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Rationalizing Size, Value, and Momentum Effects with a CAPM2

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Abstract

This paper introduces a new Capital Asset Pricing Model (CAPM^2) that in accordance with recent decision theory models can price the key factors that drive risky choice behavior: (i) goal achievement (importance of the overall probability of obtaining positive payoffs), (ii) loss aversion (losses loom larger than gains), and (iii) preference for security/potential (downside risk aversion and preference for upside potential). These three factors are also the key drivers of size, value, and momentum anomalies. Moreover, zero cost portfolios that take long (short) positions on securities with higher (lower) loadings on the three CAPM^2 factors deliver positive and statistically significant risk adjusted returns.

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1 Introduction

The traditional mean-variance Capital Asset Pricing Model (CAPM) developed by Sharpe (1964) and Lintner (1965) has long served as the backbone of academic finance, and it has been used in numerous important applications. However, several studies indicate that the cross-section of stock returns cannot be explained by market beta alone, as predicted by the CAPM. In particular, the CAPM fares poorly in explaining the high risk premiums of portfolios formed by small cap, high book-to-market securities (Fama and French, 1992), and momentum winners (Jegadeesh and Titman, 1993). In response to the empirical failure of the standard mean-variance CAPM, researchers have examined the performance of alternative models of asset prices. For example, Harvey and Siddique (2000) and Dittmar (2002) study the three-moment CAPM, which takes into account skewness in addition to mean and variance. In contrast, the literature on downside risk considers semivariance, rather than variance, as the primary source of risk (e.g., Post and Levy, 2005; Ang, Chen, and Xing, 2006). These alternative asset pricing models fit stock return data better than the standard mean-variance CAPM, but size, value, and momentum effects remain.

A fundamental limitation of the existing asset pricing models is the poor descriptive ability of the utility functions from which they are derived. The extensive literature on decision making under risk clearly shows that utility functions displaying asymmetries in the domains of gains and losses can explain risky choice behavior significantly better than standard utility functions (e.g., Kahneman and Tversky, 1979; Levy and Levy, 2002; Diecidue and Van de Ven, 2008; Piccioni, 2014). However, despite the extensive use of asymmetric utility functions in the decision theory literature, little effort has been devoted to studying their implications on the cross section of stock returns.

The purpose of this paper, then, is to examine whether asset pricing models based on asymmetric utility functions can shed light on the sources of risk that drive the cross section of stock returns. In order to do so, I derive a new Capital Asset Pricing Model by approximating the utility function of the representative agent with Continuous Asymmetric Polynomial Models (CAPM^2).[1]

Testing the CAPM^2 on size value and momentum portfolios, I find that the cross-section of stock returns can be rationalized by a utility function displaying (i) a positive shift at zero, (ii) a higher slope in the negative domain, and (iii) concavity for losses and mild convexity for gains.[2] These findings are consistent with the results of Piccioni (2014), who shows that a utility function with the same three characteristics explains how people face risk, because it captures three fundamental factors that drive risky choice behavior: (i) "Goal Achievement", i.e., the importance of the overall probability of obtaining positive payoffs (consistent with a utility function with a positive shift at zero); (ii) "Loss Aversion", i.e., the empirical observation that losses loom larger than gains (consistent with a utility function steeper in the negative domain); and (iii) "preference for Security/Potential", i.e., downside risk aversion and preference for upside potential (consistent with a utility function concave for losses and convex for gains).

The results of this paper show that Goal Achievement, Loss Aversion, and preference for Security/Potential are also the fundamental factors driving size, value, and

[1] I approximate utility with polynomial models displaying different parameter values in the positive and negative domains. In other words, the utility function is approximated with separate polynomial expansions in the domains of gains and losses. The utility function is continuously differentiable because I use a continuous weighting function, rather than an indicator function, to weight the upside and downside components of the utility function. The risk-free rate is the reference point, or target return, used to distinguish between gains and losses.

[2] As shown in Figure 1, the pricing kernel of the CAPM^2 displays (i) a bump around zero, (ii) a higher level in the negative domain, and (iii) it is decreasing for losses and slightly increasing for gains.

momentum anomalies. In fact, small cap, high book-to-market securities, and momentum winners deliver (i) lower returns when market returns are close to zero (implying higher loadings on Goal Achievement), (ii) lower average returns during market downturns (implying higher loadings on Loss Aversion), and (iii) higher covariance with market returns during market downturns and lower covariance with market returns during rising markets (implying higher loadings on preference for Security/Potential). Moreover, size, value, and momentum factors do not load when they are tested on the CAPM^2, even when tests are conducted on size, value, and momentum portfolios.

Running a wide range of model comparison tests, I find that the CAPM^2 yields a significant improvement with respect to the existing asset pricing models.[3] Moreover, the CAPM^2 is the most parsimonious model that can explain size, value, and momentum anomalies, since the three CAPM^2 factors are all jointly necessary to explain the cross-section of stock returns.

The main predictions of the CAPM^2 are also confirmed out of sample. In fact, zero cost portfolios that take long (short) positions on securities with the highest (lowest) loadings on the three CAPM^2 factors—Goal Achievement, Loss Aversion, and preference for Security/Potential—deliver positive and statistically significant returns, on a risk adjusted basis.

The first CAPM^2 factor, Goal Achievement, quantifies the importance that investors attach to the overall probability of obtaining positive portfolio returns. The theory section of the paper shows that, in contrast with existing models, Continuous Asymmetric Polynomial Models are able to take into account the effect of goal seeking behavior on asset prices. This result is obtained by including a positive constant in

[3] The CAPM^2 nests several existing asset pricing models (such as mean-variance CAPM, the three-moment CAPM, and downside risk models), allowing for model comparison tests.

the upside component of the utility function, and by guaranteeing continuity at zero. In this way, the utility function displays a positive shift around zero. Since marginal utility is high around zero, the CAPM^2 predicts higher risk premiums for securities paying off less when market returns are close to zero. This prediction, confirmed by the data, implies higher risk premiums for securities that do not contribute to increase the overall probability of obtaining positive returns in well-diversified portfolios.

The second CAPM^2 factor is Loss Aversion. Since the CAPM^2 pricing kernel has a higher level in the domain of losses, the model predicts higher risk premiums for securities delivering lower average returns during market downturns.[4] This prediction is consistent with the intuition, extensively studied in the decision theory literature, that agents are more sensitive to losses than to gains of the same amount.

Finally, the third CAPM^2 factor is preference for Security/Potential, i.e., preference for securities that covary less with the market portfolio during market downturns and covary more with the market portfolio during rising markets.[5] Preference for Security/Potential implies preference for positive skewness, a factor already priced by the three-moment CAPM. However, the CAPM^2 can capture preference for positive asymmetry more effectively than the three-moment CAPM. Since stocks are more correlated during market downturns, investing in the market portfolio yields a small reduction in downside risk at the cost of a relatively large reduction in upside potential. Confirming the results of Post, Vliet, and Levy (2008), I find that the three-moment CAPM cannot

[4] Consistent with Loss Aversion, the parameter of the linear term of the CAPM^2 utility function in the negative domain is higher than the parameter of the linear term in the positive domain (both linear terms are positive). Therefore the utility function has a higher slope in the domain of losses, and the pricing kernel has a higher level (a higher constant term) in the negative domain.

[5] The parameter of the quadratic term of the CAPM^2 utility function in the negative domain is negative, and the parameter of the quadratic term in the positive domain is slightly positive. Therefore, the CAPM^2 pricing kernel has a negative linear term in the negative domain (implying downside risk aversion), and a positive linear term in the positive domain (implying preference for upside potential).

rationalize the efficiency of the market portfolio because it cannot predict both strong downside risk aversion and mild preference for upside potential at the same time. The CAPM[2], in contrast, has such flexibility.

The remainder of the paper is organized as follows. In section 2, I examine the theory behind the asset pricing models tested in this study. Section 3 outlines the empirical analysis, including the estimation methods and model comparison tests. Section 4 reports the estimation results, and section 5 concludes.

2 Theory

2.1 The Investment Problem

I analyze investor preferences by testing if asset pricing models defined on asymmetric utility functions can rationalize the efficiency of the market portfolio. To focus my attention on the role of preferences, I adhere to the main assumptions of the standard CAPM. I consider a single period, portfolio-based, representative investor model of a frictionless and competitive capital market that satisfies the following assumptions:

1. The investment universe consists of N risky assets with return $r \in \mathbb{R}^N$, and a risk-free asset with return $r_F \in \mathbb{R}_+$. The returns $r \in \mathbb{R}^N$ are treated as random variables with continuous joint cumulative distribution function $G : \mathbb{R}^N \to [0, 1]$.

2. The representative investor constructs a portfolio by choosing portfolio weights $w \in \mathbb{R}^N$, so as to maximize the expectation of a utility function $u : \mathbb{R} \to \mathbb{R}$, differentiable and strictly increasing. The weight assigned to the risk-free asset is $1 - w' i$, with $i \in \mathbb{R}^N$ vector of ones.

3. The utility function is defined on portfolio returns in excess of the risk-free rate.[6]

These assumptions are common to several works like, for example, Post and Levy (2005), Post, Vliet, and Levy (2008), and Shumway (1997). The investment problem is:

$$\max_{w \in \mathbb{R}^N} E\big(u(r_{w,t+1})\big) \tag{1}$$

with

$$r_{w,t+1} = w'\, r_{t+1} + (1 - w'\, i)\, r_{_{F,t}} - r_{_{F,t}} = w'\, r^e_{t+1} \tag{2}$$

and $r^e_{t+1} = r_{t+1} - i\ r_{_{F,t}}$. The value-weighted market portfolio $w_{_M} \in \mathbb{R}^N$ must represent the optimal solution of the investment problem. The well-known Euler equation gives the First Order Conditions (FOC) for optimality:

$$E(m_{t+1}\, r^e_{t+1}) = 0 \tag{3}$$

where $m_{t+1} = u'(r_{w_M,t+1})$ is the stochastic discount factor, or pricing kernel, that prices all securities under the law of one price and is non-negative under the condition of no arbitrage.[7]

Assumption 3 does not impose global concavity of the utility function (it imposes only differentiability and monotonicity). Therefore, the FOC are no longer sufficient for optimality. In fact, we may wrongly classify a minimum or a local maximum (which also satisfy the FOC) as the global maximum. It follows that we need to verify that the

[6] The risk-free rate represents the natural reference point for the problem at hand. Moreover, unreported estimation results using a reference point equal to $r_{_F} + \tau$ clearly show that $\tau = 0$.

[7] Appendix B provides more details about the efficiency of the market portfolio; the connection of the pricing kernel defined in this paper and consumption-based models; and unconditional and conditional expectations models.

Second Order Conditions (SOC) are satisfied, by checking that the following matrix

$$E\big(u''(r_{w_M,t+1})\; r^e_{t+1}\, r^{e\,\prime}_{t+1}\big) \tag{4}$$

is negative definite. Checking the SOC represents an improvement with respect to several works of the asset pricing literature. As pointed out by Post, Vliet, and Levy (2008), several works of the empirical asset pricing literature estimate models that actually imply local convexity of the utility function, but fail to verify whether local risk-seeking leads to violations of the SOC for optimality. On the contrary, in this paper I explicitly analyze the implications of local convexity of the utility function, since the most recent decision theory models display local risk-seeking. Therefore, during the empirical analysis I verify that the SOC in (4) hold for all models under analysis.

2.2 Continuous Asymmetric Utility Functions

The first contribution of this paper is to show that, in contrast with existing asset pricing models, models based on continuous asymmetric utility functions can price the factors that are necessary to explain risky choice behavior. Continuous asymmetric utility functions can be defined as follows:

$$u(r_{M}) = \tilde{u}_{-}(r_{M})\,\big(1 - F\big) + \tilde{u}_{+}(r_{M})\,F \tag{5}$$

with $r_{M} = r_{w_M,t+1}$, to simplify notation. The utility function is continuously differentiable because it is the weighted sum of two continuous functions: $\tilde{u}_{-}(r_{M})$ reflects the risk attitude in the domain of losses, while $\tilde{u}_{+}(r_{M})$ captures the risk attitude in the domain of gains. The weighting function F is a strictly increasing and continuous function, with values between zero and one. For example, we can choose F as the cumulative distribution function, evaluated at r_M, of a normal random variable with mean zero and variance σ^2. If σ^2 is low enough, F is similar to an indicator function equal to one if $r_M > 0$. As discussed in section 2.3.1, the continuity of the utility function is necessary to price all factors that are studied in the decision making literature.

The utility function in (5) can be rewritten as the sum of a global component (standard utility function applied to both domains of gains and losses, $u_G(r_M)$) and an upside component (applied only to the domain of gains, $u_+(r_M)$):

$$u(r_M) = u_G(r_M) + u_+(r_M)\, F \tag{6}$$

This formulation is equivalent to the weighted sum of upside and downside components,[8] but it has the advantage of nesting standard utility functions, by setting $u_+(r_M) = 0$. Therefore, by testing the significance of $u_+(r_M)$ we can assess the validity, in the asset pricing context, of the asymmetries in utility advocated in several models of the decision theory literature.

2.3 CAPM^2 Utility Function

The CAPM^2 is derived by approximating $u_G(r_M)$ and $u_+(r_M)$ with quadratic polynomials:

$$u(r_M) = \theta_1\, r_M + \theta_2\, r_M^2 + \left(\theta_0^+ + \theta_1^+\, r_M + \theta_2^+\, r_M^2\right) F \tag{7}$$

[8] Models (5) and (6) are equivalent if $\tilde{u}_-(r_M) = u_G(r_M)$ and $\tilde{u}_+(r_M) = u_G(r_M) + u_+(r_M)$:

$$u_G(r_M) + u_+(r_M)\, F = u_G(r_M)\left(1 - F\right) + \left(u_G(r_M) + u_+(r_M)\right) F = \tilde{u}_-(r_M)\left(1 - F\right) + \tilde{u}_+(r_M)\, F.$$

Intuition and preference theory (Piccioni, 2014; Diecidue and Van de Ven, 2008; Levy and Levy, 2002; Tversky and Kahneman, 1992) suggest the following sign restrictions:[9]

$$\theta_1 \geq 0, \qquad \theta_2 \leq 0, \qquad \theta_0^+ \geq 0, \qquad \theta_1^+ \leq 0, \qquad \theta_2^+ \geq 0 \tag{8}$$

Consistent with mainstream asset pricing models, we have standard sign restrictions for the global component of the utility function: the linear term has a positive parameter ($\theta_1 \geq 0$, implying monotonicity),[10] and the quadratic term has a negative parameter ($\theta_2 \leq 0$, implying risk aversion).

2.3.1 CAPM^2 Utility Function: Goal Achievement

The constant term in $u_+(r_M)$ is positive—$\theta_0^+ \geq 0$—in order to comply with goal seeking behavior.[11] In fact, if we include a positive constant in the upside component of the utility function, when we compute expected utility we get:

$$E\big(\theta_0^+ F\big) \approx E\big(\theta_0^+ I^+\big) = \theta_0^+ E\big(I^+\big) = \theta_0^+ P(r_M > 0) \tag{9}$$

where I^+ is an indicator function equal to one if $r_M > 0$. The parameter θ_0^+ quantifies the importance of goal seeking behavior, or Goal Achievement: the importance that investors attach to the overall probability of obtaining positive portfolio returns.

The ability of pricing Goal Achievement represents the main innovation of the

[9] Estimation results obtained without imposing any sign restrictions confirm the parameters signs in (8).

[10] Imposing $\theta_1 \geq 0$ alone is not a sufficient condition for monotonicity. If θ_2 and/or θ_1^+ are sufficiently negative, the utility function may be decreasing in some intervals. Any violation of monotonicity is investigated in the empirical analysis.

[11] Parameter θ_0^+ has to be positive. Both intuition and the results of the empirical decision making literature point toward this direction (e.g., Diecidue and Van de Ven, 2008). Furthermore, besides contradicting goal seeking behavior, a negative θ_0^+ would lead to violations of monotonicity.

CAPM^2. It's important to point out that in order to assess the effect of goal seeking behavior on asset prices, we need to consider continuous utility functions with a positive shift at zero, rather than non-continuous utility functions with a discrete jump. Continuity is required because risk premiums depend upon the covariance between marginal utility and the excess returns of each security (see section 2.4 below). If we were to use an indicator function to distinguish between gains and losses, the constant parameter θ_0^+ would not appear in the FOC (the first derivative of $\theta_0^+ I^+$ is zero), and it would not be possible to price Goal Achievement. If, instead, we use a continuous weighting function F to distinguish between gains and losses, we end up having an additional term equal to $\theta_0^+ f$ in the pricing kernel (where f is the first derivative of the weighting function F). This additional term produces a bump in the stochastic discount factor (as highlighted in Figure 2). As pointed out in section 2.4, since marginal utility is high around zero, the CAPM^2 predicts higher risk premiums for securities that do not contribute to increase the overall probability of obtaining positive returns in well-diversified portfolios.

2.3.2 CAPM^2 Utility Function: Loss Aversion

The parameter of the linear term in $u_+(r_M)$ is negative: $\theta_1^+ \leq 0$. It follows that the CAPM^2 utility function has a higher slope in the domain of losses, and the pricing kernel has a higher level (a higher constant) in the negative domain. As a consequence, the CAPM^2 predicts that investors are willing to sacrifice average portfolio returns in market booms in order to get higher average portfolio returns during market downturns. This prediction is consistent with Loss Aversion (e.g., Kahneman and Tversky, 1979), that is, the observed behavior that agents are more sensitive to losses than to gains of the same amount.

2.3.3 CAPM^2 Utility Function: Preference for Security/Potential

Finally, the parameter of the quadratic term in $u_+(r_M)$ is positive: $\theta_2^+ \geq 0$. The sign of θ_2^+ does not impose any restriction on the shape of the utility function in the positive domain. In fact, the utility function is convex in the positive domain only if θ_2^+ is large enough: $\theta_2^+ > |\theta_2|$.[12] Convexity in the domain of gains would be consistent with preference for upside potential (e.g., Post and Levy, 2005). If, instead, we have $0 < \theta_2^+ < |\theta_2|$, the utility function is concave in both domains of gains and losses, but concavity is stronger in the negative domain (similar to disappointment aversion models). This is consistent with the intuition that investors are, at the least, more averse to downside variance than upside variance. Furthermore, regardless of the relative magnitude of θ_2^+ and θ_2, imposing $\theta_2 \leq 0$ and $\theta_2^+ \geq 0$ is consistent with preference for positive skewness, as discussed in section 2.5.

Note that Prospect Theory value function (Kahneman and Tversky, 1979) would imply opposite sign restrictions on the quadratic terms of (7): $\theta_2 \geq 0$ and $\theta_2^+ \leq 0$, and $\theta_2 + \theta_2^+ \leq 0$. In fact, Prospect Theory value function is convex for losses and concave for gains, implying risk seeking behavior for losses and risk aversion for gains (which in turn implies preference for *negative* skewness). As discussed in section 4.2.3, models with Prospect Theory sign restrictions are rejected.

[12] If θ_2^+ is too large, the SOC are violated. This condition is verified during the empirical analysis.

2.4 CAPM^2 Factors and Risk Premiums

The pricing kernel of (7) is given by:

$$m = \theta_1 + 2\,\theta_2\, r_{_M} + \theta_0^+ f + \theta_1^+ \big(F + r_{_M} f\big) + \theta_2^+ \big(2\, r_{_M} F + r_{_M}^2 f\big) \tag{10}$$

where f is the first derivative of the weighting function F. Rearranging the FOC, we can write the expected excess return of each security i as:

$$E(r_i) = \frac{-cov(m\,,\, r_i)}{E(m)} \tag{11}$$

Therefore, the CAPM^2 implies

$$E(r_i) = \frac{-\theta_0^+ cov(r^+_{_{0,M}}\,,\, r_i) - \theta_1^+ cov(r^+_{_{1,M}}\,,\, r_i) - \theta_2\, cov(r_{_M}\,,\, r_i) - \theta_2^+ cov(r^+_{_{2,M}}\,,\, r_i)}{E(m)} \tag{12}$$

with:

$$r^+_{_{0,M}} = f; \qquad r^+_{_{1,M}} = F + r_{_M} f; \qquad r^+_{_{2,M}} = 2\, r_{_M} F + r_{_M}^2 f \tag{13}$$

As long as parameters signs comply with (8), the CAPM^2 implies higher risk premiums for securities delivering:

(i) *lower* returns when market returns are close to zero (higher loadings on Goal Achievement). In fact:

$$-\theta_0^+ cov(r^+_{_{0,M}}\,,\, r_i) = -\theta_0^+ cov(f\,,\, r_i) \tag{14}$$

Securities that deliver lower returns when the market portfolio returns are close to zero do not contribute to increase the overall probability of obtaining positive returns in well-diversified portfolios. Therefore, investors displaying goal seeking behavior demand higher risk premiums to hold these securities.

(ii) *lower* average returns during market downturns (higher loadings on Loss Aversion). In fact:[13]

$$-\theta_1^+ \, cov(r^+_{1,M} \, , \, r_i) \approx \theta_1^+ \, cov(I^- \, , \, r_i) \tag{15}$$

Since $\theta_1^+ \leq 0$, equation (15) implies higher risk premiums for securities that pay off less, on average, during market downturns.

(iii) *higher* covariance with market returns during market downturns and relatively *lower* covariance with market returns during rising markets (higher loadings on preference for Security/Potential). In fact:[14]

$$cov(-2\,\theta_2\, r_M - \theta_2^+\, r^+_{2,M} \, , \, r_i) \approx -2\,\theta_2\, cov(r_M\, I^-, \, r_i) - 2\,(\theta_2 \, + \, \theta_2^+)\, cov(r_M\, I^+, \, r_i) \tag{16}$$

with $I^- = 1 - I^+$. Since $\theta_2 \leq 0$ and $\theta_2^+ > 0$, the model implies that investors display, at the least, more aversion toward securities with higher downside covariance relative to upside covariance with the market. If we have $\theta_2 + \theta_2^+ > 0$ (as confirmed in the estimation results discussed in section 4.1), the model makes the stronger prediction of preference (therefore, lower risk premiums) for securities with higher covariance with market returns during rising markets.

[13] Since $r^+_{1,M} \approx I^+$, we have:
$-\theta_1^+ \, cov(r^+_{1,M} \, , \, r_i) \approx -\theta_1^+ \, cov(I^+, \, r_i) = -\theta_1^+ \, cov((1 - I^-) \, , \, r_i) = \theta_1^+ \, cov(I^- \, , \, r_i)$.

[14] Since $r^+_{2,M} \approx 2\,\theta_2^+\, r_M\, I^+$, we have:
$cov(-2\,\theta_2\, r_M - \theta_2^+\, r^+_{2,M} \, , \, r_i) \approx cov(-2\,\theta_2\, r_M - 2\,\theta_2^+\, r_M\, I^+, \, r_i) =$
$= cov(-2\,\theta_2\, r_M\, (I^- + I^+) - 2\,\theta_2^+\, r_M\, I^+, \, r_i) = -2\,\theta_2\, cov(r_M\, I^-, \, r_i) - 2\,(\theta_2 + \theta_2^+)\, cov(r_M\, I^+, \, r_i)$.

These predictions are confirmed by the data. In fact, as shown in section 4.5, securities with higher loadings on Goal Achievement, Loss Aversion, and preference for Security/Potential deliver higher average returns. The premiums are statistically significant, on a risk adjusted basis.

2.5 Standard Polynomial Models

Besides allowing to test the predictions of several models of the empirical decision making literature, the CAPM^2 can also nest the main models of the existing asset pricing literature. For example, the standard mean-variance CAPM can be obtained by setting $u_+(r_M) = 0$ in (7). Utility is therefore equal to:

$$u(r_M) = \theta_1 \, r_M + \theta_2 \, r_M^2 \tag{17}$$

The only prediction of the standard mean-variance CAPM is global risk aversion: higher risk premiums for securities delivering higher covariance with market portfolio returns. The three-moment CAPM, instead, can also price preference for positive asymmetry (or skewness), as long as $\theta_3 > 0$ in the following cubic model:

$$u(r_M) = \theta_1 \, r_M + \theta_2 \, r_M^2 + \theta_3 \, r_M^3 \tag{18}$$

The CAPM^2 can nest the three-moment CAPM if we include a cubic term in the global component of the polynomial approximation in (7). In this way, by running model comparison tests between the two models, we can verify whether preference for positive asymmetry is better captured by the three-moment CAPM or by the CAPM^2. Model comparison tests between the three-moment CAPM and the CAPM^2 are necessary because preference for positive asymmetry and preference for Security/Potential are interconnected. In fact, downside risk aversion and preference for upside potential imply preference for positive skewness. However, the results of this study emphasize the importance of breaking preference for positive asymmetry into strong downside risk aversion and mild preference for upside potential (see section 4.2.1). This result can be obtained with asymmetric quadratic models (CAPM^2), but not with cubic utility functions (three-moment CAPM).

Several works of the empirical asset pricing literature find that asset pricing models displaying strong preference for positive asymmetry can fit stock return data better than the standard mean-variance CAPM (e.g., Harvey and Siddique, 2000, and Dittmar, 2002). However, models based on cubic utility functions are not flexible enough to imply strong preference for positive asymmetry without violating the necessary SOC for optimality. For example, Dittmar (2002) finds that admissible pricing kernels (obtained from (18)) display very high values of θ_3, implying a reverse S-shaped utility function with concavity for losses and strong convexity for gains, leading to violations of the SOC for optimality. If global concavity is imposed, instead, models based on higher-order polynomials are no longer admissible for the cross-section of stock returns. In other words, if global concavity is imposed, models based on cubic utility functions (three-moment CAPM) are not able to improve with respect to models based on standard quadratic utility functions (mean-variance CAPM). This result is consistent with Tsiang (1972), who demonstrates that a quadratic function is likely to give a good approximation for any concave utility function over the typical sample range, and that higher-order polynomials are unlikely to improve the fit. The results of the empirical analysis of this paper (section 4.1) also confirm that the three-moment

CAPM cannot improve with respect to the mean-variance CAPM when the SOC for optimality (rather than the more restrictive condition of global concavity of the utility function) are imposed.

The optimality issues of the three-moment CAPM are made clear by Post, Vliet, and Levy (2008), who point out that, since stocks are more correlated during market downturns, investing in the market portfolio yields a small reduction in downside risk at the cost of a relatively large reduction in upside potential. With cubic utility functions, investors with high preference for skewness assign a relatively high weight to upside potential. Therefore, the small reduction in downside risk that they obtain by investing in the market portfolio does not sufficiently compensate them for the large reduction in upside potential. It follows that the market portfolio is optimal only for investors displaying strong downside risk aversion and mild preference for upside potential.

Models based on cubic utility functions (three-moment CAPM) are not flexible enough to predict strong downside risk aversion and mild preference for upside potential at the same time. On the other hand, as shown in section 4.2.1, such flexibility can be achieved with asymmetric quadratic utility functions (CAPM^2).

3 Empirical Analysis

3.1 Estimation Methods

The empirical analysis is focused on testing the CAPM^2 on size, value, and momentum portfolios. In order to test the significance of each CAPM^2 factor, I also study the performance of a broad set of asset pricing models nested by the CAPM^2 (see Table 1).

For every model, we can directly test the implied Capital Asset Pricing Model. Regarding the CAPM[2], the FOC for every security i are:

$$E(\theta_1\, r_i + 2\,\theta_2\, r_{_M}\, r_i + \theta_0^+\, r^+_{_{0,M}}\, r_i + \theta_1^+\, r^+_{_{1,M}}\, r_i + \theta_2^+\, r^+_{_{2,M}}\, r_i) = 0 \tag{19}$$

with

$$r^+_{_{0,M}} = f; \qquad r^+_{_{1,M}} = F + r_{_M}\, f; \qquad r^+_{_{2,M}} = 2\, r_{_M}\, F + r^2_{_M}\, f \tag{20}$$

As shown in Appendix C, rearranging (19) we get:

$$E(\tilde{r}_i) = \beta_i\, E(\tilde{r}_{_M}) \tag{21}$$

with:

$$\tilde{r}_i = \theta_1\, r_i + \theta_1^+\, r^+_{_{1,M}}\, r_i, \qquad \tilde{r}_{_M} = \theta_1\, r_{_M} + \theta_1^+\, r^+_{_{1,M}}\, r_{_M} \tag{22}$$

and

$$\beta_i = \frac{-E(2\,\theta_2\, r_{_M}\, r_i + \theta_0^+\, r^+_{_{0,M}}\, r_i + \theta_2^+\, r^+_{_{2,M}}\, r_i)}{-E(2\,\theta_2\, r_{_M}\, r_{_M} + \theta_0^+\, r^+_{_{0,M}}\, r_{_M} + \theta_2^+\, r^+_{_{2,M}}\, r_{_M})} \tag{23}$$

Equation (21) is defined on $\tilde{r}_i$, rather than r_i, in order to avoid including the left hand side variable in the right hand side (note that $\theta_1^+\, r^+_{_{1,M}}\, r_i \approx r_i$ if $r_{_M} > 0$, since $r^+_{_{1,M}} \approx I^+$). In order to get a formula defined on expected returns (rather than adjusted returns $\tilde{r}_i$), note that:

$$E(\tilde{r}_i) = E(r_i\, \tilde{\theta}_1) = E(r_i)\, E(\tilde{\theta}_1) + cov(r_i\,,\, \tilde{\theta}_1) \tag{24}$$

with $\tilde{\theta}_1 = \theta_1 + \theta_1^+\, r^+_{_{1,M}}$. Therefore, equation (21) can be written as:

$$E(r_i) = \gamma_i + \beta_i\, E(r_{_M}) \tag{25}$$

with

$$\gamma_i = \frac{-cov(r_i\,,\,\tilde{\theta}_1) + \beta_i\,cov(r_{_M}\,,\,\tilde{\theta}_1)}{E(\tilde{\theta}_1)} \tag{26}$$

Note that models not displaying Loss Aversion (i.e., models with $\theta_1^+ = 0$) have $\gamma_i = 0$ in equation (25), or, equivalently, $\tilde{r}_i = r_i$ and $\tilde{r}_{_M} = r_{_M}$ in equation (21). Note also that, since we use unconditional expectations, each model displays constant gammas and betas.

3.2 Generalized Method of Moments estimation

All models under analysis are tested on the following moment conditions:

$$E\begin{pmatrix} r - \gamma - \beta\, r_{_M} \\ m\, R_{_F} - 1 \end{pmatrix} = 0 \tag{27}$$

r is the vector of the excess returns of the N risky assets; γ and β are the vectors of CAPM gammas and betas of the risky assets; $r_{_M}$ is the excess return of the market portfolio; m is the pricing kernel; and $R_{_F}$ is the gross risk-free rate: $R_{_F} = 1 + r_{_F}$.

The last moment condition is included to identify all parameters. Furthermore, Dahlquist and Soderlind (1999) and Farnsworth et al. (1999) find that imposing this restriction on the pricing kernel is important in the context of performance valuation. The risk-free rate moment condition is derived from the FOC, as shown in Appendix D.

Equation (27) forms a set of moment conditions that can be used to test each model via Hansen's (1982) Generalized Method of Moments (GMM). The sample version of (27) is:

$$g_{_T}(\theta) = \frac{1}{T}\sum_{t=1}^{T} u_t = 0 \tag{28}$$

Where T is the number of observations, θ is the vector of parameters, and

$$u_t = \begin{pmatrix} r_t - \gamma - \beta \, r_{M,t} \\ m_t \, R_{F,t} - 1 \end{pmatrix} \tag{29}$$

Equation (29) is a system of $N + 1$ equations, with K parameters. Hansen (1982) shows that a test of model specification can be obtained by minimizing the following quadratic form:

$$J(\theta) = g_T(\theta)' \, W_T(\theta) \, g_T(\theta) \tag{30}$$

where $W_T(\theta)$ is the GMM weighting matrix. Hansen (1982) shows that the efficient weighting matrix is the inverse of the covariance matrix of the moment conditions:

$$W_T(\theta) = Var\big(g_T(\theta)\big)^{-1} \tag{31}$$

The sample orthogonality conditions (28) can be interpreted as the pricing errors obtained using the approximate utility function. If the model provides a good description of the data, then these pricing errors should be close to zero and the minimized value of equation (30) will be small. Hansen (1982) shows that $J_T = T \cdot J(\hat{\theta}_{GMM})$ has a chi-square distribution with degrees of freedom equal to the number of moment conditions minus the number of parameters (in our case, $N + 1 - K$). This statistic is commonly referred to as the "J test" or as the test of the model's "overidentifying restrictions".

The J_T statistic tests the magnitude of the weighted average of the pricing errors. However, using the efficient weighting matrix, there are two ways to get a small value of the J_T statistic: first, generate small pricing errors with a high degree of precision;

second, generate large pricing errors with even larger standard deviations. In fact, a model can achieve a low J_T by simply blowing up the covariance matrix of the moment conditions, rather than reducing the average pricing errors. It follows that when evaluating the descriptive ability of a model, it is important to check the magnitude of the average pricing errors. To this end, for each model, I check the Mean Absolute Average Error (MAAE):

$$MAAE(\hat{\theta}_{GMM}) = \frac{1}{N}\sum_{n=1}^{N}\left|g_{T,n}(\hat{\theta}_{GMM})\right| \tag{32}$$

This metric directly checks the pricing ability of each model and is particularly appealing in the CAPM context, since it gives a measure of the average distance between predicted and realized returns.

Furthermore, to ensure that any improvement in the J_T statistic is not due to volatile pricing errors, I check the R^2 of each moment condition:

$$R_i^2 = 1 - \frac{MSE_i}{TSS_i} \tag{33}$$

Where TSS_i is the Total Sum of Squares of asset i returns, while MSE_i is the Mean Squared Error of asset i. To guarantee that the estimation process does not lead to regions of the parameter space that simply blow up the errors volatility (rather than lowering the average pricing errors), I impose the non-negativity of each R_i^2 as one of the admissibility conditions that all models have to satisfy.[15]

For every model, I verify that the following admissibility conditions are satisfied:

[15] The maximum value that R_i^2 can achieve is 1 (if $MSE_i = 0$). A value of R_i^2 lower than zero implies that residuals are more volatile than the assets under analysis: $MSE_i > TSS_i$.

1. Monotonicity. Each utility function has to be globally increasing (and the corresponding pricing kernel strictly positive). This is a minimum condition that each model has to satisfy. Any violation would imply that investors prefer less to more, leading to violations of the no-arbitrage condition in asset pricing.

2. Second Order Conditions. Since global concavity is not imposed, the SOC for optimality defined in equation (4) have to be verified.

3. Hansen and Jagannathan bounds. The variance of the pricing kernel has to be sufficiently high, in order to satisfy the Hansen and Jagannathan bounds.

4. Non-negative R_i^2. The non-negativity of each R_i^2 guards against parameter estimates that blow up errors volatility rather than improving pricing ability.

3.3 Model Comparison Tests

The CAPM^2 nests all models listed in Table 1. Since each CAPM^2 factor is priced by a specific term of the polynomial approximation of the utility function, we can assess the significance of one or more factors with the following χ^2 difference test:

$$T\, g_{T}(\theta_{R})'\, W_{T}(\theta_{R})\, g_{T}(\theta_{R}) - T\, g_{T}(\theta)'\, W_{T}(\theta_{R})\, g_{T}(\theta) \sim \chi_p^2 \tag{34}$$

with θ_{R} equal to the set of optimal parameters of the restricted model, where the tested parameters are imposed to be equal to zero.

A low value of the difference test implies that the parameters set to zero are not statistically significant (since the J test of the restricted model does not rise much by imposing the restriction). The test statistic is distributed according to a chi-square distribution with p degrees of freedom (p is the number of restrictions). For every comparison, I use the weighting function of the restricted model for both restricted and unrestricted optimizations.

3.4 Data

To test the moment conditions defined in (27), we need proxies for the market portfolio and the risk-free asset. I approximate the market portfolio using the CRSP all-share index, which is the value-weighted average of all common stocks listed on the NYSE, AMEX, and NASDAQ. For the risk-free asset I use the one-month US Treasury bill.

The main results of the paper are obtained by testing each model on 30 portfolios: size, book-to-market, and momentum portfolio deciles. I focus on these assets because they represent the most studied phenomena in the literature (Fama and French, 1992; Jegadeesh and Titman, 1993). Robustness results are obtained using portfolios based on different classifications of size, book-to-market, and momentum: 25 double sorted size and book-to-market portfolios; six double sorted size and book-to-market portfolios, and six double sorted size and momentum portfolios. Furthermore, I also use the 48 industry portfolios (based on the four-digit SIC code classification) to check the results obtained with portfolios built on formation criteria that are independent of size, value, and momentum effects. For all securities under analysis, I use data at monthly frequency from July 1963 to June 2010 (564 observations) obtained from the data library on the homepage of Kenneth French.[16] All portfolios are value weighted.

[16] Micro-cap stocks are excluded. The data set starts in 1963 because the COMPUSTAT data used to construct the benchmark portfolios are biased towards big, historically successful firms for the earlier years (Fama and French, 1992). See http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/datalibrary.html.

4 Results

4.1 Main Results

Table 2 shows the estimation results of the CAPM^2 (denoted as $\text{CAPM}^2_{\text{GA,LA,SP}}$),[17] three-moment CAPM (CAPM_{SK}), and mean-variance CAPM. For each model, the table shows the J test of the "overidentifying restrictions", parameters values (with p-values in parenthesis), and goodness of fit (MAAE). All models are tested on 30 value-weighted portfolios: size, book-to-market, and momentum portfolio deciles.

Results are consistent with the main predictions of the empirical decision making literature: the cross-section of stock returns can be rationalized by asymmetric utility functions, but not by standard utility functions. In fact, the CAPM^2 (derived from the continuous asymmetric quadratic utility function defined in (7)) is admissible, while the mean-variance CAPM (standard quadratic utility) and three-moment CAPM (standard cubic utility) are rejected.[18] In particular, the CAPM^2 displays a superior fit: as shown in Figure 3, the MAAE of the CAPM^2 is 2.73 basis points,[19] as opposed to 20 basis points for both mean-variance CAPM and three-moment CAPM.

Figure 1 shows the utility function and pricing kernel of the CAPM^2. The utility function displays (i) a positive shift at zero, (ii) a higher slope in the domain of losses, and (iii) concavity for losses and mild convexity for gains. The corresponding pricing kernel displays (i) a bump around zero, (ii) a higher level in the domain of losses, and (iii) it is decreasing in the domain of losses and slightly increasing for gains. These

[17] When not specified, the simple notation "CAPM^2" refers to model $\text{CAPM}^2_{\text{GA,LA,SP}}$: the complete version of the Continuous Asymmetric Polynomial Model-Capital Asset Pricing Model, which prices all CAPM^2 factors.

[18] In particular, by imposing the SOC for optimality, the three-moment CAPM cannot improve with respect to the mean-variance CAPM.

[19] One basis point is equal to 0.0001.

three characteristics allow to price the three factors that drive risky choice behavior: (i) Goal Achievement, (ii) Loss Aversion, and (iii) preference for Security/Potential (see Piccioni, 2014).

4.2 CAPM^2 Factors

The CAPM^2 factors (Goal Achievement, preference for Security/Potential, and Loss Aversion) are all jointly necessary to explain the cross section of stock returns. In fact, model comparison tests in Table 5 highlight:

1. The joint significance of all CAPM^2 factors: highly significant χ^2 difference test of 40.70 when all CAPM^2 factors are removed from the CAPM^2, i.e., when $\text{CAPM}^2_{\text{GA,LA,SP}}$ is compared with the mean-variance CAPM;

2. The individual significance of each CAPM^2 factor: highly significant χ^2 difference tests when each CAPM^2 factor is removed from $\text{CAPM}^2_{\text{GA,LA,SP}}$.

Table 3 and Table 4 show that models pricing only one or two CAPM^2 factors are rejected. The only CAPM^2 reduced-from model that is not rejected is the model pricing Loss Aversion and preference for Security/Potential: $\text{CAPM}^2_{\text{LA,SP}}$. However, even though $\text{CAPM}^2_{\text{LA,SP}}$ is admissible, such a model is not able to drive out size, value, and momentum factors. As highlighted in section 4.3, we need all CAPM^2 factors to completely rationalize size, value, and momentum anomalies. Furthermore, we obtain a significant improvement by including Goal Achievement in $\text{CAPM}^2_{\text{LA,SP}}$: highly significant χ^2 comparison test of 22.48 for the $\text{CAPM}^2_{\text{GA,LA,SP}}/\text{CAPM}^2_{\text{LA,SP}}$ comparison, and remarkable improvement in the fit (MAAE of 2.73 basis points for $\text{CAPM}^2_{\text{GA,LA,SP}}$, as opposed to 10.52 basis points for $\text{CAPM}^2_{\text{LA,SP}}$).

4.2.1 Preference for Skewness

As pointed out in the previous section, preference for Security/Potential is one of the factors that are necessary to explain the cross-section of stock returns. Preference for Security/Potential implies preference for positive asymmetry, a factor already priced by the three-moment CAPM. Therefore, it is important to investigate whether preference for positive asymmetry is better captured by approximating utility with cubic polynomials (three-moment CAPM, which implies preference for Skewness, as long as $\theta_3 > 0$) or with asymmetric quadratic polynomials (CAPM^2, which implies preference for Security/Potential, as long as $\theta_2^+ > 0$). In order to do so, we need to run model comparison tests using models that price both preference for Security/Potential and Skewness, or either one of the two factors.

Table 6 shows the results obtained with CAPM$^2_{\text{SK;GA,LA,SP}}$ (which prices both Skewness and preference for Security/Potential), CAPM$^2_{\text{SK;GA,LA}}$ (Skewness), and CAPM$^2_{\text{GA,LA,SP}}$ (preference for Security/Potential). Even though CAPM$^2_{\text{SK;GA,LA}}$ is not rejected, it is clear that preference for positive asymmetry is better captured by preference for Security/Potential rather than Skewness. In fact, the χ^2 difference test for the CAPM$^2_{\text{SK;GA,LA,SP}}$/CAPM$^2_{\text{SK;GA,LA}}$ comparison is 4.41 (preference for Security/Potential is statistically significant after controlling for preference for Skewness), while the χ^2 difference test for the CAPM$^2_{\text{SK;GA,LA,SP}}$/CAPM$^2_{\text{GA,LA,SP}}$ comparison is very close to zero (preference for Skewness is not statistically significant after controlling for preference for Security/Potential). Moreover, the parameter of the cubic term in CAPM$^2_{\text{SK;GA,LA,SP}}$ utility function is non-significant and close to zero, while the parameter of the upside quadratic term is significant.

4.2.2 Downside Risk

Since the CAPM^2 can price preference for Security/Potential (i.e., downside risk aversion and preference for upside potential), this paper is clearly related to the downside risk literature. Several works in the downside risk literature study models that replace variance with downside variance, within the standard mean-variance CAPM framework (e.g., Ang, Chen, and Xing, 2006). This can be obtained by approximating utility with:

$$u(r_{_M}) = \theta_1 \, r_{_M} + \theta_2^- \, r_{_M}^2 \, I^- \tag{35}$$

where I^- is an indicator function equal to one if $r_{_M} < 0$. This utility function (concave for losses and linear for gains) can also be obtained by neutralizing the upside quadratic term in $\mathrm{CAPM}^2_{\mathrm{SP}}$, i.e. by setting $\theta_2^+ = -\theta_2$ in:

$$u(r_{_M}) = \theta_1 \, r_{_M} + \theta_2 \, r_{_M}^2 + \theta_2^+ \, r_{_M}^2 \, F \tag{36}$$

Model (36) is rejected: J test equal to 47.64 with p-value of 0.01, and MAAE of 21 basis points. Similar results are obtained with the other models in Table 1 if we impose $\theta_2^+ = -\theta_2$. Models that price only downside risk aversion cannot explain the cross section of stock returns.

4.2.3 Prospect Theory

Asymmetric quadratic utility functions could be used to test Prospect Theory, the most studied model in the behavioral literature. However, two important considerations have to be made on this regard. First, Prospect Theory value function is convex for losses and concave for gains, implying the following parameter signs for the quadratic terms of the utility function: $\theta_2 > 0$, $\theta_2^+ < 0$, and $\theta_2 + \theta_2^+ < 0$. These sign restrictions would imply preference for downside risk and aversion toward upside potential (and, therefore, preference for negative skewness), a counterintuitive result that is rejected by the data: all models of Table 1 are rejected if we impose Prospect Theory sign restrictions on the quadratic terms. For example, if we impose $\theta_2 \geq 0$ and $\theta_2^+ \leq 0$ in $\text{CAPM}^2_{\text{GA,LA,SP}}$, the J test is equal to 50.32 (p-value of 0.00) and the MAAE is 19.20 basis points. Moreover, the estimation of the CAPM^2 without imposing any sign restriction confirms that: $\theta_2 < 0$, $\theta_2^+ > 0$, and $\theta_2 + \theta_2^+ > 0$.

The second important consideration to be made on Prospect Theory is that a fundamental component of Kahneman and Tversky's model is the probability weighting function, according to which individuals overestimate low probabilities and underestimate higher probabilities. Since the econometric methodology used in this work cannot account for the distortions in expectations that may be produced by the probability weighting function, the full version of Prospect Theory cannot be tested in this paper.

4.3 Size, Value, and Momentum Factors

Table 7 shows the results obtained by including size, value, and momentum factors in the three models of Table 2. Size, value, and momentum factors are priced by adding three terms to the original utility functions: $\theta_{SMB}\, SMB + \theta_{HML}\, HML + \theta_{MOM}\, MOM$; where SMB, HML, MOM are the "Small-Minus-Big", "High-Minus-Low", and "Momentum" factors obtained from the data library on the homepage of Kenneth French. The parameters of size, value, and momentum factors are negative, consistent with a positive premium for securities delivering higher loadings on these factors.

The first two columns of Table 7 clearly show that size, value, and momentum factors significantly improve the performance of both mean-variance CAPM and three-moment CAPM: the two models become admissible and there's a considerable improvement in their fit. Furthermore, the parameters of size, value, and momentum factors are highly significant (jointly and individually). These results are consistent with the extensive literature documenting the importance of size, value, and momentum factors at improving the performance of existing asset pricing models.

The last column of Table 7 shows that size, value, and momentum factors do not load when they are tested on the CAPM^2, even when tests are conducted on size, value, and momentum portfolios. In fact, size, value, and momentum factors' parameters are close to zero and not significant (individually and jointly) when they are included in the CAPM^2. The CAPM^2 is the only model considered in this study that is able to drive out size, value, and momentum factors.

4.4 Robustness Tests

Tables 9, 8, and 10 show the results of several robustness tests, obtained by using different sets of test assets: 25 double sorted size and book-to-market portfolios (Table 8); 12 portfolios given by the aggregation of six double sorted size and book-to-market portfolios and six double sorted size and momentum portfolios (Table 9); and 48 industry portfolios (Table 10).

The test assets of Tables 8 and 9 have been chosen to check the results obtained using different aggregations of size, value, and momentum portfolios. The test assets of Table 10 have been chosen to check the results obtained using portfolio formation criteria independent of size, value, and momentum effects. Lo and MacKinlay (1990) and Conrad, Cooper, and Kaul (2003) argue that many of the empirical regularities observed studying portfolios built on characteristics such as size and value may be overstated due to data snooping. It follows that it is important to check that results are robust to the choice of portfolio formation strategies.

Results of Table 2 are robust to changes in tested assets. In particular, the three CAPM^2 factors are always significant (individually and jointly), and the CAPM^2 utility function has always the same shape: positive shift at zero, higher slope for losses, concavity for losses and convexity for gains. Furthermore, the CAPM^2 is always admissible and it always represents a significant improvement with respect to the other models under analysis. Note that in Table 10 the J test doesn't have enough power to reject any model, but the CAPM^2 represents a significant improvement with respect to the mean-variance CAPM and the three-moment CAPM.

4.5 Portfolios Sorted on the CAPM^2 Factors

Table 11 shows that the main predictions of the CAPM^2 are confirmed out of sample. The Table displays the results obtained with zero cost portfolios that take long (short) positions on securities with the highest (lowest) loadings on the three CAPM^2 factors: Goal Achievement, Loss Aversion, and preference for Security/Potential. Portfolios are built by following a procedure similar to Fama-French (1992): every month, all stocks in the cross section are sorted according to their loadings on each factor (i.e., their covariance with each factor over the previous sixty months). Then, I form ten value weighted portfolios and check their returns over the next month. The procedure is repeated every month. The zero cost portfolios are given by the difference of the highest and lowest deciles of every sorting.

To obtain risk adjusted returns, I consider the "alphas" of the following regression:

$$SORT_{i,t} = \alpha_i + \beta_{MKT,i} MKT_t + \beta_{SMB,i} SMB_t + \beta_{HML,i} HML_t + \beta_{MOM,i} MOM_t + \epsilon_{i,t} \tag{37}$$

Where $SORT_{i,t}$ is the return at time t of the zero cost portfolio i (obtained by sorting on one of the CAPM^2 factors), MKT_t is the return of the market portfolio at time t, SMB_t is the size factor at time t, HML_t is the value factor at time t, and MOM_t is the momentum factor at time t. Standard errors (reported in parenthesis in table 11) are obtained using Newey-West autocorrelation consistent estimation.

Goal Achievement portfolios are obtained by sorting stocks according to their covariance with $r^+_{0,M} = f$, as defined in (13). The lowest decile is formed by stocks with the highest covariance with $r^+_{0,M}$ (lowest loadings on Goal Achievement). In other words, the lowest decile consists of the securities that pay off the most when the market portfolio delivers low returns in absolute value. The CAPM^2 predicts that these securities should deliver lower average returns, since they contribute the most to increase the overall probability of obtaining positive returns in well diversified portfolios. The prediction is confirmed by the data. In fact, the average monthly return of the difference between the highest decile (highest loadings on Goal Achievement) and the lowest decile (lowest loadings on Goal Achievement) is 0.36% and highly significant.

Loss Aversion portfolios are obtained by sorting stocks according to their covariance with $r^+_{1,M} \approx I^+$. The lowest decile is formed by stocks with the lowest covariance with $r^+_{1,M}$ (lowest loadings on Loss Aversion). In other words, lowest decile consists of securities that delivers the lowest average returns when the market rises, and the highest average returns during market downturns. Since the CAPM^2 implies higher sensitivity to losses than gains of the same amount, the model predicts a positive premium for the difference between the highest and lowest decile, as confirmed by the data: monthly premium of 0.75%, highly significant.

Security/Potential portfolios are obtained by sorting stocks according to the difference of their covariances with r_M and $r^+_{2,M} \approx r_M I^+$. The lowest decile contains the stocks with the lowest difference of the two covariances (lowest loadings on Security/Potential).[20] In other words, the lowest decile contains the securities with the lowest downside risk relative to upside potential. As confirmed by the data, the CAPM^2 predicts lower returns for these securities: positive and statistically significant premium of 1.22% for the difference between the highest and lowest decile.

Finally, Table 11 reports the risk-adjusted returns of the CAPM^2 factors. After controlling for the market portfolio, and size, value, and momentum factors, the alphas of the CAPM^2 zero cost portfolios are generally positive and significant. Two points are worth mentioning. First, the Goal Achievement factor loses significance after controlling for size, value, and momentum, while Loss Aversion loses significance after controlling for the market portfolio. Second, surprisingly, Loss Aversion and Security/Potential factors increase their significance after size, value, and momentum factors are included. This is because of the negative correlation of Loss Aversion and Security/Potential factors with value and momentum factors.

[20] Similar results are obtained if we sort stock according to the difference of their covariances with $r^-_{2,M} = 2\, r_M\, (1-F) - r^2_M\, f \approx r_M\, I^-$ and $r^+_{2,M} = 2\, r_M\, F + r^2_M\, f \approx r_M\, I^+$.

5 Conclusions

This paper shows that models providing more realistic assumptions on how people face risk can improve our understanding of the cross-section of stock returns. Specifically, I find that size, value, and momentum anomalies are rationalized by a Capital Asset Pricing Model based on Continuous Asymmetric Polynomial Models (CAPM^2). The estimated CAPM^2 utility function is in line with the results of recent works of the decision making literature, which show that risky choice behavior can be rationalized by a utility function with (i) a positive shift at zero, (ii) a higher slope in the negative domain, and (iii) concavity for losses and mild convexity for gains. Piccioni (2014) points out that such a utility function can capture the key factors that are necessary to explain risky choice behavior: (i) Goal Achievement (the importance of the overall probability of obtaining positive payoffs). (ii) Loss Aversion (losses loom larger than gains), and (iii) preference for Security/Potential (downside risk aversion and preference for upside potential). The results of this work show that Goal Achievement, Loss Aversion, and preference for Security/Potential are also the key factors driving the cross section of stock returns.

The first CAPM^2 factor, Goal Achievement, is priced because the utility function of the CAPM^2 displays a positive shift at zero. Consistent with the implications of recent decision theory models, I find higher risk premiums for securities that do not contribute to increase the overall probability of obtaining positive returns in a well diversified portfolio. Moreover, the theory section of the paper shows that only Continuous Asymmetric Polynomial Models are able to take in to account the effect of goal seeking behavior on asset prices.

The second CAPM^2 factor is Loss Aversion. Because of Loss Aversion, the CAPM^2

predicts higher risk premiums for securities delivering lower average returns during market downturns. Consistent with the data on size, value, and momentum portfolios, the CAPM^2 predicts higher risk premiums for securities paying off less, on average, when investors need it the most.

The third CAPM^2 factor, preference for Security/Potential, is related to preference for positive asymmetry, a factor already captured by the three-moment CAPM. However, this study presents clear evidence that the CAPM^2 can capture preference for positive asymmetry more effectively than the three-moment CAPM. In particular, I find that the CAPM^2 can imply strong preference for positive asymmetry (which is necessary to explain the cross section of stock returns) without violating the SOC for optimality. This result is obtained because the CAPM^2 can break preference for positive asymmetry into strong downside risk aversion and mild preference for upside potential. The three-moment CAPM, in contrast, does not have such flexibility.

Size, value, and moment factors do not load when all CAPM^2 factors are taken into account. Running a wide range of model comparison tests, I find that the CAPM^2 is the most parsimonious model that can explain size, value, and momentum anomalies. In fact, the three CAPM^2 factors are all jointly necessary to explain the cross-section of stock returns. These results are consistent with Piccioni (2014), who shows that the same three factors are all jointly necessary to explain risky choice behavior.

Finally, the main predictions of the model are confirmed out of sample. In fact, zero cost portfolios with long (short) positions on securities with higher (lower) loadings on the three CAPM^2 factors (Goal Achievement, Loss Aversion, and preference for Security/Potential) deliver positive and statistically significant risk adjusted returns.

This paper can be extended in several directions. First, further research is needed to develop conditional models that are able to incorporate time varying preferences within the asymmetric utility framework. Furthermore, a deeper analysis is needed to distinguish between the CAPM^2 factors and other cross-sectional effects. For example, liquidity effects and downside risk are interconnected, because liquidity typically dries up when the largest market losses occur and, in turn, liquidity dry-up may cause or amplify these losses. It follows that an approach that is able to disentangle liquidity and downside risk is needed.

A Decision Theory: Target Utility Theory

Piccioni (2014) develops a model for risky choice behavior, called Target Utility Theory (TUT), that rationalizes several puzzles of the empirical decision making literature.

The utility function of Target Utility Theory (TUT) is given by the sum of two components: a value function that captures the "pure" utility of each payoff, and an anticipatory feeling component that captures the expected regret/rejoice. More specifically, the utility of a generic lottery X, when it's compared with another lottery Y, is defined as follows:

$$\begin{aligned} U(X_Y) = &\ \textstyle\sum_{i=1}^{N} p_i\, F^{-}_{x_i} \big(v(x_i) - \textstyle\sum_{j=1}^{J} p_j\, F^{+}_{y_j}\, r(y_j)\big) \\ + &\ \textstyle\sum_{i=1}^{N} p_i\, F^{+}_{x_i} \big(v(x_i) - \textstyle\sum_{j=1}^{J} p_j\, F^{-}_{y_j}\, r(y_j)\big) \end{aligned} \tag{38}$$

Value—$v(x_i)$—and regret/rejoice—$r(y_j)$—functions are given by:

$$v(x_i) = \begin{cases} -\lambda\,(-x_i)^{\psi} & if \quad x_i \leq 0 \\ x_i^{\psi} & if \quad x_i > 0 \end{cases} \qquad r(y_j) = \begin{cases} -\lambda\,(-y_j)^{\gamma} & if \quad y_j \leq 0 \\ y_j^{\gamma} & if \quad y_j > 0 \end{cases} \tag{39}$$

The weighting functions F^{+} and F^{-} identify gains and losses, respectively, and satisfy $F^{-} = 1 - F^{+}$. Functions F^{+} and F^{-} are equal to the c.d.f. of a normal distribution. In this way, the weighting functions are similar to indicator functions, but continuous around zero.

The value function is concave for losses and convex for gains, implying preference for security/potential (risk aversion for losses and risk seeking behavior for gains). The regret/rejoice function, instead, creates a jump in utility at the reference point, implying goal seeking behavior (the desire of achieving relevant aspiration levels). Finally, the loss aversion parameter λ takes into account the higher sensitivity to losses than gains of the same amount. It follows that the three parameters of the model—ψ , γ , λ—are all greater than one.[21] Each parameter is associated with one of the factors captured by the model: ψ controls for preference for security/potential, γ controls for goal seeking behavior, and λ controls for loss aversion.

Regret is defined as the negative feeling associated with the ex-post knowledge that a different past decision would have given a positive payoff, instead of the non-positive payoff obtained with the chosen prospect. Regret depends upon the positive outcomes of the alternative prospect, and it is experienced only when the decision maker fails to achieve the target return. Therefore, regret creates a negative shift in utility in the

[21] The regret/rejoice function is assumed to be concave for losses and convex for gains, like the value function. The spirit of the model is that the impact of any outcome increases at an increasing rate (higher marginal impact of outcomes of higher magnitude).

negative domain, equal to $-\sum_{j=1}^{J} p_j \, F_{y_j}^{+} \, r(y_j)$. Rejoice, instead, is associated with the avoidance of a loss, and it creates a positive shift in utility in the positive domain, equal to $-\sum_{j=1}^{J} p_j \, F_{y_j}^{-} \, r(y_j)$.

Because of the regret/rejoice component, the overall probabilities of realizing gains and losses play an important role in the decision making process. Since the model penalizes prospects that deliver a low probability of achieving the target return (i.e., a low probability of realizing gains), TUT implies goal seeking behavior.

To compare the performance of TUT with several models of the decision making literature, Piccioni (2014) uses a logit model to run maximum likelihood estimations over a wide range of results of the empirical decision making literature. The estimation results show that TUT represents a significant improvement with respect to: Expected Utility Theory, Original Prospect Theory (Kahneman and Tversky, 1979), Cumulative Prospect Theory (Tversky and Kahneman, 1992), SP/A theory (Lopes, 1987), Regret Theory (Loomes and Sugden, 1982), Disappointment Aversion (Gul, 1991), and Expected Utility Theory with jumps (Diecidue and Van de Ven, 2008). Furthermore, several comparison tests show that all factors implied by TUT (goal seeking behavior, loss aversion, and preference for security/potential) are important to rationalize the phenomena of the empirical decision making literature.

The utility function of TUT can be approximated with the continuous asymmetric quadratic model in (7). Note that TUT displays a negative shift in the negative domain and a positive shift in the domain of gains. However, in the context of asset pricing models estimation, only the overall difference in levels in the positive and negative domains of the utility function can be assessed, and not the precise extent of positive and negative shifts. This does not create a problem in the present work, because only the overall difference in levels in the positive and negative domains is important to determine investors' preferences. In fact, preferences are not affected by positive or negative shifts affecting the utility function as a whole. It follows that θ_0^+ contains enough information for the problem at hand.

B The Optimization Process

B.1 Efficiency of the Market Portfolio

The only two conditions imposed on the utility functions studied in this paper are differentiability and monotonicity. The use of non-globally concave utility functions may raise doubts about the efficiency of the market portfolio. However, Khanna and Kulldorff (1999) found that the mutual fund theorem "... holds for all investors, regardless of their attitude toward risk, as long as they do not prefer less to more [i.e., monotonicity]. This includes investors who are risk seekers as well as those with a combination of risk seeking and risk averse preferences" (p. 168). Moreover, Vanden (2006) finds that preferences aggregate in an economy where heterogeneous agents display utility functions with several switching points, i.e., wealth levels at which the parameters of the utility function switch to a different set of values.

Nevertheless, as pointed out by Post and Levy (2005), given the large number of investors who appear to hold the market portfolio in the form of passive mutual funds and exchange traded funds that track broad value-weighted equity indexes, it is interesting to ask what kind of utility functions could rationalize such behavior, in the face of attractive premiums offered by size, book-to-market, and momentum portfolios—the so-called "revealed preferences" approach.

B.2 Approximation of the Stochastic Discount Factor

The pricing kernel defined in equation (3) is the first derivative of the utility function of the representative agent. Even though the utility functions studied in this paper are directly defined over market portfolio returns in excess of the risk-free rate, their pricing kernels can be related to the pricing kernels studied in consumption-based models, which are equal to the marginal rate of substitution: $U'(C_{t+1})/U'(C_t)$, with C_t consumption at time t. Brown and Gibbons (1985) show the conditions under which consumption and wealth are equivalent and the marginal rate of substitution can be expressed as a function of aggregate wealth: $U'(W_{t+1})/U'(W_t)$. Several works, like Dittmar (2002) and Harvey and Siddique (2000), run Taylor approximations of $U'(W_{t+1})$, deriving a pricing kernel that is a polynomial function of the market portfolio return. The approximation of the pricing kernel in these works is equivalent to the approximation used in section 2.1.

B.3 Conditional and Unconditional Expectations

In this work, each model is estimated using unconditional expectations. However, there is a wealth of evidence that the risk/return characteristics of most securities show structural and cyclical variation, justifying the use of conditional models. The problem with conditional models is that they entail a large risk of specification error, because they have to specify how each aspect of investor preferences depends on the state of the world. As Ghysels (1998) points out, "... if the beta risk [in the capital asset pricing model with time-varying beta] is inherently misspecified, there is a real possibility that we commit serious pricing errors, potentially larger than with a constant traditional beta model" (p. 550). Ghysels finds that pricing errors of the unconditional CAPM are smaller than those of the conditional CAPM. In addition, Post and Vliet (2006) point out that the problem of imposing the regularity conditions is very severe, because we have to make sure that the utility function is well-behaved for all possible states of the world. The development of conditional models based on asymmetric utility functions with time-varying preferences is beyond the scope of the present work, and it is left for future research.

C CAPM Derivation

The derivation of the Capital Asset Pricing Model from the First Order Conditions in (3) is quite straightforward. Regarding the CAPM[2], for every security i, we have:

$$E(\theta_1 \, r_i + 2\,\theta_2 \, r_M \, r_i + \theta_0^+ \, r_{0,M}^+ \, r_i + \theta_1^+ \, r_{1,M}^+ \, r_i + \theta_2^+ \, r_{2,M}^+ \, r_i) = 0 \tag{40}$$

with

$$r_{0,M}^+ = f; \qquad r_{1,M}^+ = F + r_M \, f; \qquad r_{2,M}^+ = 2\, r_M \, F + r_M^2 \, f \tag{41}$$

We can rewrite the FOC as:

$$E(\tilde{r}_i) = -E(2\,\theta_2 \, r_M \, r_i + \theta_0^+ \, r_{0,M}^+ \, r_i + \theta_2^+ \, r_{2,M}^+ \, r_i) \tag{42}$$

with

$$\tilde{r}_i = \theta_1 \, r_i + \theta_1^+ \, r_{1,M}^+ \, r_i = r_i \,(\theta_1 + \theta_1^+ \, r_{1,M}^+) = r_i \, \tilde{\theta}_1 \tag{43}$$

We keep $\tilde{r}_i$ on the left hand side (rather than r_i) to avoid including the left hand side variable in the right hand side (note that $\theta_1^+ \, r_{1,M}^+ \, r_i \approx r_i$ if $r_M > 0$, since $r_{1,M}^+ \approx I^+$).

The same condition as in (42) holds for the market:

$$E(\tilde{r}_{_M}) = -E(2\,\theta_2\, r_{_M}\, r_{_M} + \theta_0^+\, r^+_{_{0,M}}\, r_{_M} + \theta_2^+\, r^+_{_{2,M}}\, r_{_M}) \tag{44}$$

Therefore, for every security i, we have:

$$E(\tilde{r}_i) = \beta_i\, E(\tilde{r}_{_M}) \tag{45}$$

with:

$$\tilde{r}_i = \theta_1\, r_i + \theta_1^+\, r^+_{_{1,M}}\, r_i, \qquad \tilde{r}_{_M} = \theta_1\, r_{_M} + \theta_1^+\, r^+_{_{1,M}}\, r_{_M} \tag{46}$$

and

$$\beta_i = \frac{-E(2\,\theta_2\, r_{_M}\, r_i + \theta_0^+\, r^+_{_{0,M}}\, r_i + \theta_2^+\, r^+_{_{2,M}}\, r_i)}{-E(2\,\theta_2\, r_{_M}\, r_{_M} + \theta_0^+\, r^+_{_{0,M}}\, r_{_M} + \theta_2^+\, r^+_{_{2,M}}\, r_{_M})} \tag{47}$$

Moreover, since

$$E(\tilde{r}_i) = E(r_i\, \tilde{\theta}_1) = E(r_i)\, E(\tilde{\theta}_1) + cov(r_i\,,\, \tilde{\theta}_1) \tag{48}$$

with $\tilde{\theta}_1 = \theta_1 + \theta_1^+\, r^+_{_{1,M}}$, we have:

$$E(r_i) = \gamma_i + \beta_i\, E(r_{_M}) \tag{49}$$

with

$$\gamma_i = \frac{-cov(r_i\,,\, \tilde{\theta}_1) + \beta_i\, cov(r_{_M}\,,\, \tilde{\theta}_1)}{E(\tilde{\theta}_1)} \tag{50}$$

D The Risk-Free Rate Moment Condition

The FOC defined in (3) holds for all assets. Then, for each security i we have:

$$E(m\, r_i^e) = 0 \quad \Rightarrow \quad E(m\, R_i) = E(m\, R_{_F}) = \phi \tag{51}$$

with ϕ positive (since R_i, $R_{_F}$, and m are positive) and constant (since we have unconditional expectations). Therefore, the pricing kernel estimated in this work is the scaled pricing kernel $\widetilde{m}$ that satisfies:

$$E(\widetilde{m}\, r_i^e) = 0, \quad E(\widetilde{m}\, R_i) = 1, \quad E(\widetilde{m}\, R_{_F}) = 1 \tag{52}$$

with $\widetilde{m} = m/\phi$. Since ϕ is positive and constant, all considerations made on the utility function of the CAPM[2] are still valid.

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Table 1: Models under Analysis

	Utility					Factors		
	Global Polynomial		Upside Polynomial					
	linear	quadratic	constant	linear	quadratic	Goal Achievement	Loss Aversion	Security/Potential
$\text{CAPM}^2_{\text{GA,LA,SP}}$	✓	✓	✓	✓	✓	✓	✓	✓
$\text{CAPM}^2_{\text{GA,LA}}$	✓	✓	✓	✓		✓	✓	
$\text{CAPM}^2_{\text{GA,SP}}$	✓	✓	✓		✓	✓		✓
$\text{CAPM}^2_{\text{LA,SP}}$	✓	✓		✓	✓		✓	✓
$\text{CAPM}^2_{\text{GA}}$	✓	✓	✓			✓		
$\text{CAPM}^2_{\text{LA}}$	✓	✓		✓			✓	
$\text{CAPM}^2_{\text{SP}}$	✓	✓			✓			✓
CAPM	✓	✓						

Table 1 provides a partial list of the models studied in this paper. As discussed in section 2.2, the utility function of each model is given by the sum of a standard polynomial expansion applied to both domains of gains and losses ("Global Polynomial"), and a polynomial expansion applied only to the domain of gains ("Upside Polynomial").
The table specifies the terms of the polynomial approximation that are included in each model, and which factors are priced as a consequence. "GA" refers to models pricing Goal Achievement; "LA" refers to models pricing Loss Aversion; "SP" refers to models pricing preference for Security/Potential.
Models that can price preference for positive Skewness—denoted with "SK"—are discussed in sections 2.5 and 4.2.1. Models that price size, value, and momentum factors are shown in Table 7 and discussed in section 4.3.

Table 2: Testing the CAPM^2 on Size, Value, and Momentum Portfolios

	CAPM	CAPM_{SK}	$\text{CAPM}^2_{\text{GA,LA,SP}}$
J test	48.45	46.59	7.75
	(0.01)	(0.02)	(0.99)
MAAE ($\times 10^{-4}$)	19.80	19.82	2.73
# parameters	2	3	6
Linear: θ_1	1.04	0.99	0.27
	(0.00)	(0.00)	(0.00)
Quadratic: θ_2	–8.62	–5.51	–14.36
	(0.00)	(0.00)	(0.00)
Cubic: θ_3		8.68	
		(0.99)	
Upside Constant: θ_0^+			0.04
			(0.00)
Upside Linear: θ_1^+			–0.18
			(0.00)
Upside Quadratic: θ_2^+			16.53
			(0.00)
Weighting: σ			0.01
			(0.00)

The table shows the results of the GMM estimation of the mean-variance CAPM, three-moment CAPM (CAPM_{SK}, which considers Skewness in addition to mean and variance), and the CAPM^2 ($\text{CAPM}^2_{\text{GA,LA,SP}}$: complete version of the CAPM^2, which prices Goal Achievement, Loss Aversion, and preference for Security/Potential).
Each model is tested on 30 value-weighted portfolios given by the aggregation of size, book-to-market, and momentum portfolio deciles. Data are at monthly frequency over the period July 1963-June 2010 (564 observations).
The table shows parameter estimates and the J test of the models' over-identifying restrictions (p-values are in parentheses), as well as the measure of each model's fit—the Mean Absolute Average Error—defined in equation (32) in section 3.2.

Moment conditions and econometric methodology are defined in section 3.2. The GMM weighting matrix is the efficient weighting matrix, equal to the inverse of the covariance matrix of the moment conditions (obtained with Heteroskedasticity and Autocorrelatation Consistent (HAC) estimation). The optimization process is run with the Continuously Updated GMM procedure.
All models satisfy the admissibility conditions defined in section 3.2 (monotonicity, Second Order Conditions for optimality, non-negative R^2, and Hansen-Jagannathan bounds), except the mean-variance CAPM that does not satisfy monotonicity.

Table 3: Models with a Single CAPM^2 Factor

	CAPM$^2_{\text{GA}}$	CAPM$^2_{\text{LA}}$	CAPM$^2_{\text{SP}}$
J test	48.45	46.28	46.40
	(0.01)	(0.01)	(0.01)
MAAE ($\times 10^{-4}$)	19.80	18.91	20.76
# parameters	4	4	4
Linear: θ_1	1.04	1.14	0.57
	(0.00)	(0.00)	(0.00)
Quadratic: θ_2	–8.33	–4.15	–17.05
	(0.00)	(0.00)	(0.00)
Cubic: θ_3			
Upside Constant: θ_0^+	0.01		
	(0.99)		
Upside Linear: θ_1^+		–0.23	
		(0.97)	
Upside Quadratic: θ_2^+			18.25
			(0.99)
Weighting: σ	0.00	0.01	0.00
	(0.99)	(0.99)	(0.99)

The table shows the results obtained by testing models that price only one CAPM^2 factor: CAPM$^2_{\text{GA}}$ prices only Goal Achievement; CAPM$^2_{\text{LA}}$ prices only Loss Aversion; CAPM$^2_{\text{SP}}$ prices only preference for Security/Potential. The table shows the results of the GMM estimation of the moment conditions defined in (27). Test assets and econometric methodology are the same as in Table 2.
All models satisfy the admissibility conditions defined in section 3.2 (monotonicity, Second Order Conditions for optimality, non-negative R^2, and Hansen-Jagannathan bounds), except CAPM$^2_{\text{GA}}$ that does not satisfy monotonicity.

Table 4: Models with Two CAPM^2 Factors

	$\text{CAPM}^2_{\text{GA,LA}}$	$\text{CAPM}^2_{\text{GA,SP}}$	$\text{CAPM}^2_{\text{LA,SP}}$
J test	46.28	46.40	30.23
	(0.01)	(0.01)	(0.26)
MAAE ($\times 10^{-4}$)	18.91	20.76	10.52
# parameters	5	5	5
Linear: θ_1	1.14	0.57	1.11
	(0.00)	(0.00)	(0.00)
Quadratic: θ_2	–4.16	–17.05	–13.58
	(0.00)	(0.00)	(0.00)
Cubic: θ_3			
Upside Constant: θ_0^+	0.00	0.00	
	(0.98)	(0.99)	
Upside Linear: θ_1^+	–0.23		–0.88
	(0.90)		(0.00)
Upside Quadratic: θ_2^+		18.25	16.08
		(0.99)	(0.00)
Weighting: σ	0.01	0.00	0.01
	(0.98)	(0.99)	(0.23)

The table shows the results obtained by testing models that price two CAPM^2 factors: $\text{CAPM}^2_{\text{GA,LA}}$ prices Goal Achievement and Loss Aversion; $\text{CAPM}^2_{\text{GA,SP}}$ prices Goal Achievement and preference for Security/Potential; $\text{CAPM}^2_{\text{LA,SP}}$ prices Loss Aversion and preference for Security/Potential. The table shows the results of the GMM estimation of the moment conditions defined in (27). Test assets and econometric methodology are the same as in Table 2.

All models satisfy the admissibility conditions defined in section 3.2 (monotonicity, Second Order Conditions for optimality, non-negative R^2, and Hansen-Jagannathan bounds).

Table 5: Model Comparison Tests

	$CAPM^2_{GA,LA,SP}$	$CAPM^2_{GA,LA}$	$CAPM^2_{GA,SP}$	$CAPM^2_{LA,SP}$	$CAPM^2_{GA}$	$CAPM^2_{LA}$	$CAPM^2_{SP}$	CAPM
$CAPM^2_{GA,LA,SP}$	–							
$CAPM^2_{GA,LA}$	38.53 (0.00)	–						
$CAPM^2_{GA,SP}$	38.65 (0.00)	–	–					
$CAPM^2_{LA,SP}$	22.48 (0.00)	–	–	–				
$CAPM^2_{GA}$	40.70 (0.00)	2.17 (0.14)	2.04 (0.15)	–	–			
$CAPM^2_{LA}$	38.53 (0.00)	0.00 (0.96)	–	16.05 (0.00)	–	–		
$CAPM^2_{SP}$	38.65 (0.00)	–	0.00 (0.99)	16.17 (0.00)	–	–	–	
CAPM	40.70 (0.00)	2.17 (0.34)	2.04 (0.36)	18.21 (0.00)	0.00 (0.99)	2.16 (0.14)	2.04 (0.15)	–

The table shows the χ^2 difference tests for model comparison (with p-values in parentheses) defined in equation (34) in section 3.3. The test is given by the difference between the J test of restricted (rows) and unrestricted (columns) models. Note that the comparison test can be obtained only when the unrestricted model (columns) nests the restricted model (rows). A low value of the difference test implies that the parameters set to zero are not statistically significant (since the J test of the restricted model does not rise much).

Table 6: Models with Preference for Skewness and/or Security/Potential

	$\text{CAPM}^2_{\text{GA,LA,SP}}$	$\text{CAPM}^2_{\text{SK;GA,LA}}$	$\text{CAPM}^2_{\text{SK;GA,LA,SP}}$
J test	7.75	12.16	7.75
	(0.99)	(0.98)	(0.99)
MAAE ($\times 10^{-4}$)	2.73	3.31	2.73
# parameters	6	6	7
Linear: θ_1	0.27	1.99	0.18
	(0.00)	(0.00)	(0.00)
Quadratic: θ_2	–14.36	–2.33	–15.31
	(0.00)	(0.00)	(0.00)
Cubic: θ_3		24.50	0.00
		(0.00)	(0.99)
Upside Constant: θ_0^+	0.04	0.02	0.04
	(0.00)	(0.00)	(0.00)
Upside Linear: θ_1^+	–0.18	–0.72	–0.12
	(0.00)	(0.00)	(0.00)
Upside Quadratic: θ_2^+	16.53		17.63
	(0.00)		(0.04)
Weighting: σ	0.01	0.01	0.01
	(0.00)	(0.00)	(0.00)

The table shows the estimation results of models that can price both preference for Security/Potential and Skewness ($\text{CAPM}^2_{\text{SK;GA,LA,SP}}$), or either one of the two factors ($\text{CAPM}^2_{\text{GA,LA,SP}}$ and $\text{CAPM}^2_{\text{SK;GA,LA}}$). These models are studied to test whether preference for positive asymmetry is better captured by approximating utility with higher order polynomials (three-moment CAPM: Skewness) or with asymmetric polynomials (CAPM^2: preference for Security/Potential).
The table shows the results of the GMM estimation of the moment conditions defined in (27). Test assets and econometric methodology are the same as in Table 2.
All models satisfy the admissibility conditions defined in section 3.2 (monotonicity, Second Order Conditions for optimality, non-negative R^2, and Hansen-Jagannathan bounds).

Table 7: Models including Size, Value, and Momentum Factors

	CAPM	$CAPM_{SK}$	$CAPM^2_{GA,LA,SP}$
J test	37.83	35.81	7.08
	(0.06)	(0.07)	(0.99)
MAAE ($\times 10^{-4}$)	9.31	7.06	2.73
# parameters	5	6	9
Linear: θ_1	1.07	0.89	0.34
	(0.00)	(0.00)	(0.01)
Quadratic: θ_2	–5.29	–9.04	–12.96
	(0.00)	(0.00)	(0.00)
Cubic: θ_3		69.80	
		(0.18)	
Upside Constant: θ_0^+			0.03
			(0.00)
Upside Linear: θ_1^+			–0.21
			(0.00)
Upside Quadratic: θ_2^+			15.88
			(0.00)
Weighting: σ			0.01
			(0.00)
Size	–3.55	–2.21	–0.87
	(0.01)	(0.00)	(0.77)
Value	–4.55	–7.09	–0.00
	(0.00)	(0.00)	(0.99)
Momentum	–2.47	–3.12	–0.00
	(0.00)	(0.01)	(0.99)
Joint Significance of Size, Value, and Momentum Factors			
χ^2 difference test	12.64	10.78	0.61
	(0.01)	(0.01)	(0.89)

The table shows the estimation results of models that are extended to include size, value, and momentum factors. The table shows the results of the GMM estimation of moment conditions defined in (27). Test assets and econometric methodology are the same as in Table 2.
The table also shows the results of the χ^2 difference test for the joint significance of size, value, and momentum factors (with p-values in parentheses). The difference tests are obtained by comparing the J test values of table 2 with the the J test values of table 7. A low value of the difference test implies that size, value, and momentum factors are not jointly significant. The test has three degrees of freedom.
All models satisfy the admissibility conditions defined in section 3.2 (monotonicity, Second Order Conditions for optimality, non-negative R^2, and Hansen-Jagannathan bounds), except the mean-variance CAPM that does not satisfy monotonicity.

Table 8: Robustness, Testing the CAPM^2 on 25 Size and Book-to-Market Portfolios

	CAPM	CAPM_{SK}	$\text{CAPM}^2_{\text{GA,LA,SP}}$
J test	49.53	46.93	7.96
	(0.00)	(0.00)	(0.99)
MAAE ($\times 10^{-4}$)	29.16	29.26	3.69
# parameters	2	3	6
Linear: θ_1	1.06	0.99	0.63
	(0.00)	(0.00)	(0.00)
Quadratic: θ_2	–9.60	–5.90	–9.40
	(0.00)	(0.00)	(0.00)
Cubic: θ_3		8.30	
		(0.99)	
Upside Constant: θ_0^+			0.02
			(0.00)
Upside Linear: θ_1^+			–0.38
			(0.00)
Upside Quadratic: θ_2^+			12.42
			(0.00)
Weighting: σ			0.01
			(0.00)

The table shows the results of the GMM estimation of moment conditions defined in (27). Econometric methodology and models under analysis are the same as in Table 2. The test assets are the 25 double sorted Size and Book-to-Market (value-weighted) portfolios. Data are at monthly frequency over the period July 1963-June 2010 (564 observations). All models satisfy the admissibility conditions defined in section 3.2 (monotonicity, Second Order Conditions for optimality, nonnegative R^2, and Hansen-Jagannathan bounds), except the standard mean-variance CAPM that doesn't satisfy monotonicity.

Table 9: Robustness, Testing the CAPM^2 on 12 Portfolios (Six Size and Book-to-Market, Six Size and Momentum)

	CAPM	CAPM_{SK}	$\text{CAPM}^2_{\text{GA,LA,SP}}$
J test	70.87	55.42	6.81
	(0.00)	(0.00)	(0.45)
MAAE ($\times 10^{-4}$)	28.03	28.20	3.07
# parameters	2	3	6
Linear: θ_1	1.05	0.83	0.25
	(0.00)	(0.00)	(0.00)
Quadratic: θ_2	–9.55	–9.33	–13.84
	(0.00)	(0.00)	(0.00)
Cubic: θ_3		62.57	
		(0.99)	
Upside Constant: θ_0^+			0.03
			(0.00)
Upside Linear: θ_1^+			–0.16
			(0.00)
Upside Quadratic: θ_2^+			17.83
			(0.00)
Weighting: σ			0.01
			(0.00)

The table shows the results of the GMM estimation of moment conditions defined in (27). Econometric methodology and models under analysis are the same as in Table 2. The test assets are the six double sorted size and book-to-market (value-weighted) portfolios, plus the six double sorted size and momentum (value-weighted) portfolios. Data are at monthly frequency over the period July 1963-June 2010 (564 observations). All models satisfy the admissibility conditions defined in section 3.2 (monotonicity, Second Order Conditions for optimality, nonnegative R^2, and Hansen-Jagannathan bounds), except the standard mean-variance CAPM that doesn't satisfy monotonicity.

Table 10: Robustness, Testing the CAPM^2 on 48 Industry Portfolios

	CAPM	CAPM_{SK}	$\text{CAPM}^2_{\text{GA,LA,SP}}$
J test	51.88	51.27	15.47
	(0.29)	(0.27)	(0.99)
MAAE ($\times 10^{-4}$)	18.19	18.42	6.43
# parameters	2	3	6
Linear: θ_1	1.07	0.99	1.03
	(0.00)	(0.00)	(0.00)
Quadratic: θ_2	–10.82	–6.74	–4.19
	(0.00)	(0.00)	(0.00)
Cubic: θ_3		15.57	
		(0.99)	
Upside Constant: θ_0^+			0.01
			(0.00)
Upside Linear: θ_1^+			–0.53
			(0.00)
Upside Quadratic: θ_2^+			6.08
			(0.00)
Weighting: σ			0.01
			(0.00)

The table shows the results of the GMM estimation of moment conditions defined in (27). Econometric methodology and models under analysis are the same as in Table 2. The test assets are the 48 industries portfolios (value-weighted), which are based on four-digit SIC code classification. Data are at monthly frequency over the period July 1963-June 2010 (564 observations). All models satisfy the admissibility conditions defined in section 3.2 (monotonicity, Second Order Conditions for optimality, nonnegative R^2, and Hansen-Jagannathan bounds), except the standard mean-variance CAPM that doesn't satisfy monotonicity.

Table 11: Out of Sample Results, Portfolio Sorted on CAPM^{2} Factors

		Sorting on Goal Achievement	
α	0.36%	0.31%	0.24%
	(2.12)	(1.94)	(1.36)
β_{MKT}		0.14	0.12
		(2.98)	(2.35)
β_{SML}			0.22
			(2.80)
β_{HML}			0.11
			(1.24)
β_{MOM}			-0.01
			(-0.11)

		Sorting on Loss Aversion	
α	0.75%	0.32%	1.00%
	(1.95)	(1.10)	(3.78)
β_{MKT}		1.14	0.79
		(10.48)	(11.12)
β_{SML}			0.56
			(5.98)
β_{HML}			-0.82
			(-6.89)
β_{MOM}			-0.43
			(-4.04)

	Sorting on Security/Potential		
α	1.22%	0.77%	1.46%
	(2.82)	(2.29)	(4.87)
β_{MKT}		1.20	0.77
		(11.13)	(9.85)
β_{SML}			0.86
			(9.09)
β_{HML}			-0.89
			(-6.33)
β_{MOM}			-0.43
			(-3.18)

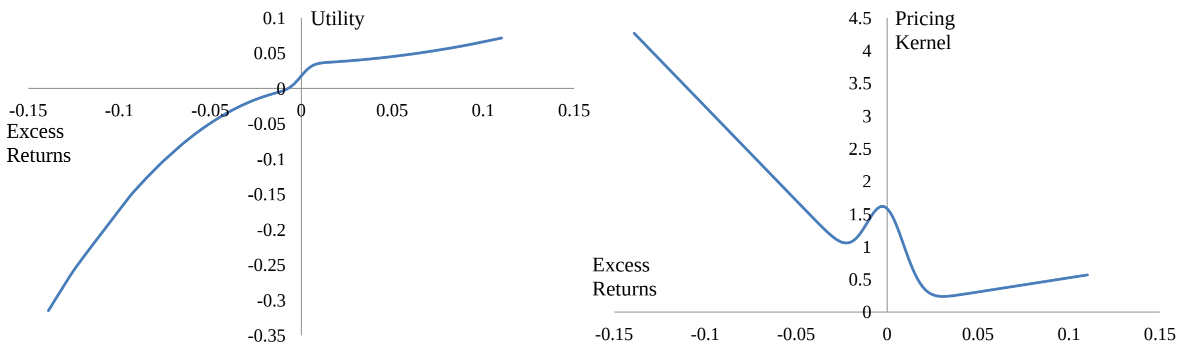
The Table shows the results obtained with zero cost portfolios that take long (short) positions on securities with the highest (lowest) loadings on the three CAPM^2 factors: Goal Achievement, Loss Aversion, and preference for Security/Potential (see sections 4.5 and 2.4). For every zero cost portfolio, the table reports the coefficients of the following regression (standard errors, in parenthesis, are obtained using Newey-West autocorrelation consistent estimation):

$$SORT_{i,t} = \alpha_i + \beta_{MKT,i}\, MKT_t + \beta_{SMB,i}\, SMB_t + \beta_{HML,i}\, HML_t + \beta_{MOM,i}\, MOM_t + \epsilon_{i,t}$$

Where $SORT_{i,t}$ is the return at time t of the zero cost portfolio i (obtained by sorting on one of the CAPM^2 factors), MKT_t is the return of the market portfolio at time t, SMB_t is the size factor at time t, HML_t is the value factor at time t, and MOM_t is the momentum factor at time t.
Zero cost portfolios are built by following a procedure similar to Fama-French (1992): every month, all stocks in the cross section are sorted according to their loadings on each factor (i.e., their covariance with each factor, as defined in section 2.4, over the previous sixty months). Then, I form ten value weighted portfolios and check their returns over the next month. Portfolios are rebalanced every month. Zero cost portfolios are given by the difference of the highest and lowest deciles of every sorting. The dataset is composed of all common stocks listed on the NYSE, AMEX, and NASDAQ, with monthly data from July 1963 to June 2010.

Figure 1: The CAPM^2

(a) CAPM^2 Utility Function **(b)** CAPM^2 Pricing Kernel



The graph shows the utility function and the pricing kernel of the CAPM^2. The utility function is approximated with the Continuous Asymmetric Polynomial Model defined in (7), with parameter estimates shown in table 2. The utility function displays:

(i) a positive shift at zero, consistent with Goal Achievement (the importance that investors attach to the overall probability of obtaining positive portfolio returns);

(ii) a higher slope in the negative domain, consistent with Loss Aversion (losses loom larger than gains);

(iii) concavity for losses and mild convexity for gains, consistent with preference for Security/Potential (downside risk aversion and preference for upside potential).

The corresponding pricing kernel displays (i) a bump around zero, (ii) a higher level in the negative domain, and (iii) it is decreasing for losses and slightly increasing for gains. As a consequence, the CAPM^2 predicts higher risk premiums for securities delivering:

(i) lower returns when market returns are close to zero (higher loadings on Goal Achievement);

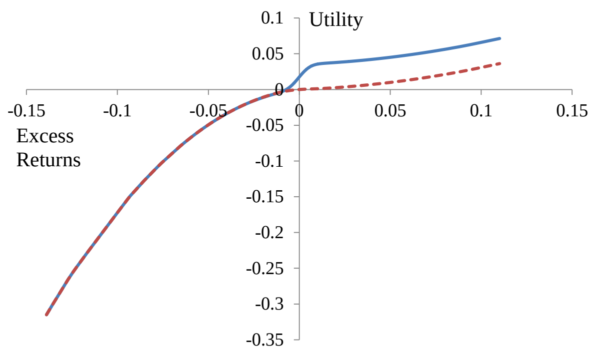
(ii) lower average returns during market downturns (higher loadings on Loss Aversion);

(iii) higher covariance with market returns during market downturns and *lower* covariance with market returns during rising markets (higher loadings on preference for Security/Potential).

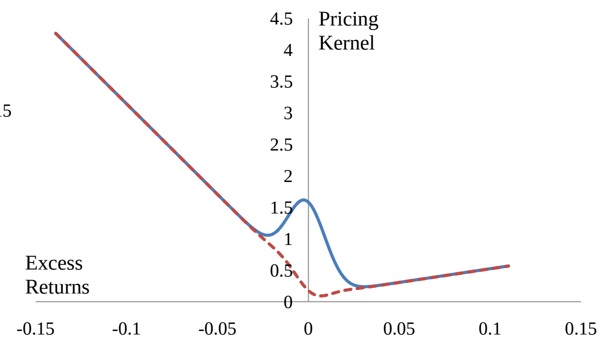
As discussed in section 4.1, the CAPM^2 can explain size, value, and momentum anomalies because small cap, high book-to-market securities, and momentum winners deliver higher loadings on Goal Achievement, Loss Aversion, and preference for Security/Potential. These results are consistent with Piccioni (2014), who shows that Goal Achievement, Loss Aversion, and preference for Security/Potential are also the key factors driving risky choice behavior.

Figure 2: The CAPM^2 and Goal Achievement

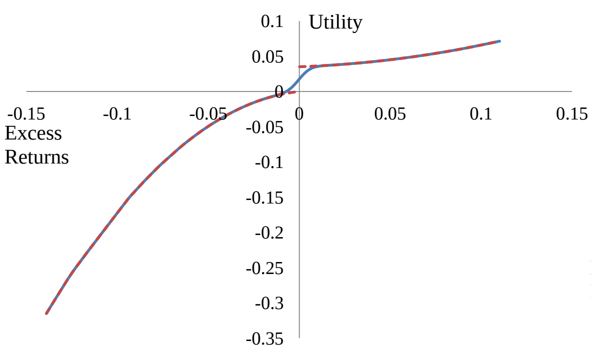
(a) Utility Function with/without θ_0^+



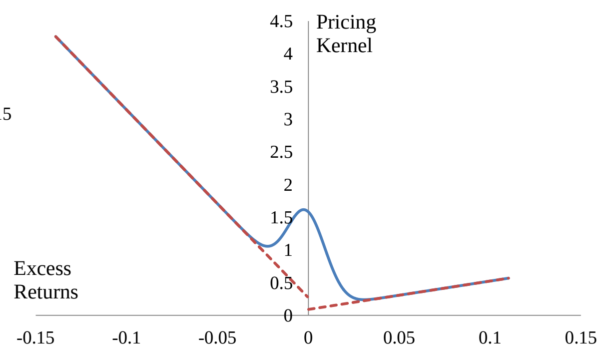
(b) Pricing Kernel with/without θ_0^+



(c) Utility Function with/without continuity



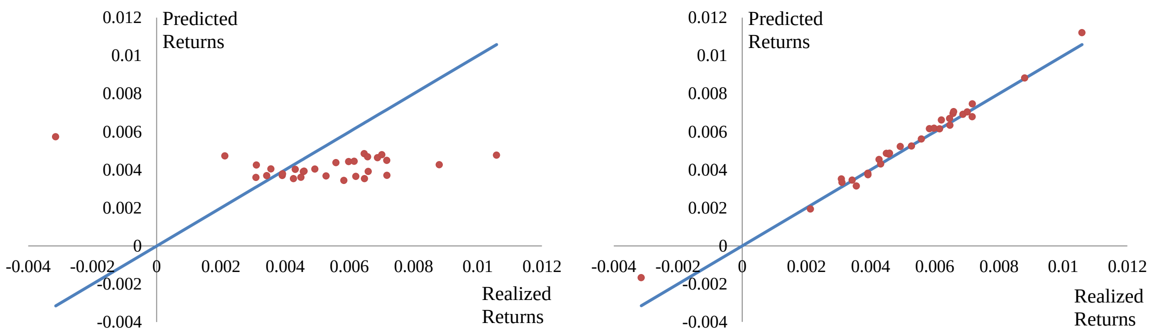
(d) Pricing Kernel with/without continuity



The graph highlights the positive shift of the utility function of the CAPM^2, and the corresponding bump of the pricing kernel. The positive shift in utility is obtained by guaranteeing continuity and by including a positive constant—θ_0^+—in the upside component of the utility function. The solid lines show the utility function and the pricing kernel of the CAPM^2. The dotted lines in (a) and (b) show the utility function and the pricing kernel when we set $\theta_0^+ = 0$; while the dotted lines in (c) and (d) show the utility function and the pricing kernel when we use an indicator function to distinguish between gains and losses.
Because of the positive shift of the utility function at zero, marginal utility is high around zero. Therefore, the CAPM^2 predicts higher risk premiums for securities paying off less when market portfolio returns are close to zero. This, in turn, implies higher risk premiums for securities that do not contribute to increase the overall probability of obtaining positive returns in well-diversified portfolios. As a consequence, the CAPM^2 can price goal seeking behavior, or Goal Achievement.

Figure 3: Predicted and Realized Returns

(a) Mean-Variance CAPM **(b)** CAPM^2



The graph shows the fit of the mean-variance CAPM and CAPM^2. A model with a perfect fit (i.e., a model whose average predicted returns are equal to the realized returns of the securities under analysis) would display all the dots (predicted returns) lying on the forty five degree line (realized returns). As shown in Table 2, the Mean Absolute Average Error (the measure of fit defined in equation (32)) is equal to 19.80 basis points for the mean-variance CAPM and 2.73 basis points for the CAPM^2.