A LANDAU–GINZBURG/CALABI–YAU CORRESPONDENCE FOR THE MIRROR QUINTIC

by

Nathan Charles Priddis

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Doctoral Committee:

Professor Yongbin Ruan, Chair
Professor Alessandro Chiodo, Institut de Mathématiques de Jussieu
Professor William Fulton
Professor Mircea Mustaţă
Associate Professor Leopoldo A. Pando Zayas

to Amy and our kids

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CHAPTER I

Introduction

Over twenty years ago, mirror symmetry led physicists to conjecture the LG/CY correspondence (see [24, 25]). It describes a deep relationship between the geometry of Calabi–Yau complete intersections and the local structure of corresponding singularities. The conjecture could not be made mathematically precise until 2007 with the development of Fan–Jarvis–Ruan–Witten (FJRW) theory (see [14]). The LG/CY correspondence is now understood as a relationship between the Gromov–Witten (GW) theory of a Calabi–Yau and the FJRW theory of the corresponding singularity (see Conjecture IV.2).

Though FJRW theory is interesting in its own right, evidence also suggests that it is easier to calculate than GW theory. For example in [18], Guéré calculates the genus zero FJRW theory in a range of cases where the corresponding GW theory is currently unknown. Thus the LG/CY correspondence provides a possible method for determining the GW theory of many Calabi–Yau's.

In genus zero the LG/CY correspondence has been proven for hypersurfaces in Gorenstein weighted projective spaces in [5, 4, 19] and for certain complete intersections in [8]. In this paper we prove a version of the genus zero LG/CY correspondence for the mirror quintic, a Calabi–Yau (CY) hypersurface in an orbifold quotient of projective space. This is the first example of the LG/CY correspondence for a space which cannot be constructed as a complete intersection in weighted projective space.

In all of the known examples, the LG/CY correspondence has been carried out in the following manner. The numbers encoding the information for FJRW theory (resp. GW theory) are packaged into a formal generating function, known as the J–function. Both FJRW theory and GW theory these theories are A–model theories. Using mirror symmetry, J–function for FJRW theory is related to another function encoding B–model data called the I–function in an explicit way. The same can be done for GW theory yielding another I–function. Using analytic continuation and a symplectic transformation, the two respective I–functions can be related in a specific way. Putting all of these relations together, gives the relation between the two J–functions.

In the remainder of the introduction we describe the relationship between these various functions using the first known example, the quintic. Then we give an overview of the LG/CY correspondence for the mirror quintic.

1.1 LG/CY for the quintic

As mentioned above, the LG/CY correspondence relies heavily on a related concept known as mirror symmetry. Given a CY–threefold X, mirror symmetry gives a relationship between the A–model of X and the B–model of its "mirror" $X^\vee$—also a CY–threefold. Roughly speaking, the A–model is determined by the Kähler structure and the B–model by the complex structure. The Kähler deformations of a three–fold X are parametrized by $H^{1,1}(X)$, whereas the dimension of the complex deformations is $h^{2,1}(X)$. Thus a first prediction of mirror symmetry is $h^{1,1}(X) = h^{2,1}(X^\vee)$.

However, mirror symmetry goes beyond that. In essence, the A–model encodes enumerative information about a space, whereas the B–model encodes the information of the variation of Hodge structures, which are determined via period integrals.

The quintic

The polynomial defining the (Fermat) quintic is

$$W = x_1^5 + \cdots + x_5^5. \tag{1.1}$$

The vanishing locus of W in projective space defines the quintic M as a subset of $\mathbb{P}^4$, i.e. $M := \{W = 0\}$.

The *mirror quintic* is the Deligne–Mumford stack

$$\mathcal{W} = [M/\bar{G}] \subset [\mathbb{P}^4/\bar{G}]$$

where we define the group $\bar{G} \cong (\mathbb{Z}/5\mathbb{Z})^3$ as the subgroup of the big torus of $\mathbb{P}^4$ acting via generators e_1, e_2, e_3:

$$\begin{aligned} e_1[x_1, x_2, x_3, x_4, x_5] &= [\zeta x_1, x_2, x_3, x_4, \zeta^{-1} x_5] \\ e_2[x_1, x_2, x_3, x_4, x_5] &= [x_1, \zeta x_2, x_3, x_4, \zeta^{-1} x_5] \\ e_3[x_1, x_2, x_3, x_4, x_5] &= [x_1, x_2, \zeta x_3, x_4, \zeta^{-1} x_5]. \end{aligned} \tag{1.2}$$

The B–model

In order to describe the LG/CY correspondence for the quintic, we need to describe both the A–model and the B–model. using the B–model, we obtain the I–functions which we can relate explicitly. In order to set up the two sides of the LG/CY correspondence, let us first consider the B–model.

As described in [22], the B–model is defined in terms of the variation of hodge structure, which we can study via period integrals. Let X be a complex variety and let X_t be a family of deformations of X parametrized by some base S. Let ω_t be a local section of $R^3\pi_*\mathbb{C} \otimes \mathcal{O}_S$; i.e. for each $t \in S$, we have $\omega_t \in H^3(X_t)$. Let $\{\gamma_i(t)\}$ be a basis of locally constant sections of $H_3(X_t)$. The *period integrals* of ω_t are defined as

$$\int_{\gamma_i(t)} \omega_t.$$

Given a choice of $\gamma_i(t)$, the corresponding period integrals satisfy the Picard–Fuchs equations of ω_t, which are defined via derivatives of ω_t.

The LG/CY correspondence arises in large part from the observation that although the space of Kähler deformations of M is contractible, the space of complex deformations of $\mathcal{W}$ is not. In fact, the deformations of $\mathcal{W}$ are given by the family of polynomials

$$W_\psi = W - \psi x_1 x_2 \ldots x_5. \tag{1.3}$$

We may define a family of Deligne–Mumford stacks

$$\mathcal{W}_\psi := \{W_\psi = 0\} \subset [\mathbb{P}^4/\bar{G}].$$

Under the action $\psi \to \alpha\psi$ with $\alpha^5 = 1$, this becomes a family of (orbifold) CY three–folds over the stack $[\mathbb{P}^1/\mathbb{Z}_5]$. It is regular away from the points $\psi = 0$, $\psi = \infty$ and $\psi^5 = 1$.

Let ω be a family of $(3,0)$ forms on W_ψ. For the quintic we can express the periods of any other 3–form as a linear combination of derivatives of the periods of ω. Hence the B–model for the quintic is determined by the periods of ω.

Taking successive derivatives of ω, we arrive at a differential equation satisfied by the periods

$$D_q^4 - 5q\prod_{m=1}^{4}(5D_q + mz) \tag{1.4}$$

where $D_q = zq\frac{\partial}{\partial q}$. This equation is known as the Picard–Fuchs equation.

We can express a solution to this equation in terms of a cohomology-valued hypergeometric function, which we will denote by $I^M(s,z)$. It is given by

$$I^M(s,z) := e^{sH/z}\Big(\sum_{k\geq 0} e^{ks}\frac{\prod_{m=1}^{5k}(5H+mz)}{\prod_{b=1}^{k}(H+bz)^5}\Big) \pmod{H^4}.$$

Here $s = -5\log\psi$. When we expand this function in powers of H, it yields a basis of solutions to the Picard–Fuchs equations for the periods of $\mathcal{W}$, which take a nice form under the identification $q = e^s$ (see Section 6.2).

This is the I–function that will be related to the GW theory as mentioned previously.

Remark I.1. In the B–model, H is a formal variable used to track the variuos linearly independent solutions of the Picard–Fuchs equation, with $H^4 = 0$. We use the notation I^M instead of $I^{\mathcal{W}}$ to match the notation of later chapters. This is justified in part by Givental's mirror theorem, which equates the function I^M with the so–called J function for M. This will be described shortly.

If we expand the periods around the point $\psi = 0$, we obtain a the Picard–Fuchs equations

$$D_t^4 - 5^5 t^{-5}\prod_{m=1}^{4}(D_t - mz) \tag{1.5}$$

where $t = \psi$ and $D_t = zt\frac{\partial}{\partial t}$. One can obtain an I–function

$$I^{(W,\langle J\rangle)}(t,z) = \sum_{k=1,2,3,4}\phi_{J^k} z^{2-k}\sum_{l\geq 0} t^{k+5l}\frac{\Gamma((k+5l)/5)^5}{\Gamma(k/5)\Gamma(k+5l)}.$$

This is the I–function that will relate to the FJRW theory.

At the moment H^k and $\phi_{\mathcal{J}^k}$ are simply book–keeping devices to keep track of the solutions to the Picard–Fuchs equations. In other words the coefficients of $\phi_{\mathcal{J}^k}$ give a basis of solutions to (1.5) and the coefficients of the variou powers of H give solutions to (1.4). In the next section we will interpret H^k and $\phi_{\mathcal{J}^k}$ as elements of the respective state spaces for GW theory and FJRW theory.

Mirror symmetry is a way of relating these I–functions with the J–functions defined by the A–model, which we now describe.

The A–model

Much of the study of mirror symmetry has focused on a correspondence between the (contractible) space of Kähler structures of M—which is also one–dimensional—and a contractible neighborhood of the point $\psi = \infty$ in the family we just described. This is the CY side of the LG/CY correspondence.

The CY A–model can be best described by GW theory. The aim of GW theory is to study a space X by looking at maps $f : C \to X$ where C is a complex curve. There is a moduli space $\overline{\mathcal{M}}_{g,n}(X, \delta)$ parametrizing the *stable maps* $f : C \to X$ where C is a complex curve of genus g with n distinct marked points and $f_*(C) = \delta \in H_2(X)$.

For each marked point, we can define an evaluation map $ev_i : \overline{\mathcal{M}}_{g,n}(X, \delta) \to X$ defined by

$$ev_i([f : C \to X]) := f(p_i),$$

where p_i is the marked point. We can pull back cohomology classes from X to the moduli space of stable maps via these evaluation maps.

Each marked point also yields a tautological class $\psi_i \in H^*(\overline{\mathcal{M}}_{g,n}(X, \delta))$, which is the first Chern class of the line bundle whose fiber over a point $[f : C \to X]$ is the cotangent line $T^*_{p_i}C$.

Taking all of this together, we can define GW invariants on X via the integrals

$$\langle \psi^{a_1}\alpha_1, \ldots, \psi^{a_n}\alpha_n \rangle^X_{g,n,\delta} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\delta)]^{vir}} \prod_{i=1}^{n} ev_i^*(\alpha_i)\psi_i^{k_i}$$

where $[\overline{\mathcal{M}}_{g,n}(X,\delta)]^{vir}$ denotes the *virtual fundamental class*.

We now specialize to the quintic M. One often organizes the GW invariants into a generating function $\mathcal{F}^M$ whereby relations between the GW invariants can be viewed as partial differential equations satisfied by $\mathcal{F}^M$.

From the generating function $\mathcal{F}^M$ we can construct Givental's J–funciton, which encodes genus zero invariants of M. Choose a basis $\{\beta_i\}_{i\in I}$ for $H^*(M)$ and let $\{\beta^i\}$ denote the dual basis. We can express a point of the state space with this basis as $\mathbf{s} = \sum_{i\in I} s^i\beta_i$. The J–function is defined as the cohomology–valued function

$$J^M(\mathbf{s}, z) := z + \mathbf{s} + \sum_{n\geq 0} \sum_{\substack{a,\delta\geq 0 \\ h\in I}} \frac{Q^\delta}{n! z^{a+1}} \langle \mathbf{s}, \ldots, \mathbf{s}, \psi^a\beta_h \rangle^M_{0,n+1,\delta} \beta^h.$$

The J–function should be viewed as a function on $H^*(M)$.

Restricting to $H^2(M)$ we obtain the *small J–function* $J^M_{small}(t,z)$. In the case of the quintic hypersurface, the genus zero GW invariants are completely determined by $J^M_{small}(t,z)$.

The mirror theorem for the Fermat quintic, as formulated by Givental (see [15]) is a correspondence between J^M_{small} and I^M. Let $H \in H^2(M)$ denote the pullback of the hyperplane class from $\mathbb{P}^4$ and let s be the dual coordinate. We can view I^M as a function on $H^2(M)$. Then Givental's mirror theorem can be stated as follows:

Theorem I.2 (Givental)**.** *After an explicit change of variables, J^M_{small} is equal to I^M.*

On the other hand, if we consider a neighborhood of $\psi = 0$, we end up on the Landau–Ginzberg (LG) side of the LG/CY correspondence. We use the term

Landau–Ginzburg model to refer to a pair (W, G) where W is nondegenerate quasi-homogeneous polynomial on $\mathbb{C}^N$ and G is a finite subgroup of $\mathrm{Aut}(W)$. We think of this data as defining a singularity $\{W = 0\} \subset [\mathbb{C}^N/G]$.

Fan, Jarvis and Ruan (see [14]) have defined a theory known as FJRW theory (named after Fan, Jarvis, Ruan and Witten) which describes the LG A–model. FJRW theory studies the singularity $\{W = 0\} \subset [\mathbb{C}^N/G]$ by considering the so–called *W–structures*, which are collections of line bundles $\{\mathcal{L}_1, \ldots \mathcal{L}_N\}$ over complex (orbifold) curves. These line bundles are roots of the log canonical bundle $\omega_{\log}$ that satisfy the condition that for each monomial of W_l of W

$$W_l(\mathcal{L}_1, \ldots, \mathcal{L}_N) \cong \omega_{\log}.$$

The $\mathcal{L}_i$ also satisfy some other conditions determined by the group G (see [14] for details).

There is a moduli space of W–structures $W_{g,n}$ parametrizing the set of W–structures over complex orbifold curves of genus g with n–marked (orbifold) points. These curves are allowed orbifold structure only at marked points and nodes.

The moduli space $W_{g,n}$ decomposes into connected components based on the multiplicities of the $\mathcal{L}_i$ at each of the (orbifold) marked points (see Section 2.2). In other words, given $h_1, \ldots, h_n \in G$, we can write

$$W_{g,n} = \sqcup W_{g,n}(h_1, \ldots h_n).$$

There is also a state space $\mathscr{H}_{W,G}$ which plays the role of $H^*(M)$ in GW theory. The state space is defined in terms of the Lefschetz thimbles of the singularity. We do not give a full definition here, but will be content to remark that $\mathscr{H}_{W,G}$ a decomposition indexed by G:

$$\mathscr{H}_{W,G} = \bigoplus_{h \in G} \mathscr{H}_h.$$

The summands $\mathscr{H}_h$ are called *sectors*. The state space can also be decomposed into the *broad* part and the *narrow* part. The narrow part of $\mathscr{H}_{W,G}$ is a direct sum of sectors which are one–dimensional and is indexed by a subset $\hat{S} \subset G$. For $h \in \hat{S}$, we choose a generator of the sector $\mathscr{H}_h$, which we denote by ϕ_h. In this dissertation, we will focus solely on the narrow part.

Each marked point also yields a tautological class $\psi_i \in H^2(W_{g,n}(h_1, \ldots, h_n))$. Similar to GW theory, we can define FJRW invariants for narrow insertions ϕ_{h_i} as

$$\left\langle \psi^{a_1}\phi_{h_1}, \ldots, \psi^{a_n}\phi_{h_n} \right\rangle_{g,n}^{(W,G)} := \frac{1}{|G|^{g-1}} \int_{[W_{g,n,G}(\mathbf{h})]^{vir}} \prod_{i=1}^{n} \psi_i^{a_i}$$

The general definition is somewhat involved, so we omit it here, and refer the reader to the original treatment in [14]. (For the case of the quintic, see [5].)

The LG model for the quintic is the pair $(W, \langle \mathcal{J} \rangle)$, where W is the Fermat quintic as in (1.1) and $\mathcal{J} = (e^{2\pi i/5}, \ldots, e^{2\pi i/5}) \in \mathrm{Aut}(W)$. For the conditions on the line bundles imply that $\mathcal{L}_i$ is a fifth root of $\omega_{\log}$ and $\mathcal{L}_1 \cong \ldots \cong \mathcal{L}_5$.

As with GW theory, we organize the invariants into a generating function $\mathcal{F}^{(W,\langle \mathcal{J} \rangle)}$, which satisfies similar differential equations to $\mathcal{F}^M$. Again we choose a basis $\{\phi_h\}_{h \in \hat{S}}$ for the narrow part of the state space. Here as before ϕ_h is a generator of $\mathscr{H}_h$ and the dual basis is $\{\phi^h\}$. We can express an element of $\mathscr{H}_{W,\langle \mathcal{J} \rangle}$ in this basis $\mathbf{t} = \sum_{h \in \hat{S}} t^h \phi_h$ and from $\mathcal{F}^{(W,\langle \mathcal{J} \rangle)}$ construct the FJRW J–function

$$J^{(W,\langle \mathcal{J} \rangle)}(\mathbf{t}, -z) = -z\phi_{\mathcal{J}} + \mathbf{t} + \sum_{n \geq 0} \sum_{\substack{a \geq 0 \\ h \in G}} \frac{1}{n!(-z)^{a+1}} \left\langle \mathbf{t}, \ldots \mathbf{t}, \psi^a \phi_h \right\rangle_{0,n+1}^{(W,\langle \mathcal{J} \rangle)} \phi^h.$$

To obtain the small J–function, we restrict to the degree 2 part of $\mathscr{H}_{W,\langle \mathcal{J} \rangle}$, which in this case is one–dimensional, generated by $\phi_{\mathcal{J}^2}$ with dual coordinate t. As with the GW theory for M, the small J–function $J_{small}^{(W,\langle \jmath \rangle)}(t,z)$ completely determines the FJRW theory for $(W, \langle \jmath \rangle)$.

In [5], Chiodo–Ruan proved a mirror theorem for the LG side similar to Theorem I.2 of Givental, relating $J_{small}^{(W,\langle \jmath \rangle)}$ with $I^{(W,\langle \jmath \rangle)}$. As on the CY side, we can view $I^{(W,\langle \jmath \rangle)}$ as a function on $\mathscr{H}^2_{W,\langle \jmath \rangle}$.

Theorem I.3 (Chiodo–Ruan)**.** *The function* $J_{small}^{(W,\langle \jmath \rangle)}$ *is equal to* $I^{(W,\langle \jmath \rangle)}$ *after an explicit change of variables.*

Remark I.4. In slight contrast to the presentation here, Chiodo–Ruan actually define the I–function via the LG A–model theory, and then prove that it satisfies the corresponding Picard–Fuchs equations. However, since their I–function provides a basis of solutions to (1.5), it gives the periods of the B–model in a neighborhood of $\psi = 0$, and so is consistent with our treatment here.

1.1.1 LG/CY correspondence

For the LG A–model, we have $J_{small}^{(W,\langle \jmath \rangle)}(t,z)$, which we would like to relate to $J^M_{small}(q,z)$ for the CY A–model. Theorem **??** and Theorem I.2 relate $J_{small}^{(W,\langle \jmath \rangle)}(t,z)$ to $I^{(W,\langle \jmath \rangle)}(t,z)$ and $J^M(q,z)$ to $I^M(q,z)$, resp. With these two theorems in place, the last piece to the LG/CY correspondence is to relate the two I–functions. This was done by Chiodo–Ruan in [5] via analytic continuation and a symplectic transformation.

The first step is to do analytic continuation on I^M from a neighborhood of $\psi = \infty$ to a neighborhood of $\psi = 0$ via the Mellin–Barnes method and the change of variables $q = t^{-5}$. After doing so, we have another function $I^{M'}$ satisfying the Picard–Fuchs equations (1.5). After identifying $H^*(M)$ with $\mathscr{H}_{W,\langle \jmath \rangle}$ via the map $H^k \mapsto \phi_{\jmath^{k+1}}$, Chiodo–Ruan proved the following:

Theorem I.5 (Chiodo–Ruan). *There is a* $\mathbb{C}[z, z^{-1}]$*–valued symplectic transformation* $\mathbb{U}$ *such that*

$$\mathbb{U}\big(I^{(W,\langle \mathcal{J} \rangle)}\big) = I^M.$$

Using Theorems I.2, I.3 and VI.1 we obtain the desired relation between respective J–functions for the GW invariants and the FJRW invariants.

Givental's symplectic formalism may make it possible to determine the higher genus LG/CY correspondence from the genus zero correspondence. Namely, it has been conjectured (see [5, Conjecture 3.2.1]) that the quantization of $\mathbb{U}$ should relate the higher genus invariants of the two respective theories. This completes the LG/CY correspondence for the quintic.

1.2 The mirror quintic

It is natural to ask whether a similar scheme can be used to relate the GW theory for $\mathcal{W}$ to the corresponding LG A–model. In other words, we would like to relate the respective J–functions, to the I–functions using mirror symmetry, and then relate the two I functions via analytic continuation and a symplectic transformation.

The LG A–model is given by the pair (W, G) where W is the Fermat quintic in (1.1) and G is the group

$$G := \langle \mathcal{J}, e_1, e_2, e_3 \rangle \cong (\mathbb{Z}_5)^4. \tag{1.6}$$

Recall e_1, e_2 and e_3 are defined by (1.2).

The main obstacle to this scheme for the mirror quintic is that $h^{2,1}(\mathcal{W}) = 101$. Thus the Picard–Fuchs equations in question are partial differential equations in 101 variables. To write these equations and provide solutions would be unfeasible. For this reason we restrict our attention to a one–dimensional subspace of the degree two part of each theory, which we will describe, each in turn. The one–dimensional subspaces we choose are arguably the most natural and most important dimension in each theory.

First note that $\mathcal{W}$ is an orbifold, so we will need to use the *Chen–Ruan cohomology* instead of the ordinary cohomology. We define this cohomology in terms of the *inertia orbifold*.

Given an orbifold $\mathcal{X}$ which is a global quotient, i.e. $\mathcal{X} = [X/\Gamma]$, where Γ is a finite group, the inertia orbifold has a simple description. Let S_Γ be the set of conjugacy classes of Γ, X^g denote the fixed point set of $g \in \Gamma$ and $C(g)$ the centralizer. Then the inertia orbifold is

$$I\mathcal{X} = \bigsqcup_{[g]\in S_\Gamma} [X^g/C(g)].$$

The orbifold $\mathcal{X}$ can be identified with the *untwisted sector*, i.e. $[X^e/\Gamma]$ in the above decomposition. Now we define

$$H^*_{CR}(\mathcal{X}) := H^*(I\mathcal{X}).$$

There is also a degree shift, which we will not describe here (see e.g. [3]).

1.2.1 B–model

For the B–model, we restrict to the one–dimensional space of deformations of M defined by

$$M_\psi := \{W_\psi = 0\} \subset \mathbb{P}^4.$$

Here W_ψ is defined as in (1.3).

As described above we want to consider the period integrals for 3–forms on M_ψ. In a neighborhood of $\psi = \infty$, these will describe the CY B–model. We choose a $(3,0)$–form ω and as before construct $I^{\mathcal{W}}(s,z)$ with $s = -5\log\psi$. The components of $I^{\mathcal{W}}(s,z)$ give a basis for the periods of ω. In other words, $I^{\mathcal{W}}$ give a basis for solutions to the Picard–Fuchs equations determined by ω and its derivatives.

In this case it is no longer true that any 3–form is a linear combination of derivatives of the periods of ω. However, there is a set of 3–forms $\{\omega_g\}$ indexed by the components of $I\mathcal{W}$, such that every 3–form can be expressed as a linear combination of $\{\omega\} \cup \{\omega_g\}$ and derivatives thereof. For each ω_g there is a function $I_g^{\mathcal{W}}$ whose components give the periods of ω_g. The corresponding Picard–Fuchs equations obtained by differentiating ω_g are as follows.

$$\begin{aligned}
& g = (0,0,0,\tfrac{1}{5},\tfrac{4}{5}) && (D_q)^2 - 5^3 q(D_q + 2z)(D_q + 3z) \\
(1.7) \qquad & g = (0,0,0,\tfrac{2}{5},\tfrac{3}{5}) && (D_q)^2 - 5^3 q(D_q + z)(D_q + 4z) \\
& g = (0,0,\tfrac{1}{5},\tfrac{1}{5},\tfrac{3}{5}) && (D_q)(D_q - z) - 5^4 q(D_q + z)(D_q + 3z) \\
& g = (0,0,\tfrac{2}{5},\tfrac{2}{5},\tfrac{1}{5}) && (D_q)(D_q - 2z) - 5^4 q(D_q + z)(D_q + 2z)
\end{aligned}$$

Since the components of $I\mathcal{W}$ are indexed by certain elements of $\bar{G}$, we have indexed these by elements of $\bar{G}$ using a convention which we will describe in Notation II.6. Any other element of $\bar{G}$ indexing a component of $I\mathcal{W}$ can be obtained by permuting the coordinates of those g listed here. Again we have made the change of variables $q = e^s$.

We can do the same near $\psi = 0$ obtaining functions $I_h^{(W,G)}$ indexed by the sectors of $\mathscr{H}_{W,G}$ satisfying the corresponding Picard–Fuchs equations:

$$
\begin{aligned}
&h = (\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{1}{5}) && (D_t)^2 - (\tfrac{5}{t})^5 (D_t - 2z)(D_t - 3z) \\
(1.8) \qquad &h = (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{4}{5}) && (D_t)^2 - (\tfrac{5}{t})^5 (D_t - z)(D_t - 4z) \\
&h = (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{4}{5}) && (D_t)(D_t + z - (\tfrac{5}{t})^5 (D_t - z)(D_t - 3z) \\
&h = (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{3}{5}, \tfrac{2}{5}) && (D_t)(D_t + 2z) - (\tfrac{5}{t})^5 (D_t - z)(D_t - 2z).
\end{aligned}
$$

1.2.2 A–model

Mirror symmetry tells us that the periods defined in a neighborhood of the point $\psi = \infty$ should be related to the GW theory of $\mathcal{W}$. Since $\mathcal{W}$ is an orbifold, we must use orbifold GW theory. To define orbifold GW theory, we modify standard GW theory, most notably by letting the evaluation maps have target $I\mathcal{W}$ instead of $\mathcal{W}$. Hence, when we define the GW invariants, $\langle \alpha_1 \psi^{k_1}, \ldots, \alpha_n \psi^{k_n} \rangle^{\mathcal{W}}_{g,n,\delta}$, we have insertions $\alpha_1, \ldots, \alpha_n \in H^*_{CR}(\mathcal{W})$.

As with the quintic, we define a J–function, $J^{\mathcal{W}}(\mathbf{s}, z)$. However, the degree two part of the cohomology of $\mathcal{W}$ has dimension 101. So we restrict $J^{\mathcal{W}}(\mathbf{s}, z)$ to the one–dimensional subspace defined by the degree 2 part of the untwisted sector $H^2(\mathcal{W})$ to obtain the small J–function $J^{\mathcal{W}}_{small}(s, z)$. Here we let s be the dual coordinate to $H \in H^2(\mathcal{W})$. After doing so, we obtain the following mirror theorem.

Theorem I.6 (Lee–Shoemaker [20]). *The function $J^{\mathcal{W}}_{small}$ is equal to $I^{\mathcal{W}}$ after an explicit change of variables.*

Remark I.7. The change of variables in this theorem is called the *mirror transformation*.

Recall that $I^{\mathcal{W}}$ does not determine all of the periods of M near $\psi = \infty$. To obtain a full correspondence, we must consider the derivatives of $J^{\mathcal{W}}(\mathbf{s}, z)$ indexed by the twisted sectors of $I\mathcal{W}$. Let $\mathbb{1}_g$ denote the fundamental class of $\mathcal{W}_g$ and s^g the dual coordinate. Then we define

$$J_g^{\mathcal{W}}(s,z) := z\frac{\partial J^{\mathcal{W}}(\mathbf{s},z)}{\partial s^g}\Big|_{s\in H^2_{un}(\mathcal{W})}$$

The mirror theorem can then be extended to these functions as well:

Theorem I.8 (Lee–Shoemaker [20]). *For each g, the function $J_g^{\mathcal{W}}$ is equal to $I_g^{\mathcal{W}}$ after applying the mirror transformation.*

On the LG side, FJRW theory gives an analogous statement near the point $\psi = 0$. With W and G defined as in (1.1) and (1.6), respectively, we define the J–function $J^{(W,G)}(\mathbf{t},z)$ in the same way as before. To define the small J–function $J^{(W,G)}_{small}(t,z)$, we restrict $J^{(W,G)}(\mathbf{t},z)$ to the one–dimensional subspace of $\mathscr{H}_{W,G}$ generated by $\phi_{\jmath^2}$ with t the dual coordinate. As with the GW theory in this case, this small J–function is not enough to give a full correspondence, so we consider the derivatives of $J^{(W,G)}$. For each $h \in S$ such that ϕ_h is of degree 2 in $\mathscr{H}_{W,G}$, let t^h be the dual coordinate. Then we define

$$J_g^{(W,G)}(t,z) := z\frac{\partial J^{(W,G)}(\mathbf{t},z)}{\partial t^h}\Big|_{t\phi_{\jmath^2}\in\mathscr{H}_{\jmath^2}}$$

One of the main theorems proved in this dissertation can be stated as follows:

Theorem I.9. *After an explicit change of variables, $J^{(W,G)}_{small}$ is equal to $I^{(W,G)}$. Furthermore, under the same change of variables $J_h^{(W,G)}$ is equal to $I_h^{(W,G)}$.*

Remark I.10. This theorem follows as a corollary from Theorem V.10, which gives an expression for a "big I–function", which is derived from the big J–function $J^{(W,G)}(\mathbf{t},z)$. The justification for the name I–function is that its derivatives satisfy the Picard–Fuchs equations given in (1.8), which is the content of Remark V.13.

However, it is not known whether this I function satisfies the proper PDE's to give the full genus zero theory for the B–model.

Remark I.11. Phrased slightly differently, Theorem I.9 says that the small FJRW J–function $J_{small}^{(W,G)}(t,z)$ satisfies, up to a change of variables, the Picard–Fuchs equations of a holomorphic $(3,0)$–form on M_ψ around $\psi = 0$ and the first derivatives $J_h^{(W,G)}(t,z)$ give solutions to the Picard–Fuchs equations of the other (non–holomorphic) families of 3–forms on M_ψ around $\psi = 0$.

1.2.3 LG/CY correspondence

The first part of the LG/CY correspondence is a state–space isomorphism, which is much more interesting for the mirror quintic than for the quintic. This gives us a way to identify $\mathscr{H}_{W,G}$ and $H^*_{CR}(\mathcal{W})$ as graded vector spaces. Chiodo–Ruan prove this isomorphism more generally in [5], but we compute it directly in Section 6.1. This isomorphism gives us a correspondence between the indexing sets for $\mathcal{W}_g$ and the sectors of $\mathscr{H}_{W,G}$. In particular, for each $I_g^{\mathcal{W}}$, there is a corresponding function $I_h^{(W,G)}$.

We do analytic continuation on the functions $I^{\mathcal{W}}$ and $I_g^{\mathcal{W}}$ to a neighborhood around $\psi = 0$ obtaining functions $I^{\mathcal{W}'}$ and $I_g^{\mathcal{W}'}$. The final step is the symplectic transformation as described in the following theorem:

Theorem I.12 ((= Theorem VI.1))**.** *There exists a linear symplectic transformation* $\mathbb{U}$ *which identifies* $I^{\mathcal{W}'}$ *and* $I_g^{\mathcal{W}'}$ *with* $I^{(W,G)}$ *and the corresponding* $I_h^{(W,G)}$*, resp.*

Givental's symplectic formalism allows one to rephrase the above theorem in a more useful form. In this setting, one can view the genus zero generating functions of GW theory and FJRW theory as generating *Lagrangian cones* $\mathscr{L}^{\mathcal{W}}$ and $\mathscr{L}^{(W,G)}$ in appropriate symplectic vector spaces. These Lagrangian subspaces completely determine the respective genus zero theories. The above theorem then identifies a certain subset of $\mathscr{L}^{\mathcal{W}}$, the *small slice* (see Definition IV.4) of $\mathscr{L}^{\mathcal{W}}$, with the corresponding (small) slice of $\mathscr{L}^{(W,G)}$.

Theorem I.13 ((= Theorem VI.4))**.** *The symplectic transformation* $\mathbb{U}$ *identifies the analytic continuation of the small slice of* $\mathscr{L}^{\mathcal{W}}$ *with the small slice of* $\mathscr{L}^{(W,G)}$.

As in the case of the quintic, it is conjectured that the quantization of $\mathbb{U}$ identifies the (analytic continuation of the) higher genus GW theory of $\mathcal{W}$ with the FJRW theory of (W, G).

Remark I.14. The material in this dissertation is a result of collaborative work with Mark Shoemaker, and can also be found in the preprint [21].

CHAPTER II

The Landau–Ginzburg Model

For the mirror quintic, the Landau–Ginzburg model is described by FJRW–theory. Here we will give a brief review of the definitions and facts we will need to describe the LG/CY correspondence. This was developed by Fan–Jarvis–Ruan in [14]. We will more closely follow the discussion in [5] . The mirror theorem for the LG model will be given in Chapter V.

2.1 State Space

A polynomial $W \in \mathbb{C}[x_1, \ldots, x_N]$ is *quasihomogeneous* of degree d with integer weights $w_1, \ldots, w_N$ if for every $\lambda \in \mathbb{C}$,

$$W(\lambda^{w_1} x_1, \ldots, \lambda^{w_N} x_N) = \lambda^d W(x_1, \ldots, x_N).$$

By rescaling the numbers $w_1, \ldots, w_N$ and d, we can require that $\gcd(w_1, \ldots, w_N) = 1$. For each $1 \leq k \leq N$, let $q_k = \frac{w_k}{d}$. The central charge of W is defined to be

$$\hat{c} := \sum_{k=1}^{N} (1 - 2q_k). \tag{2.1}$$

A polynomial is *nondegenerate* if

(i) the weights q_k are uniquely determined by W, and

(ii) the hypersurface defined by W is non–singular in projective space.

The maximal group of diagonal symmetries is defined as

$$G_{max} := \left\{ (\alpha_1, \ldots, \alpha_N) \subseteq (\mathbb{C}^*)^N \,|\, W(\alpha_1 x_1, \ldots, \alpha_N x_N) = W(x_1, \ldots, x_N) \right\}$$

Note that G_{max} always contains the *exponential grading element* $\mathcal{J} = (e^{2\pi i q_1}, \ldots, e^{2\pi i q_N})$. If W is nondegenerate, G_{max} will be finite. Define the *exponent of* W, denoted $\bar{d}$, as the order of the largest cyclic subgroup of G_{max}. In this paper, we will assume for simplicity that $\bar{d}$ is equal to the degree d of W. This does not hold in general, but will be true in the case of interest to us.

A group $G \subset G_{max}$ is *admissible* if there is a Laurent polynomial Z, quasihomogeneous with the same weights as W, having no monomials in common with W, such that the maximal group of diagonal symmetries of $W + Z$ is equal to G. Every admissible group G has the property that $\mathcal{J} \in G$. Let

$$\mathrm{SL}_W = \left\{ (\alpha_1, \ldots, \alpha_N) \in G_{max} | \prod \alpha_i = 1 \right\}.$$

If W satisfies the *Calabi–Yau* condition $\sum_{k=1}^{N} w_k = d$, then $Z = x_1 x_2 \ldots x_N$ will be quasihomogeneous with the same weights as W; thus SL_W will be admissible.

Let G be an admissible group. For $h \in G$, let $\mathbb{C}_h^N$ denote the fixed locus of $\mathbb{C}^N$ with respect to h. Let N_h be the complex dimension of the fixed locus of h. Define

$$\mathscr{H}_h := H_{N_h}(\mathbb{C}_h^N, W_h^{+\infty}; \mathbb{C})^G,$$

that is, G–invariant elements of the the middle dimensional relative cohomology of $\mathbb{C}_h^N$. Here $W_h^{+\infty} := (\Re W_h)^{-1}(\rho, \infty)$, for $\rho >> 0$. The state space is the direct sum of the "sectors" $\mathscr{H}_h$, i.e.

$$\mathscr{H}_{W,G} := \bigoplus_{h \in G} \mathscr{H}_h.$$

$\mathscr{H}_{W,G}$ is $\mathbb{Q}$–graded by the W–degree. To define this grading, first note that each element $h \in G$ can be uniquely expressed as

$$h = (e^{2\pi i \Theta_1(h)}, \dots, e^{2\pi i \Theta_N(h)})$$

with $0 \le \Theta_k(h) < 1$. The *degree–shifting number* is

$$\iota(h) := \sum_{k=1}^{N} (\Theta_k(h) - q_k).$$

For $\alpha_h \in \mathscr{H}_h$, the (real) W–degree of α_h is defined by

$$\deg_W(\alpha_h) := N_h + 2\iota(h). \tag{2.2}$$

Remark II.1. Although we will not need it in this paper, one can define a product structure on $\mathscr{H}_{W,G}$, which then becomes a graded algebra. Let $\phi_{\mathcal{J}}$ be the fundamental class in $\mathscr{H}_{\mathcal{J}}$, and note that $\deg_W(\phi_{\mathcal{J}}) = 0$. In fact $\phi_{\mathcal{J}}$ is the identity element in $\mathscr{H}_{W,G}$. This partially explains the prominence of the element $\mathcal{J}$ in the above discussion.

There is also a non–degenerate pairing

$$\langle -, - \rangle : \mathscr{H}_h \times \mathscr{H}_{h^{-1}} \to \mathbb{C},$$

which induces a symmetric non–degenerate pairing,

$$\langle -, - \rangle : \mathscr{H}_{W,G} \times \mathscr{H}_{W,G} \to \mathbb{C}.$$

2.2 Moduli of W–curves

Recall that an *n–pointed orbifold curve* is a stack of Deligne–Mumford type with at worst nodal singularities with orbifold structure only at the marked points and the nodes. We require the nodes to be *balanced*, in the sense that the action of the stabilizer group be given by

$$(x, y) \mapsto (e^{2\pi i/k} x, e^{-2\pi i/k} y).$$

Given such a curve, $\mathcal{C}$, let ω be its dualizing sheaf. The *log–canonical bundle* is

$$\omega_{\log} := \omega(p_1 + \cdots + p_n)$$

Following [5], we will consider *d–stable* curves. A d–stable curve is a proper connected orbifold curve $\mathcal{C}$ of genus g with n distinct smooth markings $p_1, \ldots, p_n$ such that

(i) the n–pointed underlying coarse curve is stable, and

(ii) all the stabilizers at nodes and markings have order d.

There is a moduli stack, $\overline{\mathcal{M}}_{g,n,d}$ parametrizing such curves. It is proper, smooth and has dimension $3g - 3 + n$. (As noted in [5], it differs from the moduli space of curves only because of the stabilizers over the normal crossings.)

Write W as a sum of monomials $W = W_1 + \cdots + W_s$, with $W_l = c_l \prod_{k=1}^{N} x_k^{a_{lk}}$. Given line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_N$ on the d–stable curve $\mathcal{C}$, we define the line bundle

$$W_l(\mathcal{L}_1, \ldots, \mathcal{L}_N) := \bigotimes_{k=1}^{N} \mathcal{L}_k^{\otimes a_{lk}}.$$

Definition II.2. A *W–structure* is the data $(\mathcal{C}, p_1, \ldots, p_n, \mathcal{L}_1, \ldots, \mathcal{L}_N, \varphi_1, \ldots \varphi_N)$, where $\mathcal{C}$ is an n–pointed d–stable curve, the $\mathcal{L}_k$ are line bundles on $\mathcal{C}$ satisfying

$$W_l(\mathcal{L}_1, \ldots, \mathcal{L}_N) \cong \omega_{\log}, \tag{2.3}$$

and for each k, $\varphi_k : \mathcal{L}_k^{\otimes d} \to \omega_{\log}^{w_k}$ is an isomorphism of line bundles.

There exists a moduli stack of W–structures, denoted by $W_{g,n}$.

Proposition II.3 (Chiodo–Ruan [5])**.** *The stack $W_{g,n}$ is nonempty if and only if $n > 0$ or $2g - 2$ is a positive multiple of d. It is a proper, smooth Deligne–Mumford stack of dimension $3g - 3 + n$. It is etale over $\overline{\mathcal{M}}_{g,n,d}$ of degree $|G_{max}|^{2g-1+n}/d^N$.*

The moduli space can be decomposed into connected components, which we now describe. Because $\mathcal{L}_k$ is a dth root of a line bundle pulled back from the coarse underlying curve, the generator of the isotropy group at p_i acts on $\mathcal{L}_k$ by multiplication by $e^{2\pi i m^i_k/d}$ for some $0 \leq m^i_k < d$. The integer m^i_k is the *multiplicity* of $\mathcal{L}_k$ at p_i, and will usually be denoted $\mathrm{mult}_{p_i}(\mathcal{L}_k)$. Equation (2.3) ensures that $(e^{2\pi i m^i_1/d}, \ldots, e^{2\pi i m^i_N/d}) \in G_{max}$. Furthermore, when we push forward the line bundle $\mathcal{L}_k$ to the coarse curve, we find it has degree

$$q_k(2g-2+n) - \sum_{i=1}^{n} \mathrm{mult}_{p_i}(\mathcal{L}_k)/d, \tag{2.4}$$

which must therefore be an integer.

Let $\mathbf{h} = (h_1, \ldots, h_n)$ denote an n–tuple of group elements, $h_i \in G_{max}$. Define $W_{g,n}(\mathbf{h})$ to be the stack of n–pointed, genus g W–curves for which $\mathrm{mult}_{p_i}(\mathcal{L}_k)/d = \Theta_k(h_i)$. The following proposition describes a decomposition of $W_{g,n}$ in terms of multiplicities:

Proposition II.4 (Fan–Jarvis–Ruan [14])**.** *The stack $W_{g,n}$ can be expressed as the disjoint union*

$$W_{g,n} = \coprod W_{g,n}(\mathbf{h})$$

with each $W_{g,n}(\mathbf{h})$ an open and closed substack of $W_{g,n}$. Furthermore, $W_{g,n}(\mathbf{h})$ is non–empty if and only if

$$h_i \in G_{max},\ i = 1, \ldots, n$$
$$q_k(2g-2+n) - \sum_{i=1}^{n} \Theta_k(h_i) \in \mathbb{Z}, \qquad k = 1, \ldots, N.$$

Suppose $G \subset G_{max}$ is an admissible group, so G is the maximal group of diagonal symmetries of $W + Z$ for some choice of quasihomogeneous Laurent polynomial Z. We define $W_{g,n,G}$ to be the stack of $(W+Z)$–curves with genus g and n

marked points. This definition does not depend on the particular choice of Z (see [14]).

Proposition II.5 (Fan–Jarvis–Ruan [14])**.** *$W_{g,n,G}$ is a proper substack of $W_{g,n}$.*

We denote the universal curve by $\pi : \mathscr{C} \to W_{g,n,G}(\mathbf{h})$, and the universal W–structure by $(\mathbb{L}_1, \ldots, \mathbb{L}_N)$.

For each substack $W_{g,n}(\mathbf{h})$, one may define a virtual cycle $[W_{g,n}(\mathbf{h})]^{vir}$ of degree

$$2\left((\hat{c}-3)(1-g)+n-\sum_{i=1}^{n}\iota(h_i)\right).$$

The virtual cycle $[W_{g,n,G}(\mathbf{h})]^{vir}$ is defined as

$$[W_{g,n,G}(\mathbf{h})]^{vir} := \frac{|G_{max}|}{|G|} i^*[W_{g,n}(\mathbf{h})]^{vir},$$

with $i : W_{g,n,G}(\mathbf{h}) \hookrightarrow W_{g,n}(\mathbf{h})$ the inclusion map.

2.3 FJRW Invariants

FJRW invariants can be defined for any pair (W, G) where W is a nondegenerate quasihomogeneous polynomial and G is an admissible group. However, the most general definition is somewhat complicated, and unnecessary for our purposes here. To simplify the exposition, we will specialize to the case of interest to us, namely $W = x_1^5 + \cdots + x_5^5$ and $G = \mathrm{SL}_W$. Note SL_W is the same as G defined in (1.6)

W is degree five with weights $w_k = 1$ for $1 \leq k \leq 5$. In this case $\mathcal{J} = (e^{2\pi i/5}, \ldots, e^{2\pi i/5})$, and $\hat{c} = 3$.

Notation II.6. By a slight abuse of notation, we will often represent a group element $h = (e^{2\pi i \Theta_1(h)}, \ldots, e^{2\pi i \Theta_N(h)})$ by

$$h = (\Theta_1(h), \ldots, \Theta_5(h)).$$

With this convention, we can write

$$G = \left\{ \left(\tfrac{r_1}{5}, \ldots, \tfrac{r_5}{5}\right) \mid \sum_{k=1}^{5} r_k \equiv 0 \pmod 5, 0 \leq r_k \leq 4 \right\}.$$

In computing the state space, we find that the only non–zero sectors are the identity sector $\mathscr{H}_e$, and those with $N_h = 0$. If $N_h = 0$ we call $\mathscr{H}_h$ a *narrow sector*. Let $\hat{S} = \{h \in G | N_h = 0\}$ denote the index set for the narrow sectors. As each narrow sector is fixed by G, the state space can be decomposed as

$$\mathscr{H}_{W,G} = \mathscr{H}_e \oplus \bigoplus_{h \in \hat{S}} \mathscr{H}_h.$$

with $\mathscr{H}_h \cong \mathbb{C}$. The elements of $\mathscr{H}_e$ have degree three. The elements of each of the narrow sectors have even W–degree. In what follows we will focus on the subspace of narrow sectors,

$$\mathscr{H}^{nar}_{W,G} := \bigoplus_{h \in \hat{S}} \mathscr{H}_h.$$

Remark II.7. As in [5] the narrow sectors form a closed theory. In other words, any invariant involving an insertion that is not narrow vanishes.

There is an obvious choice of basis $\{\phi_h\}_{h \in \hat{S}}$, where ϕ_h is the fundamental class in $\mathscr{H}_h$. Let $\{\phi^h\}$ denote the dual basis with respect to the pairing, i.e. $\phi^h = \phi_{h^{-1}}$.

The moduli space may now be described as

$$W_{g,n,G} = \left\{ (\mathcal{C}, p_1, \ldots, p_n, \mathcal{L}_1, \ldots, \mathcal{L}_5, \varphi_1, \ldots, \varphi_5) | \varphi_k : \mathcal{L}_k^{\otimes 5} \xrightarrow{\sim} \omega_{\log}, \otimes_{k=1}^{5} \mathcal{L}_k \cong \omega_{\log} \right\}.$$

For each $\mathbf{h} = (h_1, \ldots, h_n) \in (\hat{S})^n$, the virtual cycle $[W_{g,n}(\mathbf{h})]^{vir}$ has degree

$$2\left((\hat{c} - 3)(1 - g) + n - \sum_{k=1}^{n} \iota(h_k) \right) = 2n - 2\sum_{k=1}^{n} \iota(h_k).$$

We also have psi classes ψ_i for each marked point. The class ψ_i is defined as the first Chern class of the bundle whose fiber over a point is the cotangent line to the corresponding coarse underlying curve at the i–th marking.

With this in mind, we can define the FJRW invariant

$$\langle \psi^{a_1}\phi_{h_1}, \ldots, \psi^{a_n}\phi_{h_n} \rangle_{g,n}^{(W,G)} := \frac{1}{625^{g-1}} \int_{[W_{g,n,G}(\mathbf{h})]^{vir}} \prod_{i=1}^{n} \psi_i^{a_i}.$$

Extending linearly, we obtain invariants defined for any insertions in $\mathscr{H}_{W,G}^{nar}$.

The perfect obstruction theory used to define the virtual class is given by $-R\pi_*(\bigoplus_{k=1}^5 \mathbb{L}_k)^\vee$. In genus zero, the situation simplifies greatly:

Proposition II.8. *The genus zero FJRW theory for the mirror quintic is concave, and*

$$-R\pi_*\Big(\bigoplus_{k=1}^{5} \mathbb{L}_k\Big)^\vee = R^1\pi_*\Big(\bigoplus_{k=1}^{5} \mathbb{L}_k\Big)^\vee.$$

Proof. We will show that over any geometric point $(\mathcal{C}, p_1, \ldots, p_n, \mathcal{L}_1, \ldots, \mathcal{L}_5, \varphi_1, \ldots, \varphi_5)$ in the moduli space, the fiber $\bigoplus_{k=1}^5 H^0(\mathcal{C}, \mathcal{L}_k) = 0$. This then implies the result. Let $f : \mathcal{C} \to C$ denote the map from the stack $\mathcal{C}$ to the coarse underlying curve C, and let $|\mathcal{L}_k|$ denote the push forward $f_*\mathcal{L}_k$. Then $H^0(\mathcal{C}, \mathcal{L}_k) \cong H^0(C, |\mathcal{L}_k|)$, thus it suffices to show that the line bundle $|\mathcal{L}_k|$ has no global sections.

Let Γ be the dual graph to C (see [17]). Recall that each vertex v of Γ corresponds to a rational curve component C_v. Let P_v denote the set of special points (marks and nodes) on C_v and k_v the number of such points. For $\tau \in P_v$, let $\mathrm{mult}_\tau(\mathcal{L}_k)$ be the multiplicity of $\mathcal{L}_k$ at the point τ. As in equation (2.4), the degree of the push forward $|\mathcal{L}_k|_{C_v}$ can be expressed in terms of the multiplicity at each special point:

$$\begin{aligned}\deg(|\mathcal{L}_k|_{C_v}) &= \tfrac{1}{5}(k_v - 2) - \tfrac{1}{5}\sum_{\tau\in P} \mathrm{mult}_\tau(\mathcal{L}_k) \\ &= -\tfrac{2}{5} + \tfrac{1}{5}\sum_{\tau\in P} (1 - \mathrm{mult}_\tau(\mathcal{L}_k))\end{aligned}$$

Since we have restricted our consideration to narrow sectors, $\mathrm{mult}_\tau(\mathcal{L}_k) > 0$ whenever τ is not a node. If C is irreducible, we see that $\deg(|\mathcal{L}_k|)$ is negative and $H^0(C, |\mathcal{L}_k|) = 0$. If C is reducible, each component of C_v has at least one node and we obtain the following inequality:

$$\deg(|\mathcal{L}_k|_{C_v}) \leq \tfrac{1}{5}(\#\text{nodes}(C_v) - 2) < \#\text{nodes}(C_v) - 1. \tag{2.5}$$

Since we are in genus 0, Γ is a tree. Choose one of the 1–valent vertices, v. There is only one node on the corresponding rational component C_v. By equation (2.5), $\deg(|\mathcal{L}_k|_{C_v}) < 0$ so any section of $|\mathcal{L}_k|$ must vanish on C_v. Choosing a vertex attached to $t+1$ edges, (2.5) yields $\deg(|\mathcal{L}_k|_{C_v}) < t$. Therefore if a section of $|\mathcal{L}_k|_{C_v}$ vanishes at t of the nodes, we see by degree considerations that it must be identically zero on C_v.

By starting at the outer vertices of Γ and working in, the above two facts allow one to show that a section of $|\mathcal{L}_k|$ must vanish on every component of C. □

On each W–curve in $W_{g,n,G}$ we have $\otimes_{k=1}^{5} \mathcal{L}_k \cong \omega_{\log}$. This implies that $\mathcal{L}_5$ is determined by $\mathcal{L}_1$,$\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$. We will use this fact to facilitate computation.

Let $(A_4)_{g,n}$ denote the moduli space of genus g, n–marked A_4–curves corresponding to the polynomial $A_4 = x^5$. Such W–structures are often referred to as 5–spin curves. Let $(A_4^4)_{g,n}$ denote the fiber product

$$(A_4^4)_{g,n} := (A_4)_{g,n} \times_{\overline{\mathcal{M}}_{g,n,5}} (A_4)_{g,n} \times_{\overline{\mathcal{M}}_{g,n,5}} (A_4)_{g,n} \times_{\overline{\mathcal{M}}_{g,n,5}} (A_4)_{g,n}$$

Proposition II.9. *There is a surjective map*

$$s : (A_4^4)_{g,n} \to W_{g,n,G}$$

which is a bijection at the level of a point.

Proof. The map is

$$(\mathcal{L}_1, \ldots, \mathcal{L}_4, \phi_1, \ldots, \phi_4) \to$$
$$\Big(\mathcal{L}_1, \ldots, \mathcal{L}_4, \Big(\big(\bigotimes_{k=1}^{4} \mathcal{L}_k^\vee\big) \otimes \omega_{log} \Big), \varphi_1, \ldots, \varphi_4, \varphi_1^\vee \otimes \cdots \otimes \varphi_4^\vee \otimes \mathrm{id} \Big).$$

Notice that the image of this map satisfies $\bigotimes_{k=1}^{5} \mathcal{L}_k \cong \omega_{\log}$, and the fifth line bundle in the image is a fifth root of $\omega_{\log}$. Furthermore, every point in $W_{g,n,G}$ is of this form. It is clear that this map is bijective at the level of a point. This implies the proposition. $\square$

Using the previous two propositions, we can give a more useful description of the genus zero correlators. Given $\mathbf{h} = (h_1, \ldots, h_n)$, let us denote

$$A_4^4(\mathbf{h})_{g,n} :=$$
$$(A_4)_{g,n}(\Theta_1(h_1), \ldots \Theta_1(h_n)) \times_{\overline{\mathcal{M}}_{g,n,d}} \cdots \times_{\overline{\mathcal{M}}_{g,n,d}} (A_4)_{g,n}(\Theta_4(h_1), \ldots, \Theta_4(h_n)).$$

Each factor of $(A_4)_{g,n}$ is equipped with a universal A_4–structure. Abusing notation, we denote the universal line bundle over the ith factor of $(A_4^4)_{g,n}$ also by $\mathbb{L}_i$. By the universal properties of the W–structure on $W_{g,n}$, we have $s^*\mathbb{L}_k \cong \mathbb{L}_k$ for $1 \leq k \leq 4$. Define $\mathbb{L}_5$ on $(A_4^4)_{g,n}$ as the pullback $s^*\mathbb{L}_5$.

In [14] the authors show that $[W_{0,n,G}]^{vir}$ is Poincaré dual to $5c_{top}\Big(R^1\pi_*\big(\bigoplus_{i=1}^{5} \mathbb{L}_i\big)^\vee\Big)$ as a consequence of concavity. By the projection formula, we can pull the correlators back to $(A_4^4)_{0,n}$. The map s has degree 5, so we get the following expression for the genus 0 correlators:

$$\langle \psi^{a_1}\phi_{h_1}, \ldots, \psi^{a_n}\phi_{h_n} \rangle_{0,n}^{(W,G)} = 625 \int_{A_4^4(\mathbf{h})_{0,n}} \prod_{i=1}^{n} \psi_i^{a_i} \cup c_{top}\Big(R^1\pi_*\big(\bigoplus_{i=1}^{5} \mathbb{L}_i\big)^\vee\Big)$$

CHAPTER III

The Calabi–Yau Model

The CY model is defined by GW theory for $\mathcal{W}$. However, as mentioned previously, we must use orbifold GW theory. Here we introduce the mirror quintic and describe the orbifold GW theory. This discussion can be found in greater detail in [20].

3.1 The State Space

Recall the pair (W, G) from Section 2.3. Let $\bar{G}$ denote the quotient $G/\langle \mathcal{J} \rangle$. Let $\mathcal{Y}$ denote the global quotient orbifold

$$\mathcal{Y} = [\mathbb{P}^4/\bar{G}]$$

where the $\bar{G}$–action on $\mathbb{P}^4$ comes from coordinate–wise multiplication. Note that this is the same as (1.2). The mirror quintic $\mathcal{W}$ is defined as the hypersurface

$$\mathcal{W} := \{W = 0\} \subset \mathcal{Y}.$$

The correct state space for orbifold Gromov–Witten theory is *Chen–Ruan orbifold cohomology*, defined via the *inertia orbifold* (see [3]). If $\mathcal{X} = [X/H]$ is a global quotient of a nonsingular variety X by a finite group H, the inertia orbifold $I\mathcal{X}$ takes a particularly simple form. Let S_H denote the set of conjugacy classes (h) in

H, then

$$I\mathcal{X} = \coprod_{(h)\in S_H} [X^h/C(h)].$$

As a vector space, the Chen–Ruan cohomology groups $H^*_{CR}(\mathcal{X})$ of an orbifold $\mathcal{X}$ are the cohomology groups of its inertia orbifold:

$$H^*_{CR}(\mathcal{X}) := H^*(I\mathcal{X}).$$

We will now describe the Chen–Ruan cohomology of the mirror quintic $\mathcal{W}$ (for more detail, see [20]). For an element $g \in G$, denote by $[g]$ the corresponding element in $\bar{G}$ and $I(g) := \{k \in \{1,2,3,4,5\} \mid \Theta_k(g) = 0\}$. The order of this set is N_g as defined in Section II.

Fix an element $\bar{g} \in \bar{G}$. Given $g \in G$ such that $[g] = \bar{g}$, the set

$$\mathbb{P}^4_g := \left\{x_j = 0\right\}_{j \notin I(g)} \subset \mathbb{P}^4$$

is a component of the fixed locus $(\mathbb{P}^4)^{\bar{g}}$. From this we see that each element $g \in G$ such that $[g] = \bar{g}$ corresponds to a connected component $\mathcal{Y}_g$ of $I\mathcal{Y}$ associated with $\mathbb{P}^4_g \subset (\mathbb{P}^4)^{\bar{g}}$. Note that if g has no coordinates equal to zero (i.e. $\Theta_k(g) = 0$) then $\mathbb{P}^4_g$ is empty, and so is $\mathcal{Y}_g$. This gives us a convenient way of indexing components of $I\mathcal{Y}$.

Let

$$\mathcal{Y}_g = \{(x, [g]) \in I\mathcal{Y} \mid x \in [\mathbb{P}^4_g/\bar{G}]\},$$

and let S denote the set of all g such that $\Theta_k(g)$ is equal to 0 for at least one k. Then

$$I\mathcal{Y} = \coprod_{g\in S} \mathcal{Y}_g\,,$$

with each $\mathcal{Y}_g$ a connected component.

The inertia orbifold of the mirror quintic $\mathcal{W}$ can be described in terms of $I\mathcal{Y}$. The mirror quintic $\mathcal{W}$ intersects nontrivially with $\mathcal{Y}_g$ exactly when $N_g \geq 2$. (that is, $\dim \mathcal{Y}_g \geq 1$.) Let

$$\tilde{S} := \left\{ g \in G \middle| N_g \geq 2 \right\}.$$

Then

$$I\mathcal{W} = \coprod_{g \in \tilde{S}} \mathcal{W}_g \text{, where } \mathcal{W}_g = \mathcal{W} \cap \mathcal{Y}_g.$$

All nontrivial intersections are transverse, so

$$\dim(\mathcal{W}_g) = \dim(\mathcal{Y}_g) - 1 = N_g - 2.$$

For $g \in \tilde{S}$, the *age* of g is defined as

$$\text{age}(g) := \sum_{k=1}^{5} \Theta_k(g).$$

The Chen–Ruan cohomology of $\mathcal{W}$ is defined, as a graded vector space, by

$$H^*_{CR}(\mathcal{W}) := \bigoplus_{g \in \tilde{S}} H^{*-2\,\text{age}(g)}(\mathcal{W}_g).$$

In FJRW theory we were interested only in $\mathscr{H}^{nar}_{W,G}$. Similarly, here we will only be interested in the subring of $H^*_{CR}(\mathcal{W})$ consisting of classes of even (real) degree. These are the *ambient classes*. We will denote this ring as $H^{even}_{CR}(\mathcal{W})$.

Let $\mathbb{1}_g$ denote the constant function with value one on $\mathcal{W}_g$. Let $\overline{H}$ denote the class in $H^*(\mathcal{Y})$ which pulls back to the hyperplane class in $\mathbb{P}^4$ and H the induced class on $\mathcal{W}$.

A convenient basis for $H^{even}_{CR}(\mathcal{W})$ is

$$\bigcup_{g \in \tilde{S}} \{\mathbb{1}_g, \mathbb{1}_g H, \ldots, \mathbb{1}_g H^{\dim(\mathcal{W}_g)}\}.$$

Let s represent the dual coordinate to $H \in H^*_{CR}(\mathcal{W})$. We denote by $H^2(\mathcal{W})$ the subspace sH of classes in $H^2_{CR}(\mathcal{W})$ supported on the *untwisted component* $\mathcal{W} \subset I\mathcal{W}$.

3.2 Orbifold GW invariants

We can also define the orbifold GW invariants for $\mathcal{W}$ (see e.g. [1] or [2]).There exists a moduli space of stable maps from genus–g, n–marked pre–stable orbifold curves to $\mathcal{W}$ of degree $\delta \in H_2(\mathcal{W})$, which we will denote by $\overline{\mathcal{M}}_{g,n}(\mathcal{W},\delta)$. Our orbifold curves are allowed to have non-trivial orbifold structure only at the marked points and nodes. As before, the nodes must be balanced (see Section 2.2). As described in [1] and [12], although there are not generally well–defined maps

$$ev_i : \overline{\mathcal{M}}_{g,n}(\mathcal{W},\delta) \to \mathcal{W}$$

there are maps

$$ev_i^* : H^*_{CR}(\mathcal{W}) \to H^*(\overline{\mathcal{M}}_{g,n}(\mathcal{W},\delta)$$

which behave as if evaluation maps existed, just as in the non–orbifold setting. Under these maps, we can pull back cohomology classes from $\mathcal{W}$ to $\overline{\mathcal{M}}_{g,n}(\mathcal{W},\delta)$.

Similar to FJRW theory, we can define ψ classes $\psi_i \in H^*(\overline{\mathcal{M}}_{g,n}(\mathcal{W},\delta)$. The class ψ_i is the first Chern class of the line bundle whose fiber over a point is the cotangent line to the coarse underlying curve at the i–th marked point.

There is a virtual class $[\overline{\mathcal{M}}_{g,n}(\mathcal{W},\delta)]^{vir}$, which allows us to define the orbifold GW invariants for $\alpha_i \in H^*_{CR}(\mathcal{W})$

$$\langle \psi^{a_1}\alpha_1, \ldots, \psi^{a_n}\alpha_n \rangle^{\mathcal{W}}_{g,n,\delta} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathcal{W},\delta)]^{vir}} \prod_{i=1}^{n} ev_i^*(\alpha_i)\psi_i^{k_i}$$

Summing over the degree, we write

$$\langle \psi^{a_1}\alpha_1, \ldots, \psi^{a_n}\alpha_n \rangle^{\mathcal{X}}_{g,n} := \sum_{\delta} Q^{\delta} \langle \psi^{a_1}\alpha_1, \ldots, \psi^{a_n}\alpha_n \rangle^{\mathcal{X}}_{g,n,\delta},$$

where the Q^{δ} are formal Novikov variables used to guarantee convergence.

CHAPTER IV

Givental Formalism

In this chapter, we will discuss Givental's formalism, which provides the backdrop for the respective J–functions.

Let $\square$ denote a theory—either the Gromov–Witten theory of a space $\mathcal{X}$ or the FJRW theory of a pair (W, G)—with state space $\left(H^{\square}, \langle -, - \rangle_{\square}\right)$ containing the basis $\{\beta_i\}_{i \in I}$ and invariants

$$\langle \psi^{a_1}\beta_{i_1}, \ldots, \psi^{a_n}\beta_{i_n}\rangle^{\square}_{g,n}.$$

We may define formal generating functions of $\square$–invariants. Let $\mathbf{t} = \sum_{i \in I} t^i \beta_i$ represent a point of $H^{\square}$ written in terms of the basis. For notational convenience denote the formal series $\sum_{k \geq 0} \mathbf{t}_k \psi^k$ as $\mathbf{t}(\psi)$. Define the genus g generating function by

$$\mathcal{F}^{\square}_g := \sum_n \frac{1}{n!}\langle \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi)\rangle^{\square}_{g,n}.$$

Let $\mathcal{D}$ denote the *total genus descendent potential*,

$$\mathcal{D}^{\square} := \exp\left(\sum_{g \geq 0} \hbar^{g-1}\mathcal{F}^{\square}_g\right).$$

As in Gromov–Witten theory, the correlators in FJRW theory satisfy the so–called *string equation* (SE), *dilation equation* (DE), and *topological recursion relation* (TRR) (For the proof in orbifold Gromov–Witten theory see [23], in the case of

FJRW theory see [14]). These equations can be formulated in terms of differential equations satisfied by the various genus g generating functions $\mathcal{F}_g^{\square}$. We can use this extra structure to rephrase the theory in terms of Givental's *overruled Lagrangian cone*. For a more detailed exposition of what follows we refer the reader to Givental's original paper on the subject [15].

Let $\mathscr{V}^{\square}$ denote the vector space $H^{\square}((z^{-1}))$, equipped with the symplectic pairing

$$\Omega_{\square}(f_1, f_2) := \mathrm{Res}_{z=0}\langle f_1(-z), f_2(z)\rangle_{\square}. \tag{4.1}$$

$\mathscr{V}^{\square}$ admits a natural polarization $\mathscr{V}^{\square} = \mathscr{V}_+^{\square} \oplus \mathscr{V}_-^{\square}$ defined in terms of powers of z:

$$\begin{aligned} \mathscr{V}_+^{\square} &= H^{\square}[z], \\ \mathscr{V}_-^{\square} &= z^{-1}H^{\square}[[z^{-1}]]. \end{aligned}$$

We obtain Darboux coordinates $\{q_k^i, p_{k,i}\}$ with respect to the polarization on $\mathscr{V}^{\square}$ by representing each element of $\mathscr{V}^{\square}$ in the form

$$\sum_{k\geq 0}\sum_{i\in I} q_k^i \beta_i z^k + \sum_{k\geq 0}\sum_{i\in I} p_{k,i}\beta^i(-z)^{-k-1}$$

One can view $\mathcal{F}_0^{\square}$ as the generating function of a Lagrangian subspace $\mathscr{L}^{\square}$ of $\mathscr{V}^{\square}$. Let β_0 denote the unit in $H^{\square}$, and make the change of variables (the so–called Dilaton shift)

$$q_1^0 = t_1^0 - 1 \quad q_k^i = t_k^i \text{ for } (k,i) \neq (1,0).$$

Then the set

$$\mathscr{L}^{\square} := \left\{ \mathbf{p} = d_{\mathbf{q}}\mathcal{F}_0^{\square} \right\}$$

defines a Lagrangian subspace. More explicitly, $\mathscr{L}^{\square}$ contains the points of the form

$$-\beta_0 z + \sum_{\substack{k\geq 0\\ i\in I}} t^i_k \beta_i z^k + \sum_{\substack{a_1,\ldots,a_n,a\geq 0\\ i_1,\ldots,i_n,i\in I}} \frac{t^{i_1}_{a_1}\cdots t^{i_n}_{a_n}}{n!(-z)^{a+1}} \langle \psi^{a_1}\beta_{i_1},\ldots,\psi^{a_n}\beta_{i_n},\psi^a\beta_i\rangle^{\square}_{0,n+1}\beta^i.$$

Because $\mathcal{F}^{\square}_0$ satisfies the SE, DE, and TRR, $\mathscr{L}^{\square}$ will take a special form. In fact, $\mathscr{L}^{\square}$ is a cone satisfying the condition that for all $f \in \mathscr{V}^{\square}$,

$$\mathscr{L}^{\square} \cap L_f = zL_f \tag{4.2}$$

where L_f is the tangent space to $\mathscr{L}^{\square}$ at f. Equation (4.2) justifies the term overruled, as each tangent space L_f is filtered by powers of z:

$$L_f \supset zL_f \supset z^2 L_f \supset \cdots$$

and $\mathscr{L}^{\square}$ itself is ruled by the various zL_f. The codimension of zL_f in L_f is equal to $\dim(H^{\square})$.

A generic slice of $\mathscr{L}^{\square}$ parametrized by $H^{\square}$, i.e.

$$\{f(\mathbf{t})|\mathbf{t}\in H^{\square}\} \subset \mathscr{L}^{\square},$$

will be transverse to the ruling. Given such a slice, we can reconstruct $\mathscr{L}^{\square}$ as

$$\mathscr{L}^{\square} = \left\{ zL_{f(\mathbf{t})} | \mathbf{t}\in H^{\square}\right\}. \tag{4.3}$$

Givental's J–function is defined in terms of the intersection

$$\mathscr{L}^{\square} \cap -\beta_0 z \oplus H \oplus \mathscr{V}^{-}.$$

Writing things out explicitly, the J–function is given by

$$J^{\square}(\mathbf{t},z) = \beta_0 z + \mathbf{t} + \sum_{n\geq 0}\sum_{\substack{a\geq 0\\ h\in I}} \frac{1}{n! z^{a+1}} \langle \mathbf{t},\ldots,\mathbf{t},\beta_h\psi^a\rangle^{\square}_{0,n+1}\beta^h.$$

In other words, we can obtain the J–function by setting $t_k^i = 0$ whenever $k > 0$.

In [16] it is shown that the image of $J^{\square}(\mathbf{t}, -z)$ is transverse to the ruling of $\mathscr{L}^{\square}$, so $J^{\square}(\mathbf{t}, -z)$ is a function satisfying (4.3). Thus the ruling at $J^{\square}(\mathbf{t}, -z)$ is spanned by the derivatives of $J^{\square}$, i.e.

$$zL_{J^{\square}(\mathbf{t},-z)} = \{J^{\square}(\mathbf{t}, -z) + z\sum c_i(z)\frac{\partial}{\partial t^i}J^{\square}(\mathbf{t}, -z) | c_i(z) \in \mathbb{C}[z]\}. \tag{4.4}$$

By the string equation, $z\frac{\partial}{\partial t^0}J^{\square}(\mathbf{t}, z) = J^{\square}(\mathbf{t}, z)$, so (4.4) simplifies to

$$zL_{J^{\square}(\mathbf{t},-z)} = \{z\sum c_i(z)\frac{\partial}{\partial t^i}J^{\square}(\mathbf{t}, -z) | c_i(z) \in \mathbb{C}[z]\}.$$

4.1 Quantization of Symplectic operators

One of the useful tools in this formalism, is quantization of infinitesimal symplectic operators, which we will briefly introduce here. Details can be found in [9] and [10]

Suppose that T is an endomorphism of $\mathscr{V}^{\square}$ of the form

$$T = \sum_{m\geq 0} B_m z^m \tag{4.5}$$

where $B_m : H^{\square} \to H^{\square}$. Let B_m^* denote the adjoint with respect to the pairing $\langle , \rangle_{\square}$. Then the symplectic adjoint of T is

$$T^*(-z) := \sum_{m\geq 0} B_m^*(-z)^m.$$

We say T is symplectic if $\Omega_{\square}(Tf_1, Tf_2) = \Omega_{\square}(f_1, f_2)$. This is equivalent to

$$T^*(-z)T(z) = \mathrm{Id}\,.$$

We will be interested in a transformation of the form

$$T = \exp(A)$$

in which A also has the form (4.5). In this case, T being symplectic is equivalent to A being *infinitesimal symplectic*, or in other words that $\Omega(Af,g)+\Omega(f,Ag)=0$ for any $f,g\in\mathcal{V}^{\square}$. This in turn implies that

$$A_m^* = (-1)^{m+1}A_m. \tag{4.6}$$

The process of quantization then proceeds as we now describe. First we quantize quadratic monomials in the variables $\{q_k^i, p_{k,i}\}$ as follows:

$$\widehat{q_k^i q_\ell^j} = q_k^i q_\ell^j$$
$$\widehat{p_{k,i}p_{\ell,j}} = \hbar^2 \frac{\partial}{\partial q_k^i}\frac{\partial}{\partial q_\ell^j},$$
$$\widehat{q_k^i p_{\ell,j}} = \hbar q_k^i \frac{\partial}{\partial q_\ell^j}$$

In order to quantize a symplectomorphism $T=\exp(A)$ of the form (4.5) define a function h_A on $\mathcal{V}^{\square}$ by

$$h_A(f) = \frac{1}{2}\Omega_{\square}(Af,f).$$

Since h_A is a quadratic in the variables $\{q_k^i, p_{k,i}\}$, it can be quantized by the above formula. We define the quantization of A via

$$\hat{A} = \frac{1}{\hbar}\widehat{h_A},$$

and $\hat{T}$ is defined by

$$\hat{T} = \exp(\hat{A}).$$

We will illustrate this procedure with the following example.

Example IV.1. Suppose that the infinitesimal symplectic transformation A is of the form

$$A = A_m z^m,$$

where $A_m : H^{\square} \to H^{\square}$ is a linear transformation and $m > 0$.

To compute $\hat{A}$, one must first compute the Hamiltonian h_A. Let

$$f(z) = \sum_{k\geq 0}\sum_{i\in I} p_{k,i}\beta^i(-z)^{-1-k} + \sum_{\ell\geq 0}\sum_{j\in I} q^j_\ell \beta_j z^\ell \in \mathscr{V}^{\square}.$$

Then

$$\begin{aligned} h_A(f) &= \frac{1}{2}\Omega_{\square}(Af,f) \\ &= \frac{1}{2}\mathrm{Res}_{z=0}\Big\langle (-z)^m \sum_{k_1\geq 0}\sum_{i\in I} p_{k_1,i}(A_m\beta^i)\, z^{-1-k_1} + (-z)^m \sum_{\ell_1\geq 0}\sum_{j\in I} q^j_{\ell_1}(A_m\beta_j)(-z)^{\ell_1}, \\ &\qquad \sum_{k_2\geq 0}\sum_{i\in I} p_{k_2,i}\beta^i(-z)^{-1-k_2} + \sum_{\ell_2\geq 0}\sum_{j\in I} q^j_{\ell_2}\beta_j z^{\ell_2}\Big\rangle_{\square} \end{aligned}$$

Since only the z^{-1} terms contribute to the residue, the right-hand side is equal to

$$\frac{1}{2}\sum_{k\geq 0}^{m-1}\sum_{i,j\in I}(-1)^k p_{k,i}p_{m-k-1,j}\left\langle A_m\beta^i, \beta^j\right\rangle_{\square} + \frac{1}{2}\sum_{k\geq 0}\sum_{i,j\in I}(-1)^m p_{m+k,i}q^j_k\left\langle A_m\beta^i, \beta_j\right\rangle_{\square}$$
$$- \frac{1}{2}\sum_{k\geq 0}\sum_{i,j\in I} q^j_k p_{m+k,i}\left\langle A_m\beta_j, \beta^i\right\rangle_{\square}.$$

By (4.6), we have

$$\left\langle \beta^i, A_m\beta_j\right\rangle_{\square} = (-1)^{m+1}A_m\beta^i, \beta_j.$$

Thus, denoting $(A_m)^i_j = \langle A_m\beta^i, \beta_j\rangle_{\square}$ and $(A_m)^{ij} = \langle A_m\beta^i, \beta^j\rangle_{\square}$, we can write

$$h_A(f) = \frac{1}{2}\sum_{k=0}^{m-1}\sum_{i,j\in I}(-1)^k p_{k,i}p_{m-k-1,j}(A_m)^{ij} + (-1)^m\sum_{k\geq 0}\sum_{i,j\in I} p_{m+k,i}q^j_k(A_m)^i_j.$$

This implies that

$$\hat{A} = \frac{\hbar}{2}\sum_{k=0}^{m-1}\sum_{i,j\in I}(-1)^k(A_m)^{ij}\frac{\partial}{\partial q^i_k}\frac{\partial}{\partial q^j_{m-k-1}} + (-1)^m\sum_{k\geq 0}\sum_{i,j\in I}(A_m)^i_j q^j_k\frac{\partial}{\partial q^i_{m+k}}. \tag{4.7}$$

From this example, it is not difficult to deduce the general form of $\hat{A}$ for any A of the form (4.5).

4.2 The conjecture

The LG/CY correspondence was first proposed by physicists (see [24, 25]), and is given as a conjecture in [5]. It is phrased mathematically as a correspondence between Gromov–Witten invariants of a Calabi–Yau manifold, and the FJRW invariants of a specified pair (W, G). In genus 0, the correspondence can be interpreted in terms of the Lagrangian cones of the respective theories. In [5] the genus 0 conjecture is proven for the Fermat quintic using this interpretation. For simplicity we state the conjecture below only in the particular case of the *mirror quintic*.

In what follows we will use (W, G) in place of $\square$ to denote the FJRW theory of (W, G) and $\mathcal{W}$ in place of $\square$ to denote the Gromov–Witten theory of $\mathcal{W}$. The FJRW and Gromov–Witten state spaces will be $\mathscr{H}_{W,G}^{nar}$ and $H_{CR}^{even}(\mathcal{W})$ respectively. The full LG/CY correspondence may be stated as a relationship between $\mathcal{D}^{(W,G)}$ and the analytic continuation of $\mathcal{D}^{\mathcal{W}}|_{Q^d=1}$, where the latter represents the total genus descendant potential for $\mathcal{W}$ after setting the Novikov variable to one. Once Novikov variables have been set to one, the conjecture may be phrased as follows:

Conjecture IV.2 ([5]). *Let $\mathscr{V}^{(W,G)}$ and $\mathscr{V}^{\mathcal{W}}$ be the Givental spaces corresponding to the FJRW theory of (W, G) and the Gromov–Witten theory of $\mathcal{W}$.*

1. *There is a degree–preserving $\mathbb{C}[z, z^{-1}]$–valued linear symplectic isomorphism*

$$\mathbb{U} : \mathscr{V}^{(W,G)} \to \mathscr{V}^{\mathcal{W}}$$

and a choice of analytic continuation of $\mathscr{L}^{\mathcal{W}}$ such that

$$\mathbb{U}(\mathscr{L}^{(W,G)}) = \mathscr{L}^{\mathcal{W}}.$$

2. *After analytic continuation, up to an overall constant the total potential functions are related by quantization of* $\mathbb{U}$*, i.e.*

$$\mathcal{D}^{\mathcal{W}} = \hat{\mathbb{U}}(\mathcal{D}^{(W,G)}).$$

Remark IV.3. It is not guaranteed that $\mathcal{D}^{\mathcal{W}}|_{Q^d=1}$ is an analytic function. Implicit in the conjecture, however, is the claim that after setting the Novikov variables to one, $\mathcal{D}^{\mathcal{W}}$ converges in some neighborhood. Thus one must first check convergence in order to prove the LG/CY correspondence. For the purposes of this paper however, the necessary convergence will follow from the mirror theorem of [20] restated here in equation (6.1).

4.2.1 The Small Slice of $\mathscr{L}$

In [5], the LG/CY correspondence is proven by relating the respective J–functions for the two theories. A crucial point in the argument is that in the case of the quintic three–fold M, the J–function $J^M(\mathbf{s}, z)$ (and hence the full Lagrangian cone $\mathscr{L}^M$) may be recovered from the small J–function

$$J^M_{small}(s,z) := J^M(\mathbf{s}, z)|_{\mathbf{s}=s\in H^2(M)}.$$

This is no longer true for the mirror quintic.

Although calculating the big J–function for $\mathcal{W}$ appears to be a difficult problem, in [20] its derivatives $\frac{\partial}{\partial s^i} J^{\mathcal{W}}(\mathbf{s}, z)$ may be calculated at any point $sH \in H^2(\mathcal{W})$. This allows us to prove a "small" version of the LG/CY correspondence for the mirror quintic. We will phrase the theorem in analogy with Conjecture IV.2.

In order to do so we define the *small slice* of $\mathscr{L}^{\mathcal{W}}$ and $\mathscr{L}^{(W,G)}$ to be that part of the ruling coming from $sH \in H^2(\mathcal{W})$ and $t\phi_{\mathcal{J}^2} \in \mathscr{H}^2_{W,G}$ respectively:

Definition IV.4. The small slices of $\mathscr{L}^{\mathcal{W}}$ and $\mathscr{L}^{(W,G)}$ are defined by

$$\mathscr{L}^{\mathcal{W}}_{small} := \{zL_{J^{\mathcal{W}}(\mathbf{s},-z)} | \mathbf{s} = sH\}.$$

$$\mathscr{L}^{(W,G)}_{small} := \{zL_{J^{(W,G)}(\mathbf{t},-z)} | \mathbf{t} = t\phi_{\mathcal{J}^2}\}.$$

Our main theorem may then be stated as a correspondence between the small slices of the Lagrangian cones $\mathscr{L}^{(W,G)}$ and $\mathscr{L}^{\mathcal{W}}$.

Theorem IV.5. *(=Theorem VI.4) There exists a symplectic transformation $\mathbb{U}$ identifying the analytic continuation of $\mathscr{L}^{\mathcal{W}}_{small}$ with $\mathscr{L}^{(W,G)}_{small}$.*

CHAPTER V

A Landau–Ginzburg mirror theorem

In this section we prove a mirror theorem for (W, G), which will be necessary to prove the LG/CY correspondence. A similar theorem for the GW theory of $\mathcal{W}$ was proven in [20]. Fix as a basis for $\mathscr{H}^{nar}_{W,G}$ the set $\{\phi_h\}_{h\in\hat{S}}$ defined in Section 2.3.

5.1 The twisted theory

We will construct a *twisted* FJRW theory whose invariants coincide with those of (W, G) in genus zero. We first extend the state space

$$\mathscr{H}^{ext}_{W,G} := \mathscr{H}^{nar}_{W,G} \oplus \bigoplus_{h\in G\setminus\hat{S}} \phi_h\mathbb{C}.$$

Any point $\mathbf{t} \in \mathscr{H}^{ext}_{W,G}$ can be written as $\mathbf{t} = \sum_{h\in G} t^h\phi_h$. Let $i_k(h) := \langle\Theta_k(h) - \frac{1}{5}\rangle$, where $\langle - \rangle$ denotes the fractional part. Notice $i_k(h) = \frac{4}{5}$ exactly when $\Theta_k(h) = 0$. Set

$$\deg_W(\phi_h) := 2\sum_{k=1}^{5} i_k(h).$$

For $h \in \hat{S}$, this definition matches the W–degree defined in (2.2).

We extend the definition of our FJRW invariants to include insertions ϕ_h in $\mathscr{H}^{ext}_{W,G}$. Namely, set

$$\langle \psi^{a_1}\phi_{h_1}, \ldots, \psi^{a_n}\phi_{h_n} \rangle^{(W,G)}_{0,n} = 0$$

if $h_i \in G \setminus \hat{S}$.

We would like to unify our definition of the extended FJRW invariants, by re–expressing them as integrals over $(\widetilde{A}_4^4)_{0,n}$, a slight variation of $(A_4^4)_{0,n}$. We will make use of the following lemma.

Lemma V.1 (Chiodo–Ruan [5]). *Let $\mathcal{C}$ be a d–stable curve and let M be a line bundle pulled back from the coarse space. If $l|d$, there is an equivalence between two categories of lth roots $\mathcal{L}$ on d–stable curves:*

$$\left\{\mathcal{L}|\mathcal{L}^{\otimes l} \cong M\right\} \leftrightarrow \coprod_{0 \leq E < \sum lD_i} \left\{\mathcal{L}|\mathcal{L}^{\otimes l} \cong M(-E), \mathrm{mult}_{p_i}(\mathcal{L}) = 0\right\}.$$

where the union is taken over divisors E which are linear combinations of integer divisors D_i corresponding to the marked points p_i.

Proof. Let p denote the map which forgets stabilizers along the markings. The correspondence is simply $\mathcal{L} \mapsto p^* p_*(\mathcal{L})$. □

Definition V.2. For $m_1, \ldots, m_n \in \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$, consider the stack $\tilde{A}_4 \left(m_1, \ldots, m_n\right)_{g,n}$ classifying genus g, n–pointed, 5–stable curves equipped with fifth roots:

$$\widetilde{A}_4 \left(m_1, \ldots, m_n\right)_{g,n} := \left\{(\mathcal{C}, p_1, \ldots, p_n, \mathcal{L}, \varphi)|\phi : \mathcal{L}^{\otimes 5} \xrightarrow{\sim} \omega_{\log}(-\sum_{i=1}^{n} 5m_i D_i),\ \mathrm{mult}_{p_i}(\mathcal{L}) = 0\right\},$$

where the integer divisors D_i correspond to the markings p_i.

The moduli space $\widetilde{A}_4 \left(m_1, \ldots, m_n\right)_{g,n}$ also has a universal curve $\mathscr{C} \to \widetilde{A}_4$ and a universal line bundle $\widetilde{\mathbb{L}}$.

We now define an analogue of $(A_4^4)_{g,n}$, replacing $(A_4)_{g,n}$ with $(\widetilde{A}_4)_{g,n}$ in each factor. For $1 \leq i \leq n$, let $m_i = (m_{1i}, \ldots, m_{5i})$ be a 5-tuple of fractions satisfying $m_{ki} \in \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$, and $\langle \sum_{k=1}^{5} m_{ki} \rangle = 0$. Let $\mathbf{m}$ denote the $5 \times n$ matrix $(\mathbf{m})_{ki} = m_{ki}$.

Define

$$\widetilde{A}_4^4(\mathbf{m})_{g,n} := \widetilde{A}_4(m_{11},\ldots,m_{1n})_{g,n} \times_{\overline{\mathcal{M}}_{g,n,5}} \cdots \times_{\overline{\mathcal{M}}_{g,n,5}} \widetilde{A}_4(m_{41},\ldots,m_{4n})_{g,n}.$$

$\widetilde{A}_4^4(\mathbf{m})_{g,n}$ carries four universal line bundles $\widetilde{\mathbb{L}}_1,\ldots,\widetilde{\mathbb{L}}_4$ satisfying

$$(\widetilde{\mathbb{L}}_k)^{\otimes 5} \cong \omega_{\log}\left(-\sum_{i=1}^{n} 5m_{ki}D_i\right).$$

Define a fifth line bundle

$$\widetilde{\mathbb{L}}_5 := \widetilde{\mathbb{L}}_1^{\vee} \otimes \cdots \otimes \widetilde{\mathbb{L}}_4^{\vee} \otimes \omega_{\log}\left(-\sum_{i=1}^{n}\sum_{k=1}^{5} m_{ki}D_i\right).$$

One can check that $(\widetilde{\mathbb{L}}_5)^{\otimes 5} \cong \omega_{\log}(-\sum_{i=1}^{n} 5m_{5i}D_i)$.

The above moduli space yields a uniform way of defining the extended FJRW invariants for (W,G). Given $\phi_{h_1},\ldots,\phi_{h_n} \in \mathscr{H}_{W,G}^{ext}$, let

$$I(\mathbf{h}) = \begin{pmatrix} i_1(h_1)+\frac{1}{5} & \cdots & i_1(h_n)+\frac{1}{5} \\ \vdots & & \vdots \\ i_5(h_1)+\frac{1}{5} & \cdots & i_5(h_n)+\frac{1}{5} \end{pmatrix}.$$

Consider the following proposition.

Proposition V.3. *On $\widetilde{A}_4^4(I(\mathbf{h}))_{0,n}$, the sheaf $\pi_*\big(\bigoplus_{k=1}^5 \widetilde{\mathbb{L}}_k\big)$ vanishes and $R^1\pi_*\big(\bigoplus_{k=1}^5 \widetilde{\mathbb{L}}_k\big)$ is locally free. Furthermore,*

$$(5.1)\quad \left\langle \psi^{a_1}\phi_{h_1},\ldots,\psi^{a_n}\phi_{h_n}\right\rangle_{0,n}^{(W,G)} = 625\int_{\widetilde{A}_4^4(I(\mathbf{h}))_{0,n}} \prod \psi_i^{a_i} \cup c_{top}\Big(R^1\pi_*\big(\bigoplus_{k=1}^{5}\widetilde{\mathbb{L}}_k\big)^{\vee}\Big).$$

Proof. Comparing A_4 and $\widetilde{A}_4$, we see that if $m_{ki} \in \left\{\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\right\}$ for all k,i, we can identify $\widetilde{A}_4^4(\mathbf{m})_{g,n}$ with $A_4^4(\mathbf{m})_{g,n}$ via Lemma V.1. Under this identification $R^j\pi_*(\widetilde{\mathbb{L}}_k) = R^j\pi_*(\mathbb{L}_k)$. This gives (5.1) in the case $\phi_{h_1},\ldots,\phi_{h_n} \in \mathscr{H}_{W,G}^{nar}$.

To finish the proof we must consider the case where $h_i \in G \setminus \hat{S}$ for some i. In this case $(I(\mathbf{h}))_{ki} = 5$ for some k. Thus it suffices to prove that if $m_{ki} = 5$ for some i and k, then $\pi_*\big(\bigoplus_{k=1}^5 \widetilde{\mathbb{L}}_k\big) = 0$ and $c_{top}\Big(R^1\pi_*\big(\bigoplus_{k=1}^5 \widetilde{\mathbb{L}}_k\big)\Big) = 0$.

Without loss of generality assume $m_{k1} = 5$. Consider the integer divisor D_1 on $\widetilde{A}_4^4(\mathbf{m})_{0,n}$ corresponding to the first marked point. We have the following exact sequence

$$0 \to \widetilde{\mathbb{L}}_k \to \widetilde{\mathbb{L}}_k(D_1) \to \widetilde{\mathbb{L}}_k(D_1)|_{D_1} \to 0.$$

This gives rise to the long exact sequence

$$0 \to \pi_*(\widetilde{\mathbb{L}}_k) \to \pi_*(\widetilde{\mathbb{L}}_k(D_1)) \to \pi_*(\widetilde{\mathbb{L}}_k(D_1)|_{D_1})$$
$$\to R^1\pi_*(\widetilde{\mathbb{L}}_k) \to R^1\pi_*(\widetilde{\mathbb{L}}_k(D_1)) \to R^1\pi_*(\widetilde{\mathbb{L}}_k(D_1)|_{D_1}) \to 0.$$

The first two terms are 0. Indeed, consider first $\pi_*(\widetilde{\mathbb{L}}_k)$. The fiber over the point $(\mathcal{C}, p_1, \ldots, p_n, \widetilde{\mathcal{L}}_1, \ldots, \widetilde{\mathcal{L}}_5)$ is equal to $H^0(\mathcal{C}, \widetilde{\mathcal{L}}_k)$. As in Proposition II.8 we will show that $\widetilde{\mathcal{L}}_k$ has no global sections by computing its degree on each irreducible component of $\mathcal{C}$. If $\mathcal{C}$ is irreducible, $\deg(\widetilde{\mathcal{L}}_k) < 0$ and the claim follows. If not, let Γ denote the dual graph to $\mathcal{C}$, let v be a vertex corresponding to the irreducible component $\mathcal{C}_v$ and let P_v be the set of special points of $\mathcal{C}_v$. As in Proposition II.8, we obtain the inequality $\deg(\widetilde{\mathcal{L}}_k|_{\mathcal{C}_v}) < \#\text{nodes}(\mathcal{C}_v) - 1$. Again one can proceed vertex by vertex starting from outer vertices of Γ and show that the restriction of $\widetilde{\mathcal{L}}_k$ to each component has no nonzero global sections.

We can do the same with $\pi_*(\widetilde{\mathbb{L}}_k(D_1))$, with one alteration. If $\mathcal{C}$ is reducible, and v' corresponds to the irreducible component carrying the first marked point, then $\deg \widetilde{\mathcal{L}}_k(D_1)|_{\mathcal{C}_{v'}} < \#\text{nodes}(\mathcal{C}_{v'})$. But any section of $\widetilde{\mathcal{L}}_k(D_1)$ must still vanish on all other components of C, and by degree considerations it must therefore vanish on $\mathcal{C}_{v'}$.

D_1 is zero–dimensional on each fiber, so $R^1\pi_*(\widetilde{\mathbb{L}}_k(D_1)|_{D_1})$ also vanishes. The above long exact sequence above becomes

$$0 \to \pi_*\widetilde{\mathbb{L}}_k(D_1)|_{D_1} \to R^1\pi_*\widetilde{\mathbb{L}}_k \to R^1\pi_*\widetilde{\mathbb{L}}_k(D_1) \to 0.$$

Therefore

$$c_{top}(R^1\pi_*\widetilde{\mathbb{L}}_k) = c_{top}(\pi_*\widetilde{\mathbb{L}}_k(D_1)|_{D_1}) \cdot c_{top}(R^1\pi_*\widetilde{\mathbb{L}}_k(D_1)).$$

But $c_{top}(\pi_*\widetilde{\mathbb{L}}_k(D_1)|_{D_1}) = 0$, as $\widetilde{\mathbb{L}}_k(D_1)|_{D_1} \cong \mathbb{L}_k|_{D_1}$ is a fifth root of $\omega_{\log}|_{D_1}$ which is trivial. Thus $c_{top}(R^1\pi_*\widetilde{\mathbb{L}}_k) = 0$ as well. □

We may define a $\mathbb{C}^*$–equivariant generalization of the above theory. This will allow us to compute invariants which, in the non–equivariant limit coincide with the genus zero FJRW invariants above. Given a point $(\mathcal{C}, p_1, \ldots, p_n, \widetilde{\mathcal{L}}_1, \ldots, \widetilde{\mathcal{L}}_5)$ in $(\widetilde{A}_4^4)_{g,n}$, let $\mathbb{C}^*$ act on the total space of $\bigoplus_{k=1}^5 \widetilde{\mathcal{L}}_k$ by multiplication of the fiber. This induces an action on $(\widetilde{A}_4^4)_{g,n}$.

Set $R = H^*_{\mathbb{C}^*}(pt, \mathbb{C})[[s_0, s_1, \ldots]]$, the ring of power series in the variables $s_0, s_1, \ldots$ with coefficients in the equivariant cohomology of a point, $H^*_{\mathbb{C}^*}(pt, \mathbb{C}) = \mathbb{C}[\lambda]$. Define a multiplicative characteristic class $\mathbf{c}$ taking values in R, by

$$\mathbf{c}(E) := \exp\left(\sum_k s_k \operatorname{ch}_k(E)\right)$$

for $E \in K^*((\widetilde{A}_4^4)_{g,n})$.

Define the twisted state space

$$\mathscr{H}^{tw} := \mathscr{H}^{ext}_{W,G} \otimes R \cong \bigoplus_{h \in G} R \cdot \phi_h$$

and extend the pairing by

$$\left\langle \phi_{h_1}, \phi_{h_2} \right\rangle := \begin{cases} \prod_{\{k \,|\, i_k(h_1) = 4/5\}} \exp(-s_0) & \text{if } h_1 = (h_2)^{-1} \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

In this definition, the empty product is understood to be 1.

We define the symplectic vector space $\mathscr{V}^{tw} := \mathscr{H}_{tw}((z^{-1}))$, with the symplectic pairing defined as in equation (4.1).

We may also define *twisted correlators* as follows. Given $\phi_{h_1}, \ldots, \phi_{h_n}$ basis elements in $\mathscr{H}^{tw}$, define the invariant

$$\langle \psi^{a_1}\phi_{h_1}, \ldots, \psi^{a_n}\phi_{h_n} \rangle_{g,n}^{tw} := 625 \int_{\tilde{A}_4^4(I(\mathbf{h}))_{g,n}} \prod \psi_i^{a_i} \cup \mathbf{c}\Big(R\pi_*\big(\bigoplus_{k=1}^{5} \widetilde{\mathbb{L}_k}\big)\Big).$$

taking values in R. We can organize these invariants into generating functions $\mathcal{F}_g^{tw}$ and $\mathcal{D}^{tw}$ as in Section IV.

Specializing to particular values of s_d yield different twisted invariants. In particular, if $s_d = 0$ for all d, we get what is referred to as the *untwisted theory*. We will denote the generating functions of the untwisted theory by $\mathcal{F}_g^{un}$ and $\mathcal{D}^{un}$.

On the other hand, setting

$$s_d = \begin{cases} -\ln\lambda & \text{if } d = 0 \\ \frac{(d-1)!}{\lambda^d} & \text{otherwise} \end{cases} \tag{5.3}$$

we obtain the (extended) FJRW–theory invariants defined above. To see this first consider the following lemma.

Lemma V.4. *[5, Lemma 4.1.2] With s_d defined as in* (5.3)*, the multiplicative class* $\mathbf{c}(-V) = e_{\mathbb{C}^*}(V^\vee)$*. In particular, the non–equivariant limit yields the top chern class of* $V^\vee$*.*

Proof. We can check this on a line bundle, and then apply the splitting principle.

Consider a line bundle $\mathcal{L}$. Then we have

$$\begin{aligned}
\exp\left(\sum_{d\geq 0} s_d \operatorname{ch}_d(-\mathcal{L})\right) &= \exp\left(\ln\lambda \operatorname{ch}_0(\mathcal{L}) - \sum_{d>0} s_d \operatorname{ch}_d(\mathcal{L})\right) \\
&= \exp\left(\ln\lambda \operatorname{ch}_0(\mathcal{L}^\vee) - \sum_{d>0} (-1)^d s_d \operatorname{ch}_d(\mathcal{L}^\vee)\right) \\
&= \exp\left(\ln\lambda \operatorname{ch}_0(\mathcal{L}^\vee) + \sum_{d>0} (-1)^{d-1} \tfrac{(d-1)!}{\lambda^d} \operatorname{ch}_d(\mathcal{L}^\vee)\right) \\
&= \lambda \exp\left(\sum_{d>0} (-1)^{d-1} \tfrac{c_1(\mathcal{L}^\vee)^d}{d\lambda^d}\right) \\
&= \lambda \exp\left(\ln(1 + \frac{c_1(\mathcal{L}^\vee)}{\lambda}\right) \\
&= \lambda + c_1(\mathcal{L}^\vee)
\end{aligned}$$

□

By Proposition II.8, $\pi_*(\bigoplus \widetilde{\mathbb{L}}_k) = 0$ and $\mathbf{c}(R\pi_*(\widetilde{\mathbb{L}}_k)) = \mathbf{c}(-R^1\pi_*(\widetilde{\mathbb{L}}_k))$. Setting s_d as in (5.3) therefore yields

$$\mathbf{c}\Big(R\pi_*\Big(\bigoplus_{k=1}^{5} \widetilde{\mathbb{L}}_k\Big)\Big) = e_{\mathbb{C}^*}\Big(R^1\pi_*\Big(\bigoplus_{k=1}^{5} \widetilde{\mathbb{L}}_k\Big)^\vee\Big).$$

Applying Proposition V.3 we obtain the following

Corollary V.5. *After specializing s_d to the values in* (5.3)*,*

$$\lim_{\lambda\to 0} \mathcal{F}_0^{tw} = \mathcal{F}_0^{(W,G)}.$$

5.2 The I–function

We will compute twisted invariants by relating them to untwisted invariants, which we can compute directly. As before it is easy to check that $\mathcal{F}_0^{un}$ satisfies SE, DE, and TRR, (where $\phi_{\mathcal{J}}$ plays the role of the unit in this theory, as in Remark II.1) so it defines an overruled Lagrangian cone $\mathscr{L}^{un} \subset \mathscr{V}^{un}$, satisfying the same geometric properties as described in Section IV. We obtain the untwisted J–function

$$J^{un}(\mathbf{t}, -z) = -z\phi_{\jmath} + \mathbf{t} + \sum_{n\geq 0}\sum_{\substack{a\geq 0\\ h\in G}} \frac{1}{n!(-z)^{a+1}} \langle \mathbf{t}, \ldots \mathbf{t}, \psi^a \phi_h \rangle^{un}_{0,n+1} \phi^h.$$

We may similarly define $J^{tw}(\mathbf{t}, z)$ and $\mathscr{L}^{tw}$ in terms of $\mathscr{F}_0^{tw}$, but it is not obvious $\mathscr{L}^{tw}$ is a Lagrangian cone. Rather than proving this directly, we will use the methods of quantization. Let $B_d(x)$ denote the dth Bernoulli polynomial, and recall $i_k(h) = \langle \Theta_k(h) - \frac{1}{5} \rangle$.

Proposition V.6. *The symplectic transformation*

$$\Delta = \bigoplus_{h\in G} \prod_{k=1}^{5} \exp\left(\sum_{d\geq 0} s_d \frac{B_{d+1}\left(i_k(h) + \frac{1}{5}\right)}{(d+1)!} z^d \right)$$

satisfies $\mathscr{L}^{tw} = \Delta(\mathscr{L}^{un})$.

Proof. Note first that the identity $B_d(1-x) = (-1)^d B_d(x)$ implies Δ is symplectic.

The proof is the same as the proof in [5] and [7], with some slight modification. We give a sketch here. The strategy is to first relate $\mathcal{D}^{un}$ to $\mathcal{D}^{tw}$ via the quantization $\hat{\Delta}$. The desired statement then follows by taking the semiclassical limit (see [9]).

We will prove that

$$\hat{\Delta}\mathcal{D}^{un} = \mathcal{D}^{tw} \tag{5.4}$$

by viewing both sides as functions with respect to the variables s_d and showing they are both solutions to the same system of differential equations. First notice that both sides of (5.4) have the same initial condition, i.e. when $\mathbf{s} = 0$ they are equal. We will show that $\mathcal{D}^{tw}$ and $\mathcal{D}^{un}$ both satisfy

$$\frac{\partial \Phi}{\partial s_d} = \sum_{k=0}^{5} P_d^{(k)} \Phi \tag{5.5}$$

where

$$P_d^{(k)} = \frac{B_{d+1}(\frac{1}{5})}{(d+1)!}\frac{\partial}{\partial t^{\jmath}_{d+1}} - \sum_{\substack{a\geq 0\\ h\in G}} \frac{B_{d+1}(i_k(h)+\frac{1}{5})}{(d+1)!} t^h_a \frac{\partial}{\partial t^h_{a+d}}$$
$$+\frac{\hbar}{2}\sum_{\substack{a+a'=d-1\\ h,h'\in G}} (-1)^{a'}\eta^{h,h'}\frac{B_{d+1}(i_k(h)+\frac{1}{5})}{(d+1)!}\frac{\partial^2}{\partial t^h_a \partial t^{h'}_{a'}},$$

and $\eta^{h,h'}$ denotes the inverse pairing.

Using the formula derived in (4.7) one can check that $\hat{\Delta}\mathcal{D}^{un}$ satisfies (5.5). It remains to show that $\mathcal{D}^{tw}$ does as well. Substituting $\mathcal{D}^{tw}$ for Φ in (5.5) and taking the derivative with respect to s_d, we see that the equation reduces to

$$\sum_{n\geq 0}\frac{1}{n!}\Big\langle \mathbf{t}(\psi),\ldots,\mathbf{t}(\psi); \mathrm{ch}_d(R\pi_*(\widetilde{\mathbb{L}}_k))\cdot \mathbf{c}(R\pi_*(\bigoplus_l \widetilde{\mathbb{L}}_l))\Big\rangle_{0,n}$$
$$= P_d^{(k)}\mathcal{F}_g^{tw} + \frac{\hbar}{2}\sum_{\substack{a+a'=d-1\\ h,h'\in G}} (-1)^{a'}\eta^{h,h'}\frac{B_{d+1}(i_k(h)+\frac{1}{5})}{(d+1)!}\frac{\partial \mathcal{F}_g^{tw}}{\partial t^h_a}\frac{\partial \mathcal{F}_g^{tw}}{\partial t^{h'}_{a'}}$$

This equation was proven in [7], and generalized to the extended state space in [5]. It is proved using Grothendieck–Riemann–Roch to give an expression for $\mathrm{ch}_d(R\pi_*(\widetilde{\mathbb{L}}_k))$. □

It will be useful to separate the summands of $J^{un}(\mathbf{t},z)$ in terms of powers of t^h. Given a function $\mathbf{n} : G \to \mathbb{Z}_{\geq 0}$, let $J^{un}_{\mathbf{n}}(\mathbf{t},z)$ denote the $\prod_{h\in G}(t^h)^{\mathbf{n}(h)}$–summand of $J^{un}(\mathbf{t},z)$. Proposition II.4 plus a straightforward ψ–class calculation shows that the correlator $\langle \phi_{h_1},\ldots,\phi_{h_n},\psi^l\phi_h\rangle^{un}_{0,n+1} = 1$ when $i_k(h) = \langle \frac{3}{5} - \sum_m i_k(h_m)\rangle$ and $l = n-2$. It is zero otherwise. Furthermore, $i_k(h^{-1}) = \langle \frac{3}{5} - i_k(h)\rangle$. We arrive at the following pleasant formula

$$J^{un}_{\mathbf{n}}(\mathbf{t},z) = \frac{\prod_{h\in G}(t^h)^{\mathbf{n}(h)}}{z^{|\mathbf{n}|-1}\prod_{h\in G}\mathbf{n}(h)!}\phi_{h_{\mathbf{n}}},$$

with $h_{\mathbf{n}}$ defined by $i_k(h_{\mathbf{n}}) = \langle \sum_{h\in G}\mathbf{n}(h)i_k(h)\rangle$.

We conclude that

$$J^{un}(\mathbf{t},z) = \sum_{\mathbf{n}} \frac{\prod_{h\in G}(t^h)^{\mathbf{n}(h)}}{z^{|\mathbf{n}|-1}\prod_{h\in G}\mathbf{n}(h)!}\phi_{h_{\mathbf{n}}}. \tag{5.6}$$

Proposition V.6 allows us to describe $\mathscr{L}^{tw}$ in terms of $\mathscr{L}^{un}$. Combining this with Equation (5.6), we will obtain an explicit description of a slice of $\mathscr{L}^{tw}$. This will then determine $J^{tw}(\mathbf{t},z)$.

Define $D_h = t^h\frac{\partial}{\partial t_0^h}$, and put $D^k = \sum_{h\in G} i_k(h)D_h$. Notice that $D_h J^{un}_{\mathbf{n}}(\mathbf{t},z) = \mathbf{n}(h)J^{un}_{\mathbf{n}}(\mathbf{t},z)$. Consider the following functions:

$$\mathbf{s}(x) = \sum_{d\geq 0} s_d\frac{x^d}{d!}$$

$$G_y(x,z) = \sum_{l,m\geq 0} s_{l+m-1}\frac{B_m(y)}{m!}\frac{x^l}{l!}z^{m-1}.$$

These functions satisfy the following:

$$G_y(x,z) = G_0(x+yz,z)$$

$$G_0(x+z,z) = G_0(x,z)+\mathbf{s}(x)$$

Proposition V.7. *The slice defined by*

$$J^{\mathbf{s}}(\mathbf{t},z) = \prod_{k=1}^{5}\left(\exp(-G_{1/5}(zD^k,z)\right)J^{un}(\mathbf{t},z)$$

lies on $\mathscr{L}^{un}$.

Proof. This lemma appears in [11] and [5]. We give the proof again here for the purpose of completeness. Any element $f\in\mathscr{V}^{tw}$ can be written in the form

$$f = -z\phi_{\jmath} + \sum_{l\geq 0}\mathbf{t}_l z^l + \sum_{l\geq 0}\frac{\mathbf{p}_l(f)}{(-z)^{l+1}}$$

for some $\mathbf{p}_l(f) = \sum_{h\in G} p_{l,h}(f)\phi^h$. If $f\in\mathscr{L}^{un}$, then we know

$$\mathbf{p}_l(f) = \sum_{n\geq 0}\sum_{h\in G}\frac{1}{n!}\left\langle \mathbf{t}(\psi),\ldots,\mathbf{t}(\psi),\psi^l\phi_h\right\rangle^{un}_{0,n+1}\phi^h$$

The idea is to define

$$E_l(f) = \mathbf{p}_l(f) - \sum_{n\geq 0}\sum_{h\in G}\frac{1}{n!}\left\langle \mathbf{t}(\psi),\ldots,\mathbf{t}(\psi),\psi^l\phi_h\right\rangle_{0,n+1}^{un}\phi^h$$

and show that $E_l(J^{\mathbf{s}}) = 0$.

Let $\deg s_d = d+1$, and proceed by induction on the degree. Since $J^{un}(\mathbf{t},z)$ lies on $\mathscr{L}^{un}$, the degree zero terms of $E_l(J^{\mathbf{s}})$ vanish. Now assuming the degree n terms vanish, we will show that the degree $n+1$ terms vanish. Because of the vanishing up to degree n, there exists another family $\widetilde{J}^{\mathbf{s}}(\mathbf{t},-z)$ such that $E_l(J^{\mathbf{s}})$ and $E_l(\widetilde{J}^{\mathbf{s}})$ agree up to degree n. Differentiating, we obtain

$$\frac{\partial}{\partial s_d}E_l(J_{\mathbf{s}}) = d_{J^{\mathbf{s}}}E_j(z^{-1}P_dJ_{\mathbf{s}})$$

where

$$P_d = \sum_{k=1}^{5}\sum_{m=0}^{d+1}\frac{1}{m!(d+1-m)!}z^mB_m(\tfrac{1}{5})(zD^k)^{d+1-m}.$$

Up to degree n, the right hand side coincides with $d_{\widetilde{J}^{\mathbf{s}}}E_l(z^{-1}P_d\widetilde{J}^{\mathbf{s}})$, which vanishes because the term in parentheses lies on $T_{d_{\widetilde{J}^{\mathbf{s}}}\mathcal{L}^{un}}$. □

Applying Δ to $J^{\mathbf{s}}(\mathbf{t},-z)$ yields a slice of the twisted cone $\mathscr{L}^{tw}$. To facilitate computation, we express $J^{\mathbf{s}}(\mathbf{t},-z)$ in terms of monomials in the t^h variables

$$J^{\mathbf{s}}(\mathbf{t},-z) = \sum_{\mathbf{n}}\prod_{k=1}^{5}\exp\left(-G_{\frac{1}{5}}\left((\sum_{h\in G}\mathbf{n}(h)i_k(h))z,z\right)\right)J^{un}_{\mathbf{n}}(\mathbf{t},-z),$$

and express Δ as

$$\Delta = \prod_{k=1}^{5}\bigoplus_{h\in G}\exp\left(G_{\frac{1}{5}}\left(i_k(h)z,z\right)\right).$$

If we set $F_{\mathbf{n}} = \lfloor \sum_{h \in G} \mathbf{n}(h) i_k(h) \rfloor$, we can write

$$\begin{aligned}\Delta\left(J^{\mathbf{s}}(\mathbf{t},-z)\right) &= \sum_{\mathbf{n}} \prod_{k=1}^{5} \exp\Bigg(G_{\frac{1}{5}}\Big(\Big\langle \sum_{h\in G} \mathbf{n}(h) i_k(h) \Big\rangle z, z\Big) \\ &\qquad - G_{\frac{1}{5}}\Big(\sum_{h\in G} \mathbf{n}(h) i_k(h) z, z\Big)\Bigg) J^{un}_{\mathbf{n}}(\mathbf{t},z) \\ &= \sum_{\mathbf{n}} \prod_{k=1}^{5} \exp\Bigg(\sum_{0\le b<F_{\mathbf{n}}} -\mathbf{s}\Big(\tfrac{1}{5}z + \Big\langle \sum_{h\in G} \mathbf{n}(h) i_k(h)\Big\rangle z + bz\Big)\Bigg) J^{un}_{\mathbf{n}}(\mathbf{t},z)\end{aligned}$$

Now we can define the modification factor

$$M_{\mathbf{n}}(z) = \prod_{k=1}^{5} \exp\Bigg(\sum_{0\le b<F_{\mathbf{n}}} -\mathbf{s}\Big(-\tfrac{1}{5}z - \Big\langle \sum_{h\in G} \mathbf{n}(h) i_k(h)\Big\rangle z - bz\Big)\Bigg)$$

Setting s_d as in (5.3), we get

$$\begin{aligned} M_{\mathbf{n}}(z) &= \prod_{\substack{1\le k\le 5\\ 0\le b<F_{\mathbf{n}}}} \exp\left(-s_0 - \sum_{d>0} s_d \frac{\left(-\frac{1}{5}z - \langle \sum_{h\in G} \mathbf{n}(h) i_k(h)\rangle z - bz\right)^d}{d!}\right) \\ &= \prod_{\substack{1\le k\le 5\\ 0\le b<F_{\mathbf{n}}}} \exp\left(\ln\lambda + \sum_{d>0} (-1)^{d-1} \frac{\left(\frac{1}{5}z + \langle \sum_{h\in G} \mathbf{n}(h) i_k(h)\rangle z + bz\right)^d}{d}\right) \\ &= \prod_{\substack{1\le k\le 5\\ 0\le b<F_{\mathbf{n}}}} \lambda \exp\left(\ln\left(1 + \frac{\frac{1}{5}z + \langle \sum_{h\in G} \mathbf{n}(h) i_k(h)\rangle z + bz}{\lambda}\right)\right) \\ &= \prod_{\substack{1\le k\le 5\\ 0\le b<F_{\mathbf{n}}}} \left(\lambda + \tfrac{1}{5}z + \Big\langle \sum_{h\in G} \mathbf{n}(h) i_k(h)\Big\rangle z + bz\right)\end{aligned}$$

Define the I–function:

$$I^{tw}(\mathbf{t},z) := \sum_{\mathbf{n}} M_{\mathbf{n}}(z) J^{un}_{\mathbf{n}}(\mathbf{t},z) \tag{5.7}$$

By Proposition V.6, $I^{tw} \subset \mathscr{L}^{tw}$. Furthermore, we know by Corollary V.5 taking the non–equivariant limit $\lambda \mapsto 0$ recovers the FJRW invariants of (W,G). Define

$$I^{(W,G)}(\mathbf{t},z) := \lim_{\lambda\to 0} I^{tw}(\mathbf{t},z)|_{\mathbf{t}\in\mathscr{H}^{nar}_{W,G}}.$$

By Corollary V.5, the function $I^{(W,G)}(\mathbf{t},z)$ lies on $\mathscr{L}^{(W,G)}$.

5.3 The mirror theorem

To state the mirror theorem, we apply the following convention:

Notation V.8. From this point forward, we restriction to $\mathbf{t}$ of degree two in $\mathscr{H}^{nar}_{W,G}$. Let t denote the dual coordinate to $\phi_{\jmath^2}$. Then we may write

$$\mathbf{t} = t\phi_{\jmath^2} + \sum_{\substack{h \in \hat{S} \setminus \jmath^2 \\ \deg_W \phi_h = 2}} t^h \phi_h. \tag{5.8}$$

We will need the following lemma.

Lemma V.9. *For* $\mathbf{t}$ *as in* (5.8), *we may expand the I–function as*

$$I^{(W,G)}(\mathbf{t}, z) = zF(\mathbf{t})\phi_{\jmath} + \mathbf{G}(\mathbf{t}) + \mathcal{O}(z^{-1}) \tag{5.9}$$

with $F(\mathbf{t}) = F_0(t) + \mathcal{O}(2)$ *and*

$$\mathbf{G}(\mathbf{t}) = G_{\jmath^2}(t)\phi_{\jmath_2} + \sum_{\substack{h \in \hat{S} \setminus \jmath^2 \\ \deg_W \phi_h = 1}} t^h G_h(t)\phi_h + \mathcal{O}(2).$$

Here $\mathcal{O}(2)$ *denotes terms of degree at least two in the variables* $\{t^h | h \neq \jmath^2\}$.

Proof. Applying the non–equivariant limit $\lambda \mapsto 0$ to (5.7), we can write

$$I^{(W,G)}(\mathbf{t}, z) = \sum_{\mathbf{n}} \prod_{\substack{k=1,\ldots,5 \\ 0 \le m < F_{\mathbf{n}}}} \left(\left(\left\langle \sum_{h \in \hat{S}} \mathbf{n}(h) i_k(h) \right\rangle + \frac{1}{5} + m \right) z \right) \frac{\prod_h t^{\mathbf{n}(h)}}{z^{|\mathbf{n}|-1} \prod_h \mathbf{n}(h)!} \phi_{h_{\mathbf{n}}},$$

where the first sum is now over $\mathbf{n} : \hat{S} \to \mathbb{Z}_{\geq 0}$.

For a given $\mathbf{n}$, the power of z in the corresponding summand is

$$1 - \sum_{h \in \hat{S}} \mathbf{n}(h) + \sum_{k=1}^{5} \lfloor \sum_{h \in \hat{S}} \mathbf{n}(h) i_k(h) \rfloor$$

where the first two terms are the contribution from $J^{un}(\mathbf{t}, z)$ and the last sum is from the modification factor $M_{\mathbf{n}}$. Since we have restricted to $\deg_W(\phi_h) \leq 2$, we have

$$1-\sum_{h\in\hat{S}}\mathbf{n}(h)+\sum_{k=1}^{5}\lfloor\sum_{h\in\hat{S}}\mathbf{n}(h)i_k(h)\rfloor\leq 1-\sum_{h\in\hat{S}}\mathbf{n}(h)+\sum_{k=1}^{5}\sum_{h\in\hat{S}}\mathbf{n}(h)i_k(h)\leq 1.$$

Consider the coefficient of z^1. For a particular $\mathbf{n}$ to contribute to this term, it must be the case that

$$\sum_{h\in\hat{S}}\mathbf{n}(h)=\sum_{k=1}^{5}\lfloor\sum_{h\in\hat{S}}\mathbf{n}(h)i_k(h)\rfloor$$

which implies that

$$\sum_{k=1}^{5}\lfloor\sum_{h\in\hat{S}}\mathbf{n}(h)i_k(h)\rfloor=\sum_{k=1}^{5}\sum_{h\in\hat{S}}\mathbf{n}(h)i_k(h).$$

Therefore $i_k(h_{\mathbf{n}}) = \langle\sum_{h\in\hat{S}}\mathbf{n}(h)i_k(h)\rangle = 0$ for $1\leq k\leq 5$, and $h_{\mathbf{n}}=\mathcal{J}$. This gives us the first term $zF(\mathbf{t})\phi_{\mathcal{J}}$. It is clear that $F(\mathbf{t}) = F_0(t)+\mathcal{O}(2)$, because for $\langle\sum_{h\in\hat{S}}\mathbf{n}(h)i_k(h)\rangle = 0$ to hold for all k there cannot be just one of the variables t^h.

Now consider the coefficient of z^0. There are two kinds of summands we need to consider, those only in the variable t and those of the form $Ct^{h'}t^m$ for some $h'\in\hat{S}$ and $m\geq 0$. Here the last factor is the m–th power of t corresponding to the element $\phi_{\mathcal{J}^2}$.

In the first case, consider the t^{5m+l}–term. Here $\sum_{h\in\hat{S}}\mathbf{n}(h)i_k(h) = m+\frac{l}{5}$, thus the power of z in this term is $5m+1-5m-l$. Because this is zero, we arrive at $l=1$, and thus $i_k(h_{\mathbf{n}})=\frac{1}{5}$ for all k.

The exponent of z in the coefficient of $t^{h'}t^m$ is

$$\sum_{k=1}^{5}\lfloor\tfrac{m}{5}+i_k(h')\rfloor-m. \tag{5.10}$$

When restricted to $h'\in\hat{S}$, we have

$$\sum_{k=1}^{5}\lfloor\tfrac{m}{5}+i_k(h')\rfloor-m+\sum_{k=1}^{5}\langle\tfrac{m}{5}+i_k(h')\rangle=1.$$

Thus expression (5.10) is equal to 0 if and only if $\sum_{k=1}^{5}\left\langle \frac{m}{5}+i_k(h')\right\rangle = 1$. One can easily check that this implies $5|m$, therefore $\langle \sum_h \mathbf{n}(h) i_k(h)\rangle = \left\langle \frac{m}{5}+i_k(h')\right\rangle = i_k(h')$. This gives the other terms of $\mathbf{G}(\mathbf{t})$. □

Now we are prepared to state the mirror theorem.

Theorem V.10 (LG Mirror Theorem)**.** *With $F(\mathbf{t})$ and $\mathbf{G}(\mathbf{t})$ as above, and $\mathbf{t}$ as in* (5.8)*,*

$$J^{(W,G)}(\boldsymbol{\tau}(\mathbf{t}),z) = \frac{I^{(W,G)}(\mathbf{t},z)}{F(\mathbf{t})} \qquad \text{where } \boldsymbol{\tau}(\mathbf{t}) = \frac{\mathbf{G}(\mathbf{t})}{F(\mathbf{t})}. \tag{5.11}$$

Proof. Recall that the J–function is uniquely characterized by the fact that is lies on $\mathscr{L}^{(W,G)}$ and is of the form $z\phi_{\jmath} + \mathbf{t} + \mathcal{O}(z^{-1})$. The theorem follows from this fact and the previous lemma. □

Remark V.11. The function $\boldsymbol{\tau}(\mathbf{t})$ is referred to as the mirror transformation.

Let $J_h^{(W,G)}(t,z)$ denote the derivative

$$J_h^{(W,G)}(t,z) := z\frac{\partial}{\partial t^h} J^{(W,G)}(\mathbf{t},z)|_{\mathbf{t}=t}.$$

Recall by (4.4) that these functions determine the small cone $\mathscr{L}^{(W,G)}_{small}$. The rest of the section will be devoted to calculating these functions. In fact as we shall see it is sufficient to compute $J_h^{(W,G)}(\mathbf{t},z)$ for ϕ_h of degree at most two. These will determine all others.

Expand $I^{(W,G)}(\mathbf{t},z)$ in terms of powers of t^h for $h \neq \jmath^2$

$$I^{(W,G)}(\mathbf{t},z) = I_{\jmath}^{(W,G)}(t,z) + \tfrac{1}{z}\Big(\sum_h t^h I_h^{(W,G)}(t,z)\Big) + \Big(\tfrac{1}{z}\Big)^2\Big(\sum_{h_1,h_2} t^{h_1}t^{h_2} I_{h_1,h_2}^{(W,G)}(t,z)\Big) + \cdots$$

so that

$$I_h^{(W,G)}(t,z) = z\frac{\partial}{\partial t^h} I^{(W,G)}(\mathbf{t},z)|_{\mathbf{t}=t}. \tag{5.12}$$

As an immediate consequence of the previous theorem and Lemma V.9 we obtain the following corollary.

Corollary V.12. *Given $h \in \hat{S}$ with $\deg_W \phi_h \leq 2$, $\phi_h \neq \phi_{\jmath^2}$, there exist functions $F_0(t)$, $G_{\jmath^2}$, and $G_h(t)$ determined by $I_h^{(W,G)}(t,z)$ such that F_0 and G_h are invertible, and*

$$J_h^{(W,G)}(\tau(t),z) = \frac{I_h^{(W,G)}(t,z)}{G_h(t)} \quad \text{where } \tau(t) = \frac{G_{\jmath^2}(t)}{F_0(t)}$$

Proof. For $h = \jmath$ this follows by setting $\mathbf{t} = t$.

For the other h use equation (5.11), differentiate both sides with respect to t^h, and set $\mathbf{t} = t$. By equation (5.12), the left hand side equals

$$\frac{G_h(t)}{F_0(t)} J_h^{(W,G)}(\tau(t),z)$$

and the right hand side equals

$$\frac{I_h^{(W,G)}(t,z)}{F_0(t)}$$

as desired. □

Remark V.13. To justify the fact that we call Theorem V.10 and its corollary a "mirror theorem," one can check that up to a factor of t or t^2, the functions $I_h^{(W,G)}(t,z)$ satisfy the Picard–Fuchs equations (1.8) of the mirror family M_ψ around $\psi = 0$. One may check this fact directly, or it follows immediately from Theorem VI.1.

The functions $I_h(t,z)$ may be computed directly from (5.7). We list below $I_h(t,z)$ for $h \in \hat{S} \setminus \jmath^2$ satisfying $\deg(\phi_h) \leq 2$. These formulas will be needed in the next section.

(i) For $h = \jmath$,

$$tI_\jmath^{(W,G)}(t,z) = \sum_{k=1,2,3,4} \phi_{\jmath^k} z^{2-k} \sum_{l \geq 0} t^{k+5l} \frac{\Gamma((k+5l)/5)^5}{\Gamma(k/5)\Gamma(k+5l)}.$$

(ii) For $h = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{4}{5})$ and $h_1 = (\frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5}, \frac{2}{5})$,

$$\begin{aligned} I_h^{(W,G)}(t,z) =& z\phi_h \sum_{l \geq 0} t^{5l} \frac{\Gamma((1+5l)/5)^3\Gamma((3+5l)/5)\Gamma((4+5l)/5)}{\Gamma(1/5)^3\Gamma(3/5)\Gamma(4/5)\Gamma(1+5l)} \\ &+ \phi_{h_1} \frac{2}{25} \sum_{l \geq 0} t^{3+5l} \frac{\Gamma((4+5l)/5)^3\Gamma((6+5l)/5)\Gamma((7+5l)/5)}{\Gamma(4/5)^3\Gamma(6/5)\Gamma(7/5)\Gamma(4+5l)} \end{aligned}$$

(iii) For $h = (\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{1}{5})$ and $h_1 = (\frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{2}{5})$,

$$\begin{aligned} I_h^{(W,G)}(t,z) =& z\phi_h \sum_{l\geq 0} t^{5l} \frac{\Gamma((2+5l)/5)^3\Gamma((3+5l)/5)\Gamma((1+5l)/5)}{\Gamma(2/5)^3\Gamma(3/5)\Gamma(1/5)\Gamma(1+5l)} \\ &+ \ \phi_{h_1} \sum_{l\geq 0} t^{1+5l} \frac{\Gamma((3+5l)/5)^3\Gamma((4+5l)/5)\Gamma((2+5l)/5)}{\Gamma(3/5)^3\Gamma(4/5)\Gamma(2/5)\Gamma(2+5l)}. \end{aligned}$$

(iv) For $h = (\frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5})$ and $h_1 = (\frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5})$,

$$\begin{aligned} I_h^{(W,G)}(t,z) =& z\phi_h \sum_{l\geq 0} t^{5l} \frac{\Gamma((1+5l)/5)^2\Gamma((2+5l)/5)^2\Gamma((4+5l)/5)}{\Gamma(1/5)^2\Gamma(2/5)^2\Gamma(4/5)\Gamma(1+5l)} \\ &+ \frac{\phi_{h_1}}{5} \sum_{l\geq 0} t^{2+5l} \frac{\Gamma((3+5l)/5)^2\Gamma((4+5l)/5)^2\Gamma((6+5l)/5)}{\Gamma(3/5)^2\Gamma(4/5)^2\Gamma(6/5)\Gamma(3+5l)}. \end{aligned}$$

(v) For $h = (\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{5})$ and $h_1 = (\frac{2}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{3}{5})$,

$$\begin{aligned} I_h^{(W,G)}(t,z) =& z\phi_h \sum_{l\geq 0} t^{5l} \frac{\Gamma((1+5l)/5)^2\Gamma((3+5l)/5)^2\Gamma((2+5l)/5)}{\Gamma(1/5)^2\Gamma(3/5)^2\Gamma(2/5)\Gamma(1+5l)} \\ &+ \phi_{h_1} \sum_{l\geq 0} t^{1+5l} \frac{\Gamma((2+5l)/5)^2\Gamma((4+5l)/5)^2\Gamma((3+5l)/5)}{\Gamma(2/5)^2\Gamma(4/5)^2\Gamma(3/5)\Gamma(2+5l)}. \end{aligned}$$

CHAPTER VI

LG/CY Correspondence

6.1 The state space correspondence

An isomorphism between the Landau–Ginzberg state space and the cohomology of corresponding Calabi–Yau hypersurfaces is proven in [6]. In the case of the mirror quintic, the work implies in particular an isomorphism between $H^{even}_{CR}(\mathcal{W})$ and $\mathscr{H}^{nar}_{W,G}$ as graded vector spaces. We will describe the correspondence explicitly below. Recall that $H^{even}_{CR}(\mathcal{W})$ can be split into summands indexed by $g \in \tilde{S}$, where $\tilde{S}$ is composed of elements $g = (r_1, r_2, r_3, r_4, r_5) \in G$ such that at least two r_i are 0. The basis for $\mathscr{H}^{nar}_{W,G}$ on the other hand is given by $\{\phi_h\}_{h \in \hat{S}}$ where $\hat{S}$ runs over elements $h = (r_1, r_2, r_3, r_4, r_5) \in G$ such that $r_i \neq 0$ for all i.

6.1.1 $\dim(\mathcal{W}_g) = 3$

For $g = e$, map

$$\mu : H^i \mapsto \phi_{\jmath^{i+1}}.$$

6.1.2 $\dim(\mathcal{W}_g) = 1$

For $g = (0,0,0,\frac{2}{5},\frac{3}{5})$, let $h = (\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{3}{5},\frac{4}{5})$ and $h_1 = (\frac{4}{5},\frac{4}{5},\frac{4}{5},\frac{1}{5},\frac{2}{5})$, then

$$\mu : \mathbb{1}_g \mapsto \phi_h$$
$$\mathbb{1}_g H \mapsto \phi_{h_1}.$$

For $g = (0,0,0,\frac{1}{5},\frac{4}{5})$, let $h = (\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{3}{5},\frac{1}{5})$ and $h_1 = (\frac{3}{5},\frac{3}{5},\frac{3}{5},\frac{4}{5},\frac{2}{5})$, then

$$\mu : \mathbb{1}_g \mapsto \phi_h$$
$$\mathbb{1}_g H \mapsto \phi_{h_1}.$$

6.1.3 $\dim(\mathcal{W}_g) = 0$

Let

$$\mu : \mathbb{1}_g \mapsto \phi_h,$$

where,

if $g = (0,0,\frac{1}{5},\frac{1}{5},\frac{3}{5})$, $h = (\frac{1}{5},\frac{1}{5},\frac{2}{5},\frac{2}{5},\frac{4}{5})$;

if $g = (0,0,\frac{4}{5},\frac{4}{5},\frac{2}{5})$, $h = (\frac{4}{5},\frac{4}{5},\frac{3}{5},\frac{3}{5},\frac{1}{5})$;

if $g = (0,0,\frac{2}{5},\frac{2}{5},\frac{1}{5})$, $h = (\frac{1}{5},\frac{1}{5},\frac{3}{5},\frac{3}{5},\frac{2}{5})$;

if $g = (0,0,\frac{3}{5},\frac{3}{5},\frac{4}{5})$, $h = (\frac{4}{5},\frac{4}{5},\frac{2}{5},\frac{2}{5},\frac{3}{5})$.

If g is a permutation of one of the above, define the map by permuting the h coordinates accordingly. By extending the above identification linearly, we obtain a map

$$\mu : H^{even}_{CR}(\mathcal{W}) \to \mathscr{H}^{nar}_{W,G}$$

identifying the state spaces. Note that this identification preserves the grading and (up to a constant factor) preserves the pairing.

6.2 Analytic continuation of $I^{\mathcal{W}}$

Let $J^{\mathcal{W}}(\mathbf{s},z)$ denote the (big) J–function of the mirror quintic $\mathcal{W}$. Let s^g denote the dual coordinate to the fundamental class $\mathbb{1}_g$ on $\mathcal{W}_g$. We define

$$J_g^{\mathcal{W}}(s,z) := z\frac{\partial}{\partial s^g} J^{\mathcal{W}}(\mathbf{s},z)|_{\mathbf{s}=sH}.$$

For g of age at most 1, we know by [20] that

$$J_g^{\mathcal{W}}(\sigma(s),z) = \frac{I_g^{\mathcal{W}}(s,z)}{H_g(s)} \qquad \text{where } \sigma(s) = \frac{G_0(s)}{F_0(s)}, \tag{6.1}$$

where here H_g, G_0, and F_0 are explicitly determined functions, and $I_g^{\mathcal{W}}$ is given below. Let $q = e^s$, then

(i) If $g = e = (0,0,0,0,0)$,

$$I_e^{\mathcal{W}}(q,z) = zq^{H/z}\left(1 + \sum_{\langle d\rangle=0} q^d \frac{\prod\limits_{1\le m\le 5d}(5H+mz)}{\prod\limits_{\substack{0<b\le d\\ \langle b\rangle=0}}(H+bz)^5}\right).$$

(ii) If $g = (0,0,0,r_1,r_2)$,

$$I_g^{\mathcal{W}}(q,z) =$$
$$zq^{H/z}\mathbb{1}_g\left(1+\sum_{\langle d\rangle=0} q^d \frac{\prod\limits_{1\le m\le 5d}(5H+mz)}{\prod\limits_{\substack{0<b\le d\\ \langle b\rangle=0}}(H+bz)^3 \prod\limits_{\substack{0<b\le d\\ \langle b\rangle=r_2}}(H+bz) \prod\limits_{\substack{0<b\le d\\ \langle b\rangle=r_1}}(H+bz)}\right).$$

(iii) If $g = (0,0,r_1,r_1,r_2)$, let $g_1 = (\langle -r_1\rangle, \langle -r_1\rangle, 0, 0, \langle r_2 - r_1\rangle)$. Then

$$I_g^{\mathcal{W}}(q,z) =$$
$$zq^{H/z}\mathbb{1}_g\left(1+\sum_{\langle d\rangle=0} q^d \frac{\prod_{1\le m\le 5d}(5H+mz)}{\prod_{\substack{0<b\le d\\ \langle b\rangle=0}}(H+bz)^2\prod_{\substack{0<b\le d\\ \langle b\rangle=\langle \bar{3}r_2\rangle}}(H+bz)^2\prod_{\substack{0<b\le d\\ \langle b\rangle=\langle \bar{2}r_1\rangle}}(H+bz)}\right)$$
$$+\,zq^{H/z}\mathbb{1}_{g_1}\left(\sum_{\langle d\rangle=r_1} q^d \frac{\prod_{1\le m\le 5d}(5H+mz)}{\prod_{\substack{0<b\le d\\ \langle b\rangle=r_1}}(H+bz)^2\prod_{\substack{0<b\le d\\ \langle b\rangle=0}}(H+bz)^2\prod_{\substack{0<b\le d\\ \langle b\rangle=r_2}}(H+bz)}\right)$$

We will analytic continue each of the above I–functions from $q = 0$ to $t = q^{-1/5} = 0$ using the Mellon–Barnes method as in [5].

6.2.1 $g = e = (0,0,0,0,0)$

The I–function $I_e^{\mathcal{W}}$ is identical to the I–function in [5, Equation (47)], after reinterpreting H as the hyperplane class in $H^2(\mathcal{W})$. We recall their analytic continuation and symplectic transformation in 6.3.1.

6.2.2 $g = (0,0,0,r_1,r_2)$

The Gamma function satisfies

$$\Gamma(z+n)/\Gamma(z) = (z)(z+1)\cdots(z+n-1)$$

and consequently

$$\Gamma(1+x/z+l)/\Gamma(1+x/z) = z^{-l}\prod_{k=1}^{l}(x+kz).$$

With this we can rewrite our I–functions. In the present case we obtain

$$I_g^{\mathcal{W}}(q,z) = z\mathbb{1}_g q^{H/z}\cdot$$
$$\sum_{\langle d\rangle=0} q^d \frac{\Gamma(1+5H/z+5d)\Gamma(1+H/z)^3\Gamma(r_1+H/z)\Gamma(r_2+H/z)}{\Gamma(1+5H/z)\Gamma(1+H/z+d)^3\Gamma(r_1+H/z+d)\Gamma(r_2+H/z+d)}.$$

The function $1/(e^{2\pi is}-1)$ has simple poles at each integer with residue 1. From this we can rewrite the function as a contour integral

$$
\begin{aligned}
I_g^{\mathcal{W}}(q,z) =& z\mathbb{1}_g q^{H/z}\frac{\Gamma(1+H/z)^3\Gamma(r_1+H/z)\Gamma(r_2+H/z)}{\Gamma(1+5H/z)}\cdot \\
&\int_C \frac{1}{e^{2\pi is}-1}q^s\frac{\Gamma(1+5H/z+5s)}{\Gamma(1+H/z+s)^3\Gamma(r_1+H/z+s)\Gamma(r_2+H/z+s)}.
\end{aligned}
$$

where the curve C goes from $+i\infty$ to $-i\infty$ and encloses all nonnegative integers to the right.

By closing the curve to the left, we obtain an expansion in terms of $t=q^{-1/5}$. The Gamma function has poles at nonpositive integers, so we obtain a sum of residues at $s=-1-l$ for $l\geq 0$ and $s=-H/z-m/5$ for $m\geq 1$. In this case, at negative integers, the residue is a multiple of H^2, and so vanishes on $\mathcal{W}_g$. The residue similarly vanishes at $s=-H/z-m/5$ when m is congruent to 0, $5r_1$, or $5r_2$. For the remaining values of m, we use

$$
\operatorname{Res}_{s=-H/z-m/5}\Gamma(1+5H/z+5s) = -\frac{1}{5}\frac{(-1)^m}{\Gamma(m)},
$$

to obtain

$$
\begin{aligned}
I_g^{\mathcal{W}'}(t,z) = z\mathbb{1}_g&\frac{\Gamma(1+H/z)^3\Gamma(r_1+H/z)\Gamma(r_2+H/z)}{5\Gamma(1+5H/z)}\cdot \\
\sum_{\substack{0<m\\ m\not\equiv 0,5r_1,5r_2}} &\frac{(-\xi)^m 2\pi i}{e^{-2\pi iH/z}-\xi^m}\frac{t^m}{\Gamma(m)\Gamma(1-m/5)^3\Gamma(r_1-m/5)\Gamma(r_2-m/5)}.
\end{aligned}
$$

Here the prefactor of $q^{H/z}$ cancels with a term in each residue. Note that $\Gamma(r_1-m/5)=\Gamma(1-r_2-m/5)$. Recalling the identity $\Gamma(x)\Gamma(1-x)=\pi/\sin(\pi x)$, we simplify the above expression as

$$
\begin{aligned}
I_g^{\mathcal{W}'}(t,z) &= z\mathbb{1}_g \frac{\Gamma(1+H/z)^3\Gamma(r_1+H/z)\Gamma(r_2+H/z)}{5\Gamma(1+5H/z)} \cdot \\
\sum_{\substack{0<m \\ m\not\equiv 0,5r_1,5r_2}} & \frac{(-\xi)^m 2\pi i}{e^{-2\pi i H/z}-\xi^m} \frac{t^m \pi^{-5}\Gamma(m/5)^3\Gamma(r_1+m/5)\Gamma(r_2+m/5)}{\Gamma(m)(\sin(\pi m/5))^{-3}(\sin(\pi(r_1+m/5)))^{-1}(\sin(\pi(r_2+m/5)))^{-1}} \\
&= z\mathbb{1}_g \frac{\Gamma(1+H/z)^3\Gamma(r_1+H/z)\Gamma(r_2+H/z)}{5\Gamma(1+5H/z)} \cdot \\
\sum_{\substack{0<k<5 \\ k\not\equiv 0,5r_1,5r_2}} & \left(\frac{(-\xi)^k 2\pi i}{e^{-2\pi i H/z}-\xi^k} \frac{1}{\Gamma(1-k/5)^3\Gamma(1-(r_1+k/5))\Gamma(1-(r_2+k/5))} \cdot \right. \\
& \left. \sum_{l\geq 0} t^{k+5l} \frac{\Gamma((k+5l)/5)^3\Gamma(r_1+(k+5l)/5)\Gamma(r_2+(k+5l)/5)}{\Gamma(k/5)^3\Gamma(r_1+k/5)\Gamma(r_2+k/5)\Gamma(k+5l)} \right).
\end{aligned}
$$

6.2.3 $g = (0,0,r_1,r_1,r_2)$

Let $g_1 = (\langle -r_1\rangle, \langle -r_1\rangle, 0, 0, \langle r_2 - r_1\rangle)$. Re–writing $I_g^{\mathcal{W}}(t,z)$ in terms of Gamma functions yields

$$
\begin{aligned}
I_g^{\mathcal{W}}(q,z) &= \\
& z\mathbb{1}_g \Gamma(1-r_1)^2\Gamma(1-r_2) \left(\sum_{\langle d\rangle=0} q^d \frac{\Gamma(1+5d)}{\Gamma(1+d)^2\Gamma(1-r_1+d)^2\Gamma(1-r_2+d)} \right) \\
& + \mathbb{1}_{g_1}\Gamma(r_1)^2\Gamma(r_2) \left(\sum_{\langle d\rangle=r_1} q^d \frac{\Gamma(1+5d)}{\Gamma(1+d)^2\Gamma(1-r_1+d)^2\Gamma(1-r_2+d)} \right) \\
&= z\mathbb{1}_g \Gamma(1-r_1)^2\Gamma(1-r_2) \int_C \frac{1}{e^{2\pi i s}-1} q^s \frac{\Gamma(1+5s)}{\Gamma(1+s)^2\Gamma(1-r_1+s)^2\Gamma(1-r_2+s)} \\
& + \mathbb{1}_{g_1}\Gamma(r_1)^2\Gamma(r_2) \int_C \frac{1}{e^{2\pi i (s-r_1)}-1} q^s \frac{\Gamma(1+5s)}{\Gamma(1+s)^2\Gamma(1-r_1+s)^2\Gamma(1-r_2+s)}
\end{aligned}
$$

Analytic continuing along the other side using the same method as above we obtain

$$
\begin{aligned}
I_g^{\mathcal{W}'}(t,z) =& z\frac{\mathbb{1}_g}{5}\Gamma(1-r_1)^2\Gamma(1-r_2)\cdot \\
&\sum_{0<k<5|k\equiv 5r_1,5r_2}\left(\frac{(-\xi)^k 2\pi i}{1-\xi^k}\frac{1}{\Gamma(1-k/5)^2\Gamma(1-(r_1+k/5))^2\Gamma(1-(r_2+k/5))}\cdot\right. \\
&\left.\sum_{l\geq 0} t^{k+5l}\frac{\Gamma((k+5l)/5)^2\Gamma(r_1+(k+5l)/5)^2\Gamma(r_2+(k+5l)/5)}{\Gamma(k/5)^2\Gamma(r_1+k/5)^2\Gamma(r_2+k/5)\Gamma(k+5l)}\right) \\
&+\mathbb{1}_{g_1}\Gamma(r_1)^2\Gamma(r_2)\cdot \\
&\sum_{0<k<5|k\equiv 5r_1,5r_2}\left(\frac{(-1)^k\xi^{k+5r_1}2\pi i}{1-\xi^{k+5r_1}}\frac{1}{\Gamma(1-k/5)^2\Gamma(1-(r_1+k/5))^2\Gamma(1-(r_2+k/5))}\cdot\right. \\
&\left.\sum_{l\geq 0} t^{k+5l}\frac{\Gamma((k+5l)/5)^2\Gamma(r_1+(k+5l)/5)^2\Gamma(r_2+(k+5l)/5)}{\Gamma(k/5)^2\Gamma(r_1+k/5)^2\Gamma(r_2+k/5)\Gamma(k+5l)}\right).
\end{aligned}
$$

6.3 The symplectic transformation

In this section we will compute the symplectic transformation by considering each function $I_g^{\mathcal{W}'}(t,z)$ separately.

6.3.1 $g = e$

Here we recall calculations from [5], and the symplectic transformation which they compute. Analytic continuation of $I_e^{\mathcal{W}}(t,z)$ yields

$$I_e^{\mathcal{W}'}(t,z) = z\frac{\Gamma(1+H/z)^5}{5\Gamma(1+5H/z)}\sum_{k=1,2,3,4}\frac{(-\xi)^k 2\pi i}{e^{-2\pi i H/z}-\xi^k}\frac{1}{\Gamma(1-k/5)^5}\sum_{l\geq 0} t^{k+5l}\frac{\Gamma((k+5l)/5)^5}{\Gamma(k/5)\Gamma(k+5l)}.$$

On the other hand

$$tI_{\jmath}^{(W,G)}(t,z) = \sum_{k=1,2,3,4}\phi_{\jmath^k}z^{2-k}\sum_{l\geq 0} t^{k+5l}\frac{\Gamma((k+5l)/5)^5}{\Gamma(k/5)\Gamma(k+5l)}.$$

Thus the transformation

$$\mathbb{U}_{\jmath^k}:\phi_{\jmath^k}\mapsto z^{k-1}\frac{\Gamma(1+H/z)^5}{\Gamma(1+5H/z)}\frac{(-\xi)^k 2\pi i}{e^{-2\pi i H/z}-\xi^k}\frac{1}{\Gamma(1-k/5)^5}$$

sends $\frac{t}{5}I_{\jmath}^{(W,G)}(t,z)$ to $I_e^{\mathcal{W}'}(t,z)$.

6.3.2 $g = (0,0,0,\frac{2}{5},\frac{3}{5})$

In this case

$$
\begin{aligned}
I_g^{\mathcal{W}'}(t,z) = z\mathbb{1}_g &\frac{\Gamma(1+H/z)^3\Gamma(2/5+H/z)\Gamma(3/5+H/z)}{5\Gamma(1+5H/z)}\cdot \\
&\Bigg(\frac{(-\xi)2\pi i}{e^{-2\pi i H/z}-\xi}\frac{1}{\Gamma(1-1/5)^3\Gamma(1-3/5)\Gamma(1-4/5)}\cdot \\
&\sum_{l\geq 0} t^{1+5l}\frac{\Gamma((1+5l)/5)^3\Gamma((3+5l)/5)\Gamma((4+5l)/5)}{\Gamma(1/5)^3\Gamma(3/5)\Gamma(4/5)\Gamma(1+5l)} \\
&+\frac{(-\xi)^4 2\pi i}{e^{-2\pi i H/z}-\xi^4}\frac{1}{\Gamma(1-4/5)^3\Gamma(1-6/5)\Gamma(1-7/5)}\cdot \\
&\sum_{l\geq 0} t^{4+5l}\frac{\Gamma((4+5l)/5)^3\Gamma((6+5l)/5)\Gamma((7+5l)/5)}{\Gamma(4/5)^3\Gamma(6/5)\Gamma(7/5)\Gamma(4+5l)}\Bigg).
\end{aligned}
$$

Using the relation $\Gamma(1+x) = x\Gamma(x)$, we can rewrite the last summand, which gives us

$$
\begin{aligned}
I_g^{\mathcal{W}'}(t,z) = z\mathbb{1}_g &\frac{\Gamma(1+H/z)^3\Gamma(2/5+H/z)\Gamma(3/5+H/z)}{5\Gamma(1+5H/z)}\cdot \\
&\Bigg(\frac{(-\xi)2\pi i}{e^{-2\pi i H/z}-\xi}\frac{1}{\Gamma(1-1/5)^3\Gamma(1-3/5)\Gamma(1-4/5)}\cdot \\
&\sum_{l\geq 0} t^{1+5l}\frac{\Gamma((1+5l)/5)^3\Gamma((3+5l)/5)\Gamma((4+5l)/5)}{\Gamma(1/5)^3\Gamma(3/5)\Gamma(4/5)\Gamma(1+5l)} \\
&+\frac{(-\xi)^4 2\pi i}{e^{-2\pi i H/z}-\xi^4}\frac{1}{\Gamma(1-4/5)^3\Gamma(1-1/5)\Gamma(1-2/5)}\cdot \\
&\frac{2}{25}\sum_{l\geq 0} t^{4+5l}\frac{\Gamma((4+5l)/5)^3\Gamma((6+5l)/5)\Gamma((7+5l)/5)}{\Gamma(4/5)^3\Gamma(6/5)\Gamma(7/5)\Gamma(4+5l)}\Bigg).
\end{aligned}
$$

If $h = (\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{3}{5},\frac{4}{5})$ and $h_1 = (\frac{4}{5},\frac{4}{5},\frac{4}{5},\frac{1}{5},\frac{2}{5})$, we see that the transformation

$$
\begin{aligned}
\mathbb{U}_h : \phi_h \mapsto \mathbb{1}_g &\frac{\Gamma(1+H/z)^3\Gamma(2/5+H/z)\Gamma(3/5+H/z)}{\Gamma(1+5H/z)}\cdot \\
&\frac{(-\xi)2\pi i}{e^{-2\pi i H/z}-\xi}\frac{1}{\Gamma(1-1/5)^3\Gamma(1-3/5)\Gamma(1-4/5)}, \\
\mathbb{U}_{h_1} : \phi_{h_1} \mapsto z\mathbb{1}_g &\frac{\Gamma(1+H/z)^3\Gamma(2/5+H/z)\Gamma(3/5+H/z)}{\Gamma(1+5H/z)}\cdot \\
&\frac{(-\xi)^4 2\pi i}{e^{-2\pi i H/z}-\xi^4}\frac{1}{\Gamma(1-4/5)^3\Gamma(1-1/5)\Gamma(1-2/5)}
\end{aligned}
$$

sends $\frac{t}{5} I_h^{(W,G)}(t,z)$ to $I_g^{\mathcal{W}'}(t,z)$.

6.3.3 $g = (0,0,0,\frac{1}{5},\frac{4}{5})$

$$
\begin{aligned}
I_g^{\mathcal{W}'}(t,z) = z\mathbb{1}_g & \frac{\Gamma(1+H/z)^3\Gamma(1/5+H/z)\Gamma(4/5+H/z)}{5\Gamma(1+5H/z)} \cdot \\
& \Bigg(\frac{(-\xi)^2 2\pi i}{e^{-2\pi i H/z} - \xi^2} \frac{1}{\Gamma(1-2/5)^3\Gamma(1-3/5)\Gamma(1-1/5)} \\
& \quad \left(-\tfrac{1}{5}\right) \sum_{l\geq 0} t^{2+5l} \frac{\Gamma((2+5l)/5)^3\Gamma((3+5l)/5)\Gamma((1+5l)/5)}{\Gamma(2/5)^3\Gamma(3/5)\Gamma(1/5)\Gamma(1+5l)} \\
& \quad + \frac{(-\xi)^3 2\pi i}{e^{-2\pi i H/z} - \xi^3} \frac{1}{\Gamma(1-3/5)^3\Gamma(1-4/5)\Gamma(1-2/5)} \cdot \\
& \quad \left(-\tfrac{1}{5}\right) \sum_{l\geq 0} t^{3+5l} \frac{\Gamma((3+5l)/5)^3\Gamma((4+5l)/5)\Gamma((2+5l)/5)}{\Gamma(3/5)^3\Gamma(4/5)\Gamma(2/5)\Gamma(2+5l)} \Bigg).
\end{aligned}
$$

If $h = (\frac{2}{5},\frac{2}{5},\frac{2}{5},\frac{3}{5},\frac{1}{5})$ and $h_1 = (\frac{3}{5},\frac{3}{5},\frac{3}{5},\frac{4}{5},\frac{2}{5})$, the transformation

$$
\begin{aligned}
\mathbb{U}_h : \phi_h \mapsto \mathbb{1}_g & \frac{\Gamma(1+H/z)^3\Gamma(1/5+H/z)\Gamma(4/5+H/z)}{\Gamma(1+5H/z)} \cdot \\
& \frac{-(\xi)^2 2\pi i}{e^{-2\pi i H/z} - \xi^2} \frac{1}{\Gamma(1-2/5)^3\Gamma(1-3/5)\Gamma(1-1/5)},
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{U}_{h_1} : \phi_{h_1} \mapsto z\mathbb{1}_g & \frac{\Gamma(1+H/z)^3\Gamma(1/5+H/z)\Gamma(4/5+H/z)}{\Gamma(1+5H/z)} \cdot \\
& \frac{(\xi)^3 2\pi i}{e^{-2\pi i H/z} - \xi^3} \frac{1}{\Gamma(1-3/5)^3\Gamma(1-4/5)\Gamma(1-2/5)}
\end{aligned}
$$

sends $\frac{t^2}{25} I_h^{(W,G)}(t,z)$ to $I_g^{\mathcal{W}'}(t,z)$.

6.3.4 $g = (0, 0, \frac{1}{5}, \frac{1}{5}, \frac{3}{5})$

Letting $g_1 = (\frac{4}{5}, \frac{4}{5}, 0, 0, \frac{2}{5})$,

$$
\begin{aligned}
I_g^{\mathcal{W}'}(t,z) = z\frac{\mathbb{1}_g}{5}\Gamma(1-1/5)^2\Gamma(1-3/5)\cdot \\
&\left(\frac{(-\xi)2\pi i}{1-\xi}\frac{1}{\Gamma(1-1/5)^2\Gamma(1-2/5)^2\Gamma(1-4/5)}\cdot\right. \\
&\sum_{l\geq 0} t^{1+5l}\frac{\Gamma((1+5l)/5)^2\Gamma((2+5l)/5)^2\Gamma((4+5l)/5)}{\Gamma(1/5)^2\Gamma(2/5)^2\Gamma(4/5)\Gamma(1+5l)} \\
&+\frac{(-\xi)^3 2\pi i}{1-\xi^3}\frac{1}{\Gamma(1-3/5)^2\Gamma(1-4/5)^2\Gamma(1-1/5)}\cdot \\
&\left.\left(-\frac{1}{5}\right)\sum_{l\geq 0} t^{3+5l}\frac{\Gamma((3+5l)/5)^2\Gamma((4+5l)/5)^2\Gamma((6+5l)/5)}{\Gamma(3/5)^2\Gamma(4/5)^2\Gamma(6/5)\Gamma(3+5l)}\right) \\
+\frac{\mathbb{1}_{g_1}}{5}\Gamma(1/5)^2\Gamma(3/5)\cdot \\
&\left(\frac{(-1)\xi^2 2\pi i}{1-\xi^2}\frac{1}{\Gamma(1-1/5)^2\Gamma(1-2/5)^2\Gamma(1-4/5)}\cdot\right. \\
&\sum_{l\geq 0} t^{1+5l}\frac{\Gamma((1+5l)/5)^2\Gamma((2+5l)/5)^2\Gamma((4+5l)/5)}{\Gamma(1/5)^2\Gamma(2/5)^2\Gamma(4/5)\Gamma(1+5l)} \\
&+\frac{(-1)^3\xi^4 2\pi i}{1-\xi^4}\frac{1}{\Gamma(1-3/5)^2\Gamma(1-4/5)^2\Gamma(1-1/5)}\cdot \\
&\left.\left(-\frac{1}{5}\right)\sum_{l\geq 0} t^{3+5l}\frac{\Gamma((3+5l)/5)^2\Gamma((4+5l)/5)^2\Gamma((6+5l)/5)}{\Gamma(3/5)^2\Gamma(4/5)^2\Gamma(6/5)\Gamma(3+5l)}\right).
\end{aligned}
$$

Letting $h = (\frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5})$ and $h_1 = (\frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5})$, the transformation

$$
\begin{aligned}
\mathbb{U}_h : \phi_h \mapsto \mathbb{1}_g\Gamma(1-1/5)^2\Gamma(1-3/5)\cdot \\
&\frac{(-\xi)2\pi i}{1-\xi}\frac{1}{\Gamma(1-1/5)^2\Gamma(1-2/5)^2\Gamma(1-4/5)} \\
&+\mathbb{1}_{g_1}\frac{\Gamma(1/5)^2\Gamma(3/5)}{z}\cdot \\
&\frac{(-1)\xi^2 2\pi i}{1-\xi^2}\frac{1}{\Gamma(1-1/5)^2\Gamma(1-2/5)^2\Gamma(1-4/5)},
\end{aligned}
$$

$$\begin{aligned}\mathbb{U}_{h_1}:\phi_{h_1}\mapsto z\mathbb{1}_g&\Gamma(1-1/5)^2\Gamma(1-3/5)\cdot\\&\frac{(\xi)^3 2\pi i}{1-\xi^3}\frac{1}{\Gamma(1-3/5)^2\Gamma(1-4/5)^2\Gamma(1-1/5)}\\&+\mathbb{1}_{g_1}\Gamma(1/5)^2\Gamma(3/5)\cdot\\&\frac{\xi^4 2\pi i}{1-\xi^4}\frac{1}{\Gamma(1-3/5)^2\Gamma(1-4/5)^2\Gamma(1-1/5)},\end{aligned}$$

sends $\frac{t}{5}I_h^{(W,G)}(t,z)$ to $I_g^{\mathcal{W}'}(t,z)$.

6.3.5 $g=(0,0,\frac{2}{5},\frac{2}{5},\frac{1}{5})$

Letting $g_1=(\frac{3}{5},\frac{3}{5},0,0,\frac{4}{5})$,

$$\begin{aligned}I_g^{\mathcal{W}'}(t,z)=z\frac{\mathbb{1}_g}{5}&\Gamma(1-2/5)^2\Gamma(1-1/5)\cdot\\&\left(\frac{(-\xi)^2 2\pi i}{1-\xi^2}\frac{1}{\Gamma(1-2/5)^2\Gamma(1-4/5)^2\Gamma(1-3/5)}\cdot\right.\\&\sum_{l\geq 0}t^{2+5l}\frac{\Gamma((2+5l)/5)^2\Gamma((4+5l)/5)^2\Gamma((3+5l)/5)}{\Gamma(2/5)^2\Gamma(4/5)^2\Gamma(3/5)\Gamma(2+5l)}\\&\frac{(-\xi)2\pi i}{1-\xi}\frac{1}{\Gamma(1-1/5)^2\Gamma(1-3/5)^2\Gamma(1-2/5)}\cdot\\&\left.\sum_{l\geq 0}t^{1+5l}\frac{\Gamma((1+5l)/5)^2\Gamma((3+5l)/5)^2\Gamma((2+5l)/5)}{\Gamma(1/5)^2\Gamma(3/5)^2\Gamma(2/5)\Gamma(1+5l)}\right)\\&+\frac{\mathbb{1}_{g_1}}{5}\Gamma(2/5)^2\Gamma(1/5)\cdot\\&\left(\frac{(-1)^2\xi^4 2\pi i}{1-\xi^4}\frac{1}{\Gamma(1-2/5)^2\Gamma(1-4/5)^2\Gamma(1-3/5)}\cdot\right.\\&\sum_{l\geq 0}t^{2+5l}\frac{\Gamma((2+5l)/5)^2\Gamma((4+5l)/5)^2\Gamma((3+5l)/5)}{\Gamma(2/5)^2\Gamma(4/5)^2\Gamma(3/5)\Gamma(2+5l)}\\&+\frac{(-1)\xi^3 2\pi i}{1-\xi^3}\frac{1}{\Gamma(1-1/5)^2\Gamma(1-3/5)^2\Gamma(1-2/5)}\cdot\\&\left.\sum_{l\geq 0}t^{1+5l}\frac{\Gamma((1+5l)/5)^2\Gamma((3+5l)/5)^2\Gamma((2+5l)/5)}{\Gamma(1/5)^2\Gamma(3/5)^2\Gamma(2/5)\Gamma(1+5l)}\right).\end{aligned}$$

Letting $h = (\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{5})$ and $h_1 = (\frac{2}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{3}{5})$, the transformation

$$\begin{aligned}\mathbb{U}_h : \phi_h \mapsto & \mathbb{1}_g \Gamma(1-2/5)^2\Gamma(1-1/5)\cdot \\ & \frac{(-\xi)2\pi i}{1-\xi}\frac{1}{\Gamma(1-1/5)^2\Gamma(1-3/5)^2\Gamma(1-2/5)} \\ & + \mathbb{1}_{g_1}\frac{\Gamma(2/5)^2\Gamma(1/5)}{z}\cdot \\ & \frac{(-1)\xi^3 2\pi i}{1-\xi^3}\frac{1}{\Gamma(1-1/5)^2\Gamma(1-3/5)^2\Gamma(1-2/5)},\end{aligned}$$

$$\begin{aligned}\mathbb{U}_{h_1} : \phi_{h_1} \mapsto & z\mathbb{1}_g \Gamma(1-2/5)^2\Gamma(1-1/5)\cdot \\ & \frac{(-\xi)^2 2\pi i}{1-\xi^2}\frac{1}{\Gamma(1-2/5)^2\Gamma(1-4/5)^2\Gamma(1-3/5)} \\ & + \mathbb{1}_{g_1}\Gamma(2/5)^2\Gamma(1/5)\cdot \\ & \frac{(-1)^2\xi^4 2\pi i}{1-\xi^4}\frac{1}{\Gamma(1-2/5)^2\Gamma(1-4/5)^2\Gamma(1-3/5)},\end{aligned}$$

sends $\frac{t}{5}I_h^{(W,G)}(t,z)$ to $I_g^{\mathcal{W}'}(t,z)$.

6.3.6 Putting things together

The above calculations define a map

$$\mathbb{U}_h : \phi_h \to \mathscr{V}^{\mathcal{W}}$$

for each $h \in \hat{S}$. Extending linearly, we may define the transformation $\mathbb{U}$,

$$\mathbb{U} := \bigoplus_{h \in \hat{S}} \mathbb{U}_h : \mathscr{V}^{(W,G)} \to \mathscr{V}^{\mathcal{W}}.$$

Expressing $\mathbb{U}$ in terms of the bases

$$\{\phi_h\}_{h\in\hat{S}} \text{ and } \{\mathbb{1}_g, \mathbb{1}_g H, \ldots, \mathbb{1}_g H^{\dim(\mathcal{W}_g)}\}_{g\in\tilde{S}},$$

$\mathbb{U}$ takes the form of a block matrix which is zero away from the diagonal blocks. The first diagonal block (corresponding to the non–twisted sector of $\mathcal{W}$) is size 4×4 and all others are 2×2. Each block is nonsingular, thus $\mathbb{U}$ is also.

Furthermore, one can check via a direct calculation on blocks that $\mathbb{U}$ is symplectic. This proves the following.

Theorem VI.1. *There is a* $\mathbb{C}[z,z^{-1}]$*–valued degree–preserving symplectic transformation* $\mathbb{U}$ *identifying* $\mathscr{V}^{(W,G)}$ *with* $\mathscr{V}^{\mathcal{W}}$*. Furthermore, for* $h \in \hat{S}$ *satisfying* $\deg(\phi_h) \leq 2$*,*

$$\mathbb{U}\left(c_h \cdot I_h^{(W,G)}(t,z)\right) = I^{\mathcal{W}'}_{\mu^{-1}(h)}(t,z)$$

where c_h *is the factor* $\frac{t}{5}$ *or* $\frac{t^2}{25}$ *depending on* h*.*

6.4 The main theorem

By equation (4.4), the slice $\mathscr{L}^{\mathcal{W}} \cap L_{J^{\mathcal{W}}(s,-z)}$ of the ruling is generated by $z\frac{\partial}{\partial s^i}J^{\mathcal{W}}(s,z)$ where i runs over a basis of $H^{even}_{CR}(\mathcal{W})$. Thus the small slice $\mathscr{L}^{\mathcal{W}}_{small}$ of the Lagrangian cone is completely determined by the first derivatives of $J^{\mathcal{W}}(\mathbf{s},z)$ evaluated at points $sH \in H^2(\mathcal{W})$. This implies the following:

Lemma VI.2. *The small slice of the Lagrangian cone* $\mathscr{L}^{\mathcal{W}}$ *is determined by*

$$\{I_g^{\mathcal{W}}(q,z)\}_{\{g\in\tilde{G}|\deg \mathbb{1}_g \leq 2\}}.$$

Proof. For $s \in H^2(\mathcal{W})$, each point of $\mathscr{L}^{\mathcal{W}} \cap L_{J^{\mathcal{W}}(s,-z)}$ is of the form

$$z\sum_{i\in I} c_i(s,z)\frac{\partial}{\partial s^i}J^{\mathcal{W}}(s,-z).$$

where I is a choice of basis for $H^{even}_{CR}(\mathcal{W})$. By choosing a particular basis, we will show that such linear combinations are completely determined by the set $\{I_g^{\mathcal{W}}(q,z)\}_{\{g\in\tilde{G}|\deg \mathbb{1}_g \leq 2\}}$.

The main result of [20] states that after choosing suitable coordinates (i.e. the mirror transformation) the I–functions $I_g^{\mathcal{W}}$ and their derivatives give the rows of the solution matrix of $\nabla_s^{\mathcal{W}}$ for $\mathcal{W}$ when restricted to $H^2(\mathcal{W})$. Here $\nabla_s^{\mathcal{W}}$ denotes the

Dubrovin connection, defined in terms of the quantum cohomology of $\mathcal{W}$ (see [13] and [20]). We summarize the content of the theorem below. Consider the subset of $H_{CR}^{even}(\mathcal{W})$ given by

$$\{(\nabla_s^{\mathcal{W}})^k 1\}_{0\leq k\leq 3} \cup \{\mathbb{1}_g, \nabla_s^{\mathcal{W}} \mathbb{1}_g\}_{\deg(\mathbb{1}_g)=2}.$$

We can check that this set forms a basis by using properties of the J–function. Note first that the elements of $\{1\} \cup \{\mathbb{1}_g\}_{\deg(\mathbb{1}_g)=2}$ are linearly independent. For $s \in H_{un}^2(\mathcal{W})$, $\nabla_s^{\mathcal{W}} \mathbb{1}_g = \frac{1}{z} H *_s \mathbb{1}_g$ is a degree four class supported in a particular component of $I\mathcal{W}$. If $g = (0,0,0,r_1,r_2)$, $\frac{1}{z} H *_s \mathbb{1}_g$ is a multiple of $\mathbb{1}_g H$, and if $g = (0,0,r_1,r_1,r_2)$, $\frac{1}{z} H *_s \mathbb{1}_g$ is a multiple of $\mathbb{1}_{g_1}$ where $g_1 = (\langle -r_1\rangle, \langle -r_1\rangle, 0, 0, \langle r_2 - r_1\rangle)$. We can check that these multiples are non–zero by observing that the periods of $\frac{1}{z} H *_s \mathbb{1}_g$ are obtained as the coefficients of $\frac{d}{ds} J_g^{\mathcal{W}}(s,z)$ (see Definition 1.4 in [20]) which are nonzero by (6.1). This shows that $\{\mathbb{1}_g, \nabla_s^{\mathcal{W}} \mathbb{1}_g\}_{\deg(\mathbb{1}_g)=2}$ are linearly independent. Similarly, $(\nabla_s^z)^k 1$ is a nonzero class of degree k supported on the non twisted sector. We conclude that

$$\{(\nabla_s^{\mathcal{W}})^k 1\}_{0\leq k\leq 3} \cup \{\mathbb{1}_g, \nabla_s^{\mathcal{W}} \mathbb{1}_g\}_{\deg(\mathbb{1}_g)=2}$$

is a set of $204 = \dim(H_{CR}^{even}(\mathcal{W}))$ linearly independent elements and thus forms a basis.

By definition, for $\mathbb{1}_g$ of degree at most 2,

$$z\frac{\partial}{\partial s^g} J^{\mathcal{W}}(\mathbf{s}, -z)|_s = J_g^{\mathcal{W}}(s, -z).$$

Because the J–function satisfies the quantum differential equation (equation 5 in [16]), if $s^{g'}$ is the dual coordinate to $\nabla_s^{\mathcal{W}} \mathbb{1}_g$, we have the following

$$z\frac{\partial}{\partial s^{g'}} J^{\mathcal{W}}(\mathbf{s}, -z)|_s = \frac{d}{ds} J_g^{\mathcal{W}}(s, -z).$$

Similarly if s^k is dual to $(\nabla_s^{\mathcal{W}})^k 1$,

$$z\frac{\partial}{\partial s^k} J^{\mathcal{W}}(\mathbf{s}, -z)|_s = \left(\frac{d}{ds}\right)^k J_e^{\mathcal{W}}(s, -z).$$

Therefore, for $s \in H^2(\mathcal{W})$, $\mathscr{L}^{\mathcal{W}} \cap L_{J^{\mathcal{W}}(s,-z)}$ is completely determined by the $\mathbb{C}[z]$–span of $\{J_g^{\mathcal{W}}(s, -z)\}_{\mathrm{age}(g)\leq 2}$.

But, by the mirror theorem (6.1), the span of $J_g^{\mathcal{W}}(s, -z)$ is equal to the span of $I_g^{\mathcal{W}}(\sigma^{-1}(s), -z)$ where σ is the mirror map. □

In FJRW theory, we have the analogous result.

Lemma VI.3. *The small slice of the Lagrangian cone $\mathscr{L}^{(W,G)}$ is determined by*

$$\{I_h^{(W,G)}(t, z)\}_{\{h\in\hat{S}|\,\deg(\phi_h)\leq 2, h\neq \jmath^2\}}.$$

Proof. The proof is essentially the same as in the previous lemma. □

Theorem VI.4. *The symplectic transformation $\mathbb{U}$ identifies the analytic continuation of the small slice of $\mathscr{L}^{\mathcal{W}}$ with the small slice of $\mathscr{L}^{(W,G)}$.*

Proof. The result follows immediately from Theorem VI.1 and the previous two lemmas. □

Remark VI.5. Theorem VI.4 proves the first part of Conjecture IV.2 restricted to the small parameters $sH \in H_{CR}^{even}(\mathcal{W})$ and $t\phi_{\jmath^2} \in \mathscr{H}_{W,G}^{nar}$ (see 4.2.1). Note that although we have restricted all calculations to the small parameters, this is enough to completely determine $\mathbb{U}$.

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