-orbits in $[\mathbf{G}/\mathbf{Z_G}(x)](k)$. This set is in bijection with coker$[G \to [\mathbf{G}/\mathbf{Z_G}(x)](k)]$ =ker$[H^1(\mathrm{Gal}(\bar{k}/k), \mathbf{Z_G}(x)) \to H^1(\mathrm{Gal}(\bar{k}/k), \mathbf{G})]$. More explicitly, let $y \in {}^{\mathbf{G}}x \cap V$. Then $y = g.x$ where $g \in \mathbf{G}(\bar{k})$. For $\sigma \in$Gal$(\bar{k}/k)$, we have $g.x = y = {}^{\sigma}y = {}^{\sigma}(g.x) = {}^{\sigma}g.{}^{\sigma}x = {}^{\sigma}g.x$ therefore $x = (g^{-1})({}^{\sigma}g).x$, so we get a cocyle $a_{\sigma} \in \ker[H^1(\mathrm{Gal}(\bar{k}/k), \mathbf{Z_G}(x)) \to H^1(\mathrm{Gal}(\bar{k}/k), \mathbf{G})]$.

When k is p-adic or finite, $H^1(\mathrm{Gal}(\bar{k}/k, \mathbf{G})$ is finite. For more details, see Serre's book [Ser] for a proof of this result for fields of cohomological dimension ≤ 1, which, in particular, covers both k finite and k p-adic. We shall call the $\mathbf{G}(\bar{k})$-orbits on $\mathbf{V}(\bar{k})$ 'geometric orbits' and the G-orbits in V 'rational orbits.' In the case when $\mathbf{V} = \mathbf{G}$ or $\mathbf{V} = \mathfrak{g}$ with $\mathbf{G}$ acting by conjugation and the adjoint action respectively, we shall say say 'x is a geometric conjugate of y' when $\exists g \in \mathbf{G}(\bar{k})$ s.t. $g.x = y$, and when $y \in {}^{\mathbf{G}}x \cap V$ we shall say 'x is a rational, geometric conjugate' of y. When x is strongly regular semi simple, we may say that 'y is stably conjugate to x.' This definition agrees with the language of Kottwitz [Kot1, Kot2, Kot3] because centralizers of strongly regular elements are connected.

For reductive, p-adic groups we have the following beautiful result of Kottwitz: $H^1(\mathrm{Gal}(\bar{k}/k), \mathbf{G}) \cong (\pi_0(\mathbf{Z}(\hat{\mathbf{G}})^{\Gamma}))^D$. We shall make great use of the Kottwitz isomorphism both to compute the order of these kernels and to write down transfer factors later. For a proof, see Kottwitz [Kot1]. We shall also use the fact that for inclusions of maximal k-tori into reductive k-groups the Kottwitz isomorphism is functorial-Kottwitz [Kot2 Theorem 1.2].

Lemma IV.1. *Let K be a pro-cyclic group with free generator that acts on groups A, B, and C. Let $0 \to A \to B \to C \to 0$ be an exact sequence of K-groups with $im(A) < B$ a normal subgroup. Then the induced map $H^1(K, B) \to H^1(K, C)$ is surjective.*

Proof. :Let σ be a topological generator for K. Let π be the surjective map $B \to C$. Say c_τ is a cocycle in $H^1(K, C)$. Then for all $n \in \mathbb{N}$, $c_{\sigma^n} = c({}^{\sigma}c)({}^{\sigma^2}c)...({}^{\sigma^{n-1}}c)$. Fix any lift b of c_σ. Since σ topologically generates K with no relations, we can define a cocycle in $H^1(K, B)$ by setting $b_{\sigma^n} = b({}^{\sigma}b)({}^{\sigma^2}b)...({}^{\sigma^{n-1}}b)$ and extending continuously to K. As π is a K-equivariant map, $\pi({}^{\sigma^j}b) = {}^{\sigma^j}c$, so $\pi(b_{\sigma^n}) = c_{\sigma^n}$. As $\pi(b_\tau) = c_\tau$ on a dense subset of K, $\pi(b_\tau) = c_\tau$ and our lemma is proved.

□

We will need to use this basic computation in several places:

- When considering unramified tori in a p-adic reductive group (section 8.1).
- When considering nilpotent orbits in the Lie algebra of a p-adic reductive group (section 8.2).
- When considering the stable conjugates of a strongly regular semi simple element in the Lie algebra (sections 9.1 and 9.2).

4.2 Unramified Tori in a Reductive p-adic Group

In this section we follow DeBacker [D3]. Say T is an unramified, maximal k-torus in G. By this we shall mean $T = \mathbf{T}(k)$, $\mathbf{T}$ is a maximal k-torus in $\mathbf{G}$, $\mathbf{T} \subset \mathbf{G}$ as k varieties, and $\mathbf{T}$ splits over an unramified extension. Take our fixed alcove $A_G \subset \mathcal{B}(G)$. Every rational conjugacy class of unramified tori of G contains a representative $\mathbf{T}$ such that the image of $T \cap G(k)_F$ in G_F is a maximal, minisotropic $\mathfrak{f}$-torus in G_F for some facet $F \subset \bar{A}_G$. Thus, to exhaust all possibilities, we need to list pairs (T, F) where F is a facet in A_G and T is an unramified torus lifting an elliptic, maximal $\mathfrak{f}$-torus in G_F. To count each rational conjugacy class once, we say two facets F_1 and F_2 are equivalent if for some (hence any) apartment $\mathcal{A}$ containing both F_1 and F_2, the smallest affine subspace in $\mathcal{A}$ containing ${}^gF_1$ contains F_2 for some $g \in G$.

We must now classify maximal tori in a reductive algebraic group $\mathbf{H}$ defined over $\mathfrak{f}$. Over $\mathfrak{F}$ all maximal tori are conjugate, so fix a maximally $\mathfrak{f}$-split torus $\mathbf{S}$. The stabilizer of $\mathbf{S}$ is $N_{\mathbf{H}}(\mathbf{S})$, thus we get $\ker[H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), N_{\mathbf{H}}(\mathbf{S})) \to H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathbf{H})]$ is in bijection with the $\mathfrak{f}$ conjugacy classes of maximal $\mathfrak{f}$ tori in $\mathbf{H}$. As $H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathbf{H}) = 0$ for all reductive, connected $\mathfrak{f}$ groups $\ker[H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), N_{\mathbf{H}}(\mathbf{S})) \to H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathbf{H})] = H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), N_{\mathbf{H}}(\mathbf{S}))$. Consider the exact sequence:

$$0 \to \mathbf{S} \to N_{\mathbf{H}}(\mathbf{S}) \to \mathbf{W} \to 0$$

Taking $\mathrm{Gal}(\mathfrak{F}/\mathfrak{f})$ fixed points we get:

$$0 \to \mathbf{S}(\mathfrak{f}) \to N_{\mathbf{H}}(\mathbf{S})(\mathfrak{f}) \to \mathbf{W}(\mathfrak{f}) \to H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathbf{S}) \to$$

$H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), N_{\mathbf{H}}(\mathbf{S})) \to H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathbf{W})$

By Lang's theorem, $H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathbf{S}) = 0$, therefore $H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), N_{\mathbf{H}}(\mathbf{S})) \to H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathbf{W})$ is injective. As $\mathrm{Gal}(\mathfrak{F}/\mathfrak{f})$ is pro-cyclic, Lemma IV.1 shows the map $H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), N_{\mathbf{H}}(\mathbf{S})) \to H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathbf{W})$ is also surjective. Combining these facts, $H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), N_{\mathbf{H}}(\mathbf{S})) \cong H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathbf{W})$.

In practice we can compute $H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathbf{W})$ explicitly, so the problem of classifying rational conjugacy classes of unramified, maximal k-tori in G is effectively solved.

4.3 Nilpotent elements in a reductive p-adic group

We again follow a paper of DeBacker [D2]. Let E be a nilpotent element in the Lie algbra of a reductive p-adic group G. Fix an alcove $C \subset \mathcal{B}(G)$. Then there is some $g \in G$ such that ${}^gE \in \mathfrak{g}(k)_F$ for some $F \subset \bar{C}$.

Note that if $E \in \mathfrak{g}(k)_F$ is nilpotent, then the image of E in $\mathsf{L}_F(\mathfrak{f})$ is also nilpotent. Furthermore, the preimage of any nilpotent in $\mathsf{L}_F(\mathfrak{f})$ intersects a unique nilpotent orbit of $\mathfrak{g}(k)_F$ of minimal dimension. Thus, for each pair (F, e) with $F \subset \mathcal{B}(G)$ a facet and $e \in \mathcal{N}(\mathsf{L}_F)(\mathfrak{f})$ a nilpotent of L_F we can produce a lift in $\mathcal{N}(\mathfrak{g})(k)/G$. We let $\mathcal{O}_{(F,e)} \subset \mathcal{N}(\mathfrak{g})(k)$ be the G-orbit of the lift.

However, two different pairs can yield the same nilpotent orbit in $\mathfrak{g}(k)$. For a fixed apartment $\mathcal{A}$ in the building $\mathcal{B}(G)$ and a subset $X \subset \mathcal{A}$ let $A(\mathcal{A}, X)$ be the smallest affine subspace of $\mathcal{A}$ containing X. For two facets F and F' we say that F is associated to F' if there is an apartment $\mathcal{A}$ containing both F and F' with $A(\mathcal{A}, gF) = A(\mathcal{A}, F')$ for some $g \in G$. In this case we get natural maps $\mathfrak{g}(K)_{gF} \cap \mathfrak{g}(K)_{F'} \to \mathsf{L}_{gF}(\mathcal{F})$ and $\mathfrak{g}(K)_{gF} \cap \mathfrak{g}(K)_{F'} \to \mathsf{L}_{F'}(\mathcal{F})$ that are surjective maps with kernel $\mathfrak{g}(K)^+_{gF} \cap \mathfrak{g}(K)^+_{F'}$ which provide an isomorphism $\mathsf{L}_{gF} \cong \mathsf{L}_{F'}$. For nilpotents $e \in \mathcal{N}(\mathsf{L}_F)(\mathfrak{f})$ and $e' \in \mathcal{N}(\mathsf{L}_{F'})$ we say e is associated to e' if ${}^g e = e'$ after we identify L_{gF} with $\mathsf{L}_{F'}$ via the above isomorphism. Likewise for functions $f \in C_c^\infty(\mathsf{L}_F(\mathfrak{f}))$ and $f' \in C_c^\infty(\mathsf{L}_{F'}(\mathfrak{f}))$.

Theorem IV.2. *(DeBacker)[D2]:$\mathcal{N}(\mathfrak{g})(k)/G(k)$ is in bijection with the set of associativity classes of pairs (F, e).*

As the closure of an alcove is a fundamental domain for $\mathcal{B}(G)$ under the G-action, we may find representatives for the associativity classes of pairs of the form (F, e) with $F \subset \bar{A}$. For each representative we compute the geometric nilpotent orbits in $\mathsf{L}_F(\mathfrak{F})$ via Bala-Carter theory[C]. Fix $E \in \mathsf{L}_F(\mathfrak{F})$. Then the rational orbits in the geometric class are in bijection with $\ker[H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathbf{Z}_{\mathsf{G}_F}(E)) \to H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathsf{G}_F)]$. Again, $H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathsf{G}_F) = 0$ and we need only compute $H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), \mathbf{Z}_{\mathsf{G}_F}(E))$. By a theorem of Steinberg, connected algebraic groups over $\mathfrak{f}$ have trivial Galois cohomology, hence we will be able to pass to the component group. In all cases we need we can compute this explicitly.

4.4 Rational Classes in a Stable Strongly Regular Semi Simple Orbit

Say X is a strongly regular semi-simple element in $\mathfrak{g}$. That is $\mathbf{Z}_{\mathbf{G}}(X) = \mathbf{T}$ some maximal torus in $\mathbf{G}$. Then, as before, the rational classes in the stable conjugacy class of X will be in bijection with the set $\ker[H^1(\mathrm{Gal}(\bar{k}/k), \mathbf{T}) \to H^1(\mathrm{Gal}(\bar{k}/k), \mathbf{G})]$.

By Steinberg's theorem for any connective, reductive k-group $\mathbf{H}$ we have that $H^1(\mathrm{Gal}(\bar{k}/K), \mathbf{H}) = 0$. By inflation restriction we get $0 \to H^1(\Gamma_{un}, \mathbf{H}(K)) \to H^1(\Gamma, \mathbf{H}) \to H^1(\mathrm{Gal}(\bar{k}/K), \mathbf{H})^{\Gamma_{un}} = 0$, so $H^1(\Gamma, \mathbf{H}) = H^1(\Gamma_{un}, \mathbf{H}(K))$. Thus we shall need to compute $\ker[H^1(\Gamma_{un}, \mathbf{T}(K)) \to H^1(\Gamma_{un}, \mathbf{G}(K))]$. We shall always do this explicitly.

CHAPTER V

Endoscopic Groups

In this chapter we define endoscopic groups and transfer factors. We also compute the endoscopic groups of G_2. More theory for endoscopy will follow in chapter 12, but we shall only need to work with one group at a time until then. We shall work with k-split groups to avoid needless technicalities involving twisting to a quasi-split inner form. Most of this is out of the papers of Kottwitz [Kot1, Kot2]. For a view on how the theory develops without assuming G k-quasi split see the paper of Kazhdan and Varshavsky [KazV].

5.1 Endoscopic Groups and Transferring Conjugacy Classes

For any k-split, k-group $\mathbf{G}$ with root datum $(X^*(\mathbf{G}), \Phi_{\mathbf{G}}, X_*(\mathbf{G}), \check{\Phi}_{\mathbf{G}})$, the dual group $\hat{\mathbf{G}}$ is the $\mathbb{C}$-group with root datum given by $(X_*(\mathbf{G}), \check{\Phi}_{\mathbf{G}}, X^*(G)_{\mathbf{G}}, \Phi_{\mathbf{G}})$ with L-action given by Γ acting trivially. Let $\mathbf{S}$ be a maximal k-split, k-torus of $\mathbf{G}$. Pick $w \in W(\mathbf{G})$ and $s \in \mathrm{Hom}(X_*(\mathbf{G}), \mathbb{C}^\times) \cong \hat{S}^{\Gamma}_{\mathbf{G}}$. We shall call such a pair (w, s) an endoscopic pair for G. Let $\check{\Phi}_H = \{\alpha \in \check{\Phi}_G | s(\alpha) = 1\}$, let $\sigma_H = w \circ \sigma_G$, and let $\Phi_H = \check{\Phi}^*_H$. Then the quadruple $(X^*(\mathbf{G}), \Phi_H, X_*(\mathbf{G}), \check{\Phi}_H)$ with Γ action given by having Frob_k act by σ_H is the root datum of an unramified, reductive k-group $\mathbf{H}$ and we say 'H is an endoscopic group of G.' Let $\mathbf{S}_H$ be a maximally k-split, maximal k-torus of

$\mathbf{H}$ such that the root datum constructed above is the root datum of $\mathbf{H}$ with respect to the torus $\mathbf{S_H}$, and let $\mathfrak{s}_\mathbf{H}$ be its Lie algebra. Then the data $(H, \mathbf{S_H}, s)$ is an endoscopic triple of G. Note this triple now depends on a choice of $\mathbf{S_G} < \mathbf{G}$ to make sense. We call H an unramified, endoscopic group of G if $(\mathbb{R} \otimes_\mathbb{Z} \Phi_\mathbf{G})^{W(\Phi_H) \rtimes < \sigma_\mathbf{H} >} = 0$.

Note $W(\mathbf{H}) < W(\mathbf{G})$ and that the identification of $X^*(\mathbf{G})$ with $X^*(\mathbf{H})$ is Γ-equivariant. Given an endoscopic triple, we have an isomorphism $\mathfrak{s}_G \cong \mathfrak{s}_H$ given by our identification of $X^*(\mathbf{G})$ with $X^*(\mathbf{H})$. By Chevalley's theorem, semi-simple classes in $\mathfrak{g}$ are classified by $\mathfrak{s}_\mathbf{G}/W(\mathbf{G})$ and semi-simple classes in $\mathfrak{h}$ are classified by $\mathfrak{s}_\mathbf{H}/W(\mathbf{H})$. Since $\Phi_\mathbf{H} \subset \Phi_\mathbf{G}$, $W(\mathbf{H}) < W(\mathbf{G})$. Combining the isomorphism $\mathfrak{s}_\mathbf{G} \cong \mathfrak{s}_\mathbf{H}$ with the inclusion of Weyl groups, we get a map $\mathcal{A}_{H/G} : H^{s.s.} \to G^{s.s.}$, called the transfer map. A lemma of Langlands and Shelstad [LanShe, 1.3A] shows this map is Γ-equivariant. Note that $\mathcal{A}_{H/G}$ is finite to one, with degree $[W(\mathbf{H}) : W(\mathbf{G})]$ on regular elements in H that map to regular elements in G. Note further that $\mathcal{A}_{H/G}$ is a map of conjugacy classes, not of elements. As we are working on geometric conjugacy classes and all split, maximal k-tori of G (resp. H) are conjugate by Groethendieck's theorem, our choices of maximal tori (and hence our choices of endoscopic triple from our endoscopic pair) are irrelevant for the definition of $\mathcal{A}_{H/F}$.

If X_G represents a conjugacy class in $\mathfrak{g}$ and X_H represents a conjugacy class in $\mathfrak{h}$ such that the transfer map sends the class of X_H to the class of X_G we call X_G an image of X_H. In the case that X_G is an image of X_H and X_G is (strongly) regular semi-simple we shall call X_H G-(strongly) regular. Denote by $\mathfrak{h}^{G-reg}$ the set of G-regular semi simple elements in $\mathfrak{h}$ and by $\mathfrak{h}^{G-s.reg}$ the set of G-strongly regular semi simple elements.

5.2 Transfer Factors on the Lie Algebra

We would like to use the transfer map $\mathcal{A}_{H/G}$ to map orbital integrals on $\mathfrak{h}$ to orbital integrals on $\mathfrak{g}$. Unfortunately, the obvious map taking functions in $C_c^\infty(\mathfrak{h})$ to functions in $C_c^\infty(\mathfrak{g})$ given by the formula $f'(X) = \Sigma_{Y s.t. \mathcal{A}_{H/G}(Y)=X} f(Y)$ fails to preserve smoothness of functions. Particularly, smoothness fails at the identity. To fix this difficulty we modify the obvious map by adding what are called transfer factors. These transfer factors are only well defined after a choice of a k-splitting of G. Our transfer factors will be a product of two factors:

$$\Delta_G^H(X_H, X_G) = \Delta_{G,I}^H(X_H, X_G)\Delta_{G,II}^H(X_H, X_G)$$

Both factors will vanish unless X_G is an image of X_H. The first factor is a character on the Galois cohomology classifying the stable conjugates of X_G. We have $s \in \hat{T}_G^\Gamma$ and we can identify $\pi_0(\hat{T}_G^\Gamma) = H^1(\Gamma, T)^D$ via the Tate-Nakayama pairing. Note that this factor depends not only on the group H, but also on the s that is used in its construction.

The second factor will be defined from the root data of both G and H and is designed to make the transfer factor invariant under different choices of tori and splitting. However, Kottwitz showed in [Kot3] that the k-splitting of G can be chosen such that $\Delta_G^H(X_H, X_G) = 1$ whenever X_G lies in a fixed Kostant section of G and X_G is an image of X_H. Thus, we shall never need to directly compute $\Delta_{G,II}^H(X_H, X_G)$, as we shall only need to compare the rational class X_G lies to our fixed Kostant section[Kos, Kot3]. For completeness we define $\Delta_H^G(X_H, X_G)$ in complete generality, but the uninterested reader may freely skip to section 5.3. See [LanShe] for details.

For $\alpha \in \Phi_G$ let k_α be the field of definition of α and let $k_{\pm\alpha}$ be the field of definition of the set $\{\alpha, -\alpha\}$. Then $k \subset k_{\pm\alpha} \subset k_\alpha$. If $[k_\alpha : k_{\pm\alpha}] = 2$ we say that α is a symmetric root. Call the set of symmetric roots Φ^{sym}. A collection $\{a_\alpha \in \bar{k}^\times | \alpha \in \Phi_G, a_{\alpha^\sigma} = a_\alpha^\sigma, a_{-\alpha} = -a_\alpha\}$ is called an a-data. Fix any a-data for G. Let χ_α be the character of $k_{\pm\alpha}^\times$ associated to the extension $k_\alpha/k_{\pm\alpha}$ by local class field theory. Let $\Delta_{G,I}^H(X_H, X_G) =< \lambda(S_G), s_{S_G} >$, where $\lambda(S_G)$ is the image of $\lambda(S_G^{sc})$ in $H^1(\Gamma, S_G)$ under the map induced by the canonical homomorphism $S_G^{sc} \to S_G$ (S^{sc} being the k-points of the split torus in the simply connected cover of $\mathbf{G}$ lying over $\mathbf{S}$) and $\lambda(S_G)$ is the invariant defined by Langlands and Shelstad in [LanShe]. Let $\Delta_{G,II}^H(X_H, X_G) = \Pi_{\alpha \in \Phi^{sym}} \chi_\alpha(\frac{\alpha(X_G)}{a_\alpha})$. The definition of Δ_G^H does not depend on the a-data used.

Much of the material in Langlands' and Shelstad's work on endoscopy does not come up on the Lie algebra. Particularly, we are skipping discussion of the notion of a χ-data and three additional multiplicands in the transfer factors for reductive p-adic groups. One factor related to the Weyl discriminant will later get baked into our nilpotent orbital as a normalizing factor. We will not mention them further, as we are only interested in the theory of endoscopy for the Lie algebra in this work.

5.3 The Elliptic Unramified Endoscopic Groups of G_2

We now specialize back to the case where $\mathbf{G}$ is the split k-group of type G_2. Recall we have fixed a hyperspecial vertex and an alcove containing it, hence a choice of positive roots Δ_{G_2}. Let α be the short root and β the long root in Δ_{G_2}. There are four possibilities for the isomorphism class of an endoscopic group. If the roots included in Φ_H are $\{\pm\alpha, \pm(3\alpha + 2\beta)\}$ then the isomorphism class of the resulting endoscopic group is SO_4. A second possibility for Φ_H is $\{\pm\beta, \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\}$ with the resulting endoscopic group of type PGL_3. The third possibility is all roots of G_2, in which case the endoscopic group is isomorphic to G_2. The fourth possibility is no roots in Φ_H, in which case one gets an unramified torus. All other possibilities end up not being elliptic. The tori have only the trivial nilpotent orbit, which is automatically stable. Furthermore, our homogeneity theorem in chapter 12 and some dimension counting will show that the trivial orbit for a torus will always transfer to the regular orbit in $\mathcal{N}(\mathfrak{g}_2)$, so we shall not pay much attention to tori that are elliptic, unramified endoscopic groups of G_2. This leaves us with three cases to work with in general.

When discussing the original group G_2 we shall discuss roots as before with no superscript. However when we wish to look at an endoscopic group of G_2, we will label the root with the Lie type of the endoscopic group in question as a superscript. For example, α would be a short root of G_2 while α^{SO_4} would be a root of SO_4. Here is the dictionary between roots in on $\mathfrak{g}_2$ and our choice of Δ_H:

Root on $\mathfrak{g}_2$	Root on $\mathfrak{pgl}_3$
$3\alpha+\beta$	α^{PGL_3}
β	β^{PGL_3}

Root on $\mathfrak{g}_2$	Root on $\mathfrak{so}_4$
α	α^{SO_4}
$3\alpha+2\beta$	β^{SO_4}

We shall also now construct various Chevalley bases analogous to those in section 1. To do this in a compatible way we will need to identify a maximal k-split k-torus and a Borel k-subgroup of each endoscopic group that maps in a nice way to that of $\mathbf{G}_2$. For each $\mathbf{H}$ we have our k-split k-torus $\mathbf{S_H}$ that maps via transfer to our fixed $\mathbf{S}_{\mathbf{G}_2}$ selected in section 1. For each $\mathbf{H}$ we have an inclusion $i_{\mathbf{H}} : \Phi_{\mathbf{H}} \hookrightarrow \Phi_{\mathbf{G}_2}$. We set $\Delta_{\mathbf{H}} = i^{-1}(i(\phi_{\mathbf{H}}) \cap \Delta_{\mathbf{G}})$. Fix a hyperspecial vertex $x_{0,\mathbf{H}} \in \mathcal{A}(\mathbf{S_H})$. Our choice of a system of positive roots determines an alcove $A_{\mathbf{H}}$ and a Chevalley basis $\{X_\phi^{\mathbf{H}}, H_\delta^{\mathbf{H}}\}$ of $\mathfrak{h}$ =Lie($\mathbf{H}$) consisting of vectors in $\mathfrak{h}_{x_{0,\mathbf{H}}}(k)$. For every $F \subset \bar{A}_{\mathbf{H}}$ facet, we can construct a Chevalley basis of L_F contained in $\mathsf{L}_F(\mathfrak{f})$, $\{\bar{X}_\phi^{\mathbf{H},F}, \bar{H}_\delta^{\mathbf{H},F}\}$, consisting of images of elements of $\mathfrak{h}_F(k)$ of the form $\{\varpi^{i_{\phi,F}} X_\phi^{\mathbf{H}}, H_\delta^{\mathbf{H}}\}$ where $i_{\phi,F} \in \{0, \pm 1\}$.

CHAPTER VI

Generalized Green Functions and Gelfand-Graev Characters

In this section we establish the facts needed about representations of reductive groups defined over $\mathfrak{f}$. We then lift many of these results to $\mathfrak{g}(k)$.

6.1 Deligne Lusztig Representations and their Characters

We will follow the conventions laid out by Lusztig. We refer to Lusztig [Lus1, Lus3, Lus4] and Carter [C]. Let $\mathbf{G}$ be a reductive $\mathfrak{f}$ group. Let $\mathbf{T} < \mathbf{B} < \mathbf{G}$ be a maximal $\mathfrak{f}$-torus contained inside a Borel $\mathfrak{f}$-group. Let $\mathbf{U} < \mathbf{B}$ be the maximal unipotent subgroup of $\mathbf{B}$. Let Θ be a character of $\mathbf{T}(\mathfrak{f})$. Let $L : \mathbf{G} \to \mathbf{G}$ be the Lang isogeny, given by the equation $L(g) = g^{-1}\mathrm{Frob}_{\mathfrak{f}}(g)$. Let $\tilde{X}_{\mathbf{T}}$ be the $\mathfrak{f}$-variety $L^{-1}(\mathbf{U}) \subset \mathbf{G}$. Deligne and Lusztig constructed a representation $R^{\Theta}_{T,G}$ of $\mathbf{G}(\mathfrak{f})$ on the vector space $\oplus H^i_{et}(\tilde{X}_{\mathbf{T}}, \mathbb{Q}_l)$ with character $R^{\Theta}_{\mathbf{T},\mathbf{G}}(g) = \Sigma_i(-1)^i tr(g, H^i_{et}(\tilde{X}_{\mathbf{T}}, \mathbb{Q}_l)_{\Theta})$. These characters are called Deligne-Luztig characters.

We say a set of functions $\mathcal{C}$ separates classes under an equivalence relation if the characteristic function of each equivalence class lies in the $\mathbb{C}$-linear span of $\mathcal{C}$. The Deligne-Lusztig characters separate rational, semisimple classes in $\mathbf{G}(\mathfrak{f})$. Furthermore, for Θ regular, the Deligne-Lusztig character differs from an irreducible character of $\mathbf{G}(\mathfrak{f})$ by at most a sign that can be explicitly computed. These characters take values in the roots of unity in $\mathbb{Q}_l$, so after identifying the roots of unity in $\mathbb{Q}_l$ with those of $\mathbb{C}$ we may (and will!) consider the Deligne-Lusztig characters as complex valued functions.

Let $\mathcal{U}(\mathbf{G})$ be the unipotent variety of $\mathbf{G}$. For $u \in \mathcal{U}(\mathbf{G})(\mathfrak{f})$, the value of $R^{\Theta}_{\mathbf{T},\mathbf{G}}(u)$ does not depend on Θ, thus we have the Green functions $Q_{\mathbf{T}}(u) = R^{1}_{\mathbf{T},\mathbf{G}}(u)$ defined on the $\mathfrak{f}$-points of unipotent variety $\mathcal{U}(\mathbf{G})(\mathfrak{f})$. Due to our assumptions on the characteristic of $\mathfrak{f}$, we can (and again will!) identify the unipotent variety of $\mathbf{G}$ with the nilpotent variety of $\mathfrak{g}$ via the $\mathbf{G}$-equivariant logarithm map. As our interest is focused on the Lie algebras, we shall from now on think of Green functions as defined on $\mathcal{N}(\mathfrak{f})$, the $\mathfrak{f}$-nilpotent variety in $\mathfrak{g}(\mathfrak{f})$, rather than $\mathcal{U}(\mathfrak{f})$. When we wish to specify which group we are computing the Green function on we shall include a superscript, for example $Q^{\mathbf{G}}_{\mathbf{T}}$ vs $Q^{\mathbf{H}}_{\mathbf{T}}$. See Cart [C] for details.

For $\mathfrak{f}$-tori $\mathbf{T}$, $\mathbf{T}' \subset \mathbf{G}$, let $N(\mathbf{T},\mathbf{T}') = \{g \in G | {}^{g}\mathbf{T} = \mathbf{T}'\}$.

Theorem VI.1 (C 7.6.2)**.** $\frac{1}{|\mathbf{G}(\mathfrak{f})|}\Sigma_{n\in\mathcal{N}(\mathbf{G})(\mathfrak{f}}Q_{\mathbf{T}}(n)Q_{\mathbf{T}'}(-n) = \frac{|N(\mathbf{T},\mathbf{T}')|}{|\mathbf{T}(\mathfrak{f})||\mathbf{T}'(\mathfrak{f})|}$. *In particular, two non conjugate Tori produce orthogonal functions on* $\mathcal{N}(\mathfrak{f})$*.*

It will be important for us to understand the relationship between the Green functions of a reductive group and the Green functions of a Levi factor of a parabolic subgroup of that reductive group. Let $\mathbf{T}$ be a $\mathfrak{f}$-torus contained in a Levi $\mathfrak{f}$-subgroup $\mathbf{L}$ of a parabolic $\mathfrak{f}$-subgroup $\mathbf{P} < \mathbf{G}$. Let $\mathbf{P} = \mathbf{L}\mathbf{U}$ be the Levi decomposition of $\mathbf{P}$ and let $\pi : P \to L$ be the projection map. Let θ be a complex character of $\mathbf{T}(\mathfrak{f})$. For any character $\psi \in \mathbf{L}(\mathfrak{f})^D$ let $(\psi)_P$ be the composition $\psi \circ \pi$.

Proposition VI.2 (C 7.4.4)**.** $R^G_{T,\theta} = Ind^G_P(R^L_{T,\theta})_P$

Proposition VI.3 (C 8.4.1)**.** *Let $X \in \mathcal{N}(\mathfrak{f})$ be a regular nilpotent element. Then for all $\mathfrak{f}$-tori $\mathbf{T}$, $Q_{\mathbf{T}}(X) = 1$.*

6.2 Lusztig Functions

While the Green functions separate rational, semisimple classes, they are not sufficient to separate rational nilpotent classes in $\mathfrak{g}(\mathfrak{f})$. To do so we must introduce functions coming from Lusztig's character sheaves. We shall limit ourselves to a concrete, explicit definition. For a field E and a reductive E-Lie algebra $\mathfrak{h}$ we say that $X \in \mathcal{N}(\mathfrak{h})(E)$ is E-distinguished if X does not lie in any Levi E-subalgebra of a E-Parabolic of $\mathfrak{h}$. If a nilpotent orbit contains a E-distinguished element we call the orbit E-distinguished as well.

For $X \in \mathfrak{g}$, let the component group of X be $C_X = Z_{\mathbf{G}}(X)/[Z_{\mathbf{G}}(X)^0]$. Notice that C_X is in bijection with $H^1(\Gamma_{\mathfrak{f}}, Z_{\mathbf{G}}(X))$ which in turn parameterizes the rational classes in the geometric class of X.

A cuspidal local system of $\mathbf{G}$ is a pair $(\mathcal{O}_X, \chi)$ consisting of a $\mathfrak{f}$-distinguished $\mathbf{G}(\mathfrak{f})$-orbit $\mathcal{O}_X$ and a 'cuspidal' character χ of the component group of X. For us, 'cuspidal' characters will be those we list explicitly. For $c \in C_X$ let $\mathcal{O}_{X,c}$ denote the rational orbit associated to c by this identification. Recall that for a set S we have define $[S]$ to be the characteristic function of the set S. The Lusztig function $\mathcal{G}_{(\mathcal{O}_X,\chi)}$ associated to the cuspidal local system $(\mathcal{O}_X, \chi)$ is given by the equation:

$\Sigma_{c \in C_X/\sim} \chi(c)[\mathcal{O}_{X,c}]$

For this paragraph we specialize to the case of G_2, its endoscopic groups, and the various possible residue groups G_x. It is easy to see that if one fixes a X in each distinguished class that distinct cuspidal local systems produce orthogonal functions. One proves this by noticing that the Lusztig functions are supported on a single geometric orbit and on that orbit take precisely the values of a character on the component group of our fixed X. Furthermore, a case by case analysis using Proposition VI.3 and the Green polynomials defined in section 7 of [Spr] shows the Lusztig functions are orthogonal to the Green functions. Thus, by counting, we get that the set of all Green functions and the Lusztig functions associated to the cuspidal local systems separates all the rational nilpotent orbits. We shall call this collection the generalized Green functions of $\mathbf{G}$. Note that a generalized Green function of $\mathbf{G}$ is either determined by a facet and an unramified torus, or a facet and a cuspidal local system. From now on, we shall denote a generalized Green function by $\mathcal{G}_{(F,\delta)}$ where F is a facet and δ may either be a maximal unramified torus or a cuspidal local system of the residue group G_F.

6.3 Generalized Gelfand-Graev characters

We shall need one more class of characters on a reductive group over $\mathfrak{f}$. These characters will be defined by induction of a generic character of a Levi subgroup. Let $\mathbf{L} < \mathbf{G}$ be a Levi $\mathfrak{f}$-subgroup. Let $\mathbf{B}$ be a Borel $\mathfrak{f}$-subgroup of $\mathbf{L}$ with unipotent radical $\mathbf{U}$. Then $\mathbf{U}/[\mathbf{U},\mathbf{U}]$ is a $\mathfrak{f}$ vector space. We call a character $\psi \in \mathbf{U}(\mathfrak{f})^D$

generic if the composition of that character by the quotient map to $\mathbf{U}/[\mathbf{U},\mathbf{U}]$ yields a distinct nontrivial character on each simple root subspace of U (note only simple root subspaces occur in the quotient). Then the induced character $\mathrm{Ind}_U^G\psi$ is a Gelfand-Graev character. For any nilpotent $e \in \mathbf{G}(\mathfrak{f})$, because we have assumed p is good, we can find a maximal parabolic $\mathfrak{f}$-subgroup $\mathbf{P}_e$ with unipotent radial $\mathbf{U}_e$ such that $e \in \mathfrak{u}_e(\mathfrak{f})$, the $\mathfrak{f}$-points of the Lie algebra of $\mathbf{U}$. Using the Killing form and the Kirillov orbit method we associate a character called the Gelfand-Graev character to e.

As with our Green functions, we may restrict a Gelfand-Graev character to $\mathcal{U}(\mathbf{G})(\mathfrak{f})$ and use our assumptions on p to build a $\mathbf{G}(\mathfrak{f})$-equivariant logarithm map to consider our Gelfand-Graev characters as functions on the Lie algebra $\mathfrak{g}(\mathfrak{f})$.

Now let $\mathbf{G}$ be a reductive k-group. Then for $F \subset \mathcal{B}(G)$ a facet, G_F is a reductive $\mathfrak{f}$-group with Lie algegra L_F. Given a pair (F, e) with $e \in \mathcal{N}(\mathsf{G}_F)(\mathfrak{f})$ we can construct a Gelfand-Graev character $\Gamma_{(F,e)} \in C^{\mathsf{G}_F(\mathfrak{f})}(\mathcal{N}(\mathsf{G}_F)(\mathfrak{f}))$ satisfying the following formula ([DKaz1] or [Lus2]):

$$\Gamma_{(F,e)}(\bar{Z}) = \tfrac{|\mathsf{L}_F(-1)|^{\frac{1}{2}}}{|\mathsf{L}_F(\leq -1)|}\Sigma_{\bar{g}\in \mathsf{G}_F(\mathfrak{f}), {}^{\bar{g}}\bar{Z}\in \mathsf{L}_F(\leq -2)}\Lambda(B(X, {}^{g}Z)).$$

Let $h_{(F,e)} \in C(\mathfrak{g}_F/\mathfrak{g}_F^+)$ be the function $[e + \mathsf{L}_F(\leq 1)]$. Then from [Lus2]:

$$\Sigma_{\bar{g}\in \mathsf{G}(\mathfrak{f})}\hat{h}_{(F,e)}({}^{\bar{g}}\bar{Z}) = |\mathsf{L}_F(1)||\mathsf{L}_F(\mathfrak{f})|^{\frac{1}{2}}\Gamma_{(F,e)}(\bar{Z}) \text{ for all } \bar{Z} \in \mathsf{L}_F(\mathfrak{f}).$$

6.4 Lifting Generalized Green Functions to a p-adic Group

Now let $\mathbf{G}$ be a k-split, reductive p-adic group. We shall produce a basis of the G-orbits of functions in $\mathcal{D}_0$ in terms of Fourier transforms of generalized Green functions. Firstly, as all alcoves are conjugate under the action of G and functions in $\mathcal{D}_0$ are compactly supported, we may choose representative functions in $\mathcal{D}_0^0$. Note that for all $F \subset \bar{A}_G$ that $\mathfrak{g}_F^+ \subset \mathfrak{g}_{A_G}$, thus we can realize the set $\mathfrak{g}_F/\mathfrak{g}_{A_G}$ as a quotient of $\mathfrak{g}_F/\mathfrak{g}_F^+ \cong \mathsf{L}_F$.

The Fourier transform gives a G-equivariant map on $C_c^\infty(\mathfrak{g})$ that restricts to a bijective map from $C(\mathfrak{g}_F/\mathfrak{g}_{A_G})$ to $C(\mathfrak{g}_{A_G}^+/\mathfrak{g})$, so we have reduced the problem to studying $C(\mathfrak{g}_{A_G}^+/\mathfrak{g}_F^+)$. However, $C(\mathfrak{g}_{A_G}^+/\mathfrak{g}_F^+)$ is a subspace of $C(\mathfrak{g}_F/\mathfrak{g}_F^+)$ with image in $\mathsf{L}_F(\mathfrak{f})$ in bijection with $C(\mathcal{N}(\mathsf{G}_F)(\mathfrak{f})$.

As in section 3, the rational, nilpotent orbits in $\mathfrak{g}(k)$ are parameterized by pairs (F, e) where F is a facet in $\mathcal{B}(G)$, e is a $\mathfrak{f}$-distinguished nilpotent in $\mathsf{L}_F(\mathfrak{f})$, and we mod out by equivalency classes by the association relation on facets and L_F conjugacy on the nilpotent elements in the residue group.

We have already shown that the generalized Green functions of L_F separate nilpotent classes, thus their lifts to $\mathfrak{g}_F$ will span G-orbits of functions in $\mathcal{D}_0$. For p-adic groups G we shall call the collection of all pairs $(F, \mathcal{G}_F)$ where F is a facet and $\mathcal{G}_F$ is the lift of a generalized Green function of L_F the generalized Green functions of G.

However, there will be some $\mathbb{C}$-linear relations among the generalized Green functions lifting from various different L_F. We shall eliminate these relations in two steps. First, pass from all generalized Green functions of G to classes under the association relation defined in Chapter 4.3. This eliminates the possibility of redundancies given by G conjugacy associating two facets as in section 3. We further eliminate all $\mathcal{G}_F = Q_T$ such that T is not elliptic in L_F. Every generalized Green function removed is redundant, and we are left with a spanning set of the correct finite cardinality, thus the remaining functions must form a linearly independent set.

We shall also need that the Fourier transforms of the functions $h_{(F,e)}$ provide a basis of G-orbits of $\mathcal{D}_0$ when we restrict (F, e) to a set of representatives for the nilpotent orbits in $\mathfrak{g}(\mathfrak{f})$. Fortunately, this is easy to see, as $h_{(F,e)}$ is the unique function amongst our collection of $h_{(F,e)}$ that takes value 1 on $\mathcal{O}_{(F,e)}$ yet vanishes on any nilpotent orbit containing $\mathcal{O}_{(F,e)}$ in its closure, thus there can be no linear relations amongst the $h_{(F,e)}$.

CHAPTER VII

Various Families of Distributions

In this section we construct several families of distributions that shall prove useful. Many of the results follow from the work in [DKaz1], however we have to be careful that the theory works for SO_4, so we write in some detail.

7.1 Nilpotent Orbital Integrals

Our primary aim in this paper is to study nilpotent orbital integrals. For a group G this is the set of μ_X where X is a nilpotent element of $\mathfrak{g}(k)$. These integrals are in bijection with the nilpotent orbits of $\mathfrak{g}(k)$ and hence there are only finitely many of them. Nilpotent orbital integrals span $J(\mathcal{N}(\mathfrak{g})(k))$ by a theorem of Harish-Chandra.

7.2 Orbital Integrals Dual to Green functions

We shall need to associate one distribution to each Green function; and, eventually, two distributions to each Lusztig function. Unfortunately, we can only offer general definitions for distributions associated to Green functions. We shall address the Lusztig functions only for the cases that concern us in particular.

We first create a distribution for each Green function. Fix a unramified, maximal torus $\mathbf{T}$ corresponding to the parameter (F, T) with F a facet, T a maximal $\mathfrak{f}$ torus of G_F with the image of $\mathbf{T}(K)$ in $\mathsf{G}_F(\mathfrak{F})$ being $\mathsf{T}(\mathfrak{F})$. Fix $X_T \in \mathfrak{t}(k)$ mapping to $\bar{X}_T \in$Lie$\mathsf{T}(\mathfrak{f})$ such that $Z_G(X_T) = T$ and $Z_{\mathsf{G}_F(\mathfrak{f})}(\bar{X}_T) = \mathsf{T}(\mathfrak{f})$. Let $\mathcal{G}_{(F',\delta)}$ be any generalized Green function. Then we have the following Lemma:

Lemma VII.1 (DKaz1)**.** *Take the assumptions of the above paragraph. Then:*

$$\mu_{X_T}(\hat{\mathcal{G}}_{(F',\delta)}) = \begin{array}{ll} \frac{(-1)^{rk(\mathsf{T})}|\mathsf{G}_F(\mathfrak{f})||N_G(T)/T|}{|\mathsf{L}_F(\mathfrak{f})|^{1/2}|\mathsf{T}(\mathfrak{f})|} & \textit{if } (F',\delta) \sim (F,T) \\ 0 & \textit{otherwise} \end{array}$$

Proof. $\mu_{X_T}(\hat{\mathcal{G}}_{(F',\delta)}) = \int_{G/T} \hat{\mathcal{G}}_{(F',\delta)}({}^g X_T) dg$

With the measure being the quotient measure. As $\hat{\mathcal{G}}_{(F',\delta)} \in C_c^\infty(\mathfrak{g}_{F'}/\mathfrak{g}_{F'}^+)$, $\hat{\mathcal{G}}_{(F',\delta)}({}^g X_T) \neq 0$ forces ${}^g X_T \in \mathfrak{g}_{F'}$ which implies $g^{-1}F' \subset \mathcal{B}(T) \subset \mathcal{B}(G)$. Let $\mathcal{F}$ be the set of G-facets in $\mathcal{B}(T) \cap G.F'$. For each $H \in \mathcal{F}$ let X_H be the image of X_T in $\mathsf{L}_H(\mathfrak{f})$. Then $Z_{\mathsf{G}_H(\mathfrak{f})}(X_H) = \mathsf{T}_H \cong \mathsf{T}$, and we can lift T_H to an unramified, maximal torus of $\mathbf{G}$, denoted $\mathbf{T}_H$. The data $(H, \mathbf{T}_H)$ yields a corresponding toric Green function $Q^H_{\mathsf{T}_H}$. Let $\mathcal{F}^{rep}$ denote a choice of representatives for T-orbits in $\mathcal{F}$. For $H \in \mathcal{F}^{rep}$, fix $g_H \in G$ with $g_H^{-1}F' = H$. Then:

$$\mu_{X_T}(\hat{\mathcal{G}}_{(F',\delta)}) = \Sigma_{H\in\mathcal{F}^{rep}} \int_{G_H T/T} \hat{\mathcal{G}}_{(F',\delta)}({}^{g_H g} X_T) dg$$

$$= \Sigma_{H\in\mathcal{F}^{rep}} \mu(G_H T/T) \hat{\mathcal{G}}_{(F',\delta)}({}^{g_H} X_T)$$

We need to compute $\hat{\mathcal{G}}_{(F',\delta)}({}^{g_H}X_T) =$

$= \int_{\mathfrak{g}_H} \Lambda(B(X_T,Y))\mathcal{G}_{(F',\delta)}({}^{g_H}Y)dY$

$= \frac{1}{|\mathsf{L}_H(\mathfrak{f})|^{(1/2)}}\Sigma_{\bar{Y}\in\mathsf{L}_H(\mathfrak{f})}\Lambda(B(X_T,Y))\mathcal{G}_{(F',\delta)}({}^{g_H}Y)$

$= \frac{1}{|\mathsf{G}_H(\mathfrak{f})||\mathsf{L}_H(\mathfrak{f})|^{1/2}}\Sigma_{\bar{Y}\in\mathsf{L}_H(\mathfrak{f})}\Sigma_{\bar{Z}\in\mathsf{L}_H(\mathfrak{f})}\Sigma_{g\in\mathsf{G}_H(\mathfrak{f})}[\bar{X}_T]({}^{g}\bar{Z})\Lambda(B(Z,Y))\mathcal{G}_{(F',\delta)}({}^{g_H}Y)$

$= \frac{1}{|\mathsf{G}_H(\mathfrak{f})|}\Sigma_{\bar{Y}\in\mathsf{L}_H(\mathfrak{f})}\mathcal{F}(\Sigma_{g\in\mathsf{G}_H(\mathfrak{f})}[\bar{X}_T]^{g})(\bar{Y})\mathcal{G}_{(F',\delta)}({}^{g_H}Y)$

By [Kaz] $\mathcal{F}(\Sigma_{g\in\mathsf{G}_F(\mathfrak{f})}[\bar{X}^{g}])(\bar{Y}) = \frac{(-1)^{rk(\mathsf{T})}|\mathsf{T}(\mathfrak{f})|}{|\mathsf{L}_{\mathsf{T}}(\mathfrak{f})|^{1/2}}Q^F_{\mathsf{T}}(\bar{Y})$. Thus we get:

$\hat{\mathcal{G}}_{(F',\delta)}({}^{g_H}X_T) = \frac{1}{|\mathsf{G}_H(\mathfrak{f})|}\Sigma_{\bar{Y}\in\mathsf{L}_H(\mathfrak{f})}\frac{(-1)^{rk(\mathsf{T})}|\mathsf{T}(\mathfrak{f})|}{|\mathsf{L}_{\mathsf{T}}(\mathfrak{f})|^{1/2}}Q^F_{\mathsf{T}}(Y)^{g_H^{-1}}\mathcal{G}_{(F',\delta)}(Y)$

$= \frac{(-1)^{rk(\mathsf{T})}|\mathsf{T}(\mathfrak{f})|}{|\mathsf{L}_{\mathsf{T}}(\mathfrak{f})|^{1/2}}(Q^H_{\mathsf{T}}, {}^{g_H^{-1}}\mathcal{G}_{(F',\delta)})_{\mathsf{L}_H}$

$= \frac{(-1)^{rk(\mathsf{T})}|\mathsf{T}(\mathfrak{f})|}{|\mathsf{L}_{\mathsf{T}}(\mathfrak{f})|^{1/2}}(Q^F_{\mathsf{T}}, \mathfrak{r}^H_F({}^{g_H^{-1}}\mathcal{G}_{(F',\delta)}))_{\mathsf{L}_F}$

Thus we get:

$\mu_{X_T}(\hat{\mathcal{G}}_{(F,\delta)}) = \Sigma_{H\in\mathcal{F}^{rep}}\frac{(-1)^{rk(\mathsf{T})}\mu(G_HT/T)|\mathsf{T}(\mathfrak{f})|}{|\mathsf{L}_{\mathsf{T}}(\mathfrak{f})|^{1/2}}(Q^F_{\mathsf{T}}, \mathfrak{r}^H_F({}^{g_H^{-1}}\mathcal{G}_{(F',\delta)}))_{\mathsf{L}_F}.$

As $\mathcal{G}_{(F',\delta)}$ is cuspidal there is an apartment $\mathcal{A}$ and $g\in G$ with $A(\mathcal{A},F) = A(\mathcal{A},gF')$. Thus to prevent $(Q^F_{\mathsf{T}}, \mathfrak{r}^H_F({}^{g_H^{-1}}\mathcal{G}_{(F',\delta)}))_{\mathsf{L}_F}$ from vanishing we get that $(F, Q^F_T) \sim (F',\delta)$. Now we consider the case when $(F',\delta)\sim(F,Q^F_T)$.

Combining VI.1 and VI.2 gives:

$\mu_{X_T}(\hat{\mathcal{G}}_{(F,\delta)}) = \Sigma_{H\in\mathcal{F}^{rep}}\frac{(-1)^{rk(\mathsf{T})}\mu(G_HT/T)|N_{\mathsf{G}}(\mathsf{T}(\mathfrak{f}))|}{|\mathsf{L}_{\mathsf{T}}(\mathfrak{f})|^{1/2}|\mathsf{T}(\mathfrak{f})|}.$

We compute $\mu(G_FT/T)$ and $|\mathcal{F}^{rep}|$.

$\mu(G_F T/T) = \frac{|\mathsf{G}_F(\mathfrak{f})||\mathsf{L}_T(\mathfrak{f})|^{(1/2)}}{|\mathsf{L}_F(\mathfrak{f})|^{(1/2)}|\mathsf{T}(\mathfrak{f})|}$

$|\mathcal{F}^{rep}| = \frac{|N_G(T)/T|}{|N_{\mathsf{G}_F}(\mathsf{T})(\mathfrak{f})/\mathsf{T}(\mathfrak{f})|}$

Thus $\mu_{X_T}(\hat{\mathcal{G}}_{(F',\delta)}) = \frac{(-1)^{rk(\mathsf{T})}|\mathsf{G}_F(\mathfrak{f})||N_G(T)/T|}{|\mathsf{L}_F(\mathfrak{f})|^{1/2}|\mathsf{T}(\mathfrak{f})|}$, finishing our proof.

□

From our work in section 6.4, we have that for any $X' \in \mathfrak{t}$ satisfying the same properties as X, $\mathrm{res}_{\mathcal{D}_0^0}\hat{\mu_X} = \mathrm{res}_{\mathcal{D}_0^0}\hat{\mu_{X'}}$. Let $D_{(F,T)} = \hat{\mu_X}$. Note that this determines a distribution in $J(\mathfrak{g}(k))$ that depends on X, but that the restriction of this distribution to $\mathcal{D}_0$ depends only on the data (F,T).

7.3 Distributions Dual to Lusztig functions

We shall now define the first of two distributions to be associated to a Lusztig function. Say $\mathcal{G}_{(F,\delta)}$ is a Lusztig function with δ a cuspidal local system of G_F. Consider the distribution $T_{(F,\delta)} \in J(\mathfrak{g}(k))$ given by the formula:

$$T_{(F,\delta)}(f) = \int_G \int_{\mathfrak{g}} f({}^g Z)\mathcal{G}_{(F,\delta)}(Z)dZdg.$$

Theorem III.4 forces this integral to always converge for any $f \in C_c^\infty(\mathfrak{g})$ as $\mathcal{G}_{(F,\delta)}$ is a cuspidal function.

Lemma VII.2. *Given (F,δ) and (F',δ') with corresponding cuspidal generalized*

Green function $\mathcal{G}_{(F',\delta')}$ we have:

$$T_{(F,\delta)}(\hat{\mathcal{G}}_{(F',\delta')}) = \begin{cases} \frac{|\mathsf{G}_F(\mathfrak{f})|^2}{|\mathsf{L}_F(\mathfrak{f})|}(\hat{\mathcal{G}}_{(F,\delta)}, \mathcal{G}_{(F,\delta)})_F & \textit{if } (F,\delta) \sim (F',\delta') \\ 0 & \textit{otherwise} \end{cases}$$

Unfortunately, the stability of the distributions $T_{(F,\delta)}$ is difficult to study, so we shall execute a more elaborate construction, case-by-case for G_2 and its endoscopic groups. The rough idea is that we shall replace convolution against a lift of $\mathcal{G}_{(F,\delta)}$ with convolution against the characteristic function of a particular set of semi-simple elements with each of these elements having a totally ramified torus as its centralizer. The miracle is that when you look at the resulting distribution's restriction to $\mathcal{D}_0^0$, it agrees with our $T_{(F,\delta)}$ up to a constant.

For a cuspidal local system on $\mathsf{G}_F(\mathfrak{f})$ given by the pair $(\bar{X}, \chi)$, we can construct a $\mathfrak{sl}_2$ triple $\{\bar{X}, \bar{H}, \bar{Y}\}$ from the Jacobson-Morosov theorem and our original assumptions on characteristic. We also have a lift $\{X, H, Y\}$ to $\mathfrak{g}_F$, cocharacters μ and $\bar{\mu}$, and filtrations of L_F and $\mathfrak{g}$. Let P be the distinguished parabolic of G_F associated to $\bar{X}$ and $\mathbf{P}$ a parabolic k-subgroup of $\mathbf{G}$ whose image in G_F is P. Note $\mathrm{Lie}(\mathsf{P})(\mathfrak{f}) = \mathsf{L}_F(\geq 0)$. Furthermore, let M be the group whose Lie algebra has $\mathfrak{f}$-points $\mathsf{L}_F(0)$ and N be the group whose Lie algebra has $\mathfrak{f}$-points $\mathsf{L}_F(> 0)$. Then $\mathsf{MN} = \mathsf{P}$ gives a Levi decomposition of P. We likewise get $\mathbf{P} = \mathbf{MN}$. Note that all these notions depend on X, even though it is absent from the notation.

First for we work with G_2 and PGL_3. Note that for every cuspidal local system for either of these groups, the nilpotent in the data defining it is the lift of a distinguished element of G_v where v is a vertex in F_0. As $\mathfrak{g}_2$ is simple, among the affine roots whose zero sets intersect F_0 in codimension 1, there is precisely one affine root for which v does not lie in the zero set and that takes positive values on F_0. Call that affine root Ψ_v.

Let $B(X,\chi) = X + \mathfrak{g}_{\Psi_v}(K) \setminus \mathfrak{g}^+_{\Psi_v}(K) \subset \mathfrak{g}(K)$. One can verify explicitly that the centralizer of any element of $B(X,\chi)$ is an elliptic K-torus in $G(K)$ that splits over a totally ramified extension by embedding $\mathfrak{g}_2$ and computing minimal polynomials.

Note the fact that X is distinguished implies $C(X) = \mathbf{M}_X$. We now identify these two groups. For $m \in C(X)$ we fix $g_m \in \mathbf{M}_X \cap \mathbf{G}(K)_F$ with $\sigma(g_m)^{-1}g_m = m$. Set $A(X,\chi,g_m) = \{{}^{g_m}Y | Y \in B(X,\chi) \text{ with } {}^cY = \sigma(Y)\}$. Note that $A(X,\chi,g_m) \cong R_K^\times$, so we get a Haar measure on $A(X,\chi,g_m)$. Let S_m be the centralizer in G of any element of $A(X,\chi,g_m)$ and S_m^0 be its parahoric. Let $\sim$ be the relation of σ-conjugacy on $\mathbf{M}_X$. Note $H^1(\Gamma, \mathbf{M}_X) \cong \mathbf{M}_X/\sim$.

We now define the distributions of interest:

$$D_{(F,\delta)} = \Sigma_{\mathsf{M}_X/\sim} \frac{\chi(m)|S_m/S_m^0|}{|Z_{\mathsf{G}_F(\mathfrak{f})}({}^{g_m}X)|} \int_{A(X,\chi,g_m)} \mu_Y dY.$$

For SO_4 the nilpotent associated to a cuspidal local system is still a lift from a distinguished nilpotent of the residue group at a vertex, however now two affine roots do not vanish on the vertex in question as the closure of an alcove is a polysimplicial complex rather than a simplicial complex. We simply adapt by working in each simple factor and writing an integral over a space isomorphic to $R^\times \times R^\times$.

More precisely, let the two affine roots vanishing at v be $\psi_{v,1}$ and $\psi_{v,2}$. We let $B(X,\chi)^{SO_4} = X + \mathfrak{g}_{\Psi_{v,1}}(K)\backslash\mathfrak{g}^+_{\Psi_{v,1}}(K) + \mathfrak{g}_{\Psi_{v,2}}(K)\backslash\mathfrak{g}^+_{\Psi_{v,2}}(K)$. Then let $A^{SO_4}(X,\chi,g_m) = \{{}^{g_m}Y | Y \in B^{SO_4}(X,\chi) \text{ with } {}^cY = \sigma(Y)\}$. All other notation remains the same and we get:

$$D_{(F,\delta)} = \Sigma_{\mathsf{M}_X/\sim} \frac{\chi(m)|S_m/S_m^0|}{|Z_{\mathsf{G}_F(\mathfrak{f})}({}^{g_m}\bar{X})|} \int_{A^{SO_4}(X,\chi,g_m)} \mu_Y dY.$$

7.4 Relating the two Distributions Dual to a Lusztig Function

These arguments are identical to those of DeBacker and Kazhdan, however there they are not thinking of the group SO_4 so we repeat them as a precaution.

Fix (F,δ) a cuspidal local system of G. Then, in every case with which we are concerned, F is a vertex in $\mathcal{B}(G)$. As before we get $B(F,\delta)$ with the centralizer of any element of $B(F,\delta)$ yielding a torus $\mathbf{T}_{(F,\delta,Y)}$. We get a corresponding $y = \mathcal{B}(T_{(F,\delta,Y)}(K) \cap \mathcal{B}(G) = F + \frac{\hat{\mu}}{2h} \in \mathcal{B}(G)$, where h is the Coxeter number of G.

We will need two auxiliary functions. We first define $I_{(F,\delta)}(Z) \in C^\infty_C(\mathfrak{g}_0)$:

$$I_{(F,\delta)}(Z) = \int_{G_y} \Sigma_{\bar{m}\in\mathsf{M}_e/\sim} \frac{\chi(m)}{|C_{\mathsf{G}_F(\mathfrak{f})}({}^{g_m}e)|} \int_{A^G(F,\chi,g_m)} \Lambda(B(Z,{}^jY))dY dj$$

Note $G_F^+ \leq G_y$, thus we can define:

$$J_{(F,\delta)}(Z) = \Sigma_{\bar{i}\in\mathsf{G}_F(\mathfrak{f})} I_{(F,\delta)}({}^iZ)$$

Lemma VII.3. $\forall Z \in \mathfrak{g}_0 \setminus \mathfrak{g}_F$, $I_{(F,\delta)}(Z) = 0$.

Proof. Fix $Z \in \mathfrak{g}_0 \setminus \mathfrak{g}_F$. Suppose $Z \notin \mathfrak{g}_{y,-(1/h)}$. For all $m \in \mathbf{M}_X(K)$ every element of $A^G(X, \chi, g_m)$ is 'good' of depth 1/h in the sense of Adler and Roche, hence $\int_{G_y} \Lambda(B(Z, {}^jY))dj = 0$ from [AD 6.3.3]. Thus we need only consider when $Z \in \mathfrak{g}_{(y,-(1/h)} \setminus \mathfrak{g}_F$. Then we can write:

$$I_{(F,\delta)}(Z) = \frac{\mu(\mathfrak{g}_y^+)}{|\mathfrak{f}^\times|} \Sigma_{l \in \mathsf{M}(\mathfrak{f})} \Sigma_{m \in M_X(K)/\sim} \frac{\chi(m)}{|C_{\mathsf{G}_F(\mathfrak{f})}({}^{g_m}e)|} \Lambda(B(Z, {}^{lg_m}X)) *$$
$$\Sigma_W \Lambda(B(Z, {}^lW)).$$

Where by summing over $\bar{W}$ we mean summing over $(\mathfrak{g}_{\psi_F} \setminus \mathfrak{g}^+_{\psi_F})/\mathfrak{g}^+_{\psi_F}$. As the final term does not involve $\bar{l}$ it suffices to show that:

$$\Sigma_{m \in M_X(K)/\sim} \frac{\chi(m)}{|C_{\mathsf{G}_F(\mathfrak{f})}({}^{g_m}e)|} \Lambda(B(Z, {}^{lg_m}X)) = 0.$$

Note $\mathfrak{g}_y \subset \mathfrak{g}_{y,-(1/h)} \subset \mathfrak{g}_F + \mathfrak{g}_{\psi_F}$ and $\mathfrak{g}_{y,-1/h}/\mathfrak{g}_y \cong \mathsf{L}_F(-2) \oplus \mathfrak{g}_{-\psi_F}/\mathfrak{g}^+_{-\psi_F}$, hence $\mathfrak{g}_{-\psi_F} \not\subset \mathfrak{g}_F$ and $\mathfrak{g}^+_{-\psi_F} \subset \mathfrak{g}_y \subset \mathfrak{g}_F$. $Z \notin \mathfrak{g}_F$ then implies $Z = Z_{-\dot{\psi}_F} + Z'$ with $Z' \in \mathfrak{g}_F$, where $Z_{-\dot{\psi}_F}$ denotes the projection of Z onto the $\mathfrak{g}_{-\dot{\psi}_F}$. We need to look at the Fourier transform of:

$$\Sigma_{m \in \mathsf{M}_e/\sim} \frac{\chi(m)|Z_{\mathsf{M}(\mathfrak{f})}({}^{g_m}e)|}{|Z_{\mathsf{G}_F(\mathfrak{f})}({}^{g_m}e)|} [{}^{\mathsf{M}(\mathfrak{f})g_m}e]$$

By results of Lusztig, this Fourier transform is supported on the $\mathfrak{f}$-rational points of the Zariski dense M-orbit in $\mathsf{L}_F(-2)$. Thus we need only show the image of Z_{-2}, the projection of Z onto $\mathfrak{g}_{-2}$, does not have image lying in this orbit.

Suppose Z_{-2} lies in this orbit. Then the centralizer of $Z_{-2} + Z_{-\dot{\psi}_F}$ must be a maximal k-torus which splits over a tamely ramified extension. However, $Z_{-\dot{\psi}_F} \notin \mathfrak{g}_y$ and $Z_{-2} \notin \mathfrak{g}_y$, thus the coset $Z + \mathfrak{g}_Y = Z_{-2} + Z_{-\dot{\psi}_F} + \mathfrak{g}_y$ must contain no nilpotents. However, $Z \in \mathfrak{g}_0$ and $\mathfrak{g}_0 \subset \mathcal{N}(k) + \mathfrak{g}_y$ [AD]. Contradiction.

$\square$

Lemma VII.4. *For* $Z \in \mathfrak{g}_0$*,* $J_{(F,\delta)}(Z) = \frac{\mu(G_y)}{\mu(G_F^+)}\hat{\mathcal{G}}_{(F,\delta)}(Z)$*.*

Proof. WLOG $Z \in \mathfrak{g}_F$.

$$J(Z) = \mu(G_y)\Sigma_{\bar{i}\in G_F/G_F^+}\Sigma_{m\in \mathsf{M}_e/\sim}\frac{\chi(m)}{|Z_{\mathsf{G}(\mathfrak{f})}({}^{g_m}e)|}\int_{A^G(F,\chi,g_m)}\Lambda(B(Z,{}^iY))dY$$

$$= \mu(G_y)\Sigma_{m\in \mathsf{M}_e/\sim}\chi(m)\Sigma_{\bar{W}\in \mathsf{L}_F(\mathfrak{f})}\Lambda(B(Z,W))[{}^{\mathsf{G}_F(\mathfrak{f})g_m}e](\bar{W})$$

$$= \frac{\mu(G_y)}{\mu(G_F^+)}\hat{\mathcal{G}_{(F,\delta)}}.$$

$\square$

Lemma VII.5 (DKaz1)**.** *:* $res_{\mathcal{D}_{0+}}T_{(F,\delta)} = \frac{|\mathsf{G}_F(\mathfrak{f})|}{|\mathsf{L}_F(\mathfrak{f})|^{\frac{1}{2}}}res_{\mathcal{D}_{0+}}D_{(F,\delta)}$

Proof. Fix $f \in \mathcal{D}_{0^+}$.

$$D_{(F,\delta)}(f) = \Sigma_{\mathsf{M}_X/\sim}\frac{\chi(m)|S_m/S_m^0|}{|Z_{\mathsf{G}_F(\mathfrak{f})}({}^{g_m}\bar{X})|}\int_{A(X,\chi,g_m)}\mu_Y(f)dY$$

$$= \Sigma_{\mathsf{M}_X/\sim}\frac{\chi(m)}{|Z_{\mathsf{G}_F(\mathfrak{f})}({}^{g_m}\bar{X})|}\int_{A(X,\chi,g_m)}\int_G f({}^jY)djdY$$

$$= \Sigma_{\mathsf{M}_X/\sim}\frac{\chi(m)}{|Z_{\mathsf{G}_F(\mathfrak{f})}({}^{g_m}\bar{X})|}\int_{A(X,\chi,g_m)}\Sigma_{\bar{j}\in G/G_F}\int_{G_F} f({}^{ji}Y)didY$$

$$= \frac{\mu(G_F^+)}{\mu(G_y)} \Sigma_{\bar{j} \in G/G_F} \int_{\mathfrak{g}} \hat{f}(-({}^j Z)) J(Z) dZ$$

$$= \Sigma_{\bar{j} \in G/G_F} \int_{\mathfrak{g}} f({}^j Z) \mathcal{G}_{(F,\delta)} dZ$$

$$= \frac{1}{\mu(G_F)} \int_G \int_{\mathfrak{g}} f({}^j Z) \mathcal{G}_{(F,\delta)}(Z) dZ dj$$

$$= \frac{|\mathsf{L}_F(\mathfrak{f})|^{\frac{1}{2}}}{|\mathsf{G}_F(\mathfrak{f})|} T_{(F,\delta)}(f) \ \square$$

□

Now we finally get:

Theorem VII.6. *The following equation holds:*

$$D_{(F,\delta)}(\hat{\mathcal{G}}_{(F',\delta')}) = \begin{array}{ll} \frac{|\mathsf{G}_F(\mathfrak{f})|}{|\mathsf{L}_F(\mathfrak{f})|^{\frac{1}{2}}} (\mathcal{G}_{(F,\delta)}, \mathcal{G}_{(F,\delta)})_{\mathsf{L}_F} & (F',\delta') \sim (F,\delta) \\ 0 & \textit{otherwise} \end{array}$$

Proof. It will suffice to show $\hat{D}_{(F,\delta)}(Z) = \frac{|\mathsf{L}_F(\mathfrak{f})|^{\frac{1}{2}}}{|\mathsf{G}_F(\mathfrak{f})|} \hat{T}_{(F,\delta)}(Z)$. As the Fourier transform maps $C_c^\infty(\mathfrak{g}_0)$ to $\mathcal{D}_{0^+}$ we need only show $\mathrm{res}_{\mathcal{D}_{0^+}} T_{(F,\delta)} = \frac{|\mathsf{G}_F(\mathfrak{f})|}{|\mathsf{L}_F(\mathfrak{f})|^{\frac{1}{2}}} \mathrm{res}_{\mathcal{D}_{0^+}} D_{(F,\delta)}$. This is Lemma VII.5.

□

CHAPTER VIII

Explicit Structure Theory

Here we work out the Weyl groups, unramified tori, rational nilpotent orbits, and cuspidal local systems of G_2 and its endoscopic groups.

8.1 Weyl Groups

We now shall work out some structure theory we shall need to continue. The Weyl group of G_2, $\mathbf{W}_{G_2}$, is D_6, the dihedral group with 12 elements. We fix two generators for $\mathbf{W}_{G_2}$ labeled R, rotation from α to β by $\frac{\pi}{3}$, and F, reflection perpendicular to the root α. When we are considering G_2 as the base group from which we construct all our endoscopic groups and as the target for the transfer map, we shall include no subscripts. When we are considering G_2 as an endoscopic group of itself we shall add subscripts to give R_{G_2} and F_{G_2}, representing the analogous transformations on the roots of the endoscopic group. The Weyl group of PSL_3 is S_3, the dihedral group with 6 elements generated by R_{PSL_3}, rotation by $\frac{2\pi}{3}$ from α^{PSL_3} to β^{PSL_3} and F_{PSL_3} defined as reflection over β^{PSL_3}. The Weyl group of SO_4 is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with generators (F, Id) and (Id, F). The Γ action on each is trivial.

For $\mathbf{W}_{G_2}$, conjugacy class structure is as follows: $\{\mathrm{Id}\}$, $\{R, R^5\}$, $\{R^2, R^4\}$, $\{R^3\}$, $\{F, FR^2, FR^4\}$, $\{FR, FR^3, FR^5\}$. We shall refer to the various conjugacy classes by

(g) where g is the first group element in the list we provided.

For $\mathbf{W}_{PSL_3}$, the three conjugacy classes are $\{\mathrm{Id}\}$, $\{R_{PSL_3}, R^2_{PSL_3}\}$, and $\{F_{PSL_3}, F_{PSL_3}R_{PSL_3}, F_{PSL_3}R^2_{PSL_3}\}$.

$\mathbf{W}_{SO_4}$ is abelian, so we shall conflate conjugacy classes with elements. The group is generated by (F, Id) which acts by reflection over the hyperplane perpendicular to α^{SO_4} and (Id, F) which acts by reflection over the hyperplane perpendicular to $(3\alpha + 2\beta)^{SO_4}$.

We will need to discuss Weyl group elements and conjugacy classes of various G_F. All of these groups are of the same type as the k-groups whose Weyl groups were discussed above with one exception: the group $GL_2(\mathfrak{f})$. We shall continue to use the notation above with F_{GL_2} being the nontrivial element of $\mathbf{W}_{GL_2} \cong \mathbb{Z}/2\mathbb{Z}$.

8.2 Unramified Tori

We parameterize unramified tori in G_2 and its endoscopic groups. The data in our chart describes the following information respectively: the facet F in terms of the positive affine roots that vanish on it, the associated $\mathfrak{f}$-group G_F, the image of Frobenius under the cocycle for T, and the corresponding element in the Weyl group of G_2.

F	G_F	$a_{Frob_\mathfrak{f}} \in \mathbf{W}_{\mathsf{G}_F}$	$a_{Frob_\mathfrak{f}} \in \mathbf{W}_{G_2}$
$\{\alpha, \beta\}$	$G_2(\mathfrak{f})$	R	R
		R^2	R^2
		R^3	R^3
$\{1-3\alpha-\beta, \beta\}$	$Sl_3(\mathfrak{f})$	R	R^2
$\{\alpha, 1-3\alpha-2\beta\}$	$SO_4(\mathfrak{f})$	$F \times F$	R^3
$\{\alpha\}$	$GL_2(\mathfrak{f})$	F	F
$\{\beta\}$	$GL_2(\mathfrak{f})$	F	FR
	$\mathbb{G}_m(\mathfrak{f})^2$	Id	Id

Now we do the same for the elliptic, unramified endoscopic groups of G_2.

PGL_3

F	G_F	$a_{Frob_\mathfrak{f}} \in \mathbf{W}_{\mathsf{G}_F}$	$a_{Frob_\mathfrak{f}} \in \mathbf{W}_{PSL_3}$
$\{\alpha^{PSL_3}, \beta^{PSL_3}\}$	$PSL_3(\mathfrak{f})$	R	R_{PSL_3}
$\{\alpha^{PSL_3}\}$	$GL_2(\mathfrak{f})$	F	F_{PSL_3}
	$\mathbb{G}_m(\mathfrak{f})^2$	Id	Id_{PSL_3}

SO_4

F	G_F	$a_{Frob_{\mathfrak{f}}} \in \mathbf{W}_{\mathsf{G}_F}$	$a_{Frob_{\mathfrak{f}}} \in \mathbf{W}_{SO_4}$
$\{(\alpha^{SO_4}, \beta^{SO_4}\}$	$SO_4(\mathfrak{f})$	(F, F)	(F, F)
$\{\alpha^{SO_4}, 1-\beta^{SO_4}\}$	$SO_4(\mathfrak{f})$	(F, F)	(F, F)
$\{\alpha^{SO_4}\}$	$GL_2(\mathfrak{f})$	F	(Id, F)
$\{\beta^{SO_4}\}$	$GL_2(\mathfrak{f})$	F	(F, Id)
	$\mathbb{G}_m(\mathfrak{f})^2$	Id	Id

8.3 Nilpotent Orbits

We now describe the distinguished nilpotent orbits for each of the residue groups.

Case 1: $\mathbb{G}_m^2$. There is only the trivial nilpotent orbit. It lifts to the trivial orbit in $\mathfrak{g}_2$

Case 2: GL_2. From rational canonical form we get two orbits, one of which is distinguished.

A)$G_{F_{\{\alpha\}}} \cong GL_2$: $X_\alpha \in \mathfrak{g}_2(k)$ represents a lift of a regular nilpotent over $\mathfrak{f}$.

B)$G_{F_{\{\beta\}}} \cong GL_2$: $X_\beta \in \mathfrak{g}_2(k)$ represents a lift of a regular nilpotent over $\mathfrak{f}$.

Case 3: SO_4. There is one distinguished $SO_4(\mathfrak{F})$-orbit in $\mathfrak{so}_4(\mathfrak{F})$. A representative element is $\bar{X}_\alpha + \bar{X}_{-3\alpha-2\beta} \in \mathfrak{so}_4(\mathfrak{f})$. The component group of the centralizer $Z_{SO_4(\mathfrak{F})}(\bar{X}_\alpha + \bar{X}_{-3\alpha-2\beta})$ is isomorphic to the Galois module $\mathbb{Z}/2\mathbb{Z}$ with the trivial action. Then $|H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), Z_{SO_4(\mathfrak{F})}(\bar{X}_\alpha + \bar{X}_{-3\alpha-2\beta}))| = 2$. Let m and $m' \in \mathfrak{so}_4(\mathfrak{f})$ be two representatives of these classes. More concretely, fix a non-square $\epsilon \in R^\times$

whose image in $\mathfrak{f}$ is a nonsquare. Lifts of representatives of the two distinguished nilpotent orbits of $SO_4(\mathfrak{f})$ are given by $X_\alpha + \varpi X_{-3\alpha-2\beta}$ and $\epsilon X_\alpha + \varpi X_{-3\alpha-2\beta} \in \mathfrak{g}_2(k)$.

Case 4: SL_3. There is one distinguished $SL_3(\mathfrak{F})$-orbit in $\mathfrak{sl}_3(\mathfrak{F})$. A representative is given by $\bar{X}_\beta + \bar{X}_{-3\alpha-\beta} \in \mathfrak{sl}_3(\mathfrak{f})$. Then the component group of $Z_{Sl_3(\mathfrak{F})}(\bar{X}_\beta + \bar{X}_{-3\alpha-\beta})$ is $\mathbb{Z}/3\mathbb{Z}$, however the $\mathrm{Gal}(\mathfrak{F})$ action depends on whether or not $|\mu_3(k)| = |\mu_3(\mathfrak{f})| = 3$ or 1. If $|\mu_3(\mathfrak{f})| = 3$, the action is trivial and $|H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), Z_{Sl_3(\mathfrak{F})}(\bar{X}_\beta + \bar{X}_{-3\alpha-\beta}))| = 3$ with the two nontrivial cocycles both splitting over the unique cubic extension of $\mathfrak{f}$. If $|\mu_3(k)| = 1$, the action of Frobenius switches the two non identity elements of $\mathbb{Z}/3\mathbb{Z}$ and $|H^1(\mathrm{Gal}(\mathfrak{F}), Z_{Sl_3(\mathfrak{F})}(\bar{X}_\beta + \bar{X}_{-3\alpha-\beta}))| = 1$. Thus if $|\mu_3(k)| = 1$ we get only a single rational nilpotent orbit, while if $|\mu_3(k)| = 3$, we get 3 rational nilpotent orbits. We fix representatives of these $\mathfrak{f}$-orbits in $\mathsf{L}_{\{\beta,3\alpha+2\beta-1\}}(\mathfrak{f})$, n, n', and n'' with n' and n'' only occurring if $|\mu_3(k)| = 3$.

Case 5: G_2. There are two distinguished orbits in $\mathfrak{g}_2(\mathfrak{F})$. Representatives are given by the elements $\bar{X}_\alpha + \bar{X}_\beta$ (which we call e_{reg}) and $\bar{X}_\beta + \bar{X}_{3\alpha+\beta} \in \mathfrak{g}_2(\mathfrak{f})$ (which we call e_{subreg}, soon to be joined by two geometric conjugates).

A)Orbits within the geometric conjugacy class of $\bar{X}_\alpha + \bar{X}_\beta = e_{reg}$. The component group of $Z_{G_2(\mathfrak{F})}(\bar{X}_\alpha + \bar{X}_\beta)$ is trivial, hence there is only one rational orbit in $\mathfrak{g}_2(k)$ corresponding to $(\{\alpha, \beta\}, e_{reg})$ and a representative is $X_\alpha + X_\beta$.

B)Orbits within the geometric conjugacy class of $\bar{X}_\beta + \bar{X}_{3\alpha+\beta} \in \mathfrak{g}_2(\mathfrak{f})$. The component group of $Z_{G_2(\mathfrak{F})}(\bar{X}_\beta + \bar{X}_{3\alpha+\beta} \in \mathfrak{g}_2(\mathfrak{f}))$ is isomorphic to S_3 with two possible actions of $\mathrm{Gal}(\mathfrak{F}/\mathfrak{f})$. If $|\mu_3(k)| = 3$ then the image of Frobenius can be trivial, a

2-cycle, or a 3-cycle. The choice of which cycle never matters, as all cycles of the same length are conjugate, hence the corresponding cocycles are cohomologous.

If $|\mu_3(k)| = 1$ then the Frobenius of $\mathfrak{f}$ exchanges the two 3-cycles and two of the 2-cycles. All three of the 2-cycles interchange the 3-cycles through conjugation, so whichever one is fixed will force cocycles that choose different 3-cycles for the image of Frobenius to be conjugate. Furthermore, conjugating by the correct non-fixed 2-cycle will force the cocycle to be trivial. If the cocycle sends Frobenius to the fixed 2-cycle, the cocycle splits over a degree 2 extension, while if the Frobenius maps to a non-fixed 2-cycle, the cocycle splits over a degree 6 extension. These two cases are not cohomologous, yielding 3 cocycles in total.

Of these, fix representatives in $\mathsf{L}_{F_{\{\alpha,\beta\}}}(\mathfrak{f})$: $X_\beta + X_{3\alpha+\beta} = e_{sr}$, e'_{sr} and e''_{sr} with $|\mathsf{M}_{e_{sr}}(\mathfrak{f})| = 6$, $|\mathsf{M}_{e'_{sr}}(\mathfrak{f})| = 3$, $|\mathsf{M}_{e''_{sr}}(\mathfrak{f})| = 2$.

Tallying up all our nilpotent orbits we get $9 + |\mu_3(k)|$ nilpotent orbits for G_2.

For PSL_3 we have the following facets up to association labeled as before:

$\{\alpha^{PSL_3}, \beta^{PSL_3}\}$	PSL_3
$\{\alpha^{PSL_3}\}$	GL_2
$\emptyset$	G_m^2

Case 1: $\mathbb{G}_m^2$. Again only the trivial nilpotent orbit. It lifts to the trivial orbit in $\mathfrak{sl}_3$.

Case 2: GL_2. As before, from rational canonical form we get two orbits, one of which is distinguished. The element $X_{\alpha^{PSL_3}}$ is a lift of a representative of the orbit.

Case 3: PSL_3 We get one distinguished orbit. A representative is $\bar{X}_{\beta^{PSL_3}}+\bar{X}_{\beta^{PSL_3}}$. As we are now dealing with PSL_3, we get the component group of $Z_{PSL_3}(\bar{X}_{\beta^{PSL_3}} + \bar{X}_{\beta^{PSL_3}})$ is trivial. Thus only one rational orbit. The element $X_{\beta^{PSL_3}} + X_{\beta^{PSL_3}}$ provides a lift to $\mathfrak{sl}_3(k)$.

For SO_4 we have the following facets up to association labeled as before:

$\{\alpha^{SO_4}, \beta^{SO_4}\}$	SO_4
$\{\alpha^{SO_4}, 1-\beta^{SO_4}\}$	SO_4
$\{\beta^{SO_4}\}$	GL_2
$\{\alpha^{SO_4}\}$	GL_2
$\emptyset$	$\mathbb{G}_m^2$

Case 1:$\mathbb{G}_m^2$ again we have the trivial orbit lifting to the trivial orbit in $\mathfrak{so}_4(k)$.

Case 2:For the residue groups of type GL_2 we have precisely one distinguished orbit each. For the facet where α^{SO_4} vanishes it lifts to $X_{\alpha^{SO_4}}$ and for the facet where β^{SO_4} vanishes a lift is $X_{\beta^{SO_4}}$.

Case 3:For residue groups of type SO_4 there is again one distinguished orbit in $SO_4(\mathfrak{F})$, however this orbit splits into 2 rational orbits as again $|H^1(\mathrm{Gal}(\mathfrak{F}/\mathfrak{f}), Z_{SO_4(\mathfrak{F})}(e))| = 2$. For the vertex where the set $\{\alpha^{SO_4}, \beta^{SO_4}\}$ vanishes a lift is $X_{\alpha^{SO_4}} + X_{\beta^{SO_4}}$ and $\epsilon X_{\alpha^{SO_4}} + X_{\beta^{SO_4}}$ while for the facet where the set $\{\alpha^{SO_4}, 1-\beta^{SO_4}+1\}$ vanishes a lift is $X_{\alpha^{SO_4}} + \varpi X_{2\beta^{SO_4}}$ and $\epsilon X_{\alpha^{SO_4}} + \varpi X_{\beta^{SO_4}}$.

8.4 Cuspidal Local Systems

We now classify the cuspidal local systems for each of the endoscopic groups of G_2.

G_2

G_F	Nilpotent	Component Group	Character	Name
G_2	$X_\beta + X_{3\alpha+\beta}$	S_3	sgn	$C_1^{G_2}$
SO_4	$X_\alpha + \varpi X_{3\alpha+2\beta}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$sgn \times sgn$	$C_2^{G_2}$
SL_3	$X_\beta + \varpi X_{-3\alpha-2\beta}$	$\mathbb{Z}/\lvert\mu_3(\mathfrak{f})\rvert\mathbb{Z}$	ψ	$C_3^{G_2}$
SL_3	$X_\beta + \varpi X_{-3\alpha-2\beta}$	$\mathbb{Z}/\lvert\mu_3(\mathfrak{f})\rvert\mathbb{Z}$	ψ^{-1}	$C_4^{G_2}$

SO_4

G_F	Nilpotent	Component Group	Character	
SO_4	$X_{\alpha^{SO_4}} + X_{\beta^{SO_4}}$	$\mathbb{Z}/2\mathbb{Z}$	sgn	$C_1^{SO_4}$
SO_4	$X_{\alpha^{SO_4}} + \varpi X_{-\beta^{SO_4}}$	$\mathbb{Z}/2\mathbb{Z}$	sgn	$C_2^{SO_4}$

PGL_3 has none as the centralizers of distinguished, nilpotent elements are all connected.

Recall from section 7.3 the definition of $B(X,\chi)$ and $B^{SO_4}(X,\chi)$. We shall soon need to understand the isomorphism class of the centralizer of an element arising from either of those two constructions. In each case, the centralizer is a totally ramified torus $\mathbf{T} = {}^g\mathbf{S_G}$ (recall $\mathbf{S}_G$ was our k-split torus previously defined). Let the splitting field of $\mathbf{T}$ over K be E. Then $\mathrm{Gal}(E/K)$ is cyclic of order prime to p generated by an element τ. The k-isomorphism class of $\mathbf{T}$ is then determined by the image of $g^{-1}\tau g \in \mathbf{W_G}$. We first work with G_2. For $C_1^{G_2}$, we have $w = R^2$ and the extension is degree 3. For the other 3 cuspidal local systems $w = R$ and the extension is degree 6.

For SO_4, in both cases $w = F^{SO_4} \times F^{SO_4}$ and the extension is of degree 2.

CHAPTER IX

Stability for Distributions Dual to Generalized Green Functions

In this chapter we study the stability properties of the distributions constructed in chapter 7.

9.1 Stability for Distributions Dual to Green Functions

Let X be a strongly regular semi-simple element in $\mathfrak{g}(k)$ such that its centralizer is a maximal, K-split torus. We address two questions needed for our harmonic analysis:

1)How many rational orbits lie in the geometric orbit of the element in question?

2)How many of these rational orbits lie in the same torus as X?

We first address question 1. As we are dealing with strongly regular elements, stable conjugacy is the same as geometric conjugacy. Look at the exact sequence of pointed sets $0 \to \mathbf{T} \to \mathbf{G} \to \mathcal{O}_{\mathbf{X}} \to 0$ and take k fixed points to get $0 \to T \to G \to \mathcal{O}_X(k) \to H^1(\Gamma, \mathbf{T}) \to H^1(\Gamma, \mathbf{G}) \to 0$. We want $\ker H^1(\Gamma, \mathbf{T}) \to H^1(\Gamma, \mathbf{G})$.

Using that the Kottwitz isomorphism is functorial for inclusion of elliptic, maximal tori [Chapter 5 or Kot1], we get $\pi_0(Z(\hat{T})^\Gamma)^D \to \pi_0(Z(\hat{G})^\Gamma)^D$. We shall compute the kernel of this map explicitly.

We now address question 2. Rational classes in $\mathcal{O}_X(K) \cap \mathrm{Lie} S(k)$ are in bijection with $N(T) = [N_{\mathbf{G}(K)}(\mathbf{T}(K))/\mathbf{T}(K)]^\Gamma/[N_G(T)/T]$. Again, we can compute this explicitly, which we shall now do.

PGL_3:For PGL_3, at the level of GL_3 all geometrically conjugate tori are rationally conjugate by rational canonical form, thus after passing to the quotient the same is true for PGL_3, and further computation becomes unnecessary.

G_2:As G_2 is simply connected $H^1(\Gamma, \mathbf{G})$ is trivial, so we need only compute $\pi_0(Z(\hat{T})^\Gamma)^D$. To do this we look at the torsion points of the co-invariants under the Galois action on $X_*(T)$, but the Γ action factors through a Γ_{un} action as T is unramified where Frob_k acts via a Weyl group element w. Then we need to compute $\mathrm{tor}[X_*(T)/(1-w)X_*(T)]$. Let S be a two dimensional split $\mathfrak{f}$-torus. Here are the results, following our parameterization from section 8.2:

Name	G_F	Im(frob)	$\lvert H^1(\mathrm{Gal}(K/k), T(K))\rvert$	$\lvert N(T)\rvert$
$T^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_R)}$	G_2	R	1	1
$T^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{R^2})}$	G_2	R^2	3	1
$T^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{R^3})}$	G_2	R^3	4	1
$T^{G_2}_{(F_{\{\alpha,1-3\alpha-2\beta\}},\mathsf{T}_{(F,F)})}$	SO_4	$F\times F$	4	3
$T^{G_2}_{(F_{\{\beta,1-3\alpha-2\beta\}},R)}$	SL_3	R	3	2
$T^{G_2}_{(F_{\{}\alpha\},\mathsf{T}_F)}$	GL_2	F	1	1
$T^{G_2}_{(F_{\{}\beta\},\mathsf{T}_F)}$	GL_2	F	1	1
$T^{G_2}_{(F_\emptyset,\mathsf{S})}$	$\mathbb{G}_m^2$	Id	1	1

Notice that geometrically conjugate tori must have isomorphic $H^1(\mathrm{Gal}(K/k), T(K)$. Consulting our chart, one sees that only two pairs share the same order of $H^1(\mathrm{Gal}(K/k), T(K)$ with the $|N(T)|$ of the two pairs summing to $|H^1(\mathrm{Gal}(K/k), T(K)|$. Hence, those pairs are geometrically conjugate and no others; specifically, $T^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{R^2})}$ and $T^{G_2}_{(F_{\{\beta,1-3\alpha-2\beta\}},R)}$ must be stably conjugate and $T^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{R^3})}$ and $T^{G_2}_{(F_{\{\alpha,1-3\alpha-2\beta\}},\mathsf{T}_{(F,F)})}$ must also be stably conjugate, and no other pairs.

SO_4:

For SO_4 things are slightly more complicated as $|H^1(\Gamma, SO_4)| = 2$. First, the table as before:

Name	G_F	$\mathrm{Im}(\mathrm{Frob}_k)$	$\lvert H^1(\mathrm{Gal}(K/k), T(K))\rvert$	$\lvert N(T)\rvert$
$T^{SO_4}_{(F_{\{\alpha,\beta\}}\mathsf{T}_{(F,F)})}$	SO_4	$F \times F$	4	1
$T^{SO_4}_{(F_{\{\alpha,1-\beta\}}\mathsf{T}_{(F,F)})}$	SO_4	$F \times F$	4	1
$T^{SO_4}_{(F_{\{\alpha\}}\mathsf{T}_{F})}$	GL_2	F	1	1
$T^{SO_4}_{(F_{\{\beta\}}\mathsf{T}_{F})}$	GL_2	F	1	1
$T^{SO_4}_{(F_{\emptyset},\mathsf{S})}$	$\mathbb{G}_m^2$	Id	1	1

Look at the maps:

$$H^1(\Gamma, \mathbf{T}^{SO_4}_{(F_{\{\alpha,\beta\}}\mathsf{T}_{(F,F)})}) \to H^1(\Gamma, SO_4)$$

and

$$H^1(\Gamma, \mathbf{T}^{SO_4}_{(F_{\{\alpha,1-\beta\}}\mathsf{T}_{(F,F)})}) \to H^1(\Gamma, SO_4)$$

By Kottwitz's theorem for elliptic tori, both maps must be surjective. Thus we compute the order of both kernels must be 2. This forces $T^{SO_4}_{(F_{\{\alpha,\beta\}}\mathsf{T}_{(F,F)})}$ and $T^{SO_4}_{(F_{\{\alpha,1-\beta\}}\mathsf{T}_{(F,F)})}$ to be stably conjugate. No other pair is.

9.2 Stability for Distributions Dual to Lusztig Functions

Recall that $\mathbf{S_G}$ is a maximal, split k-torus of $\mathbf{G}$. Let $\mathbf{T}$ be a maximal, tamely, totally ramified k-torus. In particular, $\mathbf{T}$ is elliptic over K. Let E be the extension of K that splits $\mathbf{T}$. Then $\mathrm{Gal}(E/K)$ is cyclic with generator τ. Say $\mathbf{T} = {}^g\mathbf{S_G}$ with $g \in \mathbf{G}(E)$. Then $g^{-1}\tau(g) \in N_{\mathbf{G}(E)}(S(E))$. Let w be the class in $\mathbf{W}(E)$ that $g^{-1}\tau(g)$ passes to. Let X be a strongly regular semi-simple element of $\mathfrak{g}(k)$ that has $\mathbf{T}$ as its centralizer. As in section 2, the $G(k)$ orbits in $\mathcal{O}_X(\bar{k}) \cap \mathfrak{g}(k)$ are in bijection with

$\ker[H^1(\Gamma_{un}, \mathbf{T}) \to H^1(\Gamma, \mathbf{G}(K))]$. We do a case by case analysis again.

For PGL_3 we have no cuspidal local systems.

For G_2 simply connectedness reduces the problem to computing $H^1(\Gamma_{un}, \mathbf{T}(K))$. By a theorem of Bruhat and Tits [BT3] we get:

$$H^1(\Gamma_{un}, \mathbf{T}(K)) \cong [\mathbf{T}(K)/\mathbf{T}(K)_0]/(1-\mathrm{Frob}_k)[\mathbf{T}(K)/\mathbf{T}(K)_0].$$

We have $\mathbf{T}(K)_0 = \mathbf{T}(K)_{0^+}$ as $\mathbf{T}$ is a K-anisotropic torus. By Steinberg's theorem we have $|H^1(\mathrm{Gal}(E/K), \mathbf{T}(E)_{0+})| = 1$ and $\mathbf{T}(K)_{0^+} = \mathbf{T}(E)_{0^+}^{Gal(E/K)}$ from Adler, DeBacker [AD]. By combining these facts we get that $\mathbf{T}(K)/\mathbf{T}(K)_0 \cong [\mathbf{T}(E)/\mathbf{T}(E)_{0^+}]^{Gal(E/K)}$.

Using that $\mathbf{T}$ is K-elliptic we get:

$$[\mathbf{T}(E)/\mathbf{T}(E)_{0^+}]^{Gal(E/K)} \cong [\mathbf{T}(E)_0/\mathbf{T}(E)_{0^+}]^{Gal(E/K)} \cong$$

$$[\mathbf{S}(E)_0/\mathbf{S}(E)_{0^+}]^{w\circ\tau} = [\mathbf{S}(E)_0/\mathbf{S}(E)_{0^+}]^{w}.$$

Thus we need to compute the w-fixed points of $[\mathbf{S}(E)_0/\mathbf{S}(E)_{0^+}]$. For $C_2^{G_2}$, $C_3^{G_2}$, and $C_4^{G_2}$, every Y that occurs has $w = R$ for its centralizer, and $|[\mathbf{S}(E)_0/\mathbf{S}(E)_{0^+}]^w| = 1$. Thus for these three cases, for any $Y \in B(X, \chi)$, $\mathcal{O}_Y$ is the only rational orbit in the stable class of Y, and the distributions $D_{(F,\delta)}$ will be a sum of stable distributions, hence stable.

For the remaining case, we have $C_1^{G_2}$, $w = R^2$, and $|[\mathbf{S}(E)_0/\mathbf{S}(E)_{0+}]^w| = 3$, and the individual orbital integrals $D_{(F,\delta)}$ we have built will not be stable on their own. For this cuspidal local system, the group $C(X) \cong S_3$ with the 3-cycle in S_3 involving roots of unity. Thus, we must break our analysis into the two cases where $|\mu_3(k)| = 3$ and $|\mu_3(k)| = 1$.

Case 1:$|\mu_3(k)| = 3$. In this case, the Γ_{un}-action on $C(X)$ is trivial. There are three conjugacy classes in S_3, so we get three terms:

$$D_{(F,\delta)} = \frac{sgn(\mathrm{Id})|S_{\mathrm{Id}}/S^0_{\mathrm{Id}}|}{|Z_{\mathsf{G}_F(\mathfrak{f})}({}^{g_{\mathrm{Id}}}\bar{X})|} \int_{A(X,sgn,g_{\mathrm{Id}})} \mu_Y dY + \frac{sgn(R)|S_R/S^0_R|}{|Z_{\mathsf{G}_F(\mathfrak{f})}({}^{g_R}\bar{X})|} \int_{A(X,sgn,g_R)} \mu_Y dY$$

$$+\frac{sgn(F)|S_F/S^0_F|}{|Z_{\mathsf{G}_F(\mathfrak{f})}({}^{g_F}\bar{X})|} \int_{A(X,sgn,g_F)} \mu_Y dY =$$

$$= \frac{1*3}{6} \int_{A(X,sgn,g_{\mathrm{Id}})} \mu_Y dY + \frac{1*3}{3} \int_{A(X,sgn,g_R)} \mu_Y dY + \frac{-1*1}{2} \int_{A(X,sgn,g_F)} \mu_Y dY$$

For the third term in the above sum the resulting semi-simple elements Y have centralizers with trivial cohomology. Thus the third summand of this distribution is stable.

We now deal with the integrals tied to the classes (Id) and (R_{S_3}). For $Y \in A(X, sgn, g_{\mathrm{Id}})$, the rational classes in the stable orbit are represented by Y, ${}^{g_R}Y$, and ${}^{g_{R^2}}Y$. As both ${}^{g_R}Y$ and ${}^{g_{R^2}}Y$ occur in the second integral and the same $1:2$ ratio occurs, the sum of the first two terms in $D_{(F,\delta)}$ forms a stable distribution. As $D_{(F,\delta)}$ is now a sum of stable distributions, $D_{(F,\delta)}$ itself is stable.

Case 2:$|\mu_3(k)| = 1$.

Here we need to compute the Frob_k conjugacy classes in $C(X)$. In our notation, the classes are $\{\mathrm{Id}, R, R^2\}$, $\{F\}$, and $\{FR, FR^2\}$. The class associated to the first has $|H^1(\mathrm{Gal}(K/k, \mathbf{S}_m(K))| = 1$ with $|S_m/S_m^0| = 1$, while the other two have $|H^1(\mathrm{Gal}(K/k, \mathbf{S}_m(K))| = 3$ with $|S_m/S_m^0| = 3$. Thus we get the following equation:

$$D_{(F,\delta)} = \tfrac{1*3}{6} \int_{A(X,sgn,g_e)} \mu_Y dY + \tfrac{-1*3}{3} \int_{A(X,sgn,g_F R)} \mu_Y dY +$$
$$\tfrac{-1*3}{6} \int_{A(X,sgn,g_F)} \mu_Y dY$$

The first term has $|H^1(\mathrm{Gal}(K/k, \mathbf{S}_m(K))| = 1$, so it is stable. The latter two occur in a $1 : 2$ ratio, which is precisely what is needed to force their sum to be stable. Thus the whole distribution is stable.

For SO_4 we have two local systems to work with. Let $X = X_{\alpha^{SO_4}} + X_{(3\alpha+2\beta)^{SO_4}}$. Then $\mathbf{M}_X \cong \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ with trivial Γ-action. For $m \in \mathbf{M}_X$, either $m = 1$ or $m = -1$, and in both cases $|S_m/S_m^0| = 2$ and $|H^1(\mathrm{Gal}(K/k), \mathbf{S}_m)| = 2$, We get:

$$D_{(F,\delta^1)} = \tfrac{1*2}{2} \int_{A(X,sgn,g_e)} \mu_Y dY + \tfrac{-1*2}{2} \int_{A(X,sgn,g_F R)} \mu_Y dY$$

The other cuspidal local system yields the distribution:

$$D_{(F,\delta^2)} = \tfrac{1*2}{2} \int_{A(X',sgn,g_e)} \mu_Y dY + \tfrac{-1*2}{2} \int_{A(X',sgn,g_F R)} \mu_Y dY$$

Where $X' = X_{\alpha^{SO_4}} + \varpi X_{(3\alpha+2\beta)^{SO_4}}$. We shall prove that no nontrivial linear combination of these two distributions is stable. However, to do this we will need to explore some explicit conjugacy classes in $\mathfrak{so}_4(k)$. Let $X_{a,b,c,d} = aX^{SO_4}_{\alpha^{SO_4}} + bX^{SO_4}_{-(\alpha)^{SO_4}} + cX^{SO_4}_{(3\alpha+\beta)^{SO_4}} + dX^{SO_4}_{-(3\alpha+\beta)^{SO_4}}$.

Lemma IX.1. *Say $a, a', b, b', c, c', d,$ and $d' \in k^{\times}$. Then $X_{a,b,c,d}$ is geometrically conjugate to $X_{a',b',c',d'}$ if and only if $ab = a'b'$ and $cd = c'd'$.*

Proof. Note $X_{a,b,c,d}$ is geometrically conjugate to $X_{1,ab,1,cd}$ via $\check{\mu}_{\alpha^{SO_4}}(\sqrt{a}^{-1})\check{\mu}_{\beta^{SO_4}}(\sqrt{c}^{-1})$. Likewise, $X_{a',b',c',d'}$ is geometrically conjugate to $X_{1,a'b',1,c'd'}$. However, $\{X_{1,x,1,y} | x, y \in \bar{k}\}$ is a Kostant section and $X_{a,b,c,d}$ is regular, hence the conjugacy class of $X_{a,b,c,d}$ intersects $\{X_{1,x,1,y} | x, y \in \bar{k}\}$ precisely once.[Kos]

$\square$

Lemma IX.2. *Say $a, a', b, b', c, c', d,$ and $d' \in (R)^{\times}$. Then $X_{a,b\varpi,c,d\varpi}$ is rationally conjugate to $X_{a',b'\varpi,c',d'\varpi}$ if and only if the following two conditions are satisfied:*

A)$ab = a'b'$, $cd = c'd'$

B)One of the following four conditions is satisfied:

i)$\frac{a}{a'}$ and $\frac{c}{c'}$ are both squares,

ii)$\frac{a}{a'}$ and $\frac{c}{c'}$ are both not squares,

iii)$\frac{a}{b'}$ and $\frac{c}{d'}$ are both squares,

iv)$\frac{a}{d'}$ and $\frac{c}{d'}$ are both not squares.

Proof. Assume $X_{a,b\varpi,c,d\varpi} = {}^{g}X_{a',b'\varpi,c',d'\varpi}$ with $g \in \mathbf{SO}_4(k)$. We get $ab = a'b'$ and $cd = c'd'$ from 9.1. Let $\mathbf{T}_1 = Z_{\mathbf{G}}(X_{a,b\varpi,c,d\varpi})$ and $\mathbf{T}_2 = Z_{\mathbf{G}}(X_{a',b'\varpi,c',d'\varpi})$. Both of these are tamely ramified and their respective buildings sit inside of $\mathcal{B}(S0_4)$. Then $\mathcal{B}(T_1) \cap \mathcal{B}(SO_4) = \mathcal{B}(T_2) \cap \mathcal{B}(SO_4) = \{x\}$ where x is the center of the alcove A we have selected. Then $g.x = x$, thus $g \in \mathrm{stab}_G(x)$. We know:

$$\mathrm{stab}_G(x) =< U_\alpha(R), U_{-1-\alpha}(R), U_\beta(R), U_{-\beta-1}(R), \mathbf{S}(R) >$$
$$\cup w < U_\alpha(R), U_{-1-\alpha}(R), U_\beta(R), U_{-\beta-1}(R), \mathbf{S}(R) >.$$

We look at the action of three cocharacters $\check{\alpha}, \check{\beta}$, and $\frac{\check{\alpha}+\check{\beta}}{2}$ and the action of $\tilde{w}$, a choice of representative for w satisfying equation 4 below:

1. ${}^{\check{\alpha}(r)}X_{x,y\varpi,z,w\varpi} = X_{r^2x,r^{-2}y\varpi,z,w\varpi}$

2. ${}^{\check{\beta}(s)}X_{x,y\varpi,z,w\varpi} = X_{x,y\varpi,sz,s^{-2}w\varpi}$

3. ${}^{\frac{\check{\alpha}+\check{\beta}}{2}(q)}X_{x,y\varpi,z,w\varpi} = X_{qx,q^{-1}y\varpi,qz,q^{-1}w\varpi}$

4. ${}^{\tilde{w}}X_{x,y\varpi,z,w\varpi} \longrightarrow X_{y,x\varpi,w,z\varpi}$.

Recall S_{SO_4} is the k-points of our k-split k-torus that we previously fixed. The only element of the root subgroups that preserves the set $\{X_{x,y\varpi,z,w\varpi}|x,y,z,w \in R^\times\}$ is the identity, while the rest of $(SO_4)_x$ does preserve the set $\{X_{x,y\varpi,z,w\varpi}|x,y,z,w \in R^\times\}$, hence by intersecting the Bruhat decomposition of SO_4 with $(SO_4)_x$ we get that g lies in $S_{SO_4} \coprod wS_{SO_4}$. Recall ϵ is a fixed non square in $R^\times$. Let $S' = \check{\alpha}(k^\times)\check{\beta}(k^\times)$. Then $S_{SO_4} = S' \coprod \frac{(\check{\alpha}+\check{\beta})}{2}(\epsilon)S'$, and we get:

$$g \in S_{SO_4} \coprod wS_{SO_4} = S' \coprod \frac{(\check{\alpha}+\check{\beta})}{2}(\epsilon)S' \coprod wS' \coprod w\frac{(\check{\alpha}+\check{\beta})}{2}(\epsilon)S'.$$

These four possibilities for g correspond to the four possible conditions in B).

Now say conditions A) and B) are satisfied. If either condition B)iii or condition B)iv are satisfied, then conjugating $X_{a,b,c,d}$ by $\tilde{w}$ and leaving $X_{a',b',c',d'}$ yields a pair satisfying either condition B)i or condition B)ii. If $\frac{a}{a'} \notin (R^\times)^2$ and $\frac{c}{c'} \notin (R^\times)^2$ then replace $X_{a,b\varpi,c,d\varpi}$ with $^{\frac{\check{\alpha}+\check{\beta}}{2}(\epsilon)}$ and we get $\frac{a}{a'} \in (R^\times)^2$ and $\frac{c}{c'} \in (R^\times)^2$, thus we may assume condition B)i without loss of generality. Fix $r, s \in R^\times$ with $r^2 = \frac{a'}{a}$ and $s^2 = \frac{c'}{c}$. Then:

$$^{\check{\alpha}(r)\check{\beta}(s)}X_{a,b\varpi,c,d\varpi} = X_{aa'/a,ba/a'\varpi,cc'/c,dc/c'\varpi} = X_{a',bb'/v\varpi,c',dd'/d\varpi}$$

$$= X_{a',b',\varpi,c',d'\varpi}$$

And our Lemma is proved.

$\square$

Let $X^1_{a,b} = X_{1,a\varpi,1,b\varpi}$, $X^2_{a,b} = X_{\epsilon,a\epsilon^{-1}\varpi,1,b\varpi}$, $X^3_{a,b} = X_{1,a\varpi,b,\varpi}$ and $X^4_{a,b} = X_{\epsilon,a\epsilon^{-1}\varpi,b,\varpi}$. Then:

$$D_{(F,\delta^1)} = \int_{a,b\in R^\times} \mu_{X^1_{a,b}} - \int_{a,b\in R^\times} \mu_{X^2_{a,b}}$$

$$D_{(F,\delta^2)} = \int_{a,b\in R^\times} \mu_{X^3_{a,b}} - \int_{a,b\in R^\times} \mu_{X^4_{a,b}}$$

Lemma IX.3. *No $\mathbb{C}$-linear combination of $D_{(F,\delta^1)}$ and $D_{(F,\delta^2)}$ is stable.*

Proof. By Lemma IX.1, $X^i_{a,b}$ is geometrically conjugate to $X^j_{c,d}$ if and only if $a = c$ and $b = d$. By Lemma IX.2, $X^1_{a,b}$ is rationally conjugate to $X^2_{a,b}$ if and only if $\frac{a}{b} \notin (R^\times)^2$. Likewise, $X^3_{a,b}$ is rationally conjugate to $X^4_{a,b}$ if and only if $\frac{a}{b} \notin (R^\times)^2$. Furthermore, $X^1_{a,b}$ is rationally conjugate to $X^3_{a,b}$ if and only if a or $b \in (R^\times)^2$ and $X^2_{a,b}$ is rationally conjugate to $X^3_{a,b}$ if and only if a or $b \in (R^\times)^2$. Finally, $X^1_{a,b}$ is rationally conjugate to $X^4_{a,b}$ and $X^3_{a,b}$ is rationally conjugate to $X^2_{a,b}$ if and only if a or $b \notin (R^\times)^2$. We summarize all this data in three cases:

1. Exactly one of a or b is in $(R^\times)^2$. Then $X^i_{a,b}$ is rationally conjugate to $X^j_{a,b}$ for all i, j.

2. Both a and b are in $(R^\times)^2$. Then $X^1_{a,b}$ is rationally conjugate to $X^3_{a,b}$ and $X^2_{a,b}$ is rationally conjugate to $X^4_{a,b}$ and no other pairs are rationally conjugate.

3. Both a and b are not in $(R^\times)^2$. Then $X^1_{a,b}$ is rationally conjugate to $X^4_{a,b}$ and $X^2_{a,b}$ is rationally conjugate to $X^3_{a,b}$ and no other pair is rationally conjugate.

Let $rD_{(F,\delta^1)} + sD_{(F,\delta^w)}$ be a stable linear combination of $D_{(F,\delta^1)}$ and $D_{(F,\delta^2)}$.

Assume $a, b \in (R^\times)^2$ as in case 2. Then the occurrence of $\mu_{X^1_{a,b}}$ in $D_{(F,\delta^1)}$ must be paired with an equal sign occurrence of its geometric but not rational conjugate in the expression $rD_{(F,\delta^1)} + sD_{(F,\delta^2)}$. That means either $\mu_{X^2_{a,b}}$ or $\mu_{X^4_{a,b}}$ must occur with the same coefficient, and as $\mu_{X^2_{a,b}}$ only occurs with the coefficient r, this forces the equation $r = -s$.

Now assume $a, b \notin (R^\times)^2$ as in case 3. Then the $\mu_{X^1_{a,b}}$ in the expression $rD_{(F,\delta^1)} + sD_{(F,\delta^w)}$ has to pair with $\mu_{X^2_{a,b}}$ or $\mu_{X^3_{a,b}}$, and it cannot pair with $\mu_{X^2_{a,b}}$ because of the sign in $D_{(F,\delta^1)}$, thus we get the equation $r = s$.

Combining these we get $r = s = 0$ and the lemma is proved.

$\square$

9.3 A Basis for the Stable, Compactly Supported Distributions

We write down a basis for the stable distributions in $\mathrm{res}_{\mathcal{D}_0} J(\mathfrak{g}_0)$.

We first address a minor technical point. In order to force our combinations of distributions dual to Green functions be stable we will have to adjust the precise $X \in$Lie(T) that we use to define $D^G_{(F,\mathsf{T})}$ for each unramified torus. Our strategy is to pick representatives for each stable class, pick regular elements arbitrarily for our representatives, then pick the remaining regular elements to be stably conjugate to our fixed choices. For G_2, the tori we need to pick arbitrary X_T for are $T_{(F_{\alpha,\beta},\mathsf{T}_{R^2})}$ and $T_{(F_{\alpha,\beta},\mathsf{T}_{R^3})}$. For SO_4 we need to pick a regular element for $T_{(F_{\{\alpha^{SO_4},\beta^{SO_4}\}},\mathsf{T}_{(F,F)})}$. In every other case the regular elements are chosen to be conjugate to one of our fixed choices, or are not related by geometric conjugacy and thus cause no trouble. This forces the combination we built to be stable. We can make these adjustments because the restriction of our $D_{(F,\mathsf{T})}$ to $\mathcal{D}_0$ depends only on the unramified torus T, not on the particular X_T we choose.

First we write a stable basis for the distributions dual to generalized Green functions for PGL_3:

Lemma IX.4. *Let $X, Y \in \mathfrak{sl}_3(k)$ =Lie(PGL_3), $g \in \mathbf{PGL}_3(\bar{k})$ with ${}^gX = Y$. Then $\exists h \in \mathbf{PGL}_3(k)$ with ${}^hX = Y$.*

Proof. Let $\pi : \mathbf{GL}_3 \to \mathbf{PGL}_3$ be the quotient by the center of $\mathbf{GL}_3$. Note π is a k-rational map as the center of $\mathbf{GL}_3$ is a closed k-group. Fix $X', Y' \in \mathfrak{gl}_3(k), g' \in \mathbf{GL}_3(\bar{k})$ with $d\pi(X') = X$, $d\pi(Y') = Y$, $\pi(g') = g$. Then ${}^{g'}X' = Y' + Z$ with Z in the center of $\mathfrak{gl}_3$. As X' and Y' are both in the k-points of $\mathfrak{gl}_3$, we get that $Z \in \mathfrak{gl}_3(k)$, and as the k-points of a k-subgroup are the intersection of the $\bar{k}$-points of the subgroup with the k-points of the ambient group, we get that Z is in the k-points of the center of $\mathfrak{gl}_3$. By rational canonical form, $\exists h' \in \mathbf{GL}_3(k)$ with ${}^{h'}X' = Y' + Z$. Set $\pi(h') = h \in \mathbf{PGL}_3(k)$. Then ${}^hX = d\pi({}^{h'}X') = d\pi(Y' + Z) = Y + 0 = Y$.

$\square$

Corollary IX.5. *Every $PGL_3(k)$-invariant distribution on $\mathfrak{sl}_3(k)$ is stable.*

Proof. Lemma IX.4 shows rational conjugacy and geometric PGL_3-conjugacy agree for $\mathfrak{sl}_3$, hence for all $X \in \mathfrak{sl}_3^{reg,s.s.}(k)$, $\mu_X^{stab} = \mu_X$. Thus $\mathcal{K}^{stab} = \mathcal{K}$ (recall definitions from chapter 3). By Theorem III.1, every invariant distribution on $\mathfrak{sl}_3(k)$ kills every function in $\mathcal{K}$, thus every invariant distribution on $\mathfrak{sl}_3(k)$ kills every function in $\mathcal{K}^{stab}$, thus every invariant distribution on $\mathfrak{sl}_3$ is stable.

$\square$

The following distributions separate the generalized Green functions, and hence span $\mathrm{res}_{\mathcal{D}_0} J(\mathfrak{g}_0)$:

$$D^{PGL_3}_{(F_\emptyset,\mathsf{S})}$$

$$D^{PGL_3}_{(F_{\{\alpha\}},\mathsf{T}_F)}$$

$$D^{PGL_3}_{(F_{\{\alpha,\beta\}},\mathsf{T}_R)}$$

All of the above distributions are stable.

For SO_4 the following distributions separate generalized Green functions:

$$D^{SO_4}_{(F,\mathsf{S})}$$

$$D^{SO_4}_{(F_{\{\alpha\}},\mathsf{T}_F)}$$

$$D^{SO_4}_{(F_{\{\beta\}},\mathsf{T}_F)}$$

$$D^{SO_4}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{(F,F)})} + D^{SO_4}_{(F_{\{\alpha,1-\beta\}},\mathsf{T}_{(F,F)})}$$

$$D^{SO_4}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{(F,F)})} - D^{SO_4}_{(F_{\{\alpha,1-\beta\}},\mathsf{T}_{(F,F)})}$$

$$D_{(F_{\{\alpha,\beta\}},X_\alpha+X_\beta,sgn\times sgn)}$$

$$D_{(F_{\{\alpha,\beta-1\}},X_\alpha+X_\beta,sgn\times sgn)}$$

We have already shown that the first 4 distributions are in fact stable. We have also shown that no linear combination of the last 2 can be stable. We need to show that no linear combination of the last 3 can form a stable distribution.

Lemma IX.6. *For $r, s, t \in \mathbb{C}$ with at least one of r, s, t non-zero, the following distribution is not stable:*

$$r(D^{SO_4}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{(F,F)})} - D^{SO_4}_{(F_{\{\alpha,1-\beta\}},\mathsf{T}_{(F,F)})}) + sD_{(F_{\{\alpha,\beta\}},X_\alpha+X_\beta,sgn\times sgn)}+$$

$$tD_{(F_{\{\alpha,1-\beta\}},X_\alpha+X_\beta,sgn\times sgn)}$$

Proof. Say the above distribution is stable. Notice that there is no overlap between the various orbital integrals occurring in the distributions:

$$sD_{(F_{\{\alpha,\beta\}},X_\alpha+X_\beta,sgn\times sgn)} + tD_{(F_{\{\alpha,1-\beta\}},X_\alpha+X_\beta,sgn\times sgn)}$$

and

$$r(D^{SO_4}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{(F,F)})} - D^{SO_4}_{(F_{\{\alpha,1-\beta\}},\mathsf{T}_{(F,F)})})$$

Thus $r = 0$, reducing the problem to lemma IX.3.

$\square$

Corollary IX.7. *:The following is a basis of* $res_{\mathcal{D}_0} J^{st}((\mathfrak{so}_4)_0)$*:*

$$D^{SO_4}_{(F,\mathsf{S})}$$

$$D^{SO_4}_{(F_{\{\alpha\}},\mathsf{T}_F)}$$

$$D^{SO_4}_{(F_{\{\beta\}},\mathsf{T}_F)}$$

$$D^{SO_4}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{(F,F)})} + D^{SO_4}_{(F_{\{\alpha,1-\beta\}},\mathsf{T}_{(F,F)})}$$

For G_2 the following distributions separate generalized Green functions:

$$D^{G_2}_{(F,\mathsf{S})}$$

$$D^{G_2}_{(F_{\{\alpha\}},\mathsf{T}_F)}$$

$$D^{G_2}_{(F_{\{\beta\}},\mathsf{T}_F)}$$

$$D^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_R)}$$

$$D^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{R^2})} + 2D^{G_2}_{(F_{\{\beta,1-3\alpha-\beta\}},\mathsf{T}_R)}$$

$D^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{R^3})} + 3D^{G_2}_{(F_{\{\alpha,1-3\alpha-2\beta\}},\mathsf{T}_{(F,F)})}$

$D^{G_2}_{(F_{\{\alpha,\beta\}},e_{subreg},sgn)}$

$D^{G_2}_{(F_{\{\alpha,1-3\alpha-2\beta\}},m,sgn\times sgn)}$

$D^{G_2}_{(F_{\{\beta,1-3\alpha-\beta\}},n,\psi)}$

$D^{G_2}_{(F_{\{\beta,1-3\alpha-\beta\}},n,\psi^{-1})}$

$D^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{R^2})} - D^{G_2}_{(F_{\{\beta,1-3\alpha-\beta\}},\mathsf{T}_{R})}$

$D^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{R^3})} - D^{G_2}_{(F_{\{\alpha,1-3\alpha-2\beta\}},\mathsf{T}_{(F,F)})}$

Notice that the distributions $D^{G_2}_{(F_{\{\beta,1-3\alpha-\beta\}},n,\psi)}$ and $D^{G_2}_{(F_{\{\beta,1-3\alpha-\beta\}},n,\psi^{-1})}$ exist if and only if $|\mu_3(k)| = 3$, thus $\dim[\mathrm{res}_{\mathcal{D}_0} J((\mathfrak{g}_2)_0)] = 9 + |\mu_3(k)|$.

Lemma IX.8. *(DK):Let $r, s \in \mathbb{C}$ with at least one non-zero. Say $res_{\mathcal{D}_0} T = r(D^{G_2}_{(F_{\alpha,\beta},\mathsf{T}_{R^2})} - D^{G_2}_{(F_{\beta,3\alpha+\beta-1},\mathsf{T}_R)}) + s(D^{G_2}_{(F_{\alpha,\beta},\mathsf{T}_{R^3})} - D^{G_2}_{(F_{\alpha,3\alpha+2\beta-1},\mathsf{T}_{(F,F)})})$. Then T is not a stable distribution.*

Proof. Say T is stable. Say $r \neq 0$. As $\mathrm{res}_{\mathcal{D}_0} J((\mathfrak{g}_2)_0) = \mathrm{res}_{\mathcal{D}_0}\mathcal{N}$, we have $\hat{T}$ is represented on $C_c^\infty((\mathfrak{g}_2)_{0^+})$ by a locally integrable function on $(\mathfrak{g}_2)_{0^+}$. Extend this function by zero to all of $\mathfrak{g}_2$ and call the result $\hat{T}$.

Let $D^{G_2}_{(F_{\alpha,\beta},\mathsf{T}_{R^2})} - D^{G_2}_{(F_{\beta,3\alpha+\beta-1},\mathsf{T}_R)} = D_1$ and $D^{G_2}_{(F_{\alpha,\beta},\mathsf{T}_{R^3})} - D^{G_2}_{(F_{\alpha,3\alpha+2\beta-1},\mathsf{T}_{(F,F)})} = D_2$ with corresponding combination of generalized Green functions $\mathcal{G}_1$ and $\mathcal{G}_2$. As $\hat{\mathcal{G}}_{(F,\delta)} \in \mathcal{D}_0$ and $\mathcal{G}_{(F,\delta)} \in C_c^\infty((\mathfrak{g}_2)_{0^+})$ we have:

$$r = T(\hat{\mathcal{G}}_1) = \hat{T}(\mathcal{G}_1) = \int_{\mathfrak{g}_2} \hat{T}(X)\mathcal{G}_1(X)dx$$

By the Weyl integration formula we get:

$$r = \Sigma_{\mathfrak{h} Cartan} c_{\mathfrak{h}} \int_{\mathfrak{h}} |\eta(H)| \hat{T}\mu_H(\mathcal{G}_1)dH$$

For r to be non-zero there must be some H regular semisimple in $(\mathfrak{g}_2)_{0^+}$ with $\mu_H(\mathcal{G}_1) \neq 0$. As the Green functions involved are both cuspidal, we know H is elliptic. This gives us that $0 \neq \hat{T}(H) = r\hat{D}_1(H) + s\hat{D}_2(H)$

However from [W1] we know that $\hat{T}$ is a stable function. Looking at the formulae for D_1 and D_2 this is impossible. An analogous argument holds if we assume $s \neq 0$.

$\square$

CHAPTER X

Pairing Distributions with Gelfand-Graev Characters

This section works through the computational heart of our comparison of $\mathrm{res}_{\mathcal{D}_0} J(\mathfrak{g}_0)$ and $\mathrm{res}_{\mathcal{D}_0} J(\mathcal{N})$. Specifically, we compute $\mu_{\mathcal{O}_{(F',e')}}(h_{(F,e)})$, $D_{(F,T)}(h_{(F,e)})$ and $D_{(F,\delta)}(h_{(F,e)})$. The functions in question are defined in chapter 6 while the distributions in question are defined in chapter 7.

10.1 Pairing Nilpotent Orbital Integrals with Gelfand-Graev Characters

Lemma X.1. *(Waldspurger)[W3]:Let (F, e) parameterize a nilpotent orbit. Then:* $\mu_{\mathcal{O}_{(F,e)}}(h_{(F,e)}) = |\mathsf{L}_F(1)|^{\frac{1}{2}}$.

Lemma X.2. *(Waldspurger)[W3]:Say $\mathcal{O}_{(F,e)} \not\subset \bar{\mathcal{O}}_{(F',e')}$. Then:*

$$\mu_{\mathcal{O}_{(F',e')}}(h_{(F,e)}) = 0.$$

These are straightforward generalizations of lemmas of Waldspurger [W3].

10.2 Pairing Distributions Dual to Lusztig with Gelfand-Graev Characters

Let $\mathcal{G}_{(F,\delta)}$ be a Lusztig function. Let x be a vertex in $\mathcal{B}(G)$. Let $e \in \mathcal{N}(\mathsf{G}_x)(\mathfrak{f})$ be $\mathfrak{f}$-distinguished.

Lemma X.3. *If $F \cap {}^G x = \emptyset$, then $D_{(F,\delta)}(h_{(x,e)}) = 0$. If $F \cap {}^G x = F$, then $D_{(F,\delta)}(h_{(x,e)}) = |\mathsf{L}_x(0)|^{\frac{-1}{2}}\mathcal{G}_{(F,\delta)}(e)$.*

Proof. Fortunately for us, every cuspidal local system (F,δ) of every group we study has F a vertex.

$$D_{(F,\delta)}(h_{(x,e)}) = \frac{|\mathsf{L}_F(\mathfrak{f})|^{\frac{1}{2}}}{|\mathsf{G}_F(\mathfrak{f})|}\int_G\int_{\mathfrak{g}} h_{(x,e)}({}^gZ)\mathcal{G}_{(F,\delta)}(Z)dZdg$$
$$= \frac{|\mathsf{L}_F(\mathfrak{f})|^{\frac{1}{2}}}{|\mathsf{G}_F(\mathfrak{f})|}\int_G\int_{\mathfrak{g}_F} h_{(x,e)}({}^gZ)\mathcal{G}_{(F,\delta)}(Z)dZdg$$
$$= \frac{|\mathsf{L}_F(\mathfrak{f})|^{\frac{1}{2}}}{|\mathsf{G}_F(\mathfrak{f})|}\int_G\Sigma_{\bar{Z}\in\mathfrak{g}_F/(\mathfrak{g}_F\cap\mathfrak{g}^+_{g^{-1}x})} h_{(x,e)}({}^gZ)\int_{\mathfrak{g}_F\cap\mathfrak{g}^+_{g^{-1}x}}\mathcal{G}_{(F,\delta)}(Z+Z')dZ'dg$$

If $g^{-1}x \neq F$ then the image of $\mathfrak{g}_F \cap \mathfrak{g}_{g^{-1}x}$ in $\mathsf{L}_F(\mathfrak{f})$ is the $\mathfrak{f}$-points of the Lie algebra of a proper parabolic $\mathfrak{f}$-subgroup of G_F. The image of $\mathfrak{g}_F \cap \mathfrak{g}^+_{g^{-1}x}$ in $\mathsf{L}_F(\mathfrak{f})$ is the $\mathfrak{f}$-points of the nilradical of said Lie Algebra. As $\mathcal{G}_{(F,\delta)}$ is a cuspidal function, the innermost integral is zero. Therefore the whole expression vanishes if F and x are not in the same G-orbit in $\mathcal{B}(G)$. If they do lie in the same G-orbit, without loss of generality $F = x$. Then:

$$D_{(F,\delta)}(h_{(x,e)}) = \frac{|\mathsf{L}_x(\mathfrak{f})|^{\frac{1}{2}}}{|\mathsf{G}_x(\mathfrak{f})|}\int_{G_x}\int_{\mathfrak{g}_x} h_{(x,e)}({}^gZ)\mathcal{G}_{(x,\delta)}(Z)dZdg$$
$$= \frac{|\mathsf{L}_x(\mathfrak{f})|^{\frac{1}{2}}}{|\mathsf{G}_x(\mathfrak{f})|}\mu(G_x)\mu(\mathfrak{g}_x^+)|\mathsf{L}_x(\leq -1)|\mathcal{G}_{(x,\delta)}(e)$$
$$= |\mathsf{L}_x(0)|^{-\frac{1}{2}}\mathcal{G}_{(x,\delta)}(e).$$

□

10.3 Pairing Distributions Dual to Green Functions with Gelfand-Graev Characters

This computation has to be done in two steps. The second is rather elaborate, particularly for SO_4.

Lemma X.4. *(DeBacker-Kazhdan):Let (F, T) be a pair consisting of F a facet and T a minisotropic torus of G_F. Say $X_T \in \mathfrak{g}_F$ with the image of X_T in $\mathsf{L}_F(\mathfrak{f})$ a regular element of Lie$(\mathsf{T})(\mathfrak{f})$. Let $\mathbf{T}$ be a lift of T to $\mathbf{G}$. Let (C, e) be a pair consisting of C a facet with e a $\mathfrak{f}$-distinguished nilpotent of $\mathsf{L}_C(\mathfrak{f})$. Then $\mu_{X_T}(h_{(C,e)}) = 0$ unless there is a $g \in G$ such that $gC \subset \mathcal{B}(T)$. If there is such a g, then we may assume $C \subset \mathcal{B}(T)$ and the following equation holds:*

$$\mu_{X_T}(h_{(C,e)}) = \frac{(-1)^{rk(T)}|\mathsf{L}_C(1)|^{\frac{1}{2}}|N_G(T)/T|}{|N_{\mathsf{G}_C}(\mathsf{T})(\mathfrak{f})/\mathsf{T}(\mathfrak{f})|}(Ind_F^C Q_{\mathsf{T}}^F, \Gamma_{(C,e)})_{\mathsf{L}_C}$$

Proof. WLOG $Z_{\mathbf{G}}(X_T) = \mathbf{T}$. Let $h_C(Y) = \frac{1}{|\mathsf{G}_C(\mathfrak{f})|}\Sigma_{\bar{g}\in\mathsf{G}_C(\mathfrak{f})} h_{(C,e)}({}^gY)$. Then:

$$\mu_{X_T}(h_{(C,e)}) = \int_{G/T} h_{(C,e)}({}^gX_T)d^*g = \int_{G/T} h_C({}^gX_T)d^*g$$

With d^*g understood to be the quotient measure on G/T. Fix $g \in G$. As $h_{(C,e)} \in C(\mathfrak{g}_C/\mathfrak{g}_C^+)$, the condition $h_{(C,e)}({}^gX_T) \neq 0$ forces ${}^gX_T \in \mathfrak{g}_C$. Then $X_T \in \mathfrak{g}_{g^{-1}C}$, hence $g^{-1}C \subset \mathcal{B}(T) \subset \mathcal{B}(G)$. Set $\mathcal{F} = GC \cap \mathcal{B}(T)$. For $V \subset \mathcal{F}$ a facet of the same type as C, the centralizer in G_C of the image of X_T in $\mathsf{L}_V(\mathfrak{f})$ is naturally isomorphic to T. Thus we have a corresponding Green function we call Q^V_{T}. Furthermore, Q^V_{T} $=\mathrm{Ind}^V_F Q^F_{\mathsf{T}}$. Let $\mathcal{F}^{rep}$ be a choice of representatives for the T-orbits in $\mathcal{F}$. We may assume $C \subset \mathcal{F}$ without loss of generality. For $V \subset \mathcal{F}^{rep}$ fix $g_V \in G$ such that $g_V^{-1}C = V$. Then ${}^{g_V}G_V = G_C$, and as h_C is G_C-invariant we get the equation:

$$\mu_{X_T}(h_C) = \Sigma_{V \subset \mathcal{F}^{rep}} \int_{G_V T/T} h_C({}^{g_V g}C_T) d^*g$$
$$= \Sigma_{V \subset \mathcal{F}^{rep}} \mu_{d^*g}(G_V T/T) h_C({}^{g_V}X_T)$$

Notice $\mu_{d^*g}(G_V T/T) = \frac{|\mathsf{G}_V(\mathfrak{f})||\mathsf{L}_{\mathsf{T}}(\mathfrak{f})|^{\frac{1}{2}}}{|\mathsf{L}_V(\mathfrak{f})|^{\frac{1}{2}}|\mathsf{T}(\mathfrak{f})|}$, thus we get:

$$\mu_{X_T}(h_C) = \frac{|\mathsf{G}_V(\mathfrak{f})||\mathsf{L}_{\mathsf{T}}(\mathfrak{f})|^{\frac{1}{2}}}{|\mathsf{L}_V(\mathfrak{f})|^{\frac{1}{2}}|\mathsf{T}(\mathfrak{f})|} \Sigma_{V \in \mathcal{F}^{rep}} h_C({}^{g_V}X_T)$$

To finish we need to compute the term $h_C({}^{g_V}X_T)$. Define $h_V(X) = h_C({}^{g_y}X)$. From Fourier inversion we get:

$$h_V(X_T) = \int_{\mathfrak{g}_V} \Lambda(B(X_T, Y)) \hat{h}_V(-Y) dY$$

Furthermore:

$$(Q^F_{\mathsf{T}}, \mathfrak{r}^V_F \hat{h})_{\mathsf{L}_F} = (Q^V_{\mathsf{T}}, \hat{h})_{\mathsf{L}_V} = \frac{1}{|\mathsf{G}_V(\mathfrak{f})|} \Sigma_{\bar{X} \in \mathsf{L}_V(\mathfrak{f})} Q^V_{\mathsf{T}}(\bar{X}) \hat{h}_V(X)$$

As $\mathrm{supp}(\hat{h}) \subset \mathcal{N}(k)$, we can use the equation $\mathcal{F}(\Sigma_{g \in \mathsf{G}_F(\mathfrak{f}))}[\bar{X}_{\mathsf{T}}]^g)(\bar{X}) = \frac{(-1)^{rk\mathsf{T}}|\mathsf{T}(\mathfrak{f})|}{|\mathsf{L}_{\mathsf{T}}(\mathfrak{f})|^{\frac{1}{2}}} Q^F_{\mathsf{T}}(\bar{X})$ to get:

$$\frac{(-1)^{rk\mathsf{T}}|\mathsf{T}(\mathfrak{f})|}{|\mathsf{L}_{\mathsf{T}}(\mathfrak{f})|^{\frac{1}{2}}} (Q^F_{\mathsf{T}}, \mathfrak{r}^V_F \hat{h})_{\mathsf{L}_F} = \frac{1}{|\mathsf{G}_V(\mathfrak{f})|} \Sigma_{\bar{X} \in \mathsf{L}_V(\mathfrak{f})} \mathcal{F}(\Sigma_{g \in \mathsf{G}_F(\mathfrak{f}))}[\bar{X}_{\mathsf{T}}]^g)(\bar{X}) \hat{h}_V(X)$$

$$= \frac{1}{|\mathsf{G}_V(\mathfrak{f})||\mathsf{L}_V(\mathfrak{f})|^{\frac{1}{2}}} \Sigma_{\bar{X}\in \mathsf{L}_V(\mathfrak{f})} \Sigma_{\bar{Y}\in \mathsf{L}_V(\mathfrak{f})} \Sigma_{g\in \mathsf{G}_V(\mathfrak{f})} [\bar{X}_{\mathsf{T}}]({}^{g}\bar{Y}) \Lambda(B(X,Y)) \hat{h}_V(X)$$

Making the change of variables $(X,Y) \to ({}^{g^{-1}}X, {}^{g^{-1}}Y)$ and moving the sum over $\mathsf{G}_V(\mathfrak{f})$ to be first we get:

$$\frac{(-1)^{rk(\mathsf{T})}|\mathsf{T}(\mathfrak{f})|}{|\mathsf{L}_{\mathsf{T}}(\mathfrak{f})|^{\frac{1}{2}}} (Q^F_{\mathsf{T}}, \mathfrak{r}^V_F \hat{h})_{\mathsf{L}_F} = \mu_{dX}(\mathfrak{g}^+_V) \Sigma_{\bar{X}\in \mathsf{L}_V(\mathfrak{f})} \Lambda(B(X, X_T)) \hat{h}_V(X)$$
$$= \mu_{dX}(\mathfrak{g}^+_V) \Sigma_{\bar{X}\in \mathsf{L}_V(\mathfrak{f})} \Lambda(B(X, X_T)) \hat{h}_V(-X)$$
$$= \int_{\mathfrak{g}_V} \Lambda(B(X_T, X)) \hat{h}_V(-X) dx$$
$$= h_V(X_T)$$

Combining these computations we get:

$$\mu_{X_T}(h_C) = \frac{|\mathsf{G}_V(\mathfrak{f})||\mathsf{L}_{\mathsf{T}}(\mathfrak{f})|^{\frac{1}{2}}}{|\mathsf{L}_V(\mathfrak{f})|^{\frac{1}{2}}|\mathsf{T}(\mathfrak{f})|} \Sigma_{V\subset \mathcal{F}^{rep}} h_V(X_T)$$
$$= (-1)^{rk(\mathsf{T})} \frac{|\mathsf{G}_C(\mathfrak{f})|}{|\mathsf{L}_C(\mathfrak{f})|^{\frac{1}{2}}} \Sigma_{V\subset \mathcal{F}^{rep}} (Q^F_{\mathsf{T}}, \mathfrak{r}^V_F \hat{h})_{\mathsf{L}_F}$$
$$= (-1)^{rk(\mathsf{T})} \frac{|\mathsf{G}_C(\mathfrak{f})||\mathcal{F}^{rep}|}{|\mathsf{L}_C(\mathfrak{f})|^{\frac{1}{2}}} (Q^F_{\mathsf{T}}, \mathfrak{r}^V_F \hat{h})_{\mathsf{L}_F}$$
$$= (-1)^{rk(\mathsf{T})} |\mathsf{L}_C(1)|^{\frac{1}{2}} |\mathcal{F}^{rep}| (\mathrm{Ind}^C_F Q^F_{\mathsf{T}}, \Gamma_{(C,e)})_{\mathsf{L}_F}$$

Now we compute $|\mathcal{F}^{rep}|$ to finish.

Fix a maximal k-split torus $\mathbf{T}'$ of $\mathbf{G}$ such that $\mathcal{B}(T) = A(\mathcal{A}(\mathbf{T}', k), F)$. Say $V \subset \mathcal{F}^{rep}$ and $g \in G$ with $gC = V$. Let ${}^{g}\mathsf{T}$ denote the $\mathfrak{f}$-torus whose $\mathfrak{F}$-points are the image of ${}^{g}\mathbf{T}(K) \cap \mathbf{G}(K)$ in $\mathsf{G}_V(\mathfrak{F})$. Using our identification of G_F with G_V we retrieve a $k \in G_V$ with ${}^{kg}\mathsf{T} = \mathsf{T}$. As T is a lift of T, it is a lift of ${}^{kg}\mathsf{T}$, so we get $k' \in G^+_V$ with ${}^{k'kg}T = T$. Therefore $\mathcal{F}^{rep}$ is in bijection with multiplicative $N_{G_C}(T)/T$-orbits in $N_G(T)/T$. Thus:

$$\mu_{X_T}(h_C) = (-1)^{rk(\mathsf{T})}|\mathsf{L}_C(1)|^{\frac{1}{2}} \frac{|N_G(T)/T|}{|N_{\mathsf{G}_C}(\mathsf{T})(\mathfrak{f})/\mathsf{T}(\mathfrak{f})|}(\mathrm{Ind}_F^C Q_{\mathsf{T}}^F, \Gamma_{(C,e)})_{\mathsf{L}_F}$$

$$= \mu_{X_T}(h_{(C,e)})$$

□

Unfortunately, it is fairly elaborate to compute $(\mathrm{Ind}_F^x Q_{\mathsf{T}}^F, \Gamma_{(x,e)})_{\mathsf{L}_F}$, especially for SO_4. We do this by using a result of Kawanaka followed by using computations of Springer. However, to use the result of Kawanaka we must first discuss the notion of a special nilpotent orbit.

Let $X \in \mathcal{N}(\mathfrak{f})$ be a nilpotent element. Let $\mathcal{B}_X^G = \{\mathcal{B} \subset \mathbf{G}$ Borel subgroup$|X \in \mathfrak{b}(\mathfrak{f})\}$. Let $e(X)$ =$\dim_{\mathfrak{F}}(\mathcal{B}_X^G)$. We say the orbit $\mathcal{O}_X$ is special if $2e(X)$ =$\dim_{\mathfrak{F}}(\mathbf{Z}_{\mathbf{G}}(X))-\mathrm{rk}(\mathbf{G})$. Let Σ be the set of pairs (X, ϕ) with $X \in \mathcal{N}(\mathfrak{f})$ special and $\phi \in C_X^D$. Notice that $\mathbf{G}(\mathfrak{F})$ acts on Σ.

Theorem X.5. *(Springer)[Spr 6.10]:There is a bijection between* $\Sigma/\mathbf{G}(\mathfrak{F})$ *and* W^D.

Notice that all regular nilpotent orbits are special. This is because there is a unique Borel containing them by a theorem of Steinberg and the dimension of their centralizer is equal to the rank of the ambient group by definition.

For PGL_3, centralizers of nilpotent elements are connected, which is enough to force all orbits to be special.

We will also need that the subregular orbit in G_2 is special.

X	$e(X)$	$\dim_{\mathfrak{F}}(\mathbf{Z_G}(X)) - \mathrm{rk}(\mathbf{G})$
$X^{G_2}_{3\alpha+\beta} + X^{G_2}_{\beta}$	1	2

Theorem X.6. *(Kawanaka)[Kaw 2.3.2]: Let (F, e) be a pair consisting of a facet F with a $\mathfrak{f}$-distinguished nilpotent $e \in \mathcal{N}(\mathsf{G}_F)(\mathfrak{f})$. For adjoint groups of type A_n or exceptional type there is a decomposition $\Gamma_{(F,e)} = \gamma^0_{(F,e)} + \gamma^1_{(F,e)}$. We have $\gamma^1_{(F,e)} = 0$ if e is special. Furthermore, we have the following equation for $\gamma^0_{(F,e)}$:*

$$\gamma^0_{(F,e)} = |W_F|^{-1}\Sigma_{w\in W}(-1)^{rk(G)-rk(T_w)}|T_w(\mathfrak{f})|q^{e(e)}X^{(F,e)}_w(q)Q_{T_w}$$

The constant $e(e)$ is defined above (where we define $e(X)$. The constants $X^{(F,e)}_w(q)$ are given by Green polynomials (see [Spr] for full details) evaluated at q^{-1}. The torus T_w is the torus of G for which the corresponding cocycle takes value w on $\mathrm{Frob}_{\mathfrak{f}}$.

Notice that SO_4 is not covered by Kawanaka's formula. There is a more general formula of Lusztig that specializes to that of Kawanaka, but we shall work explicitly due to the low dimension of the groups involved.

In all the cases we wish to use Kawanaka's formula the above computations, Theorem X.5, and Theorem X.6 combine to show $\gamma^1_{(F,e)} = 0$.

The formula in theorem 10.6 looks superfically different from Kawanaka's original. Kawanaka collapses the factor of $q^{e(e)}$ into his $X^A_w(q)$ while DeBacker-Kazhdan do not, and I have followed the conventions of DeBacker-Kazhdan. If e is regular in G_F, as a Green function takes the value 1 on a regular orbit, we get $X_w^{(F,e)} = 1$. Moreover, for e regular $e(e) = 0$. The rest of the cases we will need will be for the subregular orbit in $\mathsf{G}_F \cong G_2$. Note for $e \in \mathbf{G}_2(\mathfrak{f})$ subregular we have $\dim_{\bar{\mathfrak{f}}}(\mathcal{B}_e^{G_2}) = 1$. We retrieve the green polynomials needed for G_2 directly from [DKaz1] or [Spr] and evaluate at q^{-1}:

$$X_w^{(F_{(\alpha,\beta)},e_1)} = 1 + q^{-1}(\chi(w) + 2\tau(w))$$

$$X_w^{(F_{(\alpha,\beta)},e_1')} = 1 + q^{-1}(\chi(w) - \tau(w))$$

$$X_w^{(F_{(\alpha,\beta)},e_1'')} = 1 + q^{-1}(\chi(w))$$

Where χ and τ are given by the following character table for W_{G_2}:

	sgn	χ	τ	$sgn\tau$	$\chi\tau$
Id	1	2	1	1	2
F	-1	0	1	-1	0
FR	-1	0	-1	1	0
R	1	1	-1	-1	-1
R^2	1	-1	1	1	-1
R^3	1	-2	-1	-1	2

Corollary X.7. *(Kawanaka, DeBacker, Kazhdan):Say G is split, adjoint of type A_n, E_n, F_4, or G_2. Fix (F, T) a facet with a minisotropic torus T. Say $X_T \in \mathfrak{g}_F$ with the image of X_T in $\mathsf{L}_T(\mathfrak{f})$ a regular element of $\mathsf{T}(\mathfrak{f})$. Let $\mathbf{T}$ be a lift of T to $\mathbf{G}$. Let*

(C,e) be a facet with a special, distinguished nilpotent of $\mathsf{L}_C(\mathfrak{f})$. Then $\mu_{X_T}(h_{(C,e)}) = 0$ unless there is a $g \in G$ such that $gC \subset \mathcal{B}(T)$. If there is such a g, then we may assume $C \subset \mathcal{B}(T)$ and the following equation holds:

$$\mu_{X_T}(h_{(C,e)}) = \frac{|\mathsf{L}_C(1)|^{\frac{1}{2}}|N_G(T)/T|q^{e(e)}X_{w_\mathsf{T}}^{(C,e)}}{|W_C|}$$

Proof. From Lemma X.4 we have:

$$\mu_{X_T}(h_{(C,e)}) = (-1)^{rk(\mathsf{T})}|\mathsf{L}_C(1)|^{\frac{1}{2}}\frac{|N_G(T)/T|}{|N_{\mathsf{G}_C}(\mathsf{T})(\mathfrak{f})/\mathsf{T}(\mathfrak{f})|}(\mathrm{Ind}_F^C Q_\mathsf{T}^F, \Gamma_{(C,e)})_{\mathsf{L}_F}$$

We compute the pairing $(\mathrm{Ind}_F^C Q_\mathsf{T}^F, \Gamma_{(C,e)})_{\mathsf{L}_F}$. Let w_T be a representative of the conjugacy class in W_{G_x} determined by T. Using Theorem X.6 and the fact that e is special:

$$(\mathrm{Ind}_F^C Q_\mathsf{T}^F, \Gamma_{(C,e)})_{\mathsf{L}_F} = (\mathrm{Ind}_F^C Q_\mathsf{T}^F, \gamma^0_{(C,e)})_{\mathsf{L}_F}$$
$$= (\mathrm{Ind}_F^C Q_\mathsf{T}^F, |W|^{-1}\Sigma_{w\in W}(-1)^{rk(G)-rk(T_w)}|T_w(\mathfrak{f})|q^{e(e)}X_w^{(C,e)}(q)Q_{T_w})_{\mathsf{L}_F}$$
$$= |W_C|^{-1}\Sigma_{w\in W}(-1)^{rk(G)-rk(T_w)}|T_w(\mathfrak{f})|q^{e(e)}X_w^{(C,e)}(q)(\mathrm{Ind}_F^C Q_\mathsf{T}^F, Q_{T_w})_{\mathsf{L}_F}$$

By [C], $$(\mathrm{Ind}_F^C Q_\mathsf{T}^F, Q_{T_w})_{\mathsf{L}_F} = \begin{cases} \frac{|\mathbf{N}_{\mathsf{G}_\mathbf{x}}(\mathsf{T})(\mathfrak{f})|}{|\mathsf{T}(\mathfrak{f})|^2} & \text{if } \mathrm{Ind}_F^C Q_\mathsf{T}^F = Q_{T_w} \\ 0 & \text{otherwise} \end{cases}$$

$$(\mathrm{Ind}_F^C Q_\mathsf{T}^F, \Gamma_{(C,e)})_{\mathsf{L}_F} =$$
$$= |W|^{-1}(-1)^{rk(\mathsf{G}_C)-rk(\mathsf{T})}|\mathsf{T}(\mathfrak{f})|q^{e(e)}X_{w_T}^{(C,e)}(q)(\mathrm{Ind}_F^C Q_\mathsf{T}^F, \mathrm{Ind}_F^C Q_T)_{\mathsf{L}_F}$$
$$= |W_C|^{-1}(-1)^{rk(\mathsf{G}_C)-rk(\mathsf{T})}|\mathsf{T}(\mathfrak{f})|q^{e(e)}X_{w_T}^{(C,e)}(q)\frac{|\mathbf{N}_{\mathsf{G}_\mathbf{C}}(\mathsf{T})(\mathfrak{f})|}{|\mathsf{T}(\mathfrak{f})|^2}$$
$$= \frac{(-1)^{rk(\mathsf{G}_C)-rk(\mathsf{T})}q^{e(e)}X_{w_T}^{(C,e)}(q)|\mathbf{N}_{\mathsf{G}_\mathbf{x}}(\mathsf{T})|}{|W_C||\mathsf{T}(\mathfrak{f})|}$$

Going back to our original goal:

$$\mu_{X_T}(h_{(C,e)}) = (-1)^{rk(\mathsf{T})}|\mathsf{L}_C(1)|^{\frac{1}{2}}\frac{|N_G(T)/T|}{|N_{\mathsf{G}_C}(\mathsf{T})(\mathfrak{f})/\mathsf{T}(\mathfrak{f})|}(\mathrm{Ind}_F^C Q_{\mathsf{T}}^F, \Gamma_{(C,e)})_{\mathsf{L}_F}$$
$$= (-1)^{rk(\mathsf{T})}|\mathsf{L}_C(1)|^{\frac{1}{2}}\frac{|N_G(T)/T|}{|N_{\mathsf{G}_C}(\mathsf{T})(\mathfrak{f})/\mathsf{T}(\mathfrak{f})|}\frac{(-1)^{rk(\mathsf{G}_C)-rk(\mathsf{T})}q^{e(e)}X_{w_T}^{(C,e)}(q)|\mathbf{N}_{\mathsf{G}_{\mathbf{C}}}(\mathsf{T})(\mathfrak{f})|}{|W_C||\mathsf{T}(\mathfrak{f})|}$$
$$= (-1)^{rk\mathsf{G}_C}\frac{|\mathsf{L}_C(1)|^{\frac{1}{2}}|N_G(T)/T|q^{e(e)}X_{w_{\mathsf{T}}}^{(C,e)}}{|W_C|}$$

And the corollary is proved.

□

For the groups that arise in our investigations, the rank of G_C is always 2, so the factor of $(-1)^{rk\mathsf{G}_c}$ may be discarded and we get:

$$\mu_{X_T}(h_{(x,e)}) = \frac{|\mathsf{L}_x(1)|^{\frac{1}{2}}|N_G(T)/T|q^{e(e)}X_{w_{\mathsf{T}}}^{(F,e)}}{|W_x|}$$

We explicitly compute the pairings we need in the appendix. We record the results here:

For $G = G_2$ we get the following table, with row $\frac{1}{|N_G(T)/T|}D_{(F,\mathsf{T})}$, specified by (F, T), evaluated at column $h_{(F,e)}$, specified by e:

	e_{reg}	e_{sr}	e'_{sr}	e''_{sr}	n	n'	n''	m	m''
$(F_\emptyset, \mathsf{S})$	$\frac{1}{12}$	$\frac{q+4}{12}$	$\frac{q+1}{12}$	$\frac{q+2}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$
$(F_{\{\alpha\}}, \mathsf{T}_F)$	$\frac{1}{12}$	$\frac{q+2}{12}$	$\frac{q-1}{12}$	$\frac{q}{12}$	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$
$(F_{\{\beta\}}, \mathsf{T}_F)$	$\frac{1}{12}$	$\frac{q-2}{12}$	$\frac{q+1}{12}$	$\frac{q}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$
$(F_{\{\alpha,\beta\}}, \mathsf{T}_R)$	$\frac{1}{12}$	$\frac{q-1}{12}$	$\frac{q+2}{12}$	$\frac{q+1}{12}$	0	0	0	0	0
$(F_{\{\alpha,\beta\}}, \mathsf{T}_{R^2})$	$\frac{1}{12}$	$\frac{q-1}{12}$	$\frac{q+2}{12}$	$\frac{q+1}{12}$	0	0	0	0	0
$(F_{\{\alpha,\beta\}}, \mathsf{T}_{R^3})$	$\frac{1}{12}$	$\frac{q-1}{12}$	$\frac{q+2}{12}$	$\frac{q+1}{12}$	0	0	0	0	0
$(F_{\{\beta,1-3\alpha-2\beta\}}, T_{R^2})$	0	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0
$(F_{\{\alpha,1-3\alpha-2\beta\}}, T_{R^3})$	0	0	0	0	0	0	0	$\frac{1}{4}$	$\frac{1}{4}$

These computations are not tractable for our applications to stability in chapters 11 and 13. We must construct some related stable distributions that are closer to being dual to the $h_{(F,e)}$. Notice that every stable conjugacy class of unramified tori in G_2 is parameterized by a conjugacy class in W_{G_2}. Let C_w be a collection of unramified tori forming a system of representatives for the rational classes of tori in the stable conjugacy class parameterized by w. For any $\kappa \in W_{G_2}^D$, let $D_\kappa = \Sigma_{w \in W_{G_2}} \kappa(w) \Sigma_{T \in C_w} \frac{1}{|N_G(T)/T|} D_{(F,\mathsf{T})}$. Then we get the following table for $D_\kappa(h_{(F,e)})$:

	e_{reg}	e_{sr}	e'_{sr}	e''_{sr}	n	n'	n''	m	m''
Id	1	q	q	q	1	1	1	2	2
D_{sgn}	0	0	0	0	0	0	0	-1	-1
D_χ	0	1	1	1	0	0	0	0	0
D_τ	0	2	-1	0	0	0	0	0	0
$D_{sgn\tau}$	0	0	0	0	1	1	1	0	0
$D_{\chi\tau}$	0	0	0	0	0	0	0	1	1

For $G = SO_4$ Kawanaka's results are not valid, so we do everything by hand in the appendix. We need to specify $h_{(F,e)}$ with the data (F,e) for SO_4 because there are two classes of vertices with residue group SO_4. Replace $D_{(F,T)}$ with $\frac{D_{(F,T)}}{k_{C,T}}$, as for our applications we don't need to know these evaluations beyond their behavior that is uniform in C, T, non-zero scaling. Here is the result in table format:

	$h_{(F_{\{\alpha,\beta\}},m)}$	$h_{(F_{\{\alpha,\beta\}},m')}$	$h_{(F_{\{\alpha,1-\beta\}},m)}$	$h_{(F_{\{\alpha,1-\beta\}},m')}$
$D_{(F_\emptyset,S)}$	1	1	1	1
$D_{(F_{\{\alpha\}},T_F)}$	-1	-1	-1	-1
$D_{(F_{\{\beta\}},T_F)}$	-1	-1	-1	-1
$D_{(F_{\{\alpha,\beta\}},T(F,F))}$	1	1	1	1
$D_{(F_{\{\alpha,1-\beta\}},T(F,F))}$	1	1	1	1

Note in particular that these pairings depend on only the Green function, and not on which of the four regular orbits we evaluate. This will be key in our stability computations for SO_4.

For $\kappa \in W^D_{SO_4}$ we define D_κ analogously to the case for G_2.

CHAPTER XI

Stable Nilpotent Orbital Integrals

In this section we determine which of the nilpotent orbital integrals are stable for each of the endoscopic groups of G_2.

11.1 Stable Nilpotent Orbital Integrals for PGL_3

For PGL_3 we have corollary IX.5, so all distributions are stable; in particular all distributions supported on the nilpotent cone are stable.

11.2 Stable Nilpotent Orbital Integrals for G_2

We follow DeBacker-Kazhdan [DKaz1]. The key technique of their analysis is to use homogeneity to split up the computations by dimension, using Lemma X.2 we work from highest dimension nilpotent orbit down, then use a counting argument to make sure we exhaust all possibilities.

By X.1 and X.2 distinct nilpotent k-orbits in $\mathfrak{g}(k)$ give linearly independent orbital integrals. From Theorem III.3 we know that:

$\mathrm{res}_{\mathcal{D}_0} J(\mathcal{N}) = \mathrm{res}_{\mathcal{D}_0} J((\mathfrak{g}_2)_0)$

From our computations in section 9.3 we know dim $\mathrm{res}_{\mathcal{D}_0} J^{stab}((\mathfrak{g}_2)_0) = 7 + |\mu_3(k)|$. The method of DeBacker and Kazhdan is to pick out stable nilpotent orbital integrals until we exhaust $7 + |\mu_3(k)|$ dimensions.

Firstly, the orbital integral corresponding to 0 is trivially stable.

We shall need a result on homogeniety that follows from a more general result in section 12.2. We state the version we need now and delay the proof until chapter 12.

Lemma XI.1. *Let T be a stable distribution with expansion:*

$$res_{\mathcal{D}_0} T = \Sigma_{\mathcal{O} \in \mathcal{N}(k)/G(k)} c_{\mathcal{O}}(T) \, \mathrm{res}_{\mathcal{D}_0} \, \mu_{\mathcal{O}}.$$

Then for each $i \in \mathbb{N}$, $\Sigma_{dim \mathcal{O} = i} c_{\mathcal{O}}(T) \mu_{\mathcal{O}}$ is also stable.

Proof. :See Cor XII.2.

□

Consider the distribution $D^{G_2}_{T_{(F_{\{\alpha,\beta\}},R)}}$. Our computations in chapter 8 tell us this distribution is stable. By III.3 we can write:

$$\mathrm{res}_{\mathcal{D}_0} \, D^{G_2}_{T_{(F_{\{\alpha,\beta\}},R)}} = \Sigma_{\mathcal{O} \in \mathcal{N}(k)/G_2(k)} c(D^{G_2}_{T_{(F_{\{\alpha,\beta\}},R)}}) \, \mathrm{res}_{\mathcal{D}_0} \, \mu_{\mathcal{O}}$$

Evaluating both sides of this equation at the function $h_{(F_{\{\alpha,\beta\}},e_{reg})}$ we get:

$$D^{G_2}_{T_{(F_{\{\alpha,\beta\}},R)}}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \Sigma_{\mathcal{O}\in\mathcal{N}(k)/G_2(k)} c(D^{G_2}_{T_{(F_{\{\alpha,\beta\}},R)}})\mu_{\mathcal{O}}(h_{(F_{\{\alpha,\beta\}},e_{reg})})$$

Let $C = D^{G_2}_{T_{(F_{\{\alpha,\beta\}},R)}}(h_{(F_{\{\alpha,\beta\}},e_{reg})})$. From Corollary X.7 and our Green polynomial computation for regular e we get:

$$C = D^{G_2}_{T_{(F_{\{\alpha,\beta\}},R)}}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{|L_{\{\alpha,\beta\}}(1)|^{\frac{1}{2}}||N_G(T_{(F_{\{\alpha,\beta\}},R)})/T_{(F_{\{\alpha,\beta\}},R)}|*q^0*1}{|W_{F_{\{\alpha,\beta\}}}|}$$

In particular, $C \neq 0$.

From Lemma X.2 and using the fact that $\mathcal{O}_{(F_{\{\alpha,\beta\}},e_{reg})}$ is the unique, maximal dimension, nilpotent orbit in $\mathfrak{g}_2(k)$ we get that $\mu_{\mathcal{O}}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$ unless $\mathcal{O} = \mathcal{O}_{(F_{\{\alpha,\beta\}},e_{reg})}$, in which case Lemma X.1 gives $\mu_{\mathcal{O}_{(F_{\{\alpha,\beta\}},e_{reg})}}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = |\mathsf{L}_{F_{\{\alpha,\beta\}}}(1)|^{\frac{1}{2}}$. Thus we get:

$$c_{\mathcal{O}_{reg}}(D^{G_2}_{T_{(F_{\{\alpha,\beta\}},R)}})|\mathsf{L}_{F_{\{\alpha,\beta\}}}(1)|^{\frac{1}{2}}$$

In particular $c_{\mathcal{O}_{reg}}(D^{G_2}_{T_{(F_{\{\alpha,\beta\}},R)}}) \neq 0$. Now, as $\mathcal{O}_{reg}$ is the unique orbit of maximal dimension in $\mathfrak{g}_2$, we get from Lemma XI.1 that $\mu_{\mathcal{O}_{reg}}$ is stable.

Recall our definition from section 10.3 of the stable distributions D_κ for $\kappa \in W_{G_2}$. Write $\mathrm{res}_{\mathcal{D}_0} D_{sgn} = \Sigma_{\mathcal{O}\in\mathcal{N}(k)/G_2(k)} c_{\mathcal{O}}(D_{sgn}) \,\mathrm{res}_{\mathcal{D}_0} \mu_{\mathcal{O}}$. Then by our computations of the pairing $D_{sgn}(h_{(F,e)})$ we get $c_{(F_{\{\alpha,\beta\}},e_{reg})}(D_{sgn}) = 0$, which in turn forces $c_{(F_{\{\alpha,\beta\}},e_{sr})}(D_{sgn}) = c_{(F_{\{\alpha,\beta\}},e'_{sr})}(D_{sgn}) = c_{(F_{\{\alpha,\beta\}},e''_{sr})}(D_{sgn}) = 0$. Likewise, we get

$c_{(F_{\{\beta,1-3\alpha-2\beta\}},n)}(D_{sgn}) = c_{(F_{\{\beta,1-3\alpha-2\beta\}},n')}(D_{sgn}) = c_{(F_{\{\beta,1-3\alpha-2\beta\}},n'')}(D_{sgn}) = 0.$

As $D_{sgn}(h_{(F_{\{\alpha,1-3\alpha-2\beta\}},m)}) = D_{sgn}(h_{(F_{\{\alpha,1-3\alpha-2\beta\}},m')}) = 1$, we get some combination of $c_{(F_{\{\alpha,1-3\alpha-2\beta\}},m)}(D_{sgn})$ and $c_{(F_{\{\alpha,1-3\alpha-2\beta\}},m')}(D_{sgn})$ is not 0. This combination must be stable by homogeneity.

Now we analyze the expansion:

$$\operatorname{res}_{\mathcal{D}_0} D_{(F_{\{\alpha,1-3\alpha-2\beta\}},m,sgn\times sgn)} = \Sigma_{\mathcal{O}\in\mathcal{N}(k)/G_2(k)} c_{\mathcal{O}}(D_{(F_{\{\alpha,1-3\alpha-2\beta\}},m,sgn\times sgn)}) \operatorname{res}_{\mathcal{D}_0} \mu_{\mathcal{O}}$$

Notice the same argument gives that $c_{\mathcal{O}}(D_{(F_{\{\alpha,1-3\alpha-2\beta\}},m,sgn\times sgn)}) = 0$ except for the orbits $\mathcal{O}_{(F_{\{\alpha,1-3\alpha-2\beta\}},m)}$ and $\mathcal{O}_{(F_{\{\alpha,1-3\alpha-2\beta\}},m')}$. Consider the pair:

$$c_{\mathcal{O}_{(F_{\{\alpha,1-3\alpha-2\beta\}},m)}}(D_{(F_{\{\alpha,1-3\alpha-2\beta\}},m,sgn\times sgn)})$$

and

$$c_{\mathcal{O}_{(F_{\{\alpha,1-3\alpha-2\beta\}},m')}}(D_{(F_{\{\alpha,1-3\alpha-2\beta\}},m,sgn\times sgn)})$$

This pair must be linearly independent from the pair we got for D_{sgn}. This second combination must be stable as well, hence the two distributions $\mu_{\mathcal{O}_{(F_{\{\alpha,1-3\alpha-2\beta\}},m)}}$ and $\mu_{\mathcal{O}_{(F_{\{\alpha,1-3\alpha-2\beta\}},m')}}$ must both be stable.

Similar arguments for the three stable distributions D_χ, D_τ, and $D_{(F_{\{\alpha,\beta\}},e_{sr},sgn)}$ prove that the orbital integrals $\mu_{\mathcal{O}_{(F_{\{\alpha,\beta\}},e_{sr})}}$, $\mu_{\mathcal{O}_{(F_{\{\alpha,\beta\}},e'_{sr})}}$, and $\mu_{\mathcal{O}_{(F_{\{\alpha,\beta\}},e''_{sr})}}$ are stable, likewise the distributions $D_{(F_{\{\beta,1-3\alpha-2\beta\}},n_{sr},\psi)}$, $D_{(F_{\{\beta,1-3\alpha-2\beta\}},n_{sr},\psi)}$, and $D_{sgn\tau}$ do the trick for showing the stability of the $|\mu_3(k)|$ invariant distributions $\mu_{\mathcal{O}_{(F_{\{\beta,1-3\alpha-2\beta\}},n)}}$, $\mu_{\mathcal{O}_{(F_{\{\beta,1-3\alpha-2\beta\}},n')}}$, and $\mu_{\mathcal{O}_{(F_{\{\beta,1-3\alpha-2\beta\}},n'')}}$.

We have produced $1+1+2+3+|\mu_3(k)| = 7+|\mu_3(k)|$ linearly independent stable distributions in $J(\mathcal{N})$, which is the same as the previously computed dimension of $\mathrm{res}_{\mathcal{D}_0} J^{st}((\mathfrak{g}_2)_0)$, thus the remaining two nilpotent orbital integrals $\mu_{\mathcal{O}_{(F_{\{\alpha\}},\bar{X}_\alpha)}}$ and $\mu_{\mathcal{O}_{(F_{\{\beta\}},\bar{X}_\beta)}}$ are not stable.

11.3 Stable Nilpotent Orbital Integrals for SO_4

First we observe that by our SO_4 stability computations in chapter 8, there are 4 complex dimensions of $\mathrm{res}_{\mathcal{D}_0} J^{stab}((\mathfrak{so}_4)_0)$ spanned by distributions dual to Green functions and none from Lusztig functions. As always, the nilpotent orbital integral given by the trivial orbit is stable.

We expand the following five distributions:

$$\mathrm{res}_{\mathcal{D}_0} D_{(F_\emptyset,S)} = \Sigma_{\mathcal{O}\in\mathcal{N}(k)/SO_4(k)} c_{\mathcal{O}}(D_{(F_\emptyset,S)})\, \mathrm{res}_{\mathcal{D}_0}\, \mu_{\mathcal{O}}$$

$$\mathrm{res}_{\mathcal{D}_0} D_{(F_{\{\alpha\}},T_w)} = \Sigma_{\mathcal{O}\in\mathcal{N}(k)/SO_4(k)} c_{\mathcal{O}}(D_{(F_{\{\alpha\}},T_w)})\, \mathrm{res}_{\mathcal{D}_0}\, \mu_{\mathcal{O}}$$

$$\mathrm{res}_{\mathcal{D}_0} D_{(F_{\{\beta\}},T_w)} = \Sigma_{\mathcal{O}\in\mathcal{N}(k)/SO_4(k)} c_{\mathcal{O}}(D_{(F_{\{\beta\}},T_w)})\, \mathrm{res}_{\mathcal{D}_0}\, \mu_{\mathcal{O}}$$

$$\mathrm{res}_{\mathcal{D}_0} D_{(F_{\{\alpha,\beta\}},T_(F,F))} = \Sigma_{\mathcal{O}\in\mathcal{N}(k)/SO_4(k)} c_{\mathcal{O}}(D_{(F_{\{\alpha,\beta\}},T_(F,F)})\, \mathrm{res}_{\mathcal{D}_0}\, \mu_{\mathcal{O}}$$

$$\mathrm{res}_{\mathcal{D}_0} D_{(F_{\{\alpha,1-\beta\}},T(F,F))} = \Sigma_{\mathcal{O}\in\mathcal{N}(k)/SO_4(k)} c_{\mathcal{O}}(D_{(F_{\{\alpha,1-\beta\}},T(F,F)}) \,\mathrm{res}_{\mathcal{D}_0} \mu_{\mathcal{O}}$$

We evaluate each of these at the functions $h(F_{\{\alpha,\beta\}}, m)$, $h(F_{\{\alpha,\beta\}}, m')$, $h(F_{\{\alpha,\beta-1\}}, m)$, and $h(F_{\{\alpha,1-\beta\}}, m')$. Recall our table from the end of section 10.3 that records the needed data. Notice that all four distributions evaluate to the same value on all 4 regular, rational, nilpotent k-orbits. As these orbits are maximal dimension amongst the nilpotent orbits, using the homogeniety theorem (Cor XII.2) and Lemma X.2, we get that $c_{(F_1,e_1)}(D_{(F,T)}) = c_{(F_2,e_2)}(D_{(F,T)})$ for all (F,T), and for (F_1, e_1) and (F_2, e_2) both regular. Thus $\mathrm{res}_{\mathcal{D}_0}(J^{st}((\mathfrak{so}_4)_0)\cap\mathrm{res}_{\mathcal{D}_0}\mathrm{Span}_{\mathbb{C}}\{\mu_{(F,e)}|(F,e)$ regular$\}$ $=\mathrm{Span}_{\mathbb{C}}(\mu_{\mathcal{O}_{(F_{\{\alpha,\beta\}},m)}} + \mu_{\mathcal{O}_{(F_{\{\alpha,\beta\}},m')}} + \mu_{\mathcal{O}_{(F_{\{\alpha,1-\beta\}},m)}} + \mu_{\mathcal{O}_{(F_{\{\alpha,1-\beta-1\}},m')}})$.

That is, the sum of the four regular, nilpotent orbital integrals is stable. Furthermore, any stable combination of regular, nilpotent orbital integrals is exactly a scalar times the sum of the four regular, nilpotent orbital integrals.

Now we know $\dim J^{stab}(\mathcal{N}) = 4$, and we also know:

$$J(\mathcal{N}) =$$
$$\mathbb{C}\mu_0\oplus \mathrm{Span}_{\mathbb{C}}\{\mu_{(F,e)}|(F,e) \text{ regular}\}\oplus\mathrm{Span}_{\mathbb{C}}\{\mu_{\mathcal{O}_{(F_\alpha,\bar{X}_\alpha)}}, \mu_{\mathcal{O}_{(F_\beta,\bar{X}_\beta)}}\}$$

Taking the dimension of the stable part of both sides and throwing out the two dimensions we have already found in $J(\mathcal{N})^{stab}$ we get that:

$$\dim_{\mathbb{C}}\mathrm{Span}_{\mathbb{C}}\{\mu_{\mathcal{O}_{(F_\alpha,\bar{X}_\alpha)}}, \mu_{\mathcal{O}_{(F_\beta,\bar{X}_\beta)}}\} \cap J^{stab}(\mathcal{N}) = 2$$

Hence we get that both $\mu_{\mathcal{O}_{(F_\alpha,\bar{X}_\alpha)}}$ and $\mu_{\mathcal{O}_{(F_\beta,\bar{X}_\beta)}}$ are stable.

CHAPTER XII

H-Stability and Transfer

In this chapter we define H-stability and work out some lemmas on the transfer map.

12.1 H-Stability and Transfer

Recall from chapter 5 the definition of transfer factors $\Delta_{G,H}$ and the transfer map on geometric classes $\mathcal{A}_{H/G}$. We have also defined in chapter 3 the notion of the kernel for invariant distributions $\mathcal{K} = \{f \in C_c^\infty(\mathfrak{g}) | \mu_X(f) = 0 \text{ for all } X \in \mathfrak{g}^{reg,s.s.}\}$ and for $X \in \mathfrak{g}^{reg,s.s.}$ we have stable combinations:

$$\mu_X^{stab} = \Sigma_{X'}\mu_{X'} = \Sigma_{\sigma \in H^1(\Gamma, Z_G(X))}\mu_{X_\sigma}$$

Where the sum is understood to be over a set of representatives for the rational classes in the geometric class of X.

We now generalize these notions to define a notion of stability for functions and orbital integrals on $\mathfrak{g}$ relative to a fixed endoscopic group H. For $Y \in \mathfrak{h}^{G-s.reg}$, we define a distribution $\mu_Y^{H/G} \in D(\mathfrak{g})$ by the following equation:

$$\mu_Y^{H/G} = \Sigma_X \Delta(X,Y)\mu_X$$

Where the sum is understood to be taken over a choice of representatives for the rational conjugacy classes in the geometric class $\mathcal{A}_{H/G}(\mathcal{O}_Y)$. This lets us define $\mathcal{K}_{H/G}^{stab} = \{f \in C_c^\infty(\mathfrak{g}|\mu_Y^{H/G}(f) = 0 \ \forall \ Y \in \mathfrak{h}^{G-s.reg}\}$ and dually we can define $J^{H-stab}(\mathfrak{g}) = \{T \in D((\mathfrak{g})|T(f) = 0 \forall f \in \mathcal{K}_{H/G}^{stab}\}$, the H-stable distributions on $\mathfrak{g}$. Notice that all H-stable distributions are invariant, as $\mathcal{K} \subset \mathcal{K}_{H/G}^{stab}$. Furthermore, notice that when we consider G as an endoscopic group of itself we recover our previous notion of stability.

Recall $\mathfrak{t}(X)$ is Lie$(Z_{\mathbf{G}}(X)|$. We introduce the Weyl Discriminant $D(X) = |det(adX|\mathfrak{g}/\mathfrak{t}_X)|_k$. This lets us define for $Y \in \mathfrak{h}(k)^{G-s.reg,s.s.}$:

$$\phi_Y^{Stab} = \Sigma_{Y'} D(Y')^{1/2}\mu_{Y'}$$

$$\phi_Y^{H/G} = \Sigma_X D(X)^{1/2}\Delta(X,Y)\mu_X$$

Where, again, the sum is understood to be over a choice of representatives for the rational classes in the geometric class of Y. Notice that the first distribution is defined on $\mathfrak{h}$ while the second is defined on $\mathfrak{g}$.

We have already defined in chapter 5 a map from the geometric conjugacy classes in $\mathfrak{h}$ to the geometric conjugacy classes in $\mathfrak{g}$ and the transfer factors $\Delta_{G,H}$. We now continue to define the transfer map. Given $f \in C_c^\infty(\mathfrak{g})$ and $f^H \in C_c^\infty(\mathfrak{h})$ we say that f^H is a transfer of f if and only if for all Y in $\mathfrak{h}^{G-s.reg}$ the following equation holds:

$$\phi_Y^{H/G}(f) = \phi_Y^{stab}(f^H)$$

Notice that $\mathcal{K}^{stab} \subset C_c^\infty(\mathfrak{h})$ is exactly all functions that are killed by the functionals on the right hand of the equation for every $Y \in \mathfrak{h}(k)^{s.reg,s.s.}$, while $\mathcal{K}_{H/G}^{stab} \subset C_c^\infty(\mathfrak{g})$ is exactly all functions that are killed by the functionals on the left for all $Y \in \mathfrak{h}(k)^{G-s.reg,s.s.}$. Thus $\mathcal{K}^{stab} \subset C_c^\infty(\mathfrak{h})$ is precisely the functions that are transfers of the 0 function on $\mathfrak{g}$, and 0 is a transfer for any function in $\mathcal{K}_{H/G}^{stab}$. That means if f^H and f_1^H are both transfers of f, then $f^H - f_1^H \in \mathcal{K}^{stab}$, and if f^H is a transfer of both f and f_1, then $f - f_1 \in \mathcal{K}_{H/G}^{stab}$. If $D \in J(\mathfrak{g})$ and $D^H \in J(\mathfrak{h})$ then we say that D is a transfer of D^H if the following equation holds for every f^H transfer of f:

$$D(f) = D^H(f^H)$$

Notice that if D is a transfer of D^H, then D^H must automatically be stable, while D must be H-stable. Furthermore, the distributions $\phi_Y^{H/G}$ are automatically transfers of ϕ_Y^{stab}.

12.2 H-stability and Homogeneity

For $z \in k^\times$ and $f \in C_c^\infty(\mathfrak{g})$ let ${}^z f \in C_c^\infty(\mathfrak{g})$ be given by the equation:

$${}^z f(X) = f(zX).$$

Denote by $D[n] \subset D(\mathfrak{g})$ the distributions T satisfying the equation:

$$T({}^{z^2}f) = |z|^{-n}T(f).$$

Lemma XII.1. *Say $T \in J^{H-stab}(\mathfrak{g})$. Say we have a finite collection $\{T_n\}$ with $T_n \in D[n](\mathfrak{g})$ and $res_{\mathcal{D}_0}T = \Sigma_n res_{\mathcal{D}_0}T_n$. Then T_n is H-stable for all n.*

Proof. This argument is a straight forward generalization of an argument Waldspurger[W3]. Say $f \in \mathcal{K}^{stab}_{H/G}(\mathfrak{g})$. Then as f is compactly supported and $\mathfrak{g}_{\mathcal{A}}(k)$ is an open neighborhood of zero there is some $N \in \mathbb{N}$ such that for all $z \in k^{\times}$ with $\nu(z) \geq N$, ${}^{z^2}f \in C_c(\mathfrak{g}(k)/\mathfrak{g}_{\mathcal{A}}(k)) \subset \mathcal{D}_0$. From our assumptions, for all such z we have the equation:

$$T({}^{z^2}f) = \Sigma_n T_n({}^{z^2}f) = \Sigma_n |z|^{-n}T_n(f).$$

As $f \in \mathcal{K}^{stab}_{H/G}$, ${}^{z^2}f \in \mathcal{K}^{stab}_{H/G}$. Therefore $T({}^{z^2}f) = 0$. As there are infinitely many z with distinct $|z|$ and with $|z| \geq N$, while there are only finitely many T_n, this forces the individual $T_n(f)$ to all be zero. Thus every T_n is H-stable.

$\square$

Corollary XII.2. *Say $T \in J^{H-stab}_{H/G}(\mathfrak{g})$ and $res_{\mathfrak{D}_0}T = \Sigma_{\mathcal{O} \in \mathcal{N}(k)/G(k)} c_{\mathcal{O}}(T) res_{\mathfrak{D}_0}\mu_{\mathcal{O}}$. Then $\Sigma_{\mathcal{O}'} c_{\mathcal{O}'}(T)\mu_{\mathcal{O}'}$ is H-stable where the sum is over all nilpotent orbits of a fixed dimension.*

Proof. :It is well known [H-C, Lemma 3.2] that for $X \in \mathcal{N}$, $\mu_X \in D[\dim_k \mathcal{O}_X]$.

□

Notice that when $H = G$ we recover the results of Waldspurger for stable distributions that we used in chapter 11.

Theorem XII.3. *Let D be a transfer of D^H. Say we have finite collections $\{D_n\}$ and $\{D_n^H\}$ with $D_n \in D[n](\mathfrak{g})$, $D_n^H \in D[n](\mathfrak{h})$, $res_{\mathcal{D}_0} D = \Sigma_n res_{\mathcal{D}_0} D_n$, and $res_{\mathcal{D}_0} D = \Sigma_n res_{\mathcal{D}_0} D_n$. Let $e_G = dim_k \mathfrak{g} - rk(\mathfrak{g})$, likewise e_H. Then $D_{n+e_G-e_H}$ is a transfer of D_n^H for all n.*

Proof. Again, a generalization of Waldspurger's results in [W3]. Note that all D_n are H-stable and all D_n^H are stable by Lemma XII.1. Let f^H be a transfer of f. Then $\phi_Y^{stab}(f^H) = \phi_Y^{H/G}(f)$ for all $Y \in \mathfrak{h}^{G-reg}$. Observe that for all $z \in k^\times$, $D(z^2X)^{1/2} = |det(adz^2X|\mathfrak{g}/\mathfrak{t}_{z^2X})|_F^{1/2} = |z|^{e_G} D(X)$. Notice that the map $f \to {}^z f$ commutes with the adjoint action and notice that transfer factors also ignore scaling. Thus we get:

$$\phi_{z^2Y}^{stab}({}^{z^2} f^H) = |z|^{e_H} \phi_Y^{stab}(f^H) = |z|^{e_H} \phi_Y^{H/G}(f) =$$

$$|z|^{e_H - e_G} \phi_{zY}^{H/G}({}^{z^2} f) = \phi_{zY}^{H/G}(|z|^{e_H - e_G}[{}^{z^2} f])$$

Thus ${}^{z^2} f^H$ is a transfer of $|z|^{e_H-e_G}[{}^{z^2} f]$. As both f and f^H are compactly supported in $C_c^\infty(\mathfrak{g})$ and $C_c^\infty(\mathfrak{h})$ respectively, we can find an $n \in \mathbb{N}$ such that for all $z \in k^\times$ with $\nu(z) \geq n$, ${}^{z^2} f \in \mathcal{D}_0(\mathfrak{g})$ and ${}^{z^2} f^H \in \mathcal{D}_0(\mathfrak{h})$. As before, $\forall z \in k^\times$ such that

$\nu(z) \geq n$, we have the two equations:

$$D(|z|^{e_H - e_G z^2} f) = \Sigma_n D_n(|z|^{e_H - e_G}[^{z^2} f]) = \Sigma_n |z|^{e_H - e_G - n} D_n(f)$$

$$D^H(^{z^2} f^H) = \Sigma_n D_n^H(^{z^2} f^H) = \Sigma_n |z|^{-n} D_n^H(f^H).$$

Taking the difference of these two equations, sorting terms by degree in $|z|$, and using the fact that D is a transfer of D^H we get:

$$0 = D(^{z^2} f) - D^H(^{z^2} f^H) = \Sigma_n |z|^{-n}[D_{n+e_G-e_H}(f) - D_n^H(f^H)]$$

Using the infinitely many z we get infinitely many linear relations on the finitely many possibly nonzero expressions $[D_{n+e_G-e_H}(f) - D_n^H(f^H)]$, hence all $[D_{n+e_G-e_H}(f) - D_n^H(f^H)] = 0$, or $D_{n+e_G-e_H}(f) = D_n^H(f^H)$. As we have made no assumptions on our pair consisting of f^H a transfer of f, we get that $D_{n+e_G-e_H}$ is a transfer of D_n^H.

$\square$

This theorem will let us talk about transferring nilpotent orbital integrals by working with the $T_{(F,\delta)}$.

12.3 The Transfer Map on the Nilpotent Cone

Recall from section 1 that for $x, y \in \mathcal{B}(G)$ and $r \in \mathbb{R}$ non-negative we have the depth r Moy-Prasad filtrations of the Lie algebra $\mathfrak{g}_{x,r}$ and the set $\mathfrak{g}_r = \cup_{x \in \mathcal{B}(G)} \mathfrak{g}_{x,r}$. Note that $\mathfrak{g}_r$ is a union of the open sets $\mathfrak{g}_{x,r}(k)$, hence is open. As $\mathfrak{g}_r(k) = \cap_{x \in \mathcal{B}(G)}[\mathfrak{g}_{x,r}+$

$\mathcal{N}(k)]$ and each $[\mathfrak{g}_{x,r} + \mathcal{N}(k)]$ is closed, we get that $\mathfrak{g}_r$ is closed. Thus we get two projection maps $\pi_{G,r} : C_c^\infty(\mathfrak{g}) \to C_c^\infty(\mathfrak{g}_r) \subset C_c^\infty(\mathfrak{g})$ and $\pi_{G,r}^\perp : C_c^\infty(\mathfrak{g}) \to C_c^\infty(\mathfrak{g} \setminus \mathfrak{g}_r) \subset C_c^\infty(\mathfrak{g})$ with $\pi_{G,r}(f) = f|_{\mathfrak{g}_r}$ and $\pi_{G,r}^\perp(f) = f|_{\mathfrak{g}\setminus\mathfrak{g}_r}$. Note $\mathrm{Id} = \pi_{G,r} + \pi_{G,r}^\perp$ gives a decomposition of the identity as a sum of two idempotents. Additionally, $\mathfrak{g}_r$ and $\mathfrak{g} \setminus \mathfrak{g}_r$ are $G(k)$-invariant as for $g \in G(k)$, ${}^g\mathfrak{g}_{x,r} = \mathfrak{g}_{gx,r}$. Notice that if $X \in \mathfrak{g}_{x,r}$ and $X \in \mathfrak{g}_{y,r}$ then $X \in \mathfrak{g}_{z,r}$ for all z in the geodesic between x and y.

Lemma XII.4. *Say H is an endoscopic group of G. Say $Y \in \mathfrak{h}_r^{G-s.reg}$. Say $X \in \mathcal{A}_{H/G}(\mathcal{O}_Y)(k)$. Then $X \in \mathfrak{g}_r$.*

Proof. As $Y \in \mathfrak{h}_r^{G-s.reg}5$ we have some Y' geometrically conjugate to Y with $Y' \in \mathfrak{s}_H(E)$, E/k some finite Galois extension and $\mathfrak{s}_H$ the split maximal torus in $\mathfrak{h}$ that is the Lie algebra of the torus $\mathbf{S}_H$ that the root datum of H is calculated with respect to. Let e_E be the ramification degree of E. As E is also a p-adic field we can define R_E, the ring of integers in E with uniformizer ϖ_E chosen with $\varpi_E^{e_E} = \varpi_k$. As $Y \in \mathfrak{h}_r$, we get that $Y' \in \mathfrak{s}_H(\varpi_E^{re_E} R_E)$. As in chapter 1, let $\mathfrak{s}_G$ be the torus in $\mathfrak{g}$ that is the Lie algebra of the torus $\mathbf{S}_G$ that we computed the root datum of G with. We identify $\mathfrak{s}_G$ with $\mathfrak{s}_H$ as before, and let X' be an image of Y' lying in $\mathfrak{s}_G(\varpi_E^{re_E} R_E)$. Then $\mathcal{O}_{X'}(\bar{k}) = \mathcal{A}_{H/G}(\mathcal{O}_Y)(\bar{k}) = \mathcal{O}_X(\bar{k})$. As $X' \in \mathfrak{s}_G(\varpi_E^{re_E} R_E)$ and depth is invariant under conjugation we get $X' \in \cup_{x \in \mathcal{B}(G)(E)} \mathfrak{g}_{x,re_E}(E)$, hence $X \in \cup_{x \in \mathcal{B}(G)(E)} \mathfrak{g}_{x,re_E}(E)$. Hence $X \in \mathfrak{g}_r$ as desired.

$\square$

Given a G-invariant closed and open subspace $V \subset \mathfrak{g}(k)$ and $f \in C_c^\infty(\mathfrak{g})$, $f|_V$

extended by zero is also in $C_c^\infty(\mathfrak{g})$. Given $D \in J(\mathfrak{g})$ we can define $D|_V \in J(\mathfrak{g})$ via the equation $D|_V(f) = D(f|_V)$.

Lemma XII.5. *Say $D^H \in J(\mathcal{N}(H))$. Let D be a transfer of D^H. Then the distribution $D|_{\mathfrak{g}_r} \in J(\mathfrak{g}_r) \subset J(\mathfrak{g})$ is also a transfer of D^H.*

Proof. Let f^H be a transfer of f. By Lemma 12.4 $\pi_{H,r}(f^H)$ (resp. $\pi^\perp_{H,r}(f^H)$) is a transfer of $\pi_{G,r}(f)$ (resp. $\pi^\perp_{G,r}(f)$). As $\mathcal{N}(H)(k) \subset \mathfrak{h}_0$ and $D^H \in J(\mathcal{N}(H)(k))$ we get $0 = D^H(\pi^\perp_{H,r}(f^H)) = D(\pi^\perp_{G,r}(f))$. Thus $D^H(f^H) = D(f) = D(\pi_{G,r}(f)) + D(\pi^\perp_{G,r}(f)) = D(\pi_{G,r}(f)) = D|_{\mathfrak{g}_r}(f)$.

$\square$

For $D \in J(\mathfrak{g})$ and $f \in C_c^\infty(\mathfrak{g})$ define $D_\infty(f) = lim_{r\to\infty} D|_{\mathfrak{g}_r}(f)$, where it is currently understood that $lim_{r\to\infty} D|_{\mathfrak{g}_r}(f)$ may not converge.

Theorem XII.6. *For $D \in J(\mathfrak{g})$:*

A)$lim_{r\to\infty} D|_{\mathfrak{g}_r}(f)$ always converges.

B)The map $f \to D_\infty(f)$ defines a distribution $D_\infty \in J(\mathcal{N})$.

C)If D is a transfer of $D^H \in J^{stab}(\mathcal{N}(\mathfrak{h}))$, then D_∞ is a transfer of D^H.

Proof. For arbitrary $f \in C_c^\infty(\mathfrak{g})$, $\exists L \subset \mathfrak{g}$ compact, open such that for all $x \in \mathfrak{g}$, $l \in L$, $f(x + l) = f(x)$. As $supp(f)$ is compact, $supp(f)$ is covered by a finite collection of disjoint sets $\{x_i + L\}_{i \in I}$. Then $D(f) = \Sigma_{i \in I} D(f(x_i)[x_i + L])$. As $\cap_{r \in \mathbb{R}} \mathfrak{g}_r = \mathcal{N}$ and if $A < B$ we have $\cap_{r \leq B} \mathfrak{g}_r \subset \cap_{r \leq A} \mathfrak{g}_r$ we get that there is some r_f such that the collection $J \subset I$ of indices j such that $(x_j + L) \cap \mathcal{N} \neq \emptyset$ is the same as the indices j' such that $D_{r_f}(f(x_{j'})[x_{j'} + L]) \neq 0$. Then $D_\infty(f) = D|_{\mathfrak{g}_{r_f}}(f)$, and hence $D_\infty(f)$ is always a well defined complex number. Hence A) is proved.

We now prove B). Observe that $\cap_{r \in \mathbb{R}} \mathfrak{g}_r = \mathcal{N}$ implies that D_∞ is always supported on $\mathcal{N}$. Furthermore for $f \in C_c^\infty(\mathfrak{g})$, $g \in G$ we have $D_\infty(f - {}^g f) = D|_{\mathfrak{g}_{r_f}}(f - {}^g f) = 0$ as $D|_{\mathfrak{g}_{r_f}} \in J(\mathfrak{g}_{r_f})$.

To prove C), let $f \in C_c^\infty(\mathfrak{g})$ be a transfer of $f^H \in C_c^\infty(\mathfrak{h})$. Fix r such that $D_\infty(f) = D|_{\mathfrak{g}_r}(f)$. By Lemma XII.5, $D|_{\mathfrak{g}_r}$ is a transfer of D^H, therefore $D_\infty(f) = D|_{\mathfrak{g}_r}(f) = D^H(f^H)$.

$\square$

This theorem proves that if we have D^H a stable, nilpotent distribution in $J(\mathcal{N}(\mathfrak{h}))$ that transfers to any distribution of $\mathfrak{g}$, we can find a nilpotent transfer.

CHAPTER XIII

Nilpotent Endoscopy for $\mathfrak{g}_2$

13.1 Sketch of What's Coming

In this chapter we compute the image of the transfer map. Our strategy is as follows:

1. For G_2, all the work was done explicitly in our stability computations, as transfer from G_2 to itself is just the identity map on stable distributions.

2. In each remaining case, we first take the stable combinations D_i^H where i is an arbitrary index that we explicitly write down (roughly; the decreasing split rank of the involved tori). We transfer these stable combinations of orbital integrals to combinations $D_i^{G_2}$.

3. We look at the dimensions of the stable, nilpotent orbital integrals and use the homogeneous transfer theorem to compute the dimension of their image.

4. We write each $\mathrm{res}_{\mathcal{D}_0} D_i^H = \Sigma_j a_{i,j} \mu_j$, where j runs over an index for the stable

combinations of nilpotent orbital integrals in $\mathfrak{h}$. The matrix $a_{i,j}$ is invertible so that we can isolate each individual μ_j with a non-zero coefficient. We do the analogous work in $\mathfrak{g}_2$.

5. We use dimension arguments to compute the full image for PGL_3, and most of the image for SO_4, however part of the image lies in the span of the subregular nilpotent orbits in $\mathfrak{g}_2$ for SO_4, so dimensionality arguments do not suffice.

6. We use our embedding $W_{SO_4} \hookrightarrow W_{G_2}$ and twist the various stable distributions on $\mathfrak{so}_4$ by characters in $W^D_{SO_4}$ to separate their behavior on the subregular orbits in $\mathfrak{g}_2$ to finish our computation of the image.

13.2 Transfer from G_2

Recall from chapter 5 the definition of an Endoscopic group and transfer factors. Recall from chapter

For $H \cong G_2$ in order to get the root system of H to be of type G_2 we require that $s(\check{\alpha}) = 1$ for all $\check{\alpha} \in \phi_{\check{G_2}}$, hence $s = \mathrm{Id}$. Consequently, all the transfer factors $\Delta_{G,H}(X_G, X_H) = 1$ if X_G is an image of X_H and 0 otherwise. We see that H-stability is the same as stability in this case, so no unstable distribution lies in the image under transfer, and that every stable distribution of G_2 is a transfer of itself.

13.3 Transfer from PGL_3

For $H \cong PGL_3$ we get $s(\check{\beta}) = s(3\check{\alpha} + \check{\beta}) = 1$, $s(\check{\alpha}) \neq 1$, thus $s(\check{\alpha}) = \xi$ or $s(\check{\alpha}) = \xi^{-1}$ where $\xi \in \mu_3(k)$ is a nontrivial cube root of unity. Thus there are two possible endoscopic datums that will lead to slightly different transfers.

We shall proceed by transferring orbital integrals and then using our homogeneity theorem to restrict to $\mathcal{D}_0$ and pass to nilpotent orbital integrals. Let Y be in $\mathfrak{pgl}_3^{G_2-s.reg}(k)$ with centralizer T_Y. By Lemma IX.4 the rational $PGL_3(k)$-orbit of Y in $\mathfrak{pgl}_3$ is the stable orbit. By Lemma VII.1, $\mathrm{res}_{\mathcal{D}_0}\mu_Y$ depends only on T_Y, not the particular element chosen, thus we shall consider the transfer torus by torus rather than worrying about more fine information. We can use the results of section 12.4 to transfer to $\mathfrak{g}_2$. Here are the results:

T_Y	Various T_X
$T_{(F_\emptyset, \mathrm{Id})}$	$T_{(\emptyset, \mathrm{Id})}$
$T_{(F_{\{\beta_{PGL_3}\}}, F)}$	$T_{(F_{\{\beta\}}, F)}$
$T_{(F_{\{\alpha_{PGL_3}, \beta_{PGL_3}\}}, R)}$	$T_{(F_{\{\alpha,\beta\}}, R^2)}$, $T_{(F_{\{\beta, 1-3\alpha-2\beta\}}, R)}$

Notice that $T_{(F_{\{\beta, 1-3\alpha-2\beta\}}, R)}$ occurs twice.

The transfer factors only come into play in the case that $Y \in \mathfrak{t}_{(F_{\{\alpha_{PGL_3}, \beta_{PGL_3}\}}, R)}$. Let $X_0 \in \mathfrak{t}_{(F_{\{\alpha,\beta\}}, R^2)}$ have Y as an image, with X_1 and X_2 being representatives of the other two rational classes that have Y as an image (recall our cohomology computations in chapter 9). Then we can choose s to get:

$\phi_Y^{PGL_3/G_2} = \phi_{X_0} + \xi\phi_{X_1} + \xi^{-1}\phi\mu_{X_2}$

Thus: $\mathrm{res}_{\mathcal{D}_0}\phi_Y^{PGL_3/G_2} =$

$\mathrm{res}_{\mathcal{D}_0}\, D^{G_2}_{T_{F_{\{\alpha,\beta\}},R^2)}} + \xi\,\mathrm{res}_{\mathcal{D}_0}\, D^{G_2}_{T_{(F_{\{\beta,1-3\alpha-2\beta\}},R)}} + \xi^{-1}\,\mathrm{res}_{\mathcal{D}_0}\, D^{G_2}_{T_{(F_{\{\beta,1-3\alpha-2\beta\}},R)}} =$

$\mathrm{res}_{\mathcal{D}_0}\, D^{G_2}_{T_{F_{\{\alpha,\beta\}},R^2)}} - \mathrm{res}_{\mathcal{D}_0}\, D^{G_2}_{T_{(F_{\{\beta,1-3\alpha-2\beta\}},R)}}$

Notice that (conveniently) our final answer does not depend on our choice of s in the endoscopic datum. Using the above table and transfer factor combinations we now produce a table of the transfers of the various $D^{PGL_3}_{(F,T)}$ in terms of $D^{G_2}_{(F',T')}$:

$D^{PGL_3}_{(F,T)}$	Transfer
$D^{PGL_3}_{(F_\emptyset,\mathsf{S})}$	$D^{G_2}_{(\emptyset,\mathsf{S})}$
$D^{PGL_3}_{(F_{\{\beta_{PGL_3}\}},\mathsf{T}_F)}$	$D^{G_2}_{(F_{\{\beta\}},\mathsf{T}_F)}$
$D^{PGL_3}_{(F_{\{\alpha_{PGL_3},\beta_{PGL_3}\}},\mathsf{T}_R)}$	$D^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{R^2})} - D^{G_2}_{(F_{\{\beta,1-3\alpha-2\beta\}},\mathsf{T}_R)}$

The orthogonality result, Lemma VII.1, immediately shows that the set of distributions occurring on the left hand side is linearly independent. Likewise for the right hand side. Lemmas X.1 and X.2 show that the stable combinations of nilpotent orbital integrals are linearly independent of each other. Hence, each of the three nilpotent orbital integrals must occur with non-zero coefficient in at least one of the transfers of the $D^{PGL_3}_{(F,T)}$s. As the combinations of $D^{G_2}_{(F,T)}$ are also linearly independent, the rank of the transfer map on nilpotent orbital integrals must be 3.

We now use Theorem XII.3. Recall the definition $e_G = \dim_k(\mathfrak{g}) - rk(\mathfrak{g})$ from theorem XII.3. Compute $e_{G_2} - e_{PGL_3} = 14 - 2 - (8-2) = 6$, so our homogeneity theorem will shift dimensions by 6. The three nilpotent orbits of PGL_3 are of dimension 6, 2, and 0, so we get transfers to nilpotent orbits in $\mathfrak{g}_2$ of possible dimensions 12, 8, and 6. These are the dimensions of the regular orbit, the orbit of X_β, and the orbit of X_α respectively. Thus, we get the following table for the transfer map for PGL_3 up to constants:

Orbit in $\mathfrak{pgl}_3$	Image in $\mathfrak{g}_2$
0	$\mathcal{O}_{(F_{\{\beta\}} X_\beta)}$
$\mathcal{O}_{(F_{\{\beta\}}, X_\beta)}$	$\mathcal{O}_{(F_{\{\alpha\}}, X_\alpha)}$
$\mathcal{O}_{(F_{\{\alpha,\beta\}}, e_{reg})}$	$\mathcal{O}_{(F_{\{\alpha,\beta\}}, e_{reg})}$

13.4 Transfer from SO_4

For $H \cong SO_4$ there are three possible embeddings of $H^\vee \to G_2^\vee$, but they are all equivalent. Our requirements for s are $s(\alpha^\vee) = s(3\alpha^\vee + 2\beta^\vee) = 1$, $s(\beta^\vee) \neq 1$, thus we get $s(\beta^\vee) = -1$.

This time the unramified elements produce the following transfer on tori:

Various T_Y	Various T_X
$T_{(F_\emptyset, \mathrm{Id})}$	$T_{(\emptyset, \mathrm{Id})}$
$T_{(F_{\{\alpha_{SO_4}\}}, F)}$	$T_{(F_{\{\alpha\}}, F)}$
$T_{(F_{\{\beta_{SO_4}\}}, F)}$	$T_{(F_{\{\beta\}}, F)}$
$T_{(F_{\{\alpha_{SO_4}, \beta_{SO_4}\}}, (F,F))}$, $T_{(F_{\{\alpha_{SO_4}, 1-\beta_{SO_4}\}}, (F,F))}$	$T_{(F_{\{\alpha,\beta\}}, R^3)}$, $T_{(F_{\{\alpha, 1-3\alpha-2\beta\}}, (F,F))}$

The last grouping contains 2 copies of each torus on the SO_4 side and contains 1 copy of the torus at $F_{\{\alpha,\beta\}}$ and 3 copies of the torus at $F_{\{\alpha,1-3\alpha-2\beta\}}$. We again only see transfer factors for the last pairing of stable classes on the list. Here, our choices force:

$$D^{SO_4}_{(F_{\{\alpha_{SO_4},\beta_{SO_4}\}},\mathsf{T}_{(F,F)})} + D^{SO_4}_{(F_{\{\alpha_{SO_4},1-\beta_{SO_4}\}},\mathsf{T}_{(F,F)})}$$

To transfer to:

$$D^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{R^3})} + D^{G_2}_{(F_{\{\alpha,1-3\alpha-2\beta\}},\mathsf{T}_{(F,F)})} - D^{G_2}_{(F_{\{\alpha,1-3\alpha-2\beta\}},\mathsf{T}_{(F,F)})} - D^{G_2}_{(F_{\{\alpha,1-3\alpha-2\beta\}},\mathsf{T}_{(F,F)})} =$$

$$D^{G_2}_{(F_{\{\alpha,\beta\}},\mathsf{T}_{R^3})} - D^{G_2}_{(F_{\{\alpha,1-3\alpha-2\beta\}},\mathsf{T}_{(F,F)})}$$

Again, this is independent of choosing equivalent endoscopic datums. Easy use of Lemma VII.1 shows the combinations of $D^{SO_4}_{(F,T)}$ on both sides are linearly independent, similarly Lemma 8.3.1 shows the stable combinations of nilpotent orbital integrals on $\mathfrak{so}_4$ are linearly independent. We get the following table:

Stable Combination	Transfer
$D^{SO_4}_{(F_\emptyset,\mathrm{Id})}$	$D^{G_2}_{(\emptyset,\mathrm{Id})}$
$D^{SO_4}_{(F_{\{\alpha_{SO_4}\}},F)}$	$D^{G_2}_{(F_{\{\alpha\}},F)}$
$D^{SO_4}_{(F_{\{\beta_{SO_4}\}},F)}$	$D^{G_2}_{(F_{\{\beta\}},F)}$

With the one additional result that:

$$D^{SO_4}_{(F_{\{\alpha_{SO_4},\beta_{SO_4}\}},(F,F))} + D^{SO_4}_{(F_{\{\alpha_{SO_4},1-\beta_{SO_4}\}},(F,F))}$$

Transfers to:

$D^{G_2}_{(F_{\{\alpha,\beta\}},R^3)} - D^{G_2}_{(F_{\{\alpha,1-3\alpha-2\beta\}},(F,F))}$

We again compute dimensions and use Theorem XII.3 to compute dimensions of the image; our stable combinations on $\mathfrak{so}_4$ are of dimension 0, 2, 2, and 4, and $\dim\mathfrak{g}_2 - \dim\mathfrak{so}_4 = 14 - 6 = 8$, so the nilpotent distributions in $\mathfrak{so}_4$ transfer to nilpotent distributions in $\mathfrak{g}_2$ of dimension 8, 10, 10, and 12. Combining this information with our linear independence results we immediately get that the 0 orbit in $\mathfrak{so}_4$ transfers to $\mathcal{O}_{(F_{\{\alpha\}},X_\alpha)}$, the stable combination consisting of regular orbits transfers to the regular orbit in $\mathfrak{g}_2$, and the remaining two orbits must go to linearly independent combinations of the subregular orbits in $\mathfrak{g}_2$.

Fortuitously we have already computed exactly what each of the $D^{G_2}_{(F,T)}$ do on the regular and subregular orbits. Our strategy follows that of DeBacker and Kazhdan: we note that the stable distributions on $\mathfrak{so}_4$ are nicely parameterized by conjugacy classes in W_{SO_4}. For a character $\kappa \in W^D_{SO_4}$, let D^{SO_4/G_2}_κ be the transfer of distribution $\Sigma_{w\in W_{SO_4}}\kappa(w)D^{stab}_w$ (see chap 10 for the definition of D^{stab}_w). Evaluating at $h_{(F,e)} \in C^\infty_c(\mathfrak{g}_2)$ for e regular or subregular, we get the following table:

	e_{reg}	e_{sr}	e'_{sr}	e''_{sr}	n	n'	n''	m	m''
$D^{SO_4/G_2}_{\mathrm{Id}}$	$\frac{1}{3}$	$\frac{q}{3}$	$\frac{q}{3}$	$\frac{q}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{2}$
$D^{SO_4/G_2}_{sgn\times\mathrm{Id}}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0
$D^{SO_4/G_2}_{\mathrm{Id}\times sgn}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	0	0
$D^{SO_4/G_2}_{sgn\times sgn}$	0	0	0	0	0	0	0	0	0

We remind the reader that if $|\mu_3(k)| = 1$ the nilpotents n' and n'' don't exist, hence in that case we take $\mu_{(F_{\{3\alpha+2\beta-1,\beta\}},n')} = \mu_{(F_{\{3\alpha+2\beta-1,\beta\}},n')} = 0$.

In particular, restricting to $\mathcal{D}_0$ and looking at the local expansions of $D_{sgn\times \mathrm{Id}}$ and $D_{\mathrm{Id}\times sgn}$, we observe that $C_{\mu_{reg}}(D_{sgn\times \mathrm{Id}}) = C_{\mu_{reg}}(D_{\mathrm{Id}\times sgn}) = 0$. This combined with our evaluations then forces $C_{(F_{\{\alpha,\beta\}},e_{sr})}(D_{\mathrm{Id}\times sgn}) = C_{(F_{\{\alpha,\beta\}},e'_{sr})}(D_{\mathrm{Id}\times sgn}) = C_{(F_{\{\alpha,\beta\}},e''_{sr})}(D_{\mathrm{Id}\times sgn}) \neq 0$, hence by homogeniety we get that $\mu_{(F_{\{\alpha,\beta\}},e_{sr})}+\mu_{(F_{\{\alpha,\beta\}},e'_{sr})}+\mu_{(F_{\{\alpha,\beta\}},e''_{sr})}$ must lie in the image of transfer for the endoscopic group SO_4.

Combining this information with the same argument for $D_{sgn\times \mathrm{Id}}$ then shows that $\mu_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}+\mu_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}+\mu_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}$ lives in the image under endoscopy from SO_4, and we are done.

We can actually do slightly better. Explicit computations on $GL_2(\mathfrak{f}) \cong \mathsf{G}_{F_{\{\alpha\}}}(\mathfrak{f}) \cong \mathsf{G}_{F_{\{\beta\}}}(\mathfrak{f})$ show the following pairings on SO_4 (again up to a non-zero factor):

	$h_{(F_{\{\alpha\}},X_\alpha)}$	$h_{(F_{\{\beta\}},X_\beta)}$
$D_{(\emptyset,S)}$	1	1
$D_{(F_{\{\alpha\}},F)}$	-1	0
$D_{(F_{\{\beta\}},F)}$	0	-1
$D_{(F_{\{\alpha,\beta\}},F\times F)}$	0	0

Which in turn shows the following:

	$h_{(F_{\{\alpha\}},X_\alpha)}$	$h_{(F_{\{\beta\}},X_\beta)}$
D_{Id}	0	0
$D_{sgn\times \mathrm{Id}}$	2	0
$D_{\mathrm{Id}\times sng}$	0	2
$D_{sgn\times sgn}$	2	2

Thus we see that $\mu_{(F_{\{\alpha\}},X_\alpha)}$ maps to a constant times the combination $\mu_{e_{sr}} + \mu_{e'_{sr}} + \mu_{e''_{sr}} + \mu_n + \mu_{n'} + \mu_{n''}$ while $\mu_{(F_{\{\beta\}},X_\beta)}$ maps to a constant times the combination $\mu_{e_{sr}} + \mu_{e'_{sr}} + \mu_{e''_{sr}}$. In both cases, we understand that $\mu_{n'}$ and $\mu_{n''}$ are both zero if $|\mu_3(k)| = 1$.

APPENDIX

In the appendix we compute the pairing $(Q_{\mathsf{T}}, \Gamma_e)_{\mathsf{L}_F}$ for G_2 and for $\mathfrak{so}_4$. For SO_4, our strategy is to start with SL_2, pass to $SL_2 \times SL_2$ via a theorem stated in Carter [C] on Groethendieck-Lefshetz trace formula on products, and finally to $SL_2 \times SL_2/\{\pm \mathrm{Id}\}$ via a theorem of Springer[Spr]. For G_2, we use Kawanaka's formula and the Green functions we have found in [Spr], closely following [DKaz1]. Everything here is elementary, explicit, and not generalizable.

.1 Pairings for SL_2

We need the Green functions on SL_2 to proceed. We shall also compute Gelfand-Graev characters and pair them so the reader can observe how the Gauss sums that occur simplify in an easy case before attempting the same argument over SO_4.

Theorem .1. *(Steinberg):Let T be a torus. Let $\Phi : X^*(\mathsf{T}) \otimes \mathbb{R} \to X^*(\mathsf{T}) \otimes \mathbb{R}$ be the action of* $\mathrm{Frob}_{\mathfrak{f}}$*. Then $|\mathsf{T}(\mathfrak{f})| = det|\operatorname{Frob}_{\mathfrak{f}} - \operatorname{Id}|$.*

Theorem .2. *Say G is a reductive $\mathfrak{f}$-group with maximal $\mathfrak{f}$-torus T. $Q_{\mathsf{T}}(0) = \epsilon_{\mathsf{T}}\epsilon_{\mathsf{G}}\frac{|\mathsf{G}(\mathfrak{f})|_{p'}}{|\mathsf{T}(\mathfrak{f})|_{p'}}$.*

Theorem .3. *Say G is a reductive $\mathfrak{f}$-group with maximal $\mathfrak{f}$-torus T. Say $e \in \mathcal{N}(\mathsf{G})(\mathfrak{f})$ is regular. Then $Q_{\mathsf{T}}(e) = 1$.*

Where by $|\mathsf{G}(\mathfrak{f})|_{p'}$ we mean the part of the $|\mathsf{G}(\mathfrak{f})|$ coprime to p. Combining this with the easy fact that $|SL_2(\mathfrak{f})| = q(q+1)(q-1)$, letting ϵ be a non-square in $\mathfrak{f}$, and suppressing any superscripts on our roots we get the following values of Green functions on the nilpotent cone:

	0	$\bar{X}_\alpha$	$\epsilon\bar{X}_\alpha$
Q_S	$(q+1)$	1	1
Q_{T_w}	$-(q-1)$	1	1

Over finite fields the Gelfand-Graev character $\Gamma_e \in C^{\mathsf{G}}(\mathsf{L})$ is given by $\Gamma_e(\bar{X}) = \frac{|\mathsf{L}|}{|\mathsf{L}(\leq -1)|}\Sigma_{g\in\mathsf{G}(\mathfrak{f}),{}^g\bar{X}\in\mathsf{L}(\leq-2)}\Lambda(B(e, {}^g\bar{X}))$. We compute the filtration associated to $e = \bar{X}_\alpha$:

$\mathrm{Span} X_{-\alpha} \oplus \mathfrak{t} \oplus \mathrm{Span} \bar{X}_{\alpha}$

Note that $\epsilon \bar{X}_{\alpha}$ produces the same filtration. As $\mathfrak{sl}_2(\leq -2)$ is 1 dimensional, $|\mathfrak{sl}_2(\leq -2)(\mathfrak{f})| = q$ while $|\mathfrak{sl}_2(\mathfrak{f})| = q(q-1)(q+1)$.

$$\Gamma_{\bar{X}_{\alpha}}(\bar{X}_{\alpha}) = \frac{1}{q} \Sigma_{r \in (\mathfrak{f}^{\times})^2} \Lambda(B(\bar{X}_{\alpha}, r\bar{X}_{-\alpha}))2q = 2I_1$$

$$\Gamma_{\bar{X}_{\alpha}}(\epsilon \bar{X}_{\alpha}) = \frac{1}{q} \Sigma_{r \in (\mathfrak{f}^{\times})^2} \Lambda(B(\bar{X}_{\alpha}, r\epsilon\bar{X}_{-\alpha}))2q = 2I_2$$

$$\Gamma_{\epsilon\bar{X}_{\alpha}}(\bar{X}_{\alpha}) = \frac{1}{q} \Sigma_{r \in (\mathfrak{f}^{\times})^2} \Lambda(B(\epsilon\bar{X}_{\alpha}, r\bar{X}_{-\alpha}))2q = 2I_2$$

$$\Gamma_{\epsilon\bar{X}_{\alpha}}(\epsilon \bar{X}_{\alpha}) = \frac{1}{q} \Sigma_{r \in (\mathfrak{f}^{\times})^2} \Lambda(B(\epsilon\bar{X}_{\alpha}, r\epsilon\bar{X}_{-\alpha}))2q = 2I_1$$

Where I_1 is one Gauss sum depending on choice of Λ and I_2 is the other that splits the sum $\Sigma_{\bar{Z} \in \mathfrak{sl}_2(-2) \setminus \{0\}} \Lambda(B(\bar{X}_{\alpha}, \bar{Z}))$. The key idea is that for the regular nilpotent this sum is a character sum over the full $\mathfrak{sl}_2(\leq -2) \setminus \{0\}$. As the sum of a non-trivial, irreducible character over a full group is 0 and the character in question is degree 1 we get $I_1 + I_2 = -1$.

Here is a table of the results:

	0	$\bar{X}_{\alpha}$	$\epsilon\bar{X}_{\alpha}$
$\Gamma_{\bar{X}_{\alpha}}$	$(q+1)(q-1)$	I_1	I_2
$\Gamma_{\epsilon\bar{X}_{\alpha}}$	$(q+1)(q-1)$	I_2	I_1

Now we compute the actual pairings by working orbit by orbit, using SL_2 invariance and our explicit results, and knowing the sizes of the conjugacy classes in $\mathfrak{sl}_2(\mathfrak{f})$:

$$\begin{aligned}
(\Gamma_{X_\alpha}, Q_S) &= \tfrac{(q+1)^2(q-1)-(q+1)(q-1)}{q(q+1)(q-1)} = 1\\
(\Gamma_{X_\alpha}, Q_T) &= \tfrac{-(q+1)(q-1)^2-(q+1)(q-1)}{q(q+1)(q-1)} = -1\\
(\Gamma_{\epsilon X_\alpha}, Q_S) &= \tfrac{(q+1)^2(q-1)-(q+1)(q-1)}{q(q+1)(q-1)} = 1\\
(\Gamma_{\epsilon X_\alpha}, Q_T) &= \tfrac{-(q+1)(q-1)^2-(q+1)(q-1)}{q(q+1)(q-1)} = -1
\end{aligned}$$

.2 Pairings for SO_4

We first compute Green functions on SO_4 from those of SL_2. Using that the Groethendieck-Lefschetz trace formula on products yields a tensor product of functions we can get Green functions of $SL_2 \times SL_2$. We then use Springer's Lemma [Spr, Lemma 5.3] stating that for a morphism of reductive groups $\phi : G \to H$, a torus $T < G$, and its image $\phi(T)$ we have the equation $Q_T = Q_{\phi(T)} \circ d\phi$ to pass to SO_4. Noticing that the total derivative of an isogeny is the identity map we get the following table for Green functions on SO_4:

	0	$\bar{X}_\alpha$	$\bar{X}_\beta$	$\bar{X}_\alpha + \bar{X}_\beta$	$\epsilon\bar{X}_\alpha + \bar{X}_\beta$
Q_S	$(q+1)^2$	$q+1$	$q+1$	1	1
$Q_{T_{(F,\mathrm{Id})}}$	$-(q-1)(q+1)$	$-(q-1)$	$q+1$	1	1
$Q_{T_{(\mathrm{Id},F)}}$	$-(q-1)(q+1)$	$q+1$	$-(q-1)$	1	1
$Q_{T_{(F,F)}}$	$(q-1)^2$	$-(q-1)$	$-(q-1)$	1	1

Notice that $SO_4 \cong SL_2 \times SL_2/\pm\mathrm{Id}$. A subgroup of $[SL_2 \times SL_2/\pm\mathrm{Id}](\mathfrak{f})$ is $SL_2(\mathfrak{f}) \times SL_2(\mathfrak{f})/\pm\mathrm{Id}$ which has $|SL_2(\mathfrak{f}) \times SL_2(\mathfrak{f})/\pm\mathrm{Id}\,| = |SL_2(\mathfrak{f})^2|/2 = \frac{q^2(q+1)^2(q-1)^2}{2}$. Lets compute the index of this subgroup. Let ϵ be a non square. Then $(a,b) \in [SL_2\times$

$SL_2/\pm\mathrm{Id}](\mathfrak{f})$ iff $(a,b)^\sigma = \pm(a,b)$ which is equivalent to $(a,b) = \mathrm{diag}(\epsilon, \epsilon^{-1}, \epsilon, \epsilon^{-1})(c,d)$ with $c, d \in \mathfrak{f}^\times$. Thus $|SO_4(\mathfrak{f})| = q^2(q+1)^2(q-1)^2$.

We write down the relevant filtrations on $\mathfrak{so}_4$ so that we can compute the values of Gelfand-Graev characters. For X_α we get:

$\mathrm{Span}(\bar{X}_{-\alpha}) \oplus \mathrm{Span}(\bar{X}_{-\beta}, \mathfrak{t}, \bar{X}_\beta) \oplus \mathrm{Span}\bar{X}_\alpha$

For $\bar{X}_\beta$ we get:

$\mathrm{Span}(\bar{X}_{-\beta}) \oplus \mathrm{Span}(\bar{X}_{-\alpha}, \mathfrak{t}, \bar{X}_\alpha) \oplus \mathrm{Span}\bar{X}_\beta$

For both $\bar{X}_\alpha + \bar{X}_\beta$ and $\epsilon\bar{X}_\alpha + \bar{X}_\beta$ we get:

$\mathrm{Span}(\bar{X}_{-\alpha}, \bar{X}_{-\beta}) \oplus \mathfrak{t} \oplus \mathrm{Span}(\bar{X}_\alpha, \bar{X}_\beta)$

We compute the values of $\Gamma_{\bar{X}_\alpha}$.
$\Gamma_{\bar{X}_\alpha}(0) = \frac{|\mathsf{L}_F(-1)|^{-\frac{1}{2}}}{|\mathsf{L}_F(\leq -1)|} |SO_4(\mathfrak{f})| = q(q+1)^2(q-1)^2$.

As $\mathsf{L}_F(\leq -2)$ consists entirely of nilpotent elements, we need only concern ourselves with nilpotent elements. If $\bar{X}_\beta$ is involved in anyway there is no hope of conjugating it away, so we get the character vanishes away from the orbit of $\bar{X}_\alpha$.

It remains to compute $\Gamma_{\bar{X}_\alpha}(\bar{X}_\alpha)$. We get $\frac{1}{q}\Sigma_{Z \in \mathfrak{so}_4(-2)\setminus\{0\}} \Lambda(B(X,Z)) * |Z_{so_4}(Z)|$. As $\mathfrak{so}_4(-2) \cong \mathbb{G}_a(\mathfrak{f})$ we get the sum of a nontrivial character over everything but the identity, so as before $\frac{1}{q}\Sigma_{Z\in\mathfrak{so}_4(-2)\setminus\{0\}}\Lambda(B(X,Z))*|Z_{so_4}(Z)| = \frac{1}{q}(-1)|Z_{SO_4(\mathfrak{f})}(\bar{X}_\alpha)|$.

$|Z_{SO_4(\mathfrak{f})}(\bar{X}_\alpha)| = 2q|SL_2(\mathfrak{f})|/2 = q^2(q+1)(q-1)$, thus:

$\Gamma_{\bar{X}_\alpha}(\bar{X}_\alpha) = \frac{1}{q}(-q^2(q+1)(q-1)) = -q(q+1)(q-1)$.

Likewise for $\bar{X}_\beta$.

For $\bar{X}_\alpha + \bar{X}_\beta$:

$\Gamma_{\bar{X}_\alpha+\bar{X}_\beta}(0) = \frac{1}{q^2}q^2(q+1)^2(q-1)^2 = (q+1)^2(q-1)^2$.

$\Gamma_{\bar{X}_\alpha+\bar{X}_\beta}(\bar{X}_\alpha) = \Gamma_{\bar{X}_\alpha+\bar{X}_\beta}(\bar{X}_\beta) = -(q+1)(q-1)$ by the same argument as before, with the adjustment that $|\mathfrak{so}_4(-2)| = q^2$ now.

$|Z_{SO_4(\mathfrak{f})}(\bar{X}_\alpha + \bar{X}_\beta)| = 2q^2$.

$\Gamma_{\bar{X}_\alpha+\bar{X}_\beta}(\bar{X}_\alpha + \bar{X}_\beta) = \frac{1}{q^2}\Sigma_{\bar{g}\in\mathsf{G}_F(\mathfrak{f}),{}^g\bar{Z}\in\mathsf{L}_F(\leq -2)}\Lambda(B(\bar{X}_\alpha + X_\beta,{}^g(\bar{X}_\alpha + \bar{X}_\beta))) =$

$\frac{1}{q^2}[\Sigma_{r\in(\mathfrak{f}^\times)^2}\Sigma_{s\in(\mathfrak{f}^\times)^2}\Lambda(B(\bar{X}_\alpha + \bar{X}_\beta, r\bar{X}_{-\alpha} + s\bar{X}_{-\beta}))+$
$\Sigma_{r\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Sigma_{s\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Lambda(B(\bar{X}_\alpha + \bar{X}_\beta, r\bar{X}_{-\alpha} + s\bar{X}_{-\beta}))]|Z_{SO_4(\mathfrak{f})}(\bar{X}_\alpha + \bar{X}_\beta)| =$

$\frac{2q^2}{q^2}[\Sigma_{r\in(\mathfrak{f}^\times)^2}\Sigma_{s\in(\mathfrak{f}^\times)^2}\Lambda(B(\bar{X}_\alpha, r\bar{X}_{-\alpha}))\Lambda(B(\bar{X}_\beta, s\bar{X}_{-\beta}))+$
$\Sigma_{r\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Sigma_{s\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Lambda(B(\bar{X}_\alpha, r\bar{X}_{-\alpha}))\Lambda(B(\bar{X}_\beta, s\bar{X}_{-\beta}))] =$

$2(I_1 + I_2)$.

$$\Gamma_{\bar{X}_\alpha+\bar{X}_\beta}(\epsilon\bar{X}_\alpha + \bar{X}_\beta) = \tfrac{1}{q^2}\Sigma_{\bar{g}\in \mathsf{G}_F(\mathfrak{f}),{}^g\bar{Z}\in \mathsf{L}_F(\leq -2)}\Lambda(B(\bar{X}_\alpha + X_\beta,{}^g(\epsilon\bar{X}_\alpha + \bar{X}_\beta))) =$$

$$2[\Sigma_{r\in(\mathfrak{f}^\times)^2}\Sigma_{s\in(\mathfrak{f}^\times)^2}\Lambda(B(\bar{X}_\alpha + \bar{X}_\beta, r\epsilon\bar{X}_{-\alpha} + s\bar{X}_{-\beta}))+$$
$$\Sigma_{r\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Sigma_{s\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Lambda(B(\bar{X}_\alpha + \bar{X}_\beta, r\epsilon\bar{X}_{-\alpha} + s\bar{X}_{-\beta}))] =$$

$$2[\Sigma_{r\in(\mathfrak{f}^\times)^2}\Sigma_{s\in(\mathfrak{f}^\times)^2}\Lambda(B(\bar{X}_\alpha, r\epsilon\bar{X}_{-\alpha}))\Lambda(B(\bar{X}_{-\beta}, sX_{-\beta}))+$$
$$\Sigma_{r\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Sigma_{s\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Lambda(B(\bar{X}_\alpha, r\epsilon\bar{X}_{-\alpha}))\Lambda(B(\bar{X}_{-\beta}, s\bar{X}_{-\beta}))] =$$

$$= 2(I_3 + I_4).$$

For $\Gamma_{\epsilon\bar{X}_\alpha+\bar{X}_\beta}$ the same arguments cover evaluation at 0, $\bar{X}_\alpha$, $\bar{X}_\beta$.

$$\Gamma_{\epsilon\bar{X}_\alpha+\bar{X}_\beta}(\bar{X}_\alpha + \bar{X}_\beta) = \tfrac{1}{q^2}\Sigma_{\bar{g}\in \mathsf{G}_F(\mathfrak{f}),{}^g\bar{Z}\in \mathsf{L}_F(\leq -2)}\Lambda(B(\epsilon\bar{X}_\alpha + \bar{X}_\beta,{}^g(\bar{X}_\alpha + \bar{X}_\beta))) =$$

$$2[\Sigma_{r\in(\mathfrak{f}^\times)^2}\Sigma_{s\in(\mathfrak{f}^\times)^2}\Lambda(B(\epsilon\bar{X}_\alpha + \bar{X}_\beta, r\bar{X}_{-\alpha} + s\bar{X}_{-\beta}))+$$
$$\Sigma_{r\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Sigma_{s\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Lambda(B(\epsilon\bar{X}_\alpha + \bar{X}_\beta, r\bar{X}_{-\alpha} + s\bar{X}_{-\beta}))] =$$

$$2[\Sigma_{r\in(\mathfrak{f}^\times)^2}\Sigma_{s\in(\mathfrak{f}^\times)^2}\Lambda(B(\epsilon\bar{X}_\alpha, rX_{-\alpha}))\Lambda(B(\bar{X}_\beta, sX_{-\beta}))+$$
$$\Sigma_{r\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Sigma_{s\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Lambda(B(\epsilon\bar{X}_\alpha, r\bar{X}_{-\alpha}))\Lambda(B(\bar{X}_\beta, s\bar{X}_{-\beta}))] =$$

$$2[\Sigma_{r\in(\mathfrak{f}^\times)^2}\Sigma_{s\in(\mathfrak{f}^\times)^2}\Lambda(B(\bar{X}_\alpha, r\epsilon\bar{X}_{-\alpha}))\Lambda(B(\bar{X}_\beta, s\bar{X}_{-\beta}))+$$
$$\Sigma_{r\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Sigma_{s\in\mathfrak{f}^\times\setminus(\mathfrak{f}^\times)^2}\Lambda(B(\bar{X}_\alpha, r\epsilon\bar{X}_{-\alpha}))\Lambda(B(\bar{X}_\beta, s\bar{X}_{-\beta}))] =$$

$= 2(I_3 + I_4)$

$\Gamma_{\epsilon\bar{X}_\alpha+\bar{X}_\beta}(\epsilon\bar{X}_\alpha + \bar{X}_\beta) = \frac{1}{q^2}\Sigma_{\bar{g}\in\mathsf{G}_F(\mathfrak{f}),{}^g\bar{Z}\in\mathsf{L}_F(\leq-2)}\Lambda(B(\epsilon\bar{X}_\alpha + \bar{X}_\beta,{}^g(\epsilon\bar{X}_\alpha + \bar{X}_\beta))) =$

$2[\Sigma_{r\in(\mathfrak{f}^\times)^2}\Sigma_{s\in(\mathfrak{f}^\times)^2}\Lambda(B(\epsilon\bar{X}_\alpha + \bar{X}_\beta, r\epsilon\bar{X}_{-\alpha} + s\bar{X}_{-\beta}))+$
$\Sigma_{r\in\mathfrak{f}^\times\backslash(\mathfrak{f}^\times)^2}\Sigma_{s\in\mathfrak{f}^\times\backslash(\mathfrak{f}^\times)^2}\Lambda(B(\epsilon\bar{X}_\alpha + \bar{X}_\beta, r\epsilon\bar{X}_{-\alpha} + s\bar{X}_{-\beta}))] =$

$2[\Sigma_{r\in(\mathfrak{f}^\times)^2}\Sigma_{s\in(\mathfrak{f}^\times)^2}\Lambda(B(\epsilon\bar{X}_\alpha, r\epsilon\bar{X}_{-\alpha}))\Lambda(B(\bar{X}_\beta, s\bar{X}_{-\beta}))+$
$\Sigma_{r\in\mathfrak{f}^\times\backslash(\mathfrak{f}^\times)^2}\Sigma_{s\in\mathfrak{f}^\times\backslash(\mathfrak{f}^\times)^2}\Lambda(B(\epsilon\bar{X}_\alpha, r\epsilon\bar{X}_{-\alpha}))\Lambda(B(\bar{X}_\beta, s\bar{X}_{-\beta}))] =$

$2[\Sigma_{r\in(\mathfrak{f}^\times)^2}\Sigma_{s\in(\mathfrak{f}^\times)^2}\Lambda(B(\bar{X}_\alpha, r\bar{X}_{-\alpha}))\Lambda(B(\bar{X}_\beta, s\bar{X}_{-\beta}))+$
$\Sigma_{r\in\mathfrak{f}^\times\backslash(\mathfrak{f}^\times)^2}\Sigma_{s\in\mathfrak{f}^\times\backslash(\mathfrak{f}^\times)^2}\Lambda(B(\bar{X}_\alpha, r\bar{X}_{-\alpha}))\Lambda(B(\bar{X}_\beta, s\bar{X}_{-\beta}))] =$

$= 2(I_1 + I_2)$

We must describe the various I_j. In particular, the expression $I_1 + I_2 + I_3 + I_4$ will occur in our pairings. Observe that this sum is the sum of the character $\lambda(Z) = \Lambda(B(\bar{X}_\alpha + \bar{X}_\beta, Z))$ over the rational points of the regular $SO_4(\mathfrak{f})$-orbit in $\mathfrak{so}_4$. Then:

$0 = \Sigma_{Z\in\mathfrak{SO}_4(<-2)}\lambda(Z) = I_1 + I_2 + I_3 + I_4 + \Sigma_{Z\in\mathfrak{f}^\times\bar{X}_\alpha}\lambda(Z) + \Sigma_{Z\in\mathfrak{f}^\times\bar{X}_\beta} + \lambda(0) =$
$I_1 + I_2 + I_3 + I_4 - 1 - 1 + 1$

Thus $I_1 + I_2 + I_3 + I_4 = 1$.

We now write tables for the values of the function $\Gamma_X \bar{Q}_T$ and the size of the nilpotent orbits prior to computing the full pairing:

	0	$\bar{X}_\alpha$	$\bar{X}_\beta$
$\lvert\mathcal{O}(\mathfrak{f})\rvert$	1	$(q+1)(q-1)$	$(q+1)(q-1)$
$\Gamma_{(SO_4,X_\alpha)}\bar{Q}_S$	$q(q+1)^4(q-1)^2$	$-q(q+1)^2(q-1)$	0
$\Gamma_{(SO_4,X_\beta)}\bar{Q}_S$	$q(q+1)^4(q-1)^2$	0	$-q(q+1)^2(q-1)$
$\Gamma_{(SO_4,X_\alpha+X_\beta)}\bar{Q}_S$	$(q+1)^4(q-1)^2$	$-(q+1)^2(q-1)$	$-(q+1)^2(q-1)$
$\Gamma_{(SO_4,\epsilon X_\alpha+X_\beta)}\bar{Q}_S$	$(q+1)^4(q-1)^2$	$-(q+1)^2(q-1)$	$-(q+1)^2(q-1)$

	$\bar{X}_\alpha+\bar{X}_\beta$	$\epsilon\bar{X}_\alpha+\bar{X}_\beta$
$\lvert\mathcal{O}(\mathfrak{f})\rvert$	$\frac{(q+1)^2(q-1)^2}{2}$	$\frac{(q+1)^2(q-1)^2}{2}$
$\Gamma_{(SO_4,X_\alpha)}\bar{Q}_S$	0	0
$\Gamma_{(SO_4,X_\beta)}\bar{Q}_S$	0	0
$\Gamma_{(SO_4,X_\alpha+X_\beta)}\bar{Q}_S$	$2(I_1+I_2)$	$2(I_3+I_4)$
$\Gamma_{(SO_4,\epsilon X_\alpha+X_\beta)}\bar{Q}_S$	$2(I_3+I_4)$	$2(I_1+I_2)$

$$(\Gamma_{X_\alpha}, Q_S)_{\mathfrak{so}_4} = \frac{q(q+1)^4(q-1)^2 - q(q+1)^3(q-1)^2}{q^2(q+1)^2(q-1)^2} = \frac{(q+1)^2-(q+1)}{q} = q+1$$

$$(\Gamma_{X_\beta}, Q_S)_{\mathfrak{so}_4} = \frac{q(q+1)^4(q-1)^2 - q(q+1)^3(q-1)^2}{q^2(q+1)^2(q-1)^2} = \frac{(q+1)^2-(q+1)}{q} = q+1$$

$$(\Gamma_{X_\alpha+X_\beta}, Q_S)_{\mathfrak{so}_4} = \frac{(q+1)^4(q-1)^2-(q+1)^3(q-1)^2-(q+1)^2(q-1)^3+2(I_1+I_2+I_3+I_4)\frac{(q+1)^2(q-1)^2}{2}}{q^2(q+1)^2(q-1)^2} = \frac{(q+1)^2-(q-1)-(q+1)-1}{q^2} = 1$$

$$(\Gamma_{\epsilon X_\alpha+X_\beta}, Q_S)_{\mathfrak{so}_4} = \frac{(q+1)^4(q-1)^2-(q+1)^3(q-1)^2-(q+1)^2(q-1)^3+2(I_1+I_2+I_3+I_4)\frac{(q+1)^2(q-1)^2}{2}}{q^2(q+1)^2(q-1)^2} = 1$$

	0	X_α	X_β
$\lvert\mathcal{O}(\mathfrak{f})\rvert$	1	$(q+1)(q-1)$	$(q+1)(q-1)$
$\Gamma_{(SO_4,X_\alpha)}\bar{Q}_{T_{(F,\mathrm{Id})}}$	$-q(q+1)^3(q-1)^3$	$q(q+1)(q-1)^2$	0
$\Gamma_{(SO_4,X_\beta)}\bar{Q}_{T_{(F,\mathrm{Id})}}$	$-q(q+1)^3(q-1)^3$	0	$-q(q+1)^2(q-1)$
$\Gamma_{(SO_4,X_\alpha+X_\beta)}\bar{Q}_{T_{(F,\mathrm{Id})}}$	$-(q+1)^3(q-1)^3$	$(q+1)(q-1)^2$	$-(q+1)^2(q-1)$
$\Gamma_{(SO_4,\epsilon X_\alpha+X_\beta)}\bar{Q}_{T_{(F,\mathrm{Id})}}$	$-(q+1)^3(q-1)^3$	$(q+1)(q-1)^2$	$-(q+1)^2(q-1)$

	$X_\alpha+X_\beta$	$\epsilon X_\alpha+X_\beta$
$\lvert\mathcal{O}(\mathfrak{f})\rvert$	$\frac{(q+1)^2(q-1)^2}{2}$	$\frac{(q+1)^2(q-1)^2}{2}$
$\Gamma_{(SO_4,X_\alpha)}\bar{Q}_{T_{(F,\mathrm{Id})}}$	0	0
$\Gamma_{(SO_4,X_\beta)}\bar{Q}_{T_{(F,\mathrm{Id})}}$	0	0
$\Gamma_{(SO_4,X_\alpha+X_\beta)}\bar{Q}_{T_{(F,\mathrm{Id})}}$	$2(I_1+I_2)$	$2(I_3+I_4)$
$\Gamma_{(SO_4,\epsilon X_\alpha+X_\beta)}\bar{Q}_{T_{(F,\mathrm{Id})}}$	$2(I_3+I_4)$	$2(I_1+I_2)$

$$(\Gamma_{X_\alpha}, Q_{T_{(F,\mathrm{Id})}})_{\mathfrak{so}_4} = \frac{-q(q+1)^3(q-1)^3+q(q+1)^2(q-1)^3}{q^2(q+1)^2(q-1)^2} = \frac{-(q+1)(q-1)+q-1}{q} = -(q-1)$$

$$(\Gamma_{X_\beta}, Q_{T_{(F,\mathrm{Id})}})_{\mathfrak{so}_4} = \frac{-q(q+1)^3(q-1)^3-q(q+1)^3(q-1)^2}{q^2(q+1)^2(q-1)^2} = \frac{-(q+1)(q-1)-(q+1)}{q} = -(q+1)$$

$$(\Gamma_{X_\alpha+X_\beta}, Q_{T_{(F,\mathrm{Id})}})_{\mathfrak{so}_4} = \frac{-(q+1)^3(q-1)^3+(q+1)^2(q-1)^3-(q+1)^3(q-1)^2+(q+1)^2(q-1)^2}{q^2(q+1)^2(q-1)^2} = -1$$

$$(\Gamma_{\epsilon X_\alpha+X_\beta}, Q_{T_{(F,\mathrm{Id})}})_{\mathfrak{so}_4} = \frac{-(q+1)^3(q-1)^3+q(q+1)^2(q-1)^3-q(q+1)^3(q-1)^2+(q+1)^2(q-1)^2}{q^2(q+1)^2(q-1)^2} = -1$$

	0	X_α	X_β
$\Gamma_{(SO_4,X_\alpha)}\bar{Q}_{T_{(\mathrm{Id},F)}}$	$-q(q+1)^3(q-1)^3$	$-q(q+1)^2(q-1)$	0
$\Gamma_{(SO_4,X_\beta)}\bar{Q}_{T_{(\mathrm{Id},F)}}$	$-q(q+1)^3(q-1)^3$	0	$q(q+1)(q-1)^2$
$\Gamma_{(SO_4,X_\alpha+X_\beta)}\bar{Q}_{T_{(\mathrm{Id},F)}}$	$-(q+1)^3(q-1)^3$	$-(q+1)^2(q-1)$	$(q+1)(q-1)^2$
$\Gamma_{(SO_4,\epsilon X_\alpha+X_\beta)}\bar{Q}_{T_{(\mathrm{Id},F)}}$	$-(q+1)^3(q-1)^3$	$-(q+1)^2(q-1)$	$(q+1)(q-1)^2$

	$X_\alpha + X_\beta$	$\epsilon X_\alpha + X_\beta$
$\Gamma_{(SO_4, X_\alpha)} \bar{Q}_{T_{(\mathrm{Id},F)}}$	0	0
$\Gamma_{(SO_4, X_\beta)} \bar{Q}_{T_{(\mathrm{Id},F)}}$	0	0
$\Gamma_{(SO_4, X_\alpha+X_\beta)} \bar{Q}_{T_{(\mathrm{Id},F)}}$	$2(I_1+I_2)$	$2(I_3+I_4)$
$\Gamma_{(SO_4, \epsilon X_\alpha+X_\beta)} \bar{Q}_{T_{(\mathrm{Id},F)}}$	$2(I_3+I_4)$	$2(I_1+I_2)$

$$(\Gamma_{X_\alpha}, Q_{T_{(\mathrm{Id},F)}})_{\mathfrak{so}_4} = \frac{-q(q+1)^3(q-1)^3 - q(q+1)^3(q-1)^2}{q^2(q+1)^2(q-1)^2} = \frac{-(q+1)(q-1)-(q+1)}{q} = -(q+1)$$

$$(\Gamma_{X_\beta}, Q_{T_{(\mathrm{Id},F)}})_{\mathfrak{so}_4} = \frac{-q(q+1)^3(q-1)^3 + q(q+1)^2(q-1)^3}{q^2(q+1)^2(q-1)^2} = \frac{-(q+1)(q-1)+(q-1)}{q} = -(q-1)$$

$$(\Gamma_{X_\alpha+X_\beta}, Q_{T_{(\mathrm{Id},F)}})_{\mathfrak{so}_4} = \frac{-(q+1)^3(q-1)^3 - (q+1)^3(q-1)^2 + (q+1)^2(q-1)^3 + (q+1)^2(q-1)^2}{q^2(q+1)^2(q-1)^2} = -1$$

$$(\Gamma_{\epsilon X_\alpha+X_\beta}, Q_{T_{(\mathrm{Id},F)}})_{\mathfrak{so}_4} = \frac{-(q+1)^3(q-1)^3 - (q+1)^3(q-1)^2 + (q+1)^2(q-1)^3 + (q+1)^2(q-1)^2}{q^2(q+1)^2(q-1)^2} = -1$$

	0	X_α	X_β
$\Gamma_{(SO_4, X_\alpha)} \bar{Q}_{T_{(F,F)}}$	$q(q+1)^2(q-1)^4$	$q(q+1)(q-1)^2$	0
$\Gamma_{(SO_4, X_\beta)} \bar{Q}_{T_{(F,F)}}$	$q(q+1)^2(q-1)^4$	0	$q(q+1)(q-1)^2$
$\Gamma_{(SO_4, X_\alpha+X_\beta)} \bar{Q}_{T_{(F,F)}}$	$(q+1)^2(q-1)^4$	$(q+1)(q-1)^2$	$(q+1)(q-1)^2$
$\Gamma_{(SO_4, \epsilon X_\alpha+X_\beta)} \bar{Q}_{T_{(F,F)}}$	$(q+1)^2(q-1)^4$	$(q+1)(q-1)^2$	$(q+1)(q-1)^2$

	$X_\alpha + X_\beta$	$\epsilon X_\alpha + X_\beta$
$\Gamma_{(SO_4, X_\alpha)} \bar{Q}_{T_{(F,F)}}$	0	0
$\Gamma_{(SO_4, X_\beta)} \bar{Q}_{T_{(F,F)}}$	0	0
$\Gamma_{(SO_4, X_\alpha+X_\beta)} \bar{Q}_{T_{(F,F)}}$	$2(I_1+I_2)$	$2(I_3+I_4)$
$\Gamma_{(SO_4, \epsilon X_\alpha+X_\beta)} \bar{Q}_{T_{(F,F)}}$	$2(I_3+I_4)$	$2(I_1+I_2)$

$$(\Gamma_{X_\alpha}, Q_{T_{(F,F)}})_{\mathfrak{so}_4} = \frac{q(q+1)^2(q-1)^4 + q(q+1)^2(q-1)^3}{q^2(q+1)^2(q-1)^2} = \frac{(q-1)^2+(q-1)}{q} = q-1$$

$$(\Gamma_{X_\beta}, Q_{T_{(F,F)}})_{\mathfrak{so}_4} = \frac{q(q+1)^2(q-1)^4 + q(q+1)^2(q-1)^3}{q^2(q+1)^2(q-1)^2} = q-1$$

$$(\Gamma_{X_\alpha+X_\beta}, Q_{T_{(F,F)}})_{\mathfrak{so}_4} = \frac{(q+1)^2(q-1)^4 + (q+1)^2(q-1)^3 + (q+1)^2(q-1)^3 - (q+1)^2(q-1)^2}{q^2(q+1)^2(q-1)^2} =$$

$$\frac{(q-1)^2+(q-1)+(q-1)+1}{q^2} = 1$$

$$(\Gamma_{\epsilon X_\alpha + X_\beta}, Q_{T_{(F,F)}})_{\mathfrak{so}_4} = \frac{(q+1)^2(q-1)^4+(q+1)^2(q-1)^3+(q+1)^2(q-1)^3+(q+1)^2(q-1)^2}{q^2(q+1)^2(q-1)^2} = 1$$

Now we compute $D_{(F,T)}(h_{(C,e)})$. Let $k_{C,T} = \frac{|N_G(T)/T|}{|N_{\mathsf{G}_C}(\mathsf{T})(\mathfrak{f})/\mathsf{T}(\mathfrak{f})|}$. Replace $D_{(F,T)}$ with $\frac{D_{(F,T)}}{k_{C,T}}$, as for our applications we don't need to know these evaluations more precisely than up to non-zero scaling that is uniform in both C and T. Here is the result in table format:

	$h_{(F_{\{\alpha,\beta\}},m)}$	$h_{(F_{\{\alpha,\beta\}},m')}$	$h_{(F_{\{\alpha,1-\beta\}},m)}$	$h_{(F_{\{\alpha,1-\beta\}},m')}$
$D_{(F_\emptyset,S)}$	1	1	1	1
$D_{(F_{\{\alpha\}},T_F)}$	-1	-1	-1	-1
$D_{(F_{\{\beta\}},T_F)}$	-1	-1	-1	-1
$D_{(F_{\{\alpha,\beta\}},T_{(F,F)})}$	1	1	1	1
$D_{(F_{\{\alpha,1-\beta\}},T_{(F,F)})}$	1	1	1	1

.3 Pairings for G_2

This computation is easy, but long. We use the notation for nilpotent orbits in $\mathfrak{g}_2$ defined in Chapter 8.

The formula of Corollary X.7 (when non-vanishing) is:

$$D_{(F,T_w)}(h_{(F,e)}) = \frac{|\mathsf{L}_x(1)|^{1/2}|N_G(T)/T|q^{e(e)}X_{T_w}^{(F,e)}}{|W_x|}$$

The relevant Green Polynomials are:

$$X_{(F,T_w)}^{(F_{G_2},e_{reg})} = 1$$

$$X_{(F,T_w)}^{(F_{G_2},e_{sr})} = 1 + q^{-1}(\chi(w) + 2\tau(w))$$

$$X_{(F,T_w)}^{(F_{G_2},e'_{sr})} = 1 + q^{-1}(\chi(w) - \tau(w))$$

$$X_{(F,T_w)}^{(F_{G_2},e''_{sr})} = 1 + q^{-1}\chi(w)$$

$$X_{(F,T_w)}^{(F_{SO_4},m)} = 1$$

$$X_{(F,T_w)}^{(F_{SO_4},m')} = 1$$

$$X_{(F,T_w)}^{(F_{SL_3},n)} = 1$$

$$X_{(F,T_w)}^{(F_{SL_3},n')} = 1$$

$$X_{(F,T_w)}^{(F_{SL_3},n'')} = 1$$

Notice that after rescaling we can replace $D_C^s t$ with $\Sigma_{(F,T)\in C} \frac{D_{(F,T)}}{|N_G(T)/T|}$ where the sum is over all conjugacy classes of tori whose corresponding Weyl conjugacy class in $W_{G_2}(\bar{k})$ is C.

Recall our list of nilpotent orbits in Chapter 8.

For the regular and subregular nilpotents in $\mathfrak{g}_2$ we have $|\mathsf{L}_x(1)| = 1$.

For all but the nilpotents e_{sr}, e'_{sr}, and e''_{sr} we have $e(e) = 0$, while for the three orbits e_{sr}, e'_{sr}, and e''_{sr} we have $e(e) = 1$.

For all four of e_{reg}), e_{sr}, e'_{sr}, and e''_{sr} we have$|W_x| = 12$. For both m and m' we have $|W_x| = 4$. For n, n', n'', we have $|W_x| = 6$.

Pairings are as follows:

$D_{(F,S)}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$

$D_{(F,S)}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{q^1(1+q^{-1}(2+2))}{12}$

$D_{(F,S)}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{q^1(1+q^{-1}(2-1))}{12}$

$D_{(F,S)}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{q^1(1+q^{-1}2)}{12}$

$D_{(F,S)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{1}{6}$

$D_{(F,S)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{1}{6}$

$D_{(F,S)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{1}{6}$

$D_{(F,S)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1}{4}$

$D_{(F,S)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{1}{4}$

$D_{(F_\alpha,T_F)}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$

$D_{(F_\alpha,T_F)}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{q^1(1+q^{-1}(0+2))}{12}$

$D_{(F_\alpha,T_F)}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{q^1(1+q^{-1}(0-1))}{12}$

$D_{(F_\alpha,T_F)}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{q^1(1+q^{-1}0)}{12}$

$D_{(F_\alpha,T_F)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = 0$

$D_{(F_\alpha,T_F)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = 0$

$D_{(F_\alpha,T_F)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = 0$

$D_{(F_\alpha,T_F)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1}{4}$

$D_{(F_\alpha,T_F)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{1}{4}$

$D_{(F_\beta,T_F)}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$

$D_{(F_\beta,T_F)}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{q^1(1+q^{-1}(0-2))}{12}$

$D_{(F_\beta,T_F)}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{q^1(1+q^{-1}(0+1))}{12}$

$D_{(F_\beta,T_F)}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{q^1(1+q^{-1}0)}{12}$

$D_{(F_\beta,T_F)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{1}{6}$

$D_{(F_\beta,T_F)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{1}{6}$

$D_{(F_\beta,T_F)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{1}{6}$

$D_{(F_\beta,T_F)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1}{4}$

$D_{(F_\beta,T_F)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{1}{4}$

$D_{(F_{\{\alpha,\beta\}},T_R)}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$

$D_{(F_{\{\alpha,\beta\}},T_R)}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{q^1(1+q^{-1}(1-2))}{12}$

$D_{(F_{\{\alpha,\beta\}},T_R)}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{q^1(1+q^{-1}(1+1))}{12}$

$D_{(F_{\{\alpha,\beta\}},T_R)}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{q^1(1+q^{-1}1)}{12}$

$D_{(F_{\{\alpha,\beta\}},T_R)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = 0$

$D_{(F_{\{\alpha,\beta\}},T_R)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = 0$

$D_{(F_{\{\alpha,\beta\}},T_R)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = 0$

$D_{(F_{\{\alpha,\beta\}},T_R)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = 0$

$D_{(F_{\{\alpha,\beta\}},T_R)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = 0$

$D_{(F_{\{\alpha,\beta\}},T_{R^2})}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$

$D_{(F_{\{\alpha,\beta\}},T_{R^2})}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{q^1(1+q^{-1}(-1+2))}{12}$

$D_{(F_{\{\alpha,\beta\}},T_{R^2})}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{q^1(1+q^{-1}(-1-1))}{12}$

$$D_{(F_{\{\alpha,\beta\}},T_{R^2})}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{q^1(1+q^{-1}(-1))}{12}$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^2})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^2})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^2})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^2})}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^2})}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^3})}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^3})}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{q^1(1+q^{-1}(-2-2))}{12}$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^3})}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{q^1(1+q^{-1}(-2+1))}{12}$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^3})}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{q^1(1+q^{-1}(-2))}{12}$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^3})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^3})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^3})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^3})}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},T_{R^3})}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = 0$$

$$D_{(F_{\{\beta,3\alpha+2\beta-1\}},T_{R^2})}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$$

$$D_{(F_{\{\beta,3\alpha+2\beta-1\}},T_{R^2})}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = 0$$

$$D_{(F_{\{\beta,3\alpha+2\beta-1\}},T_{R^2})}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = 0$$

$$D_{(F_{\{\beta,3\alpha+2\beta-1\}},T_{R^2})}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = 0$$

$$D_{(F_{\{\beta,3\alpha+2\beta-1\}},T_{R^2})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{1}{6}$$

$$D_{(F_{\{\beta,3\alpha+2\beta-1\}},T_{R^2})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{1}{6}$$

$$D_{(F_{\{\beta,3\alpha+2\beta-1\}},T_{R^2})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{1}{6}$$

$$D_{(F_{\{\beta,3\alpha+2\beta-1\}},T_{R^2})}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = 0$$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},T_{R^2})}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = 0$

$$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},T_{R^3})}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$$

$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},T_{R^3})}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = 0$

$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},T_{R^3})}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = 0$

$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},T_{R^3})}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = 0$

$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},T_{R^3})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = 0$

$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},T_{R^3})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = 0$

$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},T_{R^3})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = 0$

$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},T_{R^3})}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1}{4}$

$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},T_{R^3})}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{1}{4}$

Stable distributions evaluated at h_e:

$$D_{\mathrm{Id}}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$$

$D_{\mathrm{Id}}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{q+4}{12}$

$D_{\mathrm{Id}}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{q+1}{12}$

$D_{\mathrm{Id}}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{q+2}{12}$

$D_{\mathrm{Id}}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{1}{6}$

$D_{\mathrm{Id}}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{1}{6}$

$D_{\mathrm{Id}}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{1}{6}$

$D_{\mathrm{Id}}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1}{4}$

$D_{\mathrm{Id}}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{1}{4}$

$$D_F(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$$

$D_F(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{q+2}{12}$

$D_F(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{q-1}{12}$

$D_F(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{q}{12}$

$D_F(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = 0$

$D_F(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = 0$

$D_F(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = 0$

$D_F(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1}{4}$

$D_F(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{1}{4}$

$D_{FR}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$

$D_{FR}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{q-2}{12}$

$D_{FR}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{q+1}{12}$

$D_{FR}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{q}{12}$

$D_{FR}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{1}{6}$

$D_{FR}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{1}{6}$

$D_{FR}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{1}{6}$

$D_{FR}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1}{4}$

$D_{FR}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{1}{4}$

$D_R(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$

$D_R(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{q-1}{12}$

$D_R(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{q+2}{12}$

$D_R(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{q+1}{12}$

$D_R(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = 0$

$D_R(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = 0$

$D_R(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = 0$

$D_R(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = 0$

$D_R(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = 0$

$D_{R^2}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$

$D_{R^2}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{q+1}{12}$

$D_{R^2}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{q-2}{12}$

$D_{R^2}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{q-1}{12}$

$D_{R^2}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{1}{6}$

$D_{R^2}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{1}{6}$

$D_{R^2}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{1}{6}$

$D_{R^2}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = 0$

$D_{R^2}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = 0$

$D_{R^3}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$

$D_{R^3}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{q-4}{12}$

$D_{R^3}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{q-1}{12}$

$D_{R^3}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{q-2}{12}$

$D_{R^3}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = 0$

$D_{R^3}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = 0$

$D_{R^3}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = 0$

$D_{R^3}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1}{4}$

$D_{R^3}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{1}{4}$

Now lets look at the distributions associated to cuspidal local systems evaluated at $h_{(F,e)}$.

$$D_{(F_{\{\alpha,\beta\}},e_{sr},\chi)},(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},e_{sr},\chi)}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{1}{q^2}$$

$$D_{(F_{\{\alpha,\beta\}},e_{sr},\chi)}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{-1}{q^2}$$

$$D_{(F_{\{\alpha,\beta\}},e_{sr},\chi)}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{1}{q^2}$$

$$D_{(F_{\{\alpha,\beta\}},e_{sr},\chi)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},e_{sr},\chi)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},e_{sr},\chi)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},e_{sr},\chi)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = 0$$

$$D_{(F_{\{\alpha,\beta\}},e_{sr},\chi)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = 0$$

$$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},m,sgn\times sgn)},(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$$

$$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},m,sgn\times sgn)}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = 0$$

$$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},m,sgn\times sgn)}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = 0$$

$$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},m,sgn\times sgn)}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = 0$$

$$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},m,sgn\times sgn)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = 0$$

$$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},m,sgn\times sgn)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = 0$$

$$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},m,sgn\times sgn)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = 0$$

$$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},m,sgn\times sgn)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1}{q}$$

$$D_{(F_{\{\alpha,3\alpha+2\beta-1\}},m,sgn\times sgn)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{-1}{q}$$

$$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi)},(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$$

$$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi)}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = 0$$

$$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi)}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = 0$$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi)}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = 0$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{1}{q}$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{\xi}{q}$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi)}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{\xi^{-1}}{q}$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = 0$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi)}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = 0$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi^{-1})},(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi^{-1})}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = 0$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi^{-1})}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = 0$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi^{-1})}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = 0$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi^{-1})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{1}{q}$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi^{-1})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{\xi^{-1}}{q}$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi^{-1})}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{\xi}{q}$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi^{-1})}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = 0$

$D_{(F_{\{\beta,3\alpha+2\beta-1\}},n,\psi^{-1})}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = 0$

Notice that for all C conjugacy class in W_{G_2} we have $D_C(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = \frac{1}{12}$. Thus if we take any non-trivial character $\kappa \in {W_{G_2}}^D$ and form the new stable distribution $D_\kappa = \Sigma_{w\in W_{G_2}}\kappa w D_C$, then $D_\kappa(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$. We will use this fact to explore the stability of the various subregular orbits in $\mathfrak{g}_2(k)$ in chapter 10. We now compute a table of all the D_κ evaluated at the $h_{(F,e)}$ for regular and subregular (F,e).

$D_{\mathrm{Id}},(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 1$

$D_{\mathrm{Id}}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{(q+4)+3(q+2)+3(q-2)+2(q-1)+2(q+1)+(q-4)}{12} = q$

$$D_{\mathrm{Id}}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{(q+1)+3(q-1)+3(q+1)+2(q+2)+2(q-2)+(q-1)}{12} = q$$

$$D_{\mathrm{Id}}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{(q+2)+3(q)+3(q)+2(q+1)+2(q-1)+(q-2)}{12} = q$$

$$D_{\mathrm{Id}}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{1+3*0+3+2*0+2*1+0}{6} = 1$$

$$D_{\mathrm{Id}}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{1+3*0+3+2*0+2*1+0}{6} = 1$$

$$D_{\mathrm{Id}}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{1+3*0+3+2*0+2*1+0}{6} = 1$$

$$D_{\mathrm{Id}}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1+3*1+3*1+2*0+2*0+1}{4} = 2$$

$$D_{\mathrm{Id}}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{1+3*1+3*1+2*0+2*0+1}{4} = 2$$

$$D_{sgn}(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$$

$$D_{sgn}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{(q+4)-3(q+2)-3(q-2)+2(q-1)+2(q+1)+(q-4)}{12} = 0$$

$$D_{sgn}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{(q+1)-3(q-1)-3(q+1)+2(q+2)+2(q-2)+(q-1)}{12} = 0$$

$$D_{sgn}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{(q+2)-3q-3q+2(q+1)+2(q-1)+(q-2)}{12} = 0$$

$$D_{sgn}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{1-3*0-3+2*0+2*1+0}{6} = 0$$

$$D_{sgn}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{1-3*0-3+2*0+2*1+0}{6} = 0$$

$$D_{sgn}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{1-3*0-3+2*0+2*1+0}{6} = 0$$

$$D_{sgn}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1-3*1-3*1+2*0+2*0+1}{4} = -1$$

$$D_{sgn}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{1-3*1-3*1+2*0+2*0+1}{4} = -1$$

$$D_{\chi},(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$$

$$D_{\chi}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{2(q+4)+0*3(q+2)+0*3(q-2)+1*2(q-1)-2(q+1)-2(q-4)}{12} = 1$$

$$D_{\chi}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{2(q+1)+0*3(q-1)+0*3(q+1)+1*2(q+2)-2(q-2)-2(q-1)}{12} = 1$$

$$D_{\chi}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{2(q+2)+0*3(q)+0*3(q)+2(q+1)-2(q-1)-2(q-2)}{12} = 1$$

$$D_{\chi}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{2+0*3*0+0*3+1*2*1-2*1-2*0}{6} = 0$$

$$D_{\chi}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{2+0*3*0+0*3+1*2*1-2*1-2*0}{6} = 0$$

$$D_{\chi}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{2+0*3*0+0*3+1*2*1-2*1-2*0}{6} = 0$$

$$D_\chi(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{2+0*3*1+0*3*1+2*0-2*0-2*1}{4} = 0$$

$$D_\chi(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{2+0*3*1+0*3*1+2*0-2*0-2*1}{4} = 0$$

$$D_\tau,(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$$

$$D_\tau(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{(q+4)+3(q+2)-3(q-2)-2(q-1)+2(q+1)-(q-4)}{12} = 2$$

$$D_\tau(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{(q+1)+3(q-1)-3(q+1)-2(q+2)+2(q-2)-(q-1)}{12} = -1$$

$$D_\tau(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{(q+2)+3(q)-3(q)-2(q+1)+2(q-1)-(q-2)}{12} = 0$$

$$D_\tau(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{1+3*0-3-2*0+2*1-0}{6} = 0$$

$$D_\tau(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{1+3*0-3-2*0+2*1-0}{6} = 0$$

$$D_\tau(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{1+3*0-3-2*0+2*1-0}{6} = 0$$

$$D_\tau(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1+3*1-3*1-2*0+2*0-1}{4} = 0$$

$$D_\tau(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{1+3*1-3*1-2*0+2*0-1}{4} = 0$$

$$D_{sgn\tau},(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$$

$$D_{sgn\tau}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{(q+4)-3(q+2)+3(q-2)-2(q-1)+2(q+1)-(q-4)}{12} = 0$$

$$D_{sgn\tau}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{(q+1)-3(q-1)+3(q+1)-2(q+2)+2(q-2)-(q-1)}{12} = 0$$

$$D_{sgn\tau}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{(q+2)-3(q)+3(q)-2(q+1)+2(q-1)-(q-2)}{12} = 0$$

$$D_{sgn\tau}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{1-3*0+3-2*0+2*1-0}{6} = 1$$

$$D_{sgn\tau}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{1-3*0+3-2*0+2*1-0}{6} = 1$$

$$D_{sgn\tau}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{1-3*0+3-2*0+2*1-0}{6} = 1$$

$$D_{sgn\tau}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{1-3*1+3*1-2*0+2*0-1}{4} = 0$$

$$D_{sgn\tau}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{1-3*1+3*1-2*0+2*0-1}{4} = 0$$

$$D_{\chi\tau},(h_{(F_{\{\alpha,\beta\}},e_{reg})}) = 0$$

$$D_{\chi\tau}(h_{(F_{\{\alpha,\beta\}},e_{sr})}) = \frac{2(q+4)+0*3(q+2)+0*3(q-2)-2(q-1)-2(q+1)+2(q-4)}{12} = 0$$

$$D_{\chi\tau}(h_{(F_{\{\alpha,\beta\}},e'_{sr})}) = \frac{2(q+1)+0*3(q-1)+0*3(q+1)-2(q+2)-2(q-2)+2(q-1)}{12} = 0$$

$$D_{\chi\tau}(h_{(F_{\{\alpha,\beta\}},e''_{sr})}) = \frac{2(q+2)+0*3q+0*3q-2(q+1)-2(q-1)+2(q-2)}{12} = 0$$

$$D_{\chi\tau}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n)}) = \frac{2+0*3*0+0*3-2*0-2*1+2*0}{6} = 0$$

$$D_{\chi\tau}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n')}) = \frac{2+0*3*0+0*3-2*0-2*1+2*0}{6} = 0$$

$$D_{\chi\tau}(h_{(F_{\{3\alpha+2\beta-1,\beta\}},n'')}) = \frac{2+0*3*0+0*3-2*0-2*1+2*0}{6} = 0$$

$$D_{\chi\tau}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m)}) = \frac{2+0*3*1+0*3*1-2*0-2*0+2*1}{4} = 1$$

$$D_{\chi\tau}(h_{(F_{\{\alpha,3\alpha+2\beta-1\}},m')}) = \frac{2+0*3*1+0*3*1-2*0-2*0+2*1}{4} = 1$$

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