Construction of Invariant Scalar Amplitudes without Kinematical Singularities for Arbitrary-Spin Nonzero-Mass Two-Body Scattering Processes

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Abstract

The relativistic transition amplitude for any two-body reaction involving only particles of nonzero mass but arbitrary spin is decomposed in terms of scalar amplitudes that are regular functions in the space of scalar invariants at points corresponding to regular points of the transition matrix elements in momentum space, on the mass shell. Detailed formulas for the scalar amplitudes in terms of the original scattering matrix elements are given. The development is in the framework of analytic S-matrix theory, and is based on a partial generalization of the Hall-Wightman Theorem. The results hold on the complete (multisheeted) domain of regularity of the scattering amplitude.

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Contents

1 Introduction

The reduction of a Lorentz-invariant S matrix for arbitrary spins and nonzero masses into a set of scalar amplitudes was discussed in an earlier paper [1], within the framework of "analytic S-matrix theory" [2–4]. There are, in general, many possible ways to do such a decomposition. It may happen, however, that for a particular decomposition the scalar amplitudes will have poles at certain positions, even if the S-matrix elements themselves are regular at the corresponding points. These singularities are in a sense spurious, reflecting only the nature of the decomposition; such singularities have been called "kinematical singularities" [2]. The object of this paper is to give an explicit decomposition that is free of kinematical singularities, for the case of two-particle to two-particle transition amplitudes for particles with arbitrary spins and arbitrary, but nonzero, masses. One of the motivations for this work is to allow theoretical considerations based on the Mandelstam representation to be carried over to this general spin case.

For the case of multiparticle processes involving photons and spin-$\frac{1}{2}$ particles, Hearn [5] has given a decomposition that yields for each term of the field theoretic perturbation expansion a set of scalar amplitudes free of kinematical singularities. In the present work no reference to field theory or perturbation theory is made.

A decomposition for the simple special case of π-N scattering has been known for some time [6]; and a decomposition for the nontrivial special case of N-N scattering has been given by Goldberger, Grisaru, MacDowell, and Wong [7]. Their proof that the corresponding scalar amplitudes are free of kinematical singularities depends on rather awkward manipulations involving a partial wave expansion, and on a certain unproved generalization of the Hall-Wightman Theorem [8]. The more general discussion given here is more direct and is based on a study of the nature of domains for which generalizations of the Hall-Wightman Theorem can be proved.

The present paper gives the detailed arguments that were promised earlier [1]. Recently a paper by Hepp [9] on the same subject has appeared. His paper contains the generalizations of the results of Hall and Wightman needed in our discussion. Consequently our independent proofs of these generalizations will be omitted and Hepp's terminology adopted wherever convenient.

The main results obtained in this paper overlap to a considerable degree results obtained independently by Hepp. Our methods, however, are rather different from his. Moreover, they are based on elementary considerations and are self contained, in the sense that they do not depend on mathematical results not generally familiar to theoretical physicists. It is therefore believed that the present version should be a useful complement to Hepp's mathematically more sophisticated approach.

Our considerations deal only with the two-particle to two-particle case, whereas Hepp has given sufficient conditions for the existence of a global decomposition also in the general multiparticle case. In general, Hepp's results are in the nature of an existence proof, with the precise form of the decomposition not exactly specified. In the special case of two-particle reactions on the mass shell, his results do specify the decomposition in principle, but certain implicit relations are left unsolved. In our results all implicit relations are eliminated to give a specific decomposition, and we also obtain inversion formulas giving explicit expressions for the scalar amplitudes in terms of the original matrix elements.

Care has been taken to show that our results are valid not only on schlicht domains restricted to the mass shell, but also on locally schlicht domains over the mass shell.

In Sec. 2 the connection between the M-function formalism for the S-matrix introduced by Stapp [3] and the Dirac-spinor formalism is briefly reviewed. Sections 3 and 4 are devoted to the construction of a basis for matrices in the spin space. In effect, this "spin basis" transforms an M function into an equivalent tensor field under the proper, homogeneous, complex Lorentz group, $\mathcal{L}_+$. The spin basis is combined in Sec. 5 with tensor polynomials constructed from the available momemtum vectors to form a basis for the M function, and formulas are given for the corresponding scalar amplitudes. The derivation of the conditions under which kinematical singularities do not occur is given in Sec. 6, along with a discussion of the concepts required for the definition of holomorphic, covariant functions on domains over the mass shell. In Sec. 7 it is concluded that the conditions for a holomorphic decomposition are satisfied in analytic S-matrix theory.

Appendix A1 is devoted to the development of a generalized spinor calculus for the group $\mathcal{L}_+$. It is a review of properties of representation matrices for the group, of details of Clebsch-Gordan anaysis, and of properties of irreducible tensors that are useful for understanding the text. In Appendix A2 some additional properties of the spin basis constructed in Sec. 4 are described, and in Appendix A3 a pertinent example of a kinematical singularity is given. Appendix A4 gives a proof that the map from the space of three complex four-vectors to the corresponding space of scalar invariants is open, extending in this special case a result of Bargmann, Hall, and Wightman.

The amplitudes constructed in this paper are in general not independent when there are symmetries under C, P, and T. However, such symmetries do no affect the question of kinematical singularities.

2 The M Functions

For calculations in analytic S-matrix theory, where the dynamical content comes from analytic properites and no fields are involved, it is convenient to use the M-function formalism introduced by Stapp [3]. The M functions have just the minimum number of spinor components, whereas the corresponding quantities coming from Dirac fields have more than needed to describe the spin multiplicity. Moreover, the M functions have simple covariance properties; and no crossing matrices are required to obtain the corresponding M functions for related processes reached by analytic continuation.

Because the existing discussions of these functions are quite general and somewhat abstract, it may be useful to review the connection between the M functions and the standard field theoretic quantities of perturbation theory in a simple case [1, Appendix II]. Such an example can serve to make plausible, by analogy with field theory, why analytic properties are assigned to the M functions rather than to some other functions that could be constructed from the S matrix. It should be emphasized, however, that such an analogy was not the original motivation, either for the assignment of analytic properties, or for the general construction of the M functions. The M functions are constructed directly from the principles of analytic S-matrix theory as formulated by Stapp [3], and they have a well defined and simple algebraic connection to the S matrix no matter how it is parameterized in terms of the spins and momenta of the particles [1, 3].

For the scattering of a spin-$\frac{1}{2}$ particle with mass m and initial and final four-momenta k_i and k_f on a spin-0 particle with mass m_0 and initial and final momenta p_i and p_f, the S-matrix elements can be expressed as

$$\begin{aligned} R &\equiv S - I\,, \\ R_{ab} &= \bar{u}_a(k_f)\,\mathcal{T}\,u_b(k_i)\,, \end{aligned} \tag{2.1}$$

where $a, b = \pm\frac{1}{2}$ label the final and initial spin states. For each value of a and b, $\bar{u}_a$ and u_b are four-component row and column vectors, respectively, the $u_b(k)$ being two independent solutions of the free-particle Dirac equation and the $\bar{u}_a(k)$ the corresponding Dirac adjoints. The four-by-four matrix $\mathcal{T}$ depends on all of the momenta. The S matrix is normalized by the following choice of energy factors:

$$I \equiv \delta_{ab}\,\delta(\mathbf{k}_f - \mathbf{k}_i)\,\delta(\mathbf{p}_f - \mathbf{p}_i)\,\sqrt{m^2 + \mathbf{k}_i^2}\,\sqrt{m_0^2 + \mathbf{p}_i^2}\,. \tag{2.2}$$

It can be assumed without loss of generality that R has an energy-momentum conserving δ function factored out.

The signature of the Lorentz metric is taken to be $(+---)$; and four-vector indices have the values 0, 1, 2, and 3. The two-by-two Pauli matrices are written as a four-vector σ_μ with $\sigma_0 = I$ and with the usual Pauli matrices[1]

[1] *Cf.* Appendix 1.A for a definition.

$\boldsymbol{\sigma}$ as three-vector components. Thus,

$$k\cdot\sigma = k^\mu \sigma_\mu = k^0 + \mathbf{k}\cdot\boldsymbol{\sigma}\,. \tag{2.3}$$

In the representation of the Dirac matrices defined by[2]

$$\gamma_\mu \equiv \begin{pmatrix} 0 & \sigma_\mu \\ \widetilde{\sigma}_\mu & 0 \end{pmatrix}, \qquad \widetilde{\sigma}_\mu \equiv \sigma^\mu, \tag{2.4}$$

the upper two components of the column vector $u(k)$ transform as a two-component spinor with a lower undotted index, and the lower two components transform as a two-component spinor with an upper dotted index. Explicity, $u(k)$ has the form:

$$\begin{aligned} u_b(k) &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{k\cdot\sigma/m}\,\phi_b \\ \sqrt{k\cdot\widetilde{\sigma}/m}\,\phi_b \end{pmatrix}, \\ \bar{u}_a(k) &= \tfrac{1}{\sqrt{2}} \left(\phi_a^\dagger \sqrt{k\cdot\widetilde{\sigma}/m}\,, \;\; \phi_a^\dagger \sqrt{k\cdot\sigma/m} \right), \end{aligned} \tag{2.5}$$

where ϕ is the two-component spin vector that specifies the spin of the particle in its rest frame,

$$\tfrac{1}{2}\,\phi_b^\dagger\,\boldsymbol{\sigma}\,\phi_b = \mathbf{s}_b\,, \tag{2.6}$$

and where $\sqrt{k\cdot\sigma/m}$ and $\sqrt{k\cdot\widetilde{\sigma}/m}$ are Hermitian matrices belonging to the representations $\mathcal{D}^{\frac{1}{2},0}$ and $\mathcal{D}^{0,\frac{1}{2}}$, respectively, of a Lorentz transformation from rest to a frame where the particle has momentum k, for example,

$$\begin{aligned} &\sqrt{k\cdot\sigma/m} = \exp\left(\lambda \mathbf{k}\cdot\boldsymbol{\sigma}/2|\mathbf{k}|\right) = (m + k\cdot\sigma)\,/\sqrt{2m(m+k_0)}\,, \\ &\quad \cosh\lambda = k_0/m\,, \qquad k\cdot k = m^2\,. \end{aligned} \tag{2.7}$$

If the $u_{\pm\frac{1}{2}}$ are taken to correspond to spins in the $\pm z$ direction in the rest frame, then the M function corresponding to (2.1) is defined by[3]

$$\begin{aligned} M_{\alpha\dot\beta} &= \left(\sqrt{k_f\cdot\sigma/m}\;R\;\sqrt{k_i\cdot\sigma/m} \right)_{\alpha\dot\beta}, \\ &= \tfrac{1}{2}\left(\Phi_\alpha\,,\;\Phi_\alpha\,k_f\cdot\sigma/m\right)\,\mathcal{T}\;\begin{pmatrix} k_i\cdot\sigma/m\;\Phi_{\dot\beta} \\ \Phi_{\dot\beta} \end{pmatrix}, \end{aligned} \tag{2.8}$$

[2]For matrices, the notation is A^{T} for transpose, $A^\dagger$ for Hermitian conjugate, and A^* for complex conjugate. The notation $\widetilde{T}$ is used in this paper to indicate the space-inversion operation on the tensor indices of T.

[3]The M function is the same, no matter how the spin polarizations are chosen. For the general construction see [1].

where the relation

$$\sqrt{k\cdot\sigma/m}\,\sqrt{k\cdot\widetilde{\sigma}/m} = I \tag{2.9}$$

has been used and

$$\Phi_{\pm\frac{1}{2}} = \begin{pmatrix} \frac{1}{2} \pm \frac{1}{2} \\ \frac{1}{2} \mp \frac{1}{2} \end{pmatrix} . \tag{2.10}$$

If one writes $\mathcal{T}$ in the two-by-two block form

$$\mathcal{T} = \begin{pmatrix} \mathcal{T}^{++} & \mathcal{T}^{+-} \\ \mathcal{T}^{-+} & \mathcal{T}^{--} \end{pmatrix}, \tag{2.11}$$

then the explicit expression for M becomes

$$\begin{aligned} M_{\alpha\dot{\beta}} = {} & (\mathcal{T}^{++})_{\alpha}{}^{\beta'}\,(k_i\cdot\sigma/m)_{\beta'\dot{\beta}} + (\mathcal{T}^{+-})_{\alpha\dot{\beta}} \\ & + (k_f\cdot\sigma/m)_{\alpha\dot{\alpha}'}\,(\mathcal{T}^{--})^{\dot{\alpha}'}{}_{\dot{\beta}} \\ & + (k_f\cdot\sigma/m)_{\alpha\dot{\alpha}'}\,(\mathcal{T}^{-+})^{\dot{\alpha}'\beta'}\,(k_i\cdot\sigma/m)_{\beta'\dot{\beta}} \,. \end{aligned} \tag{2.12}$$

From this expression, the two-by-two M function corresponding to any four-by-four $\mathcal{T}$ matrix is readily obtained. In perturbation theory, it is the singularity structure of $\mathcal{T}$ that is significant; and it is evident that M is analytically related to $\mathcal{T}$. This example is easy to generalize for any number of spin-$\frac{1}{2}$ particles with nonzero masses.

Experimental relationships can be obtained directly from the M functions, bypassing the S functions entirely [3]. The projection operators

$$P(\mathbf{s}) = \tfrac{1}{2}\,(1 + \mathbf{s}\cdot\boldsymbol{\sigma}) \tag{2.13}$$

used in conjunction with S are replaced by the covariant operators

$$\begin{aligned} P(k,s) &= \sqrt{k\cdot\widetilde{\sigma}/m}\,P(\mathbf{s})\,\sqrt{k\cdot\widetilde{\sigma}/m} \\ &= \tfrac{1}{2}\,(k\cdot\widetilde{\sigma}/m - \mathbf{s}\cdot\boldsymbol{\sigma})\ , \end{aligned} \tag{2.14}$$

where s is the pseudo four-vector that reduces to $\mathbf{s}$ in the rest frame of the particle [10,11]. Stapp defines the M functions in general essentially by the requirement that the covariant operators (2.14) and their generalization to higher spin give the correlations between experimental observables when contracted with the M functions in expressions such as $\mathrm{Tr}(P_i\,M\,P_f\,M^{\dagger})$. The covariance of the M functions under the proper, orthochronous, homogeneous Lorentz group $\mathrm{L}_{+}^{\uparrow}$, which for our special example follows from (2.12), follows generally from the postulated Lorentz invariance of experimental correlations and the constructed spinor transformation character of $P(k,s)$.

The general form of the covariance condition can be expressed using the generalized spinor transformation operator Λ_S defined by the equation,

$$\begin{aligned}(\Lambda_S\, M)_{(\alpha)(\dot\beta)} &\equiv (\Lambda_S\, M)_{\alpha_1\cdots\alpha_M\dot\beta_1\cdots\dot\beta_N} \\ &= \mathrm{S}(\Lambda)_{(\alpha)(\dot\beta)}{}^{(\alpha')(\dot\beta')}\, M_{(\alpha')(\dot\beta')} \\ &= \left[\prod_i \mathrm{D}^{S_i}(A)_{\alpha_i}{}^{\alpha'_i}\right]\left[\prod_j \mathrm{D}^{S'_j}(B)_{\dot\beta_j}{}^{\dot\beta'_j}\right] M_{(\alpha')(\dot\beta')}\,,\end{aligned} \tag{2.15}$$

where A and B are the two-by-two unimodular matrices that specify, by means of Eq. (A1.6) in Appendix A1, the complex, proper, Lorentz transformation $\Lambda(A,B)$, and where the $\mathrm{D}^S \equiv \mathcal{D}^{S,0}$ are the $(2S+1)$-dimensional, irreducible representations of $\mathcal{L}_+$, described in Appendix A1.1. The indices α_i and $\dot\beta_j$ are generalized $(2S_i{+}1)$- and $(2S_j{+}1)$-valued spinor indices; (α) and $(\dot\beta)$ are the sets of α_i and $\dot\beta_j$; and the summation convention is used for relatively upper and lower repeated indices. Thus, Λ_S represents the action of the direct product group $\mathrm{SL}(2,\mathrm{C})\times\mathrm{SL}(2,\mathrm{C})$, which is related to $\mathcal{L}_+$ in the usual way by a two-to-one homomorphism, on a finite-dimensional carrier space of spinors. If one puts $B = A^*$, then Λ_S represents the real group, $\mathrm{L}_+^\uparrow$; and $\mathrm{D}^S(A^*)$ corresponds to the representation $\mathcal{D}^{0,S}$. The representations $\mathcal{D}^{S,0}$ and $\mathcal{D}^{0,S}$ of $\mathrm{L}_+^\uparrow$ are obtained by complexification of the representation $\mathcal{D}^S$ of the rotation group. One augments the angular momentum operators $\mathbf{J}$, which are infinitesimal generators for the rotation group, by the particular choices $\mathbf{K} = \mp\mathbf{J}$, respectively, for the infinitesimal generators of the velocity transformations.

Let $K = (k_1,\ldots,k_l)$ be the set of momentum vectors for the process described by the spinor-valued function $M(K)$, and let $\Lambda K \equiv (\Lambda k_1,\ldots,\Lambda k_l)$ be the transformed momenta. Then one obtains from the Lorentz invariance of physical correlations the covariance relation [3,4]

$$\Lambda_S\, M(\Lambda^{-1} K) = M(K)\,, \tag{2.16}$$

for physical K and Λ in $\mathrm{L}_+^\uparrow$. Using this and the assumption that $M(K)$ is holomorphic in some real neighborhood (on the mass shell) of some real point K, it is a consequence of a theorem of Stapp [4] that the domain of regularity of M can be covered by sheets, each of which maps onto itself under any transformation in $\mathcal{L}_+$, and on each of which (2.16) holds for any K and for any Λ in $\mathcal{L}_+$. This is summarized by the statement that M is $\mathcal{L}_+$-covariant throughout its domain of regularity, or that M is "completely" $\mathcal{L}_+$-covariant. The notions of a domain of regularity, of a sheet, and of complete $\mathcal{L}_+$ covariance are described precisely in Sec. 6.

Each $(2S{+}1)$-valued spinor index of an M function corresponds to a particle of spin S. In the spin-$\frac{1}{2}$ example given above, one of the spinor indices was dotted and one was undotted; and both were lower. This is purely conventional. The choice depends only on the way in which R is contracted with the operators $\sqrt{k\cdot\sigma/m}$ and $\sqrt{k\cdot\widetilde{\sigma}/m}$ to give M, or more generally, on the conventions for the generalized covariant operators $P(k, s)$. Simple conventions have been adopted [3] that lead to unambiguous transformations for altering the character of the spinor indices of M functions. For present purposes it will be convenient to let all indices be of the same type. It will be seen in the following section that this is no restriction on the generality of the results.

From now on, the M functions are to be considered as defined by (2.16), with Λ in $\mathcal{L}_+$. It is easily shown from this equation that the sum of the spins of an M function is an integer [3,1] (i.e., in every scattering process the total number of initial and final fermions is even); and thus the M function is equivalent to a tensor function.

3 Reduction of Spins

To simplify the problem of expanding the M functions, and hence the S matrix, in terms of scalar amplitudes, we shall first decompose them into irreducible parts. Of course this is a "trivial" procedure, even for the S matrix itself; but for the S matrix the reduction is somewhat messy, due to the somewhat complicated transformation law of the spin indices. For the M functions, because of their simple transformation law, the reduction is algebraically simple, involving only the familiar addition of spins by contracting with combinations of Clebsch-Gordan (C-G) coefficients. We follow the same procedure as that of Fano and Racah [12] for the rotation group. Details of the construction of the appropriate projection operators are given in Appendix A1.3.

As mentioned above, one needs only to consider M functions with all indices of the same type, say lower undotted. Any upper index can be lowered by contracting with the metric symbol $\{S\}$ defined in (A1.18). Any dotted index can be converted into an undotted index by contracting with one of the "metric" symbols introduced by Stapp [3,1],

$$\begin{aligned} [\,S,k\,]^{\dot\beta\alpha} &\equiv \mathrm{D}^S(k\cdot\widetilde\sigma/m)^{\dot\beta\alpha}, \\ \{S,k\}_{\alpha\dot\beta} &\equiv \mathrm{D}^S(k\cdot\sigma/m)_{\alpha\dot\beta}\,, \end{aligned} \tag{3.1}$$

where k is the momentum of the particle of spin S whose index is to be transformed, and where the matrix elements of D^S are labeled according to the index types of the argument. These spinors satisfy the orthogonality relations

$$\begin{aligned} \{S,k\}_{\alpha\dot\beta}\;[\,S,k\,]^{\dot\beta\alpha'} &= \mathrm{D}^S(k\cdot\sigma\,k\cdot\widetilde\sigma/m^2)_\alpha{}^{\alpha'} = \delta_\alpha{}^{\alpha'}, \\ [\,S,k\,]^{\dot\beta'\alpha}\;\{S,k\}_{\alpha\dot\beta} &= \delta_{\dot\beta}{}^{\dot\beta'}, \end{aligned} \tag{3.2}$$

because of the identity

$$k\cdot\sigma\,k\cdot\widetilde\sigma = k\cdot\widetilde\sigma\,k\cdot\sigma = k\cdot k = m^2. \tag{3.3}$$

In order to reduce the function $M_{(\alpha)}$ one contracts with the projection operators

$$[\,J\mathcal{J}:(S,N)\,]_\alpha{}^{(\beta)} \equiv [\,J\mathcal{J}:S_1,\ldots,S_N\,]_\alpha{}^{\beta_1\cdots\beta_N}, \tag{3.4}$$

for total spin J. The symbol $\mathcal{J}$ stands for the set of intermediate spins that occur in the reduction of $S_1,\ldots,S_N$, beginning at the left. These operators are formed by contracting successively with the C-G coefficients, as described in Appendix A1.3. The projections are single-spin functions with the same transformation law as an M function:

$$M(J\mathcal{J})_\alpha = [\,J\mathcal{J}:(S,N)\,]_\alpha{}^{(\beta)}\,M_{(\beta)}\,, \tag{3.5}$$

or in matrix notation

$$M(J\mathcal{J}) = [\,J\mathcal{J}:(S,N)\,]\,M\,. \tag{3.6}$$

Equation (3.5) or (3.6) can be inverted by contracting with the inverse projection operators

$$\begin{aligned} \{(S,N):J\mathcal{J}\}_{(\alpha)}{}^{\beta} &= \{S_1,\dots,S_N:J\mathcal{J}\}_{\alpha_1\dots\alpha_N}{}^{\beta} \\ &= [\,J\mathcal{J}:(S,N)\,]_{\beta}{}^{(\alpha)}. \end{aligned} \tag{3.7}$$

Writing the orthogonality relation (A1.37) in the form

$$\sum_{J,\mathcal{J}} \{(S,N):J\mathcal{J}\}\;[\,J\mathcal{J}:(S,N)\,] = I\,, \tag{3.8}$$

one finds that

$$M = \sum_{J,\mathcal{J}} \{(S,N):J\mathcal{J}\}\;M(J\mathcal{J})\,. \tag{3.9}$$

The summation in these equations is over all $J,\mathcal{J}$ occurring in the reduction of the spins (S,N).

It is clear that the decomposition (3.9) does not introduce any extra singularities into $M(J\mathcal{J})$ that were not already present in M. Thus the problem of expanding M in terms of scalar amplitudes without kinematical singularities is solved by finding such an expansion for the irreducible functions $M(J\mathcal{J})$. Note that because $\sum_i S_i$ is an integer, J is always an integer. The quantities $M(J\mathcal{J})$ are analogous in some respects to the familiar spherical tensors in the theory of the rotation group.

4 The Spin Basis: Conversion to an Equivalent Tensor

From now on we restrict ourselves to spinor-valued functions $M(J)$ of the four-momenta (the symbol $\mathcal{J}$ is suppressed because it has nothing to do with the transformation properties) that satisfy the covariance condition (2.16), with $\Lambda_S = \mathrm{D}^J(A)$. In this section a set of spinors labeled by tensor indices is constructed, which spans the spin space. These spinors are then combined with available momentum vectors in Sec. 5 to form a basis in the space of $M(J)$ functions.

The elements of the spin basis constructed here are just row vectors from a matrix that transforms the spinor $M(J)$ into an equivalent irreducible tensor. There are several ways of doing such a construction. For example, $M(J)$ is equivalent to a traceless tensor of rank $2J$ that is antisymmetric and selfdual in successive pairs of indices, and symmetric in the interchange of selfdual pairs of indices. The general transformations have been given by Barut, Muzinich, and Williams [1] and are written down in (A1.43) and (A1.44). Here, we find it algebraically more convenient to construct a spin basis that is somewhat specialized.

For the case $J = 1$, one can define a set of four spinors, labeled by the vector index μ,

$$\rho_\mu\,(1:v)_\alpha = [\,1\,\tfrac{1}{2}\,\tfrac{1}{2}\,]_\alpha{}^{\beta\gamma}\,\left[\rho_\mu\, v\cdot\widetilde{\sigma}\,\left\{\tfrac{1}{2}\right\}\right]_{\beta\gamma}\,, \tag{4.1}$$

where the symbol $[\,1\,\tfrac{1}{2}\,\tfrac{1}{2}\,]_\alpha{}^{\beta\gamma} = C\left(\tfrac{1}{2}\,\tfrac{1}{2}\,1;\beta\gamma\alpha\right)$ is a C-G coefficient in terms of the convention of Rose [13], $\rho_\mu = \sigma_\mu/\sqrt{2}$ is the normalized Pauli spinor, $\left\{\tfrac{1}{2}\right\}$ is the spin-$\tfrac{1}{2}$ metric symbol defined in (A1.18), and v is some four-vector valued, covariant function of the available momenta satisfying $v\cdot v \neq 0$. If the four-vector index in (4.1) is contracted with a vector w, the result is equivalent to the antisymmetric, selfdual part of the tensor $w^\mu\, v^\nu$. In particular, by using the symmetry of $[\,1\,\tfrac{1}{2}\,\tfrac{1}{2}\,]$, the antisymmetry of $\left\{\tfrac{1}{2}\right\}$, and (3.3), one easily calculates the relations

$$w\cdot\rho(1:v) + v\cdot\rho(1:w) = 0\,, \tag{4.2}$$

$$v\cdot\rho(1:v) = 0\,. \tag{4.3}$$

Equation (4.3) can be viewed as that relation among the four $\rho_\mu(1:v)$ that must exist because the spin-1 space is three-dimensional.

From various orthogonality relations involving C-G coefficients (such as those listed in Appendix A1.1 and A1.2, it is easy to verify that

$$(v\cdot v)^{-1}\,\rho_\mu\,(1:v)_\alpha\;\rho^\mu\,(1:v)^\beta = \delta_\alpha{}^\beta, \tag{4.4}$$

$$\begin{aligned}(v\cdot v)^{-1}\,\rho_\mu\,(1:v)_\alpha\;\rho_\nu\,(1:v)^\alpha &= g_{\mu\nu} - v_\mu\, v_\nu/v\cdot v \\ &\equiv h_{\mu\nu}(v)\,.\end{aligned} \tag{4.5}$$

The vector v can always be brought to the form $\left(\sqrt{v\cdot v},0,0,0\right)$ by a transformation in $\mathcal{L}_+$. In such a frame, (4.5) reduces to the ordinary three-dimensional metric symbol. The orthogonality relations (4.4) imply that the $\rho(1:v)$ spinors span the spin-1 space.

From the spinor calculus in Appendix A1, one derives the transformation law

$$\mathrm{D}^1(A)\ \rho^\mu(1:v) = \Lambda_\nu{}^\mu(A,B)\ \rho^\nu\left[1:\Lambda(A,B)v\right]\ . \tag{4.6}$$

Spinors for arbitrary integral J can be constructed by addition of spins of magnitude 1,

$$\begin{aligned}\rho^{\mu_1\cdots\mu_J}\left(J:v_1,\ldots,v_J\right)_\alpha \\ = \left[\,J\mathcal{J}:(1,J)\,\right]_\alpha{}^{(\beta)}\ \rho^{\mu_1}\left(1:v_1\right)_{\beta_1}\ \cdots\ \rho^{\mu_J}\left(1:v_J\right)_{\beta_J}\ ,\end{aligned} \tag{4.7}$$

where $\mathcal{J} = (2,3,\ldots,J\text{–}1)$ is in this case uniquely determined by our conventions for the construction of $[\,J\mathcal{J}:(1,J)\,]$ and thus can be suppressed. In condensed notation this becomes

$$\rho^{(\mu)}(J:v_1,\ldots,v_J) = [\,J:(1,J)\,]\bigotimes_J \rho^{\mu_i}(1:v_i)\,. \tag{4.8}$$

From the orthogonality relations (A1.36) and (4.4) one finds that

$$\prod_i (v_i\!\cdot\!v_i)^{-1}\ \rho_{(\mu)}\left(J:v_1,\ldots,v_J\right)_\alpha\ \rho^{(\mu)}\left(J:v_1,\ldots,v_J\right)^\beta = \delta_\alpha{}^\beta. \tag{4.9}$$

The transformation law (4.6) becomes

$$\begin{aligned}\mathrm{D}^J(A)\ \rho^{(\mu)}(J:v_1,\ldots,v_J) \\ = \Lambda_{(\nu)}{}^{(\mu)}(A,B)\ \rho^{(\nu)}[J:\Lambda(A,B)v_1,\ \ldots,\ \Lambda(A,B)v_J]\ ,\end{aligned} \tag{4.10}$$

where

$$\Lambda_{(\nu)}{}^{(\mu)} \equiv \Lambda_{\nu_1}{}^{\mu_1}\ \cdots\ \Lambda_{\nu_J}{}^{\mu_J}. \tag{4.11}$$

Equations (4.3) and (4.8) lead to

$$\left(v_i\right)_{\mu_i}\ \rho^{\mu_1\cdots\mu_i\cdots\mu_J}(J:v_1,\ldots,v_J) = 0\,. \tag{4.12}$$

There are further relations among the $\rho(J:v_1,\ldots,v_J)$. It is an easily proved algebraic fact that the projection operator $[\,J:(1,J)\,]$ is symmetric in the interchange of any pair of spin-1 indices; one can see this directly by noting that there is only one way to get a total spin of J from the addition of J spins of magnitude 1. From this fact and (4.8) it follows that $\rho(J:v_1,\ldots,v_J)$ is symmetric in the interchange of any pair of tensor indices when one simultaneously interchanges the corresponding four-vector arguments. Furthermore, because it is essentially an irreducible tensor of minimum rank (see Appendix A2), $\rho(J:v_1,\ldots,v_J)$ vanishes upon contraction of any pair of tensor indices.

It is convenient to choose all of the four-vector arguments $v_1, \ldots, v_J$ to be equal, and to write $\rho(J:v) \equiv \rho(J:v,\ldots,v)$. Then from what has just been said, $\rho(J:v)$ is symmetric and traceless in its tensor indices. By choosing the four-vector arguments to be the same, we have in a certain sense reduced the problem for the group $\mathcal{L}_+$ to an equivalent problem for the three-dimensional, proper, complex orthogonal group, $\mathrm{O}_+(3,\mathrm{C})$. In particular, when v is in its rest frame, (4.12) implies that $\rho(J:v)$ vanishes when any tensor index has the value zero; and one is left with a tensor with respect to $\mathrm{O}_+(3,\mathrm{C})$. From this one can see immediately that $\rho(J:v)$ is traceless, for ${\rho_\mu}^{\mu\mu_3\cdots\mu_J}(J:v)$ is a tensor of rank less than J corresponding to the irreducible representation $\mathcal{D}^J$ of $\mathrm{O}_+(3,\mathrm{C})$.

There are precisely $2J+1$ linearly independent, symmetric and traceless tensors of rank J with respect to $\mathrm{O}_+(3,\mathrm{C})$, and because of (4.9) there are at least $2J+1$ linearly independent spinors among the $\rho(J:v)$. Thus we have a basis for the spin-J space.

To save words, the space of tensors of rank J that vanish when any index is contracted with v will be denoted by $\mathcal{T}(J:v)$. Then $\rho(J:v)$ defines a one-to-one map from the spin-J spinor space onto the subspace of symmetric and traceless tensors $\mathcal{T}(J:v)$. Later, some use will be made of the fact that

$$S_{(\mu)(\nu)}(J:v) \equiv (v.v)^{-J}\ \rho_{(\mu)}\,(J:v)_\alpha\ \rho_{(\nu)}\,(J:v)^\alpha \tag{4.13}$$

is the projection operator from the space of Jth rank tensors onto the subspace of symmetric and traceless tensors in $\mathcal{T}(J:v)$. It is proved in Appendix A2 that $S(J:v)$ is a tensor not only with respect to $\mathcal{L}_+$ but also with respect to $\mathcal{L}$, the unrestricted, homogeneous, complex Lorentz group. For completeness, the remaining projection operators for the space $\mathcal{T}(J:v)$ are also given in Appendix A2.

5 Basis Functions for $M(J)$

Using the orthogonality relation (4.9) for $\rho(J:v)$ one readily finds that

$$M(J) = f^{(\mu)}\,\rho_{(\mu)}(J:v) \equiv f\cdot\rho(J:v)\,, \tag{5.1}$$

where

$$f \equiv (v\cdot v)^{-J}\;M(J)_\alpha\;\rho(J:v)^\alpha. \tag{5.2}$$

Clearly f is a symmetric and traceless tensor in $\mathcal{T}(J:v)$. From (5.2), f has no more singularities than $M(J)$, because $v\cdot v$ is assumed not to vanish. A set of basis functions with the same transformation law as $M(J)$ is constructed by finding tensors that span the tensor space and contracting with $\rho(J:v)$.

For two-body reactions on the mass shell, at most three independent four-momenta are available, the fourth being determined by energy-momentum conservation. In the region where the Gram determinant of the momentum vectors $K \equiv (k_1, k_2, k_3)$,

$$G(K) \equiv \det(k_i\cdot k_j)\,, \tag{5.3}$$

does not vanish, one can form a basis in the four-vector space by adding the pseudovector w, defined below, to the set K. Thus the vectors v_i form a basis, where[4]

$$\begin{aligned} v_i &= k_i\,, \quad i = 1,2,3\,, & v_4 &= w = [\,k_1, k_2, k_3\,]\,,\\ w^\mu &\equiv \epsilon^{\mu\nu\lambda\sigma}\,k_{1\nu}\,k_{2\lambda}\,k_{3\sigma}\,, & w\cdot w &= -G(K)\,. \end{aligned} \tag{5.4}$$

A reciprocal basis is formed by the vectors

$$\begin{aligned} \hat v_1 &= [\,v_2, v_3, v_4\,]/G\,, & \hat v_2 &= [\,v_3, v_1, v_4\,]/G\,,\\ \hat v_3 &= [\,v_1, v_2, v_4\,]/G\,, & \hat v_4 &= -v_4/G\,, \end{aligned} \tag{5.5}$$

which satisfy

$$v_i\cdot\hat v_j = \delta_{ij}\,, \qquad \sum_i {v_i}^\mu\,{\hat v_i}{}^\nu = \sum_i {\hat v_i}{}^\mu\,{v_i}^\nu = g^{\mu\nu}. \tag{5.6}$$

Then the monomials

$$T\,[(i,N)] \equiv v_{i_1}\otimes\ldots\otimes v_{i_N}\,, \qquad i_j = 1,2,3,4\,, \tag{5.7}$$

form a basis in the space of tensors of rank N, and a reciprocal basis is formed by

$$\widehat{T}\,[(i,N)] \equiv \hat v_{i_1}\otimes\ldots\otimes\hat v_{i_N}\,, \tag{5.8}$$

[4] The convention for the alternating symbol is $\epsilon_{0123} = -1$.

so that

$$T\,[(i,N)]\cdot\widehat{T}\,[(j,N)] = \delta_{(i,N)(j,N)}\,,$$
$$\sum_i T^{(\mu)}\,[(i,N)]\;\widehat{T}^{(\nu)}\,[(i,N)] = g^{(\mu)(\nu)}. \tag{5.9}$$

The expansion of f, when $G \neq 0$, is then

$$f = \sum_i\; a_i\; T\,[(i,J)]\;, \tag{5.10}$$

where

$$a_i = f\cdot\widehat{T}\,[(i,J)]\;. \tag{5.11}$$

Then $M(J)$ becomes

$$M(J) = \sum_i\; a_i\; T\,[(i,J)]\cdot\rho(J:v)\,. \tag{5.12}$$

Equation (5.11) defines a total of 4^J scalar functions, but from (5.12) only $2J{+}1$ of these can be algebraically independent. Indeed, (5.2) and (5.12) imply that they satisfy the linear relations

$$a_i = \sum_j\; a_j\; T\,[(j,J)]\cdot\rho(J:v)_\alpha\;\widehat{T}\,[(i,J)]\cdot\rho(J:v)_\alpha\;(v\cdot v)^{-J}. \tag{5.13}$$

It will not restrict the generality of the results, and it will simplify the discussion to choose v to be one of the available momenta, say k_3. Because of (4.12), all terms such that k_3 occurs in $T[(i,J)]$ drop out in the expansion (5.12). From the symmetry of $\rho(J:k_3)$, all terms where the vectors k_1, k_2, and w occur in $T[(i,J)]$ the same number of times, regardless of order, are equal. The terms can therefore be labeled by a partition of the integer J into three parts. Let

$$T^{(\mu)}(i,j,J{-}i{-}j) = {k_1}^{\mu_1}\,\cdots\,{k_1}^{\mu_i}\,{k_2}^{\mu_{i+1}}\,\cdots\,{k_2}^{\mu_{i+j}}\,w^{\mu_{i+j+1}}\,\cdots\,w^{\mu_J}. \tag{5.14}$$

The number of tensors $T[(i,J)]$ that contain k_1 a total of i times, k_2 a total of j times, and w a total of $J{-}i{-}j$ times is just the multinomial coefficient

$$\binom{J}{i,j} = \frac{J!}{i!\,j!\,(J{-}i{-}j)!}\,. \tag{5.15}$$

Defining

$$a(i,j,J{-}i{-}j) \equiv \binom{J}{i,j}\,f\cdot\widehat{T}(i,j,J{-}i{-}j)\,, \tag{5.16}$$

where $\widehat{T}(i,j,J-i-j)$ is obtained by replacing the vectors in (5.14) with their reciprocals, one has the expansion

$$M(J) = \sum_{0\le i+j\le J} a(i,j,J-i-j)\, T(i,j,J-i-j)\cdot\rho(J:k_3)\,. \tag{5.17}$$

From (5.16), $a(i,j,J-i-j)$ can have a singularity G^{-J} in addition to the singularities of f. An example of a holomorphic function that actually gives rise to such a "kinematical" singularity in the expansion (5.17) is given in Appendix A3.

So far, the fact that $\rho(J:k_3)$ is traceless has not been used. By using the well-known identity,

$$-\epsilon_{\mu\nu\lambda\rho}\,\epsilon_{\delta\kappa\sigma\tau} = \begin{pmatrix} g_{\mu\delta} & g_{\mu\kappa} & g_{\mu\sigma} & g_{\mu\tau} \\ g_{\nu\delta} & g_{\nu\kappa} & g_{\nu\sigma} & g_{\nu\tau} \\ g_{\lambda\delta} & g_{\lambda\kappa} & g_{\lambda\sigma} & g_{\lambda\tau} \\ g_{\rho\delta} & g_{\rho\kappa} & g_{\rho\sigma} & g_{\rho\tau} \end{pmatrix}, \tag{5.18}$$

all tensors $T(i,j,J-i-j)$ with $J-i-j>1$, i.e., with w occurring more than once, can be reduced to combinations of simpler terms [5]. Thus,

$$\begin{aligned} ww = -g\,G(K) &+ k_1\,k_1\,\left({m_2}^2{m_3}^2-\beta^2\right) + k_2\,k_2\,\left({m_1}^2{m_3}^2-\gamma^2\right) \\ &+ k_3\,k_3\,\left({m_1}^2{m_2}^2-\alpha^2\right) + (k_1\,k_2+k_2\,k_1)\,\left(\beta\gamma-\alpha{m_3}^2\right) \\ &+ (k_2\,k_3+k_3\,k_2)\,\left(\gamma\alpha-\beta{m_1}^2\right) + (k_3\,k_1+k_1\,k_3)\,\left(\alpha\beta-\gamma{m_2}^2\right) \end{aligned} \tag{5.19}$$

where $\alpha = k_1\cdot k_2$, $\beta = k_2\cdot k_3$, $\gamma = k_3\cdot k_1$, and ${m_i}^2 = k_i\cdot k_i$. When contracted with $\rho(J:k_3)$, the term proportional to the metric tensor g vanishes because ρ is traceless; only the terms proportional to $k_1\,k_1$, $(k_1\,k_2+k_2\,k_1)$, and $k_2\,k_2$ remain, because of (4.12).

Each of the $T\cdot\rho$ terms in (5.17) reduces to a linear combination of the following $2J+1$ functions:

$$\begin{aligned} Y^{(+)}(i:J) &= {k_1}^{\mu_1}\cdots{k_1}^{\mu_i}\,{k_2}^{\mu_{i+1}}\cdots{k_2}^{\mu_J} \\ &\qquad\qquad \times\rho_{(\mu)}(J:k_3)\,, \qquad i=0,\dots,J\,; \\ Y^{(-)}(i:J) &= {k_1}^{\mu_1}\cdots{k_1}^{\mu_i}\,{k_2}^{\mu_{i+1}}\cdots{k_2}^{\mu_{J-1}}\,w^{\mu_J} \\ &\qquad\qquad \times\rho_{(\mu)}(J:k_3)\,, \qquad i=0,\dots,J-1\,. \end{aligned} \tag{5.20}$$

Then

$$M(J) = \sum_{i=0}^{J} b_i^{(+)}\,Y^{(+)}(i:J) + \sum_{i=0}^{J-1} b_i^{(-)}\,Y^{(-)}(i:J)\,. \tag{5.21}$$

It will be shown in the following section that the scalar amplitudes $b_i^{(\pm)}$ do not have kinematical singularities.

Before proceeding to a discussion of analytic properties, we give some formulas for the scalar amplitudes in terms of $M(J)$. Setting

$$\begin{aligned}
&\alpha_{11} = {m_2}^2 {m_3}^2 - \beta^2, \qquad \alpha_{22} = {m_1}^2 {m_3}^2 - \gamma^2, \\
&\alpha_{12} = 2\left(\beta\gamma - \alpha {m_3}^2\right),
\end{aligned} \tag{5.22}$$

one can express $b_i^{(\pm)}$ in terms of the $a(i,j,J-i-j)$:

$$b_i^{(\pm)} = \sum_{l,m,n} a\left[m, J_\pm - m - 2l, 2l + \left(\tfrac{1}{2} \mp \tfrac{1}{2}\right)\right] \\
\times \binom{l}{n, i-m-2n} \alpha_{11}^n \, \alpha_{12}^{i-m-2n} \, \alpha_{22}^{l+m+n-i}, \tag{5.23}$$

where $J_\pm = J - (\frac{1}{2} \mp \frac{1}{2})$, and where the summation is over the ordered triple of integers (l,m,n) satisfying the conditions

$$\begin{aligned}
&\max(0, i-l-m) \le n \le \left[\frac{i-m}{2}\right], \\
&\max(0, i-2l) \le m \le \min(i, J_\pm - 2l), \\
&0 \le l \le \left[\frac{J_\pm}{2}\right],
\end{aligned} \tag{5.24}$$

where $[a/b]$ is the "integer part" of the rational number a/b.

The $b_i^{(\pm)}$ can also be expressed directly in terms of the traces

$$t^{(\pm)}(i : J) \equiv M(J)_\alpha \, Y^{(\pm)}(i : J)^\alpha. \tag{5.25}$$

A tedious, but straightforward calculation gives

$$b_i^{(\pm)} = \pm \left({m_3}^2 G\right)^{-J} \sum_j \sum_l t^{(\pm)}(j : J) \\
\times \eta_\pm(i,j,l) \, \alpha_{11}^l \, \alpha_{12}^{i+j-2l} \, \alpha_{22}^{J_\pm - i - j + l}, \tag{5.26}$$

where

$$\begin{aligned}
&\max(0, i+j-J_\pm) \le l \le \left[\frac{i+j}{2}\right], \\
&0 \le j \le J_\pm,
\end{aligned} \tag{5.27}$$

and where

$$\eta_\pm(i,j,l) = \sum_{m,n,r,s,t} 2^{2(l-m-n)-s-t} \binom{J}{s, 2r + \frac{1}{2} \mp \frac{1}{2}} \binom{s}{l-m-n} \\
\times \binom{J_\pm - s - 2r}{t - l + m + n} \binom{r}{m} \binom{r-m}{i-s-2m} \binom{r}{n} \binom{r-n}{j-t-2n}, \tag{5.28}$$

the summation ranging over

$$
\begin{aligned}
&\max(l-m-n, j-r-n) \leq t \\
&\qquad \leq \min\left(J_\pm + l - 2r - s - m - n, j - 2n\right) , \\
&\max(l-m-n, i-r-m) \leq s \\
&\qquad \leq \min\left(J_\pm - 2r, J_\pm - r - m - j + l, i - 2m\right) , \\
&\max(m, n) \leq r \\
&\qquad \leq \min\left(J_\pm + n - j, J_\pm + m - i, \left[\frac{J_\pm - l + m + n}{2}\right]\right) , \\
&\max(0, l-i+m) \leq n \leq \min\left(\left[\frac{j}{2}\right], l-m, j+m-l\right) , \\
&\max(0, l-j) \leq m \leq \min\left(\left[\frac{i}{2}\right], l\right) .
\end{aligned}
\tag{5.29}
$$

6 Holomorphy of the Scalar Amplitudes

The procedure followed in the sections above for reducing a covariant function into irreducible parts and for reducing the set of covariant polynomials $T(i,j,J-i-j)\cdot\rho$ into the set $Y^{(\pm)}(i:J)$ can be used to express any covariant polynomial in the momentum vectors (on the mass shell) in terms of the $Y^{(\pm)}(i:J)$, with coefficients that are polynomials in the scalar invariants. Because the $a(i,j,J-i-j)$ in (5.16) and (5.17) have at most a singularity G^{-J} in the domain where $M(J)$ is holomorphic, it follows that the same is true of the $b_i^{(\pm)}$ in (5.21). In this section we derive a condition for any domain over the mass shell, where $M(J)$ is holomorphic, that is necessary and sufficient if the $b_i^{(\pm)}$ are to be holomoprhic functions on the corresponding domain of scalar invariants. This means of course that for such domains the $b_i^{(\pm)}$ do not have poles when $G=0$, i.e., that they are free of kinematical singularities.

We shall see that the condition is essentially (although somewhat weaker than) the following: the domain where $M(J)$ is holomorphic must be such that, if it contains a point, then it contains all points with the same scalar invariants.

In order to derive the condition, we found it necessary first to prove an extension to the most general possible domain of that part of the Hall-Wightman Theorem which says roughly that a holomorphic function invariant under $\mathcal{L}$ is a holomorphic function of $\mathcal{L}$ invariants.[5] The Hall-Wightman Theorem was used by Wong [7] in his proof that the N-N amplitudes have no kinematical singularities. Its application there was not strictly justified, however, because the Hall-Wightman Theorem pertains to a special kind of domain, the future tube. The functions in analytic S-matrix theory, on the other hand, are defined on the mass shell, which has no points in common with the future tube. Thus, a generalization of the Hall-Wightman Theorem is needed to justify using its consequences in our problem.

As already mentioned, the required generalization was proved independently by Hepp, so that we can avoid some complication by omitting the proof and citing his result.

Aside from the Hall-Wightman Theorem, we find it necessary for our derivation to prove the existence of a decomposition of $M(J)$ into two parts equivalent to a tensor and pseudotensor, both of which are holomorphic, and to prove certain linear independence properties of the $Y^{(\pm)}(i:J)$ at some of the points where G vanishes.

Before we can proceed to the derivation, however, or even state our theorem precisely, some well-established geometrical facts must be mentioned, and the notion of covariant and holomorphic functions on locally schlicht domains over the mass shell must be explained. This will enable us to construct the concept of an "$\mathcal{L}_+$-invariant structure", which with the help of Stapp's Theorem mentioned in Sec. 2 provides a suitable framework for our discussion.

[5] A similar result was proved for the group $\mathcal{L}_+$, but in the application here, where no pseudoscalars exist, there is essentially no difference.

6.1 Complex Four-Vectors

Hall and Wightman [8] have established some geometrical properties of the space of l complex four-vectors. Let the number of independent vectors at a point $K \equiv (k_1, \ldots, k_l)$ be denoted by n, and the rank of the Gram determinant, $G(K) \equiv \det(k_i \cdot k_j)$, be denoted by r. Then $n \le 4$ and $r \le 4$. For $r = 3$ or 4, one has $n = r$. For $r = 1$ or 2, one can have $n = r$ or $n = r + 1$. For $r = 0$, one can have $n = 0$, 1, or 2. For points K with $n \ne r$, there are always points K' with the same scalar invariants, $k_i \cdot k_j = k'_i \cdot k'_j$, but for which $n = r$. The converse is also true for $r \le 2$. For any K, there is always a subset of r independent vectors with nonvanishing Gram determinant. Without loss of generality, it is assumed in the following that these vectors are $(k_1, \ldots, k_r)$.

A special case of a result of Hall and Wightman [8, Lemma2] is

Lemma 1. If K and K' are $n = r$ points having the same scalar invariants, then there exists a Λ in $\mathcal{L}$ such that $K' = \Lambda K$.

The following lemma also holds:

Lemma 2. Let K be such that $n \ne 4$, and if $n = 3$ then $r \ne 2$. Then for any K' there exists a Λ in $\mathcal{L}_+$ such that $K' = \Lambda K$ if and only if there exists an improper Λ' in $\mathcal{L}$ such that $K' = \Lambda' K$.

The proof is to note that in these cases, there is always a subspace orthogonal to the vectors of K. This follows immediately from the considerations of Hall and Wightman. One can then introduce an improper transformation in that subspace that acts as the identity on K.

The following terminology is standard:

Definition 1. For a set S of points K and a group $\mathcal{G}$, $\mathcal{G}S$ is the set of points of the form $K' = \Lambda K$, where K is in S and Λ is in $\mathcal{G}$. The set $\mathcal{G}K$ is called the *$\mathcal{G}$ orbit of* K.

Consider now a point K for which $r = 2$ and $l = 3$; $K \equiv (k_1, k_2, k_3)$. Then $G(k_1, k_2) \ne 0$, by the ordering convention established above. From this it follows that one can choose two orthonormal vectors (in the Lorentz metric), ω_1 and ω_2, in the space orthogonal to k_1 and k_2, such that k_1, k_2, ω_1, and ω_2 are linearly independent [8, footnote 7]:

$$k_i \cdot \omega_j = 0\,; \qquad \omega_i \cdot \omega_j = \delta_{ij}\,; \qquad i, j = 1, 2\,. \tag{6.1}$$

Because $G(K)$ has rank 2, k_3 must have one of the two alternative forms

$$k_3 = a\, k_1 + b\, k_2 + c\, \omega_\pm\,, \tag{6.2a}$$

where

$$\omega_\pm \equiv \omega_1 \pm i\, \omega_2\,. \tag{6.2b}$$

The vectors ω_+ and ω_- are on the light cone and are orthogonal to k_1 and k_2. Notice that the scalar invariants $k_i{\cdot}k_j$ are independent of c. Also, if $c = 0$, then $n = 2$; and if $c \neq 0$, then $n = 3$.

The transformation $\omega_\pm \to \lambda^{\pm 1}\omega_\pm$ is, for any $\lambda \neq 0$, a transformation in $\mathcal{L}_+$. Thus all points K of the above form with the same values of a and b, and for which k_3 has a nonvanishing component along ω_+, are connected by a transformation in $\mathcal{L}_+$. The same is true for points where k_3 has a nonvanishing component along ω_-. Evidently the point with $c = 0$ is a limit point of each of these two sets. Lemmas 1 and 2 applied to the subset of K consisting of its first two vectors alone ensure that one can find a transformation in $\mathcal{L}_+$ that takes any point K' having the same invariants as K to a point such that $k'_1 = k_1$ and $k'_2 = k_2$. The vector k'_3 must then have one of the two forms given by (6.2a). Thus the set of all points with the same scalar invariants as K is the union of three $\mathcal{L}_+$ orbits, two with $n = 3$ and one with $n = 2$. By considering the linear transformation that leaves k_1, k_2, and ω_1 unchanged and replaces ω_2 by $-\omega_2$, one sees that points of the two $n = 3$ orbits are related by an improper Lorentz transformation. Clearly, any point of the $n = 2$ orbit is a limit point of both of the $n = 3$ orbits. Because of the continuity of the scalar invariants, the union of all three orbits is a closed set. Moreover, this fact and the discussion above imply that the $n = 2$ orbit is itself a closed set, and that the union of the $n = 2$ orbit with either of the $n = 3$ orbits is a closed set.

An analogous discussion can be given for $r = 1$ points. In that case, K has the form

$$\begin{aligned} k_2 &= a_2\, k_1 + b_2\, \omega\,, \\ k_3 &= a_3\, k_1 + b_3\, \omega\,, \end{aligned} \tag{6.3}$$

where $\omega \cdot k_1 = \omega \cdot \omega = 0$. If either b_2 or b_3 is nonzero, then $n = 2$, and if both vanish then $n = 1$. From Lemma 2, $\mathcal{L}_+ K = \mathcal{L} K$. This does not mean, however, that there is only one $n = 2$, $\mathcal{L}_+$ orbit with the same invariants as the $n = r = 1$, $\mathcal{L}_+$ orbit. In fact, it is clear that there is a distinct orbit for each value of the ratio b_2/b_3. Any transformation that leaves k_1 unchanged and multiplies ω by a nonzero complex number can be written as a transformation in $\mathcal{L}_+$, just as before, and thus it is still true that the $\mathcal{L}_+$ orbit of the $n = r = 1$ point is closed while that of any of the $n = 2$ orbits becomes closed by adding the $n = 1$ orbit.

6.2 Holomorphy and Covariance on Domains Over the Mass Shell

The mass shell, $\mathcal{K}_l$, is defined by the equations

$$\begin{aligned} &k_i{\cdot}k_i = m_i^2\,, \qquad \sum_i k_i = 0\,, \\ &m_i^2 > 0\,, \qquad i = 1, \ldots, l+1\,, \end{aligned} \tag{6.4}$$

where the masses, m_i, are constants. The requirement $m_i \neq 0$ implies that $\mathcal{K}_l$ contains no $r = 0$ points. The topology of $\mathcal{K}_l$ is taken to be that induced from the complex number space $\mathbb{C}^{4(l+1)}$ in which it is embedded. Because of the conservation equation in (6.4), points in $\mathcal{K}_l$ can be represented by $K = (k_1, \ldots, k_l)$.

Since $\mathcal{K}_l$ is not a Euclidean space, but rather an algebraic variety, holomorphy on $\mathcal{K}_l$ must be defined in a generalized sense. The concept of holomorphy on "complex spaces", which need be only locally Euclidean at most points, has been extensively developed by mathematicians in recent years [14], but has been, as yet, little used by physicists. The generalization needed here is relatively simple, and stays close to concepts that are familiar for functions on ordinary locally schlicht domains. Certain essential properties of holomorphic functions continue to hold; in particular, a holomorphic function of a holomorphic function is holomorphic, and a holomorphic function has a unique analytic continuation.[6]

The most direct generalization of holomorphy is to functions on complex manifolds. A complex manifold of (complex) dimension d is a connected Hausdorff space[7] with a complex structure. This means that the space can be covered with open sets U_α each of which is mapped into the space $\mathbb{C}^d$ of d complex numbers by a homeomorphism (a one-to-one continuous map with a continuous inverse) $h_\alpha : U_\alpha \to \mathbb{C}^d$, which is such that if $U_\alpha \cap U_\beta$ is not empty, then $h_\alpha \circ h_\beta^{-1}$ is a holomorphic mapping of $h_\beta (U_\alpha \cap U_\beta)$ onto $h_\alpha (U_\alpha \cap U_\beta)$. A complex-valued function f is said to be holomorphic on such a manifold if and only if each of the functions $f \circ h_\alpha^{-1}$ is holomorphic on $h_\alpha (U_\alpha)$; that is, f must be holomorphic when expressed in terms of any of the "local coordinates".

Stapp has shown [4] that the "restricted mass shell", $\mathcal{K}_l'$, obtained by deleting all $n = 1$ points from $\mathcal{K}_l$, has a complex structure such that the components of the four-momenta are holomorphic in the local coordinates. With that structure, $\mathcal{K}_l'$ is a complex manifold of dimension $3l - 1$. It is easy to show that $\mathcal{K}_l'$ is dense in $\mathcal{K}_l$. Thus any continuous function on $\mathcal{K}_l$ is determined by its values on $\mathcal{K}_l'$. A function is said to be holomorphic on $\mathcal{K}_l$, according to the standard procedure for complex spaces, if it is continuous, and if it is holomorphic on $\mathcal{K}_l'$, regarded as a complex manifold with the complex structure just described.[8]

In order to introduce the notion of "multivalued" functions in a well-defined way, we define the concept of a "locally schlicht domain".[9]

Definition 2. *A locally schlicht domain D over $\mathcal{K}_l$*, or *domain over $\mathcal{K}_l$*, is a

[6]The concept of a locally schlicht domain over an arbitrary complex space is developed by G. Scheja [15].

[7]"Connected" in this paper means "arcwise connected". A Hausdorff space is a topological space such that any two distinct points lie in disjoint neighborhoods. A neighborhood of a point contains an open set containing the point.

[8]Hepp has shown [9] that $\mathcal{K}_l$ is a "normal" algebraic variety in $\mathbb{C}^{4(l+1)}$, that is, a function holomorphic in $\mathcal{K}_l$ is locally the restriction of a function holomorphic in the embedding space, $\mathbb{C}^{4(l+1)}$.

[9]A domain is a connected open set. For standard concepts having to do with locally schlicht domains, *cf.* A. S. Wightman [16].

pair, $D = (S, \Phi)$, where S is a connected Hausdorff space and Φ a map of S into $\mathcal{K}_l$ that is a local homeomorphism. If Φ is a global homeomorphism, then D is said to be *schlicht.*

In general, the notation D will be used for locally schlicht domains, and U for domains that are also schlicht. Then Φ/U, the restriction of Φ to U, is a homeomorphism, for $U \subset D$.

The symbol P denotes a point of the locally schlicht domain D over $\mathcal{K}_l$, and $K = \Phi P$ denotes its image under the mapping Φ into $\mathcal{K}_l$. The point ΛP in a schlicht domain $U \subset D$ is $(\Phi/U)^{-1}\Lambda\Phi P$.

Two schlicht subdomains of a locally schlicht domain are said to "lie over" each other if their Φ images coincide. If they also contain a common point then they are identical; this follows straightforwardly from the definitions.

A function f defined on a locally schlicht domain D is said to be holomorphic on D if and only if $f \circ (\Phi/U)^{-1}$ is holomorphic on ΦU for every schlicht subdomain U of D. Analytic continuation of functions holomorphic on locally schlicht domains proceeds, in the usual way, by addition of locally compatible function elements. The domain of holomorphy is the largest domain onto which the function can be continued with equivalent function elements identified [15, 16]. For spinor M functions, which have several components, the following terminology has been introduced by Stapp [4]:

Definition 3. The *domain of regularity,* $\mathcal{R}(M)$, of an m-component spinor function M is the largest domain over $\mathcal{K}_l$ onto which all components can be simultaneously analytically continued, with equivalent m-component function elements identified.

$\mathcal{L}_+$ covariance for domains over $\mathcal{K}_l$ is defined in terms of $\mathcal{L}_+$ covariance for schlicht domains.

Definition 4. A spinor functioin M is $\mathcal{L}_+$*-covariant on a schlicht domain* U if and only if it satisfies

$$M(P) = \Lambda_S^{-1}\, M(\Lambda P) \tag{6.5}$$

whenever P and ΛP are in U and Λ is in $\mathcal{L}_+$.

Definition 5. A spinor function M is $\mathcal{L}_+$*-covariant in a domain* D over $\mathcal{K}_l$ if and only if it is $\mathcal{L}_+$-covariant on some schlicht subdomain of D.

The very weak requirement imposed by Definition 5 will soon be shown to be equivalent, when M is holomorphic, to a very strong requirement. For this purpose we introduce

Definition 6. An $\mathcal{L}_+$*-invariant structure* $\mathcal{S}$ of a domain D over $\mathcal{K}_l$ is a set of $\mathcal{L}_+$-invariant sheets whose union is D. A *sheet* is a schlicht domain that cannot be properly contained in any schlicht domain. An $\mathcal{L}_+$*-invariant schlicht domain* U is one that satisfies $\mathcal{L}_+\Phi U = \Phi U$, i.e., its image in $\mathcal{K}_l$ is a union of $\mathcal{L}_+$ orbits.

The intersection of any two sheets U_1 and U_2 of $\mathcal{S}$ is $\mathcal{L}_+$-invariant; for if P is a point of $U_1 \cap U_2$ there must be an $\mathcal{L}_+$-invariant, connected, open neighborhood of P contained in U_1 that lies over an $\mathcal{L}_+$-invariant neighborhood of P contained in U_2. These two schlicht domains both contain P and hence are identical, according to an earlier remark. Hence the entire orbit $\mathcal{L}_+\Phi P$ is in $\Phi\left(U_1 \cap U_2\right))$.

Definition 7. Let M be a spinor function defined on a domain D over $\mathcal{K}_l$. Then M is *completely $\mathcal{L}_+$-covariant on D* if and only if D has an $\mathcal{L}_+$-invariant structure $\mathcal{S}$ and M is $\mathcal{L}_+$-covariant on the schlicht domain U for every sheet U in $\mathcal{S}$.

Stapp has proved [4]

Lemma 3. If M is $\mathcal{L}_+$-covariant in a domain $D \subset \mathcal{R}(M)$ over $\mathcal{K}_l$, then M is completely $\mathcal{L}_+$-covariant on $\mathcal{R}(M)$.

Actually, Stapp's result is somewhat stronger, requiring only $\mathrm{L}_+^\uparrow$ covariance on a domain in $\mathcal{R}(M)$, which can even be real. This ensures that the scattering functions of analytic S-matrix theory are completely $\mathcal{L}_+$-covariant on their domains of regularity, as indicated in Sec. 2.

Because of Lemma 3 and certain special properties of domains with $l = 3$, it turns out that the problem of decomposing $M(J)$ on locally schlicht domains can be solved completely by considering separately each sheet in the $\mathcal{L}_+$-invariant structure. We now develop the result that makes this possible.

First, it will be remarked that the orbit $\mathcal{L}_+P$ of a point P in a domain with an $\mathcal{L}_+$-invariant structure $\mathcal{S}$ is well defined; it is the inverse image of the $\mathcal{L}_+$ orbit of $K = \Phi P$ with respect to any of the sheets in $\mathcal{S}$ that contain P. By a previous remark on the $\mathcal{L}_+$ invariance of the intersection of sheets in $\mathcal{S}$, all such orbits coincide.

Definition 8. Let M be $\mathcal{L}_+$-covariant in a domain $D \subset \mathcal{R}(M)$ over $\mathcal{K}_l$. Then D is said to be *weakly I_+-saturated in $\mathcal{R}(M)$* if and only if every $r \neq n$, $\mathcal{L}_+$ orbit of a point in D has at least one $r = n$ limit point[10] in $\mathcal{R}(M)$.

The terminology "I_+-saturated", chosen to conform with that of Hepp [9], is associated with the map $I_+ : \mathcal{K}_l \to \mathcal{M}_{l+}$. Here $\mathcal{M}_{l+}$ is the space of scalar and pseudoscalar invariants, $k_i \cdot k_j$ and $k_{i_1} \cdot [\, k_{i_2}, k_{i_3}, k_{i_4}\,]$, that correspond to points of $\mathcal{K}_l$. The corresponding map for the group $\mathcal{L}$ is $I : \mathcal{K}_l \to \mathcal{M}_l$, where $\mathcal{M}_l$ is the space of scalar invariants. For $l = 3$, the maps I_+ and I are the same, because all pseudoscalars vanish. The following is essentially Hepp's definition:

Definition 9. A subset S of a schlicht domain over $\mathcal{K}_l$ is said to be *I_+-saturated* if and only if $I_+^{-1}\left[I_+(\Phi S\right] = \Phi S$.

For domains with an $\mathcal{L}_+$-invariant structure in spaces with $l = 3$, the following lemma is true:

[10] The term "limit point" as used in this paper does not apply to points at infinity.

Lemma 4. Let M be $\mathcal{L}_+$-covariant in $D \subset \mathcal{R}(M)$ over $\mathcal{K}_3$, and let D be weakly I_+-saturated in $\mathcal{R}(M)$. Then for every U in the $\mathcal{L}_+$-invariant structure $\mathcal{S}$ there is an I_+-saturated domain V such that $D \cap U \subset V \subset U$.

For the proof we require another lemma, originally proved by Bargmann, Hall, and Wightman for the map I and for $n = r$ points.[11]

Lemma 5. The I_+ image of a neighborhood of a point K of $\mathcal{K}_3$ is a neighborhood of $I_+(K)$ in $\mathcal{M}_{3+}$; i.e., the map $I_+ : \mathcal{K}_3 \to \mathcal{M}_{3+}$ is open.

The proof of Lemma 5 is given in Appendix A4. The fact that the lemma is not restricted to $n = r$ points is a rather special property of $l = 3$ spaces.

To prove Lemma 4, let U be an element of $\mathcal{S}$ that has points in common with D. (If $D \cap U$ is empty, there is nothing to prove.) Since U is open and $U = \mathcal{L}_+ U$, the domain $\mathcal{L}_+(D \cap U)$ is already saturated with respect to $n = r$ points, because the $\mathcal{L}_+$ image of a neighborhood of an $n = r$ point contains, by virtue of Lemmas 1 and 2, all $n = r$ points with the same invariants, hence limit points of all $n \neq r$ points with these invariants, and hence, by the openness and $\mathcal{L}_+$ invariance, *all* points with these invariants. There remains the question of whether $\mathcal{L}_+(D \cap U)$ is saturated with respect to $n \neq r$ points. Since $r \leq l = 3$, this can only occur for $r = 1$ or 2.

Let P be an $n \neq r$ point of $D \cap U$. Because D is weakly I_+-saturated in $\mathcal{R}$, there is an $n = r$ limit point P_0 of $\mathcal{L}_+ P$ in $\mathcal{R}$. Clearly P_0 is at least on the boundary of the $\mathcal{L}_+$-invariant sheet U. Let U_0 be a sheet of $\mathcal{S}$ that contains P_0. Then $U \cap U_0$ is nonempty, and from the discussion following Definition 6 it is a schlicht, $\mathcal{L}_+$-invariant, open set. Thus $U \cap U_0$ contains $\mathcal{L}_+ P$.

One can now construct a neighborhood N_0 of P_0 with the following properties:

(i) $N_0 \subset U_0$;

(ii) $N_0 = \mathcal{L}_+ N_0$;

(iii) For every P' in N_0 there is a P'' in $D \cap U$ such that $I_+(\Phi P') = I_+(\Phi P'')$;

(iv) Every $r = 3$ point of N_0 is also in $\mathcal{L}_+(D \cap U)$.

It is trivial to find a neighborhood N_0' of P_0 satisfying (i) and (ii) because U_0 is open and $\mathcal{L}_+$-invariant. To satisfy (iii), choose a neighborhood $N \subset D \cap U$ of P such that $\mathcal{L}_+ N$ is contained in $U \cap U_0$. This is always possible, because $\mathcal{L}_+ P$ is in $U \cap U_0$, and because $U \cap U_0$ is $\mathcal{L}_+$-invariant and open. Then because of Lemma 5 and the continuity of the map I_+, $(\Phi/U_0)^{-1} \left\{ \Phi N_0' \cap I_+^{-1} \left[I_+(\Phi N) \right] \right\} \equiv N_0$ is a neighborhood of P_0 contained in U_0. From the fact that $I_+(A \cap B) \subset I_+(A) \cap I_+(B) \subset I_+(A)$, it follows that $I_+(\Phi N_0) \subset I_+(\Phi N)$. Thus (iii) is satisfied because $N \subset D \cap U$.

[11] *Cf.* [8, Lemma 3]. The form of the statement here, when restricted to $n = r$ points, is slightly different, but the proof is not affected.

Property (iv) is already satisfied by N_0. This follows because for every $r = 3$ point P' of N_0 there is a corresponding point P'' of N with $I_+(\Phi P') = I_+(\Phi P'')$. By Lemmas 1 and 2, $\Phi P'$ and $\Phi P''$ are connected by a transformation in $\mathcal{L}_+$. Now by construction both N and N_0 are contained in the $\mathcal{L}_+$-invariant sheet U_0, and hence P' and P'' are on the same $\mathcal{L}_+$ orbit. But P'' is in $N \subset D \cap U$, and hence P' is in $\mathcal{L}_+(D \cap U)$.

It will now be shown that the domain V obtained by adding to $\mathcal{L}_+(D \cap U)$ all possible neighborhoods N_0 constructed in this way is a schlicht domain contained in U.

First we shall show that V is schlicht. If V is not schlicht, then it must contain at least two distinct points Q_0 and Q_0' that lie over each other, $\Phi Q_0 = \Phi Q_0'$. Since $\mathcal{R}$ is a Hausdorff space, there must then be two disjoint neighborhoods W_0 and W_0' of Q_0 and Q_0', respectively, contained in V. Because $\mathcal{R}$ is locally schlicht, one can choose W_0 and W_0' to be schlicht and to lie over each other. But if W_0 and W_0' lie over each other, then any $r = 3$ points must be common to both, since *all* $r = 3$ points of V are in the schlicht domain $\mathcal{L}_+(D \cap U)$. Now every schlicht open set over $\mathcal{K}_3$ contains $r = 3$ points, because $G(K)$ is holomorphic everywhere on $\mathcal{K}_3$; and if it vanishes for any open set it vanishes everywhere. Thus W_0 and W_0' cannot be disjoint, which is a contradiction. Therefore V is schlicht.

Moreover, the same argument shows that no point of $V - U$ can lie over U. But U is a sheet, i.e., a maximal schlicht domain. Thus V is contained in U, for otherwise $U \cup V$ would be a schlicht domain properly containing U.

By construction, V is $\mathcal{L}_+$-invariant. Moreover, for every $n \neq r$ point of V there is by (iii) a point of $D \cap U$ with the same invariants, and hence, by construction, an $n = r$ point of V with the same invariants. Because V is open, it follows by the same argument used at the beginning of the proof that $\Phi V = I_+^{-1}[I_+(\Phi V)]$. Thus Lemma 4 is proved.

Because V is contained in U one can write $V = (\Phi/U)^{-1} I_+^{-1}[I_+(\Phi V)]$. Because $I_+(\Phi V) = I_+[\Phi(D \cap U)]$, we also have $V = (\Phi/U)^{-1} I_+^{-1}\{I_+[\Phi(D \cap U)]\}$. This I_+-saturated domain $V \subset U$ we shall denote simply by $I_+^{-1} \circ I_+(D \cap U)$; it is the "$I_+$ saturation" of $D \cap U$.

6.3 Condition for the Absence of Kinematical Singularities

Hall and Wightman have proved, among other things, that an $\mathcal{L}$-invariant function f holomorphic on the future tube has the form $f = f' \circ I$, with f' holomorphic on the I image of the future tube. A partial generalization of their result, for $l = 3$ and the group $\mathcal{L}_+$, is needed for the proof of our basic theorem on kinematical singularities. $\mathcal{L}_+$ invariance is defined by setting $\Lambda_S = 1$ in Definitions 4, 5, and 7.

Lemma 6. Let f be $\mathcal{L}_+$-invariant in a domain $D \subset \mathcal{R}(f)$ over $\mathcal{K}_3$. Then for every sheet U in the $\mathcal{L}_+$-invariant structure of $\mathcal{R}(f)$,

$$f/D \cap U = f' \circ I_+/D \cap U \,, \tag{6.6}$$

with f' holomorphic on $I_+(D \cap U)$,[12] if and only if D is weakly I_+-saturated in $\mathcal{R}(f)$.

That the condition of weak I_+ saturation is necessary is trivial, because I_+ is a holomorphic map and (6.6) defines an invariant analytic continuation of f onto the domain $I_+^{-1} \circ I_+(D \cap U)$. To show that the condition is sufficient, we note that Hepp has independently proved the result for schlicht, I_+-saturated domains [9]. Lemma 6 then follows by applying Lemma 4 and Lemma 3.[13]

The basic result on the absence of kinematical singularities to be proved in the remaining sections is the following:

Theorem 1. Let $M(J)$ be $\mathcal{L}_+$-covariant in a domain $D \subset \mathcal{R}[M(J)]$ over $\mathcal{K}_3$. Then $M(J)$ has a unique decomposition on D,

$$M(J) = \sum_{i=0}^{J} b_i^{(+)}\, Y^{(+)}(i:J) + \sum_{i=0}^{J-1} b_i^{(-)}\, Y^{(-)}(i:J)\,, \tag{6.7a}$$

with $b_i^{(\pm)}$ holomorphic on D, where for every U in the $\mathcal{L}_+$-invariant structure of $\mathcal{R}[M(J)]$,

$$b_i^{(\pm)}/D \cap U = b_i'^{(\pm)} \circ I_+/D \cap U\,, \tag{6.7b}$$

with $b_i'^{(\pm)}$ holomorphic on $I_+(D \cap U)$, if and only if D is weakly I_+-saturated in $\mathcal{R}[M(J)]$.

Again, that weak I_+ saturation is a necessary condition is trivial, because (6.7b) with (6.7a) defines a covariant analytic continuation onto $V = I_+^{-1} \circ I_+(D \cap U)$, the $Y^{(\pm)}(i:J)$ being polynomials.

As for the converse, note that Lemma 4 implies that $D \cap U \subset V \subset U$, and that Lemma 3 implies that $M(J)$ has a covariant analytic continuation onto V. It is enough to prove the result for each I_+-saturated, schlicht domain V. The fact that the $b_i^{(\pm)}$ in (6.7a) are uniquely defined and holomorphic on D follows at once from the existence of a unique, holomorphic decomposition on each V and from the fact that these schlicht domains cover D, and hence overlap. The uniqueness, of course, follows at once from the linear independence of the $Y^{(\pm)}(i:J)$ at $r = 3$ points.

Without loss of generality, for the remainder of the proof we write $M(J)$ for the analytic continuation of $M(J)$, restricted to V. The details of the proof are given in the following sections. Here we outline the basic steps.

[12] The set $\mathcal{M}_{l+}$, which contains $I_+(D \cap U)$, is an algebraic variety (*cf.* [8, 9]), and for the general case, one would have to define holomorphy for functions on this variety just as was necessary for functions on $\mathcal{K}_l$. However, $\mathcal{M}_{3+}$ is an ordinary Euclidean space of two complex dimensions, characterized for example by the Mandelstam variables s, t, and u, with $s + t + u = \sum_i m_i^2$, so that no generalization of holomorphy is required.

[13] One can also prove this by following rather directly the original methods of Bargmann, Hall, and Wightman. See Sec. 6.7 for remarks about the case of arbitrary l.

First, we show that $M(J)$ can be written as the sum of two functions holomorphic on V, $M_{\pm}(J)$, equivalent in a sense to a tensor and a pseudotensor, which can be expanded, respectively, in terms of $Y^{(\pm)}(i:J)$. The discussion in Sec. 5 implies that the scalar amplitudes $b_i^{(\pm)}$ in this expansion are holomorphic on V except possibly for poles of the form $G(K)^{-J}$, at points with $r<3$. For points with $r=2$ and $n=3$, we show that the $Y^{(+)}(i:J)$ are linearly independent, and so are the $Y^{(-)}(i:J)$, although the two sets are not independent of each other. From these facts, we show that the $b_i^{(\pm)}$ are continuous at such points; and with the help of Lemmas 5 and 6 we show that they are also single-valued functions of the invariants at such points. This extends the definition of the scalar amplitudes to all $r=n=2$ points, and by means of Lemmas 5 and 6 we are then able to show that they are holomorphic functions of the invariants for all except $r=1$ points. The $r=1$ points are shown to be isolated in the space of invariants, and they are easily handled by a standard theorem on analytic continuation.

6.4 Tensor-Pseudotensor Decomposition

Because $V = I_+^{-1} \circ I_+(D \cap U)$ is I_+-saturated, and because of Lemma 2, if K is in V then so is $\widetilde{K} \equiv \left(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3\right)$, where $\tilde{k}^\mu \equiv k_\mu$. Then any tensor f under $\mathcal{L}_+$ that is holomorphic on V can be decomposed into unique tensor and pseudotensor parts under $\mathcal{L}$, both holomorphic on V, namely,

$$f_{\pm}(K) \equiv \tfrac{1}{2}\left[f(K) \pm \tilde{f}(\widetilde{K})\right], \qquad \tilde{f}^{(\mu)} \equiv f_{(\mu)}\,. \tag{6.8}$$

From $\mathcal{L}_+$ covariance and simple algebra, one finds that

$$\Lambda f_+(K) = f_+(\Lambda K)\,, \qquad \Lambda f_-(K) = f_-(\Lambda K)\det(\Lambda)\,, \tag{6.9}$$

for any Λ in $\mathcal{L}$.

By means of (5.2), $M(J)$ is converted into a symmetric and traceless tensor f in the space $\mathcal{T}(J:k_3)$. Writing

$$M_{\pm}(J) \equiv f_{\pm}\cdot\rho(J:k_3)\,, \tag{6.10}$$

with $f_{\pm}$ defined by (6.8), one has

$$M(J) = M_+(J) + M_-(J)\,. \tag{6.11}$$

It is straightforward to see that, along with f, $f_{\pm}$ are both symmetric and traceless and that both are in $\mathcal{T}(J:k_3)$. Thus, the projection operator $S(J:k_3)$, defined in (4.13), acts as the identity on $f_{\pm}$, and one has

$$f_{\pm} = M_{\pm}(J)_\alpha\, \rho(J:k_3)^\alpha\, {m_3}^{-2J}\,. \tag{6.12}$$

The $M_{\pm}(J)$ can be expanded individually by (5.21). The coefficients in these expansions are scalars under $\mathcal{L}$, because of Lemma 6, and the fact that no pseudoscalars can be formed from the three momentum vectors. It is shown in Appendix A2 that the projection operator $S(J\,{:}\,k_3)$ is a tensor under $\mathcal{L}$. Thus, the expressions

$$f_\pm(i\,{:}\,J) \equiv Y^{(\pm)}(i\,{:}\,J)_\alpha\ \rho(J\,{:}\,k_3)^\alpha\, {m_3}^{-2J} \tag{6.13}$$

are, respectively, tensors and pseudotensors under $\mathcal{L}$, from (5.20). The expansions therefore take the form

$$M_\pm(J) = \sum_i b_i^{(\pm)}\, Y^{(\pm)}(i\,{:}\,J)\,, \qquad i = 0, 1, \ldots, J - (\tfrac{1}{2} \mp \tfrac{1}{2})\,. \tag{6.14}$$

6.5 Independence of the $Y^{(\pm)}(i\,{:}\,J)$

The final ingredient for the proof of Theorem 1 is

Lemma 7. If K is an $n = 3$ point, the corresponding spinors (for fixed J) $Y^{(+)}(i\,{:}\,J)$ are a linearly independent set, and so are the $Y^{(-)}(i\,{:}\,J)$.

From the construction in Sec. 5, this lemma is trivial if $r = n = 3$, when all of the $Y^{(\pm)}(i\,{:}\,J)$ are linearly independent. We are interested here in the case $r = 2$ and $n = 3$. Then the $Y^{(+)}$ are not independent of the $Y^{(-)}$, because when k_3 depends, for example, on ω_+, as in (6.2a), the vector w occurring in $Y^{(-)}$ is proportional to ω_+, and is thus not linearly independent of the k_i. For the proof of Lemma 7, however, it is just as easy to consider all $n = 3$ points at once, and not just those with $r = 2$.

Let the $Y^{(+)}$ be considered first. If there exist complex numbers c_i such that

$$\sum_i c_i\, Y^{(+)}(i\,{:}\,J) = 0\,, \tag{6.15}$$

then

$$\begin{aligned} &\sum_i c_i\, Y^{(+)}(i\,{:}\,J)_\alpha\ \rho_{(\mu)}\,(J\,{:}\,k_3)^\alpha\ m_3^{-2J} \\ &\qquad = \sum_i c_i\ k_1^{\nu_1} \cdots k_1^{\nu_i}\, k_2^{\nu_{i+1}} \cdots k_2^{\nu_J}\ S_{(\nu)(\mu)}(J\,{:}\,k_3) \\ &\qquad = 0\,. \end{aligned} \tag{6.16}$$

By a transformation in $\mathcal{L}_+$, k_3 can be put into its rest frame; and then only the three-vector part of any vector contributes when contracted with $S(J\,{:}\,k_3)$, because of (4.12). In such a frame, we can just as well let μ and ν represent three-vector indices. Then contracting every μ index of $S(J\,{:}\,k_3)$ in (6.16) with the same three-vector $\mathbf{x}$, we get

$$
\begin{aligned}
S^{(\nu)}{}_{(\mu)}(J:k_3)\; x^{\mu_1}\cdots x^{\mu_J} &= x^{\nu_1}\cdots x^{\nu_J} \\
&\quad + \eta_1\,\mathbf{x}\cdot\mathbf{x} \sum_{P(i)} \delta^{\nu_{i_1}\nu_{i_2}}\; x^{\nu_{i_3}}\cdots x^{\nu_{i_J}} \\
&\quad + \eta_2\,(\mathbf{x}\cdot\mathbf{x})^2 \sum_{P(i)} \delta^{\nu_{i_1}\nu_{i_2}}\;\delta^{\nu_{i_3}\nu_{i_4}}\; x^{\nu_{i_5}}\cdots x^{\nu_{i_J}} \\
&\quad + \cdots\,,
\end{aligned} \tag{6.17}
$$

where the η_i are numerical constants, $\delta^{\mu\nu}$ is the three-dimensional Kronecker δ, $(i_1,\ldots,i_J)$ is a permutation of $(1,\ldots,J)$, and the summations are over all such permutations. Equation (6.17) is proved by noting that, in the rest frame of k_3, $S(J:k_3)$ is the projection operator for symmetric and traceless tensors under the group $\mathrm{O}_+(3,\mathrm{C})$.[14] The right-hand side is then obtained by consulting any standard reference on the decomposition of a tensor into irreducible parts.[15]

Defining $\chi_1=\mathbf{k}_1\cdot\mathbf{x}$, $\chi_2=\mathbf{k}_2\cdot\mathbf{x}$, $\chi_3=\sqrt{\mathbf{x}\cdot\mathbf{x}}$, we get from (6.16), after contracting with $\mathbf{x}$,

$$
\begin{aligned}
0 = \sum_i c_i\;\Big[{\chi_1}^i\,{\chi_2}^{J-i} &+ {\chi_3}^2\;\mathcal{P}_i^{J-2}(\chi_1,\chi_2) \\
&+ {\chi_3}^4\;\mathcal{P}_i^{J-4}(\chi_1,\chi_2)+\cdots\Big]\,,
\end{aligned} \tag{6.18}
$$

where $\mathcal{P}_i^N(\chi_1,\chi_2)$ is a homogeneous polynomial of degree N in χ_1 and χ_2. By hypothesis, $\mathbf{k}_1$ and $\mathbf{k}_2$ are independent in the rest frame of k_3; and therefore the variables χ_i are independent as $\mathbf{x}$ varies. The polynomial (6.18) is identically zero, and hence the coefficient of each distinct term must vanish. Therefore, each c_i vanishes, and the $Y^{(+)}(i:J)$ are linearly independent.

The only difference when the $Y^{(-)}$ are considered is that one has a polynomial of the form

$$
\begin{aligned}
0 = \sum_i c_i\;\Big[{\chi_1}^i\,{\chi_2}^{J-1-i}\,d &+ {\chi_3}^2\;\mathcal{P}_i^{J-3}(\chi_1,\chi_2)\;d \\
&+ {\chi_3}^4\;\mathcal{P}_i^{J-5}(\chi_1,\chi_2)\;d+\cdots\Big]\,,
\end{aligned} \tag{6.19}
$$

where

$$
d \equiv \mathbf{k}_1\times\mathbf{k}_2\cdot\mathbf{x} = \sqrt{G(\mathbf{k}_1,\mathbf{k}_2,\mathbf{x})}\,. \tag{6.20}
$$

Considered as a function of the χ_i, the region where $d\neq 0$ is an open set, and on that open set (6.19) can be divided by d. The resulting equation is therefore valid for all χ_i. Again, the c_i vanish and the $Y^{(-)}(i:J)$ are linearly independent.

[14] *Cf.* Appendix A2.

[15] *Cf.*, for the real orthogonal groups, [17, Chap. 10, Secs. 5-7]. The procedure is the same for the complex groups.

6.6 Completion of the Proof

We have seen that the scalar amplitudes $b_i^{(\pm)}$ in (5.21), and hence in (6.14), are holomorphic on V except for the possibility of poles, $G(K)^{-J}$. Because V is I_+-saturated, Lemma 6 implies that

$$b_i^{(\pm)} = b_i'^{(\pm)} \circ I_+ \,, \tag{6.21}$$

where the $b_i'^{(\pm)}$ are at least meromorphic on $I_+(V)$, the only possible poles being those just mentioned, but with G considered as a function of invariants. Thus the $b_i'^{(\pm)}$ in (6.21) are well defined, and regular for $r = 3$ points in V.

If G vanishes, its rank is one or two. The rank two case will be considered first. Let V' represent the domain obtained by deleting all $r = 1$ points from V. V' is a domain because every point of rank r has a neighborhood consisting entirely of points of rank r or greater. V' is clearly also I_+-saturated. Let V'' denote the domain obtained by deleting all $n = 2$ points from V'.[16] Then $I_+(V') = I_+(V'')$, because for every $n = r = 2$ point K' of V' there is an $n = 3$ point K'' of V'' with $I_+(K') = I_+(K'')$.

Lemma 7 implies that the $Y^{(+)}(i\!:\!J)$ are a linearly independent set on V'', and so are the $Y^{(-)}(i\!:\!J)$. The discussion of Sec. 6.4 implies that $M_\pm(J)$ are each holomorphic on V, and that at least for $r = 3$ points of V'' they can be expanded respectively in terms of $Y^{(\pm)}(i\!:\!J)$. It will now be shown that the coefficients $b_i^{(\pm)}$ of this expansion have a continuous extension to all of V'', which is then unique because the $r = 3$ points of V'' are dense in V''.

Consider for the moment the complex scalar product in the $(2J + 1)$-dimensional vector space, e.g.,

$$\Big\langle Y^{(+)}(i\!:\!J),\, M_+(J) \Big\rangle \equiv \sum_{\alpha=-J}^{J} Y^{(+)}(i\!:\!J)_\alpha{}^{*}\, M_+(J)_\alpha \,. \tag{6.22}$$

At any point of V'' one can compute the projections $P_\pm\, M_\pm(J)$ of $M_\pm(J)$, respectively, onto the subspaces spanned by $Y^{(\pm)}(i\!:\!J)$,

$$P_\pm\, M_\pm(J) = \sum_i c_i^{(\pm)}\, Y^{(\pm)}(i\!:\!J) \,. \tag{6.23}$$

The coefficients $c_i^{(\pm)}$ are uniquely determined by solving

$$\Big\langle Y^{(\pm)}(j\!:\!J),\, M_\pm(J) \Big\rangle = \sum_i c_i^{(\pm)}\, \Big\langle Y^{(\pm)}(j\!:\!J),\, Y^{(\pm)}(i\!:\!J) \Big\rangle \,. \tag{6.24}$$

The determinant of either of the $(\pm)$ matrices on the right side of (6.24) does not vanish anywhere on V'' because of the independence of the $Y^{(\pm)}(i\!:\!J)$, respectively. Because $M_\pm(J)$ are continuous on V'', the $c_i^{(\pm)}$ are continuous,

[16] Again V'' is a domain because every point with given n has a neighborhood containing no points of smaller n.

and because of the uniqueness of the decomposition, $c_i^{(\pm)} = b_i^{(\pm)}$ at all $r = 3$ points. Thus the $b_i^{(\pm)}$ have a continuous extension onto V''.

Moreover, they are functions of invariants; i.e., they can be written in the form (6.21), everywhere on V''. To see this, let K and K' be two $r = 2$ points of V'' such that $I_+(K) = I_+(K')$. Then Lemma 5 and the fact that every neighborhood contains $r = 3$ points imply that there are sequences K_m and K'_m of $r = 3$ points converging respectively to K and K' such that for all m, $I_+(K_m) = I_+(K'_m)$. The continuity of the $b_i^{(\pm)}$ on V'' and the validity of (6.21) at $r = 3$ points yield the result.

Applying Lemma 5 again, one sees that the $b_i'^{(\pm)}$ are continuous on the domain $I_+(V') = I_+(V'')$. The $b_i'^{(\pm)}$ are holomorphic on $I_+(V')$ except at the zeroes of G, and they are continuous there. Because of a basic theorem on removable singularities [18, p. 173], the $b_i'^{(\pm)}$ are therefore holomorphic on $I_+(V')$.

Only the $r = 1$ points remain. These points satisfy

$$k_i \cdot k_j = \pm m_i m_j \,, \tag{6.25}$$

and thus they are isolated in the domain $I_+(V)$. It is a consequence of a standard theorem that a function of more than one complex variable holomorphic on a domain always has an analytic continuation to isolated points of the domain [18, p. 71]. Thus, the $b_i'^{(\pm)}$ have a continuation onto $I_+(V)$, and (6.21) defines holomorphic functions on V.

Equation (6.7a) is therefore a holomorphic expansion without the kinematical singularities G^{-J}, and Theorem 1 is proved.

6.7 Alternative Methods of Proof and an Alternative Statement of Theorem 1

In the proof of Theorem 1, strong use was made of Stapp's result, which, fortified by Lemma 4, allowed the problem to be reduced to one of schlicht domains. This served as a device that simplified the discussion considerably, but it is not essential. In an earlier version of the proof that did not use Stapp's Theorem, it was first necessary to construct a generalization of the map I_+, which relates weakly I_+-saturated domains over $\mathcal{K}_l$ to corresponding domains over $\mathcal{M}_{l+}$, and to show that the generalized map is open for $n = r$ points. This is rather straightforward to do, given the proof of Bargmann, Hall, and Wightman for the map I on schlicht domains [8]. Then the partial generalization of the Hall-Wightman Theorem (Lemma 6) for weakly I_+-saturated domains over $\mathbb{C}^{4(l+1)}$ or over $\mathcal{K}_l$ (for arbitrary l) can be proved directly, essentially by a detailed inspection of Hall and Wightman's original proof.[17]

[17] The statement of the result in Lemma 6 is somewhat special due to the use of Lemma 4. In the general statement one removes the restriction that Eq. (6.6) must hold for every $\mathcal{L}_+$-invariant sheet and uses instead the generalized map I_+. Hepp has given an elegant abstract proof for saturated domains that is valid for all of the classical complex groups, to

The delicate point is then to establish that the M functions over $\mathcal{K}_3$ have a decomposition into unique tensor and pseudotensor parts (this is not globally true for arbitrary l without additional assumptions), a result that follows at once in the present version from the fact that our domain can be covered by schlicht, saturated domains where M is $\mathcal{L}_+$-covariant and holomorphic. To give a direct proof without Stapp's Theorem requires a fairly sophisticated local property of $\mathcal{L}_+$ orbits, namely, that for any neighborhood N of a point K_0, there is a neighborhood N' of K_0 contained in N such that for any two points K and K' in N', with $K' = \Lambda K$ and Λ in $\mathcal{L}_+$, there is a connected arc in $\mathcal{L}_+$ from the identity to Λ whose image in the $\mathcal{L}_+$ orbit of K and K' lies in N. The author has proved this result for neighborhoods of $r \geq 2$ points (which is actually sufficient for the application being discussed here), but the proof for the general case, achieved by Stapp [4], is difficult. In fact, this result is the key ingredient in the proof of Stapp's Theorem.[18]

Having established these points, the proof of Theorem 1 proceeds as before. The discussion is at least superficially more complicated. It amounts essentially to proving that part of Stapp's Theorem that is really needed. No real simplification results from permitting all domains to be locally schlicht and not insisting that the problem be reduced to schlicht domains, unless one assumes at the outset that the domain can be covered by saturated domains where M is holomorphic and $\mathcal{L}_+$-covariant. Given Stapp's Theorem, one can show that such a covering always exists for domains that are weakly I_+-saturated in the domain of regularity; and Lemma 4 says something stronger for the case $l = 3$, namely, that for any covering by $\mathcal{L}_+$-invariant sheets, there are subdomains of the sheets that already constitute an I_+-saturated covering.

That part of Hepp's work [9] that corresponds to the case of four four-vectors on the mass shell studied here is a proof for I_+-saturated domains, and for irreducible representations $\mathcal{D}^{j_1,j_2}$ with $|j_1 - j_2| \leq 1$. His method is first to prove the existence of a local holomorphic decomposition for I_+-saturated domains in the space of l four-vectors (on or off the mass shell), and arbitrary j_1 and j_2, and then for the special case $l = 3$ to use the linear independence properties of his spanning set of polynomials to show that there is a local holomorphic decomposition that is also unique, and hence global, when $|j_1 - j_2| \leq 1$.[19]

By way of contrast, our method is to first construct a polynomial basis with convenient linear independence properties, and then to use these properties

appear in *Math. Annalen*. He has shown, moreover, that the spaces of invariants are normal algebraic varieties, so that one has holomorphy in the stronger sense described in footnote 8.

[18] **Post-thesis note:** The assertion about "local $\mathcal{L}_+$-connectedness" for $r \geq 2$ points is correct; but only a few months after this was written, R. Jost constructed a counterexample for $n = 2$, $r = 1$ points. R. Seiler characterized the points for which it is true, and P. Minkowski and D. N. Williams extended the proof of Stapp's Theorem to the remaining points [19].

[19] For functions restricted to the mass shell, one can use the "metric" spinors in (3.1) to convert dotted indices to undotted indices and back again. By using such manipulations with the Clebsch-Gordan series one can eventually expand, without introducing singularities, a holomorphic function covariant under any representation in terms of holomorphic functions covariant under representations of the form $\mathcal{D}^{j,j}$. Thus, on the mass shell, Hepp's condition $|j_1 - j_2| \leq 1$ for $l = 3$ is not a restriction in principle.

with the generalized Hall-Wightman Theorem to get the result.

Our principal result can be stated rather simply by defining a map $J_+ : \mathcal{K}_l \to \mathcal{M}_{l+}$, which is the same as I_+ for $n = r$ points, but which maps $n \neq r$ points into the empty set. Thus J_+ "sees" only $n = r$ points. By virtue of Lemmas 1 and 2, there is only one $n = r$, $\mathcal{L}_+$ orbit for each point of $\mathcal{M}_{l+}$. Thus, in a way, J_+ is a one-to-one correspondence between orbits and invariants. An examination of Hall and Wightman's proof of the openness of the map I for $n = r$ points [8, Lemma 3] shows that in every neighborhood of an $n = r$ point one can find a neighborhood of the point which contains, for every $n \neq r$ point, an $n = r$ point with the same I (and hence I_+) image. Thus every neighborhood of an $n = r$ point contains a neighborhood N of the point such that $\mathcal{L}_+N$ is I_+-saturated. It has already been mentioned that the map $I_+ : \mathcal{K}_l \to \mathcal{M}_{l+}$ is open for $n = r$ points, and thus it follows that the J_+ image of an open set is also open, and from continuity that the $I_+^{-1} \circ J_+$ image of an open set is open. Thus, for any schlicht domain U, $I_+^{-1} \circ J_+(U)$ is a saturated domain. It is obtained from $\mathcal{L}_+U$ by deleting all $n \neq r$ $\mathcal{L}_+$ orbits whose limit points are not in $\mathcal{L}_+U$.

This procedure of deletion is well-defined even for nonschlicht domains D. Hence, although in this paper we have defined the maps I_+ and J_+ only for schlicht domains, we shall use the symbol $I_+^{-1} \circ J_+(D)$ to represent the "I_+-saturated part" of $\mathcal{L}_+D$. From the remarks just made, the I_+-saturated part of $\mathcal{L}_+D$ is a domain whenever D is a domain. Then an immediate consequence of Theorem 1 is the following:

Theorem 2. Let $M(J)$ be $\mathcal{L}_+$-covariant in a domain $D \subset \mathcal{R}[M(J)]$ over $\mathcal{K}_3$. Then the decomposition (6.7a) holds with unique invariant amplitudes $b_i^{(\pm)}$ defined and holomorphic on $I_+^{-1} \circ J_+(D)$, and for any schlicht domain $U \subset D$, these amplitudes have the form $b_i^{(\pm)} = b_i'^{(\pm)} \circ J_+$, with the $b_i'^{(\pm)}$ holomorphic on $J_+(U)$. In particular, one can let $D = \mathcal{R}[M(J)]$, and one can let U be any sheet in the $\mathcal{L}_+$-invariant structure.

This theorem is equivalent to Theorem 1, when taken together with Lemma 4. It can be summarized by the statement that the invariant amplitudes are uniquely defined and holomorphic over the J_+ image of any domain (over $\mathcal{K}_3$) on which $M(J)$ is regular.

7 Conclusion

Equations (3.9) and (6.7a) provide a global decomposition of the S matrix for two-particle reactions that is free of kinematical singularities for any weakly I_+-saturated domain on which the M functions are holomorphic. According to Stapp's postulate of minimal analyticity [3], for any scattering process there is a sheet, called the physical sheet, contained in the domain of regularity of M, such that all physical points are on the boundary of the sheet, and such that the boundary is defined by equations in the scalar invariants. This physical sheet is therefore I_+-saturated, and the decomposition theorem applies. The postulate of maximal analyticity [2, 3] says that the M functions are holomorphic everywhere except for those singularities demanded by unitarity. Although it is not as yet clear precisely how this postulate is to be formulated, one can take as a provisional interpretation that it shall imply that singularities of the M functions can occur only at points determined by the Landau equations [3, 20]. Again, these are equations involving scalar invariants. Thus, the domain obtained by omitting the Landau singularities is I_+-saturated, and the theorem again applies.

It is possible that the domain of regularity of an M function contains some, but not all, of the points corresponding to a certain solution of the Landau equations; for these equations do not guarantee the existence of a singularity. If this were to occur then the domain of regularity would not be saturated. At least for dispersion relations in the space of scalar invariants, however, this seems to be of little practical importance. There, the singularities in the scalar invariants are the important consideration, and the fact that *some* of the corresponding points in the vector variables might be regular is irrelevant.

A1 Spinor Calculus for Representations of the Complex Lorentz Group

In this appendix are collected the basic relationships of a direct generalization of the ordinary two-component spinor calculus [21,22] to a calculus for arbitrary finite-dimensinal representations of the proper, homogeneous, complex Lorentz group, $\mathcal{L}_+$. This generalized spinor calculus was used in a previous work [1], is useful in the present work, and appears to be of value in many problems involving higher spins. The development is essentially notational, and serves to define the quantities used in the text.

A1.1 Representations of $\mathcal{L}_+$

The complex homogeneous Lorentz group, $\mathcal{L}$, is the group of all complex four-by-four matrices satisfying the equation

$$\Lambda^{\mathrm{T}} \, G \, \Lambda = G \,, \tag{A1.1}$$

where

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

The proper complex homogeneous Lorentz group, $\mathcal{L}_+$, is that part of $\mathcal{L}$ connected to the identity, the set of unimodular four-by-four matrices satisfying (A1.1).

Representations of the two-by-two unimodular group, $\mathrm{SL}(2, \mathrm{C})$, can be used for the construction of a spinor calculus for $\mathcal{L}_+$, making use of the well-known two-to-one homomorphism between $\mathrm{SL}(2, \mathrm{C}) \times \mathrm{SL}(2, \mathrm{C})$ and $\mathcal{L}_+$. This homomorphism can be expressed in terms of the two-by-two Pauli matrices

$$\sigma_\mu = (I, \boldsymbol{\sigma}) \,, \qquad \widetilde{\sigma}_\mu \equiv \sigma^\mu = (I, -\boldsymbol{\sigma}) \,, \tag{A1.2}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A1.3}$$

They satisfy the transformation law

$$A \, \sigma^\mu \, B^{\mathrm{T}} \equiv \Lambda_\nu{}^\mu(A, B) \, \sigma^\nu \tag{A1.4}$$

where A and B are arbitrary two-by-two unimodular matrices, and the $\Lambda(A, B)$ so defined are the corresponding complex Lorentz transformations. By using the orthogonality relation

$$\tfrac{1}{2} \, \mathrm{Tr} \, (\sigma_\mu \, \widetilde{\sigma}_\nu) = g_{\mu\nu} \,, \tag{A1.5}$$

the correspondence can be written in the equivalent form

$$\Lambda_{\mu\nu}(A, B) = \tfrac{1}{2}\ \mathrm{Tr}\left(\widetilde{\sigma}_\mu\, A\, \sigma_\nu\, B^{\mathrm{T}}\right)\ . \tag{A1.6}$$

It is clear from (A1.4) or (A1.6) that $\Lambda(A, B) = \Lambda(-A, -B)$.

Two-component spinor indices (with values $\pm\frac{1}{2}$) transforming according to the two-by-two unimodular transformation A are customarily written as lower undotted, those transforming by B as lower dotted, and those transforming by the contragredient transformations A^{-1T} and B^{-1T}, respectively, as upper undotted and upper dotted. The summation convention is used for the invariant contraction of upper with lower indices of the same type. The raising and lowering metric spinors are $C^{-1\,\alpha\beta} = C^{-1\,\dot{\alpha}\dot{\beta}}$, and $C_{\alpha\beta} = C_{\dot{\alpha}\dot{\beta}}$, where

$$C^{-1} = -C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\,\sigma_2\,. \tag{A1.7}$$

To raise or lower an index one always contracts with the right index of C^{-1} or C, respectively.[20] These operations give quantities having the correct transformation law because of the identity for an arbitrary two-by-two matrix, M,

$$C^{-1}\,M^{\mathrm{T}}\,C = M^{-1}\ \det(M)\,. \tag{A1.8}$$

According to these conventions and (A1.4), the matrix elements of σ_μ should be written $\sigma_{\mu\ \alpha\dot{\beta}}$. Using (A1.8) the matrix $\widetilde{\sigma}_\mu$ can be considered to be defined by the relation

$$\widetilde{\sigma}_\mu = C^{-1}\,\sigma_\mu^{\mathrm{T}}\,C\,; \tag{A1.9}$$

and hence the matrix elements of $\widetilde{\sigma}_\mu$ should be written $\widetilde{\sigma}_\mu{}^{\dot{\alpha}\beta}$, corresponding to the transformation law

$$B^{-1\mathrm{T}}\,\widetilde{\sigma}^\mu\,A^{-1} = \Lambda_\nu{}^\mu(A, B)\ \widetilde{\sigma}^\nu\,. \tag{A1.10}$$

The σ_μ and $\widetilde{\sigma}_\mu$ satisfy also a second orthogonality relation,

$$\tfrac{1}{2}\,\sigma_{\mu\,\alpha\dot{\beta}}\ \widetilde{\sigma}^{\mu\,\dot{\beta}'\alpha'} = \delta_\alpha{}^{\alpha'}\ \delta_{\dot{\beta}}{}^{\dot{\beta}'}\ . \tag{A1.11}$$

For higher spins, a certain class of irreducible representations of $\mathrm{SL}(2,\mathrm{C})\times \mathrm{SL}(2,\mathrm{C})$ will be defined by their action on the space of homogeneous polynomials [11]

$$X\,(j_1, j_2)_{\alpha\dot{\beta}} = \frac{\left(\xi_{\frac{1}{2}}\right)^{j_1+\alpha}\left(\xi_{-\frac{1}{2}}\right)^{j_1-\alpha}\left(\eta_{\dot{\frac{1}{2}}}\right)^{j_2+\dot{\beta}}\left(\eta_{-\dot{\frac{1}{2}}}\right)^{j_2-\dot{\beta}}}{\left[(j_1+\alpha)!\,(j_1-\alpha)!\left(j_2+\dot{\beta}\right)!\left(j_2-\dot{\beta}\right)!\right]}\ , \tag{A1.12}$$

[20] Alternatively, an index can be lowered by contracting with the left index of C^{-1}.

where $\alpha = j_1, j_1 - 1, \ldots, -j_1$, $\dot\beta = j_2, j_2 - 1, \ldots, -j_2$, and where ξ and η are two-component spinors transforming according to

$$\xi' = A\,\xi\,, \qquad \eta' = B\,\eta\,. \tag{A1.13}$$

The $(2j_1+1)(2j_2+1)$-dimensional irreducible representations, $\mathcal{D}^{j_1,j_2}$, where j_1 and j_2 are nonnegative half integers, are defined by the equation

$$X'\,(j_1, j_2)_{\alpha'\dot\beta'} = \mathcal{D}^{j_1,j_2}(A, B)_{\alpha'\dot\beta'}{}^{\alpha\dot\beta}\, X\,(j_1, j_2)_{\alpha\dot\beta}\ , \tag{A1.14}$$

where the left-hand side is obtained by substituting (A1.13) into (A1.12).

Equations (A1.12) and (A1.14) lead to the direct product decomposition

$$\mathcal{D}^{j_1,j_2}(A, B) = \mathcal{D}^{j_1,0}(A) \otimes \mathcal{D}^{j_2,0}(B)\,. \tag{A1.15}$$

The irreducible representations of the proper, orthochronous, homogeneous Lorentz group, $\mathrm{L}_+^\uparrow$, are obtained by setting $B = A^*$. Henceforth, the notation

$$\mathrm{D}^j(A) \equiv \mathcal{D}^{j,0}(A) \tag{A1.16}$$

is used. This notation should not be confused with the standard notation for representations of the rotation group, $\mathcal{D}^j$, which are unitary. It is true, however, that for unitary-unimodular two-by-two matrices, U, which correspond to rotations, $\mathrm{D}^j(U) = \mathcal{D}^j(U)$.

The calculus for arbitrary spins is constructed by close analogy with the spin-$\frac{1}{2}$ case. The spinor indices have values $j, j–1, \ldots, -j$. Indices transforming by $\mathrm{D}^j(A)$ and $\mathrm{D}^j(B)$ are written as lower undotted and dotted, respectively, and indices transforming by the contragredient transformations $\mathrm{D}^j(A)^{-1\mathrm{T}}$ and $\mathrm{D}^j(B)^{-1\mathrm{T}}$ as upper undotted and dotted, respectively. Contraction of upper and lower indices of the same type is clearly an invariant operation. Equation (A1.8) generalizes to

$$\mathrm{D}^j(C^{-1})\,\mathrm{D}^j(A)^{\mathrm{T}}\,\mathrm{D}^j(C) = \mathrm{D}^j(A)^{-1}, \tag{A1.17}$$

where the group property of the representation matrices, the fact that A is unimodular, and the identity $\mathrm{D}^j(A^{\mathrm{T}}) = \mathrm{D}^j(A)^{\mathrm{T}}$ have been used. Thus one can define raising and lowering metric spinors

$$\begin{aligned}
&[\,j\,]^{\alpha\beta} = [\,j\,]^{\dot\alpha\dot\beta} = \mathrm{D}^j(C^{-1})^{\alpha\beta} = (-1)^{j-\alpha}\,\delta_\alpha{}^{-\beta},\\
&\{j\}_{\alpha\beta} = \{j\}_{\dot\alpha\dot\beta} = \mathrm{D}^j(C)_{\alpha\beta} = (-1)^{j+\alpha}\,\delta_\alpha{}^{-\beta},\\
&[\,j\,]_{\alpha\beta} = (-1)^{2j}\,\{j\}_{\alpha\beta} = \{j\}_{\beta\alpha}\,,
\end{aligned} \tag{A1.18}$$

where one contracts on the right index of [j] for raising or of {j} for lowering. These are just the familiar unitary matrices $d^j(\pm\pi)$, representing three-dimensional rotations by $\mp\pi$ about the y axis [23, p. 59].

The types of the spinor indices of the matrices D^j are taken to be the same as those of their arguments, except that they are of course $(2j+1)$-valued instead of 2-valued.

One notes in passing that

$$\xi^{\alpha}\,\eta_{\alpha} = (-1)^{2j}\,\xi_{\alpha}\,\eta^{\alpha}. \tag{A1.19}$$

The irreducible representations of $\mathcal{L}_+$ discussed so far are characterized by two half integers. The general finite-dimensional irreducible representations of $\mathcal{L}_+$ are characterized by four half integers, two for each occurrence of $\mathrm{SL}(2,\mathrm{C})$ in the direct product, $\mathrm{SL}(2,\mathrm{C})\times\mathrm{SL}(2,\mathrm{C})$. This follows because all irreducible representations of a group that is the direct product of two groups are obtained by taking direct products of irreducible representations of the two component groups. In this case they can be written

$$\mathcal{D}^{j_1,j_2,j_3,j_4}(A,B) = \mathrm{D}^{j_1}(A)\,\otimes\,\mathrm{D}^{j_2}(B)\,\otimes\,\mathrm{D}^{j_3}(A^*)\,\otimes\,\mathrm{D}^{j_4}(B^*)\,. \tag{A1.20}$$

The corresponding spinor calculus has eight index types rather than the four already discussed. Although the notation becomes cumbersome, the generalization is straightforward.

Those representations that depend on A^* or B^*, however, require no special consideration in analytic S-matrix theory or in field theory. This is because covariance under $\mathcal{L}_+$ in physical theories is generally a consequence of covariance under $\mathrm{L}_+^{\uparrow}$ and analytic properties [4, 24, 25]. Only representations depending on A and B arise under those circumstances. Of course, one may have occasion to consider the complex conjugate functions, which transform according to the complex conjugate representations; but this situation is trivially handled without complicating the spinor calculus with extra index types.

The total number of incoming and outgoing fermions in any scattering process is even. The sum of the spins of the incoming and outgoing particles is therefore an integer, and the corresponding M function transforms according to a tensorial representation of $\mathcal{L}_+$. The tensorial representations are obtained by combining Clebsh-Gordan (C-G) coefficients with tensor products of terms of the form $A\otimes A$, $A\otimes B$, and $B\otimes B$. Because they correspond to tensorial representations, these quantities are polynomials in Λ. The explicit dependence is given by[21]

$$
\begin{aligned}
A_\alpha{}^\beta \, A_{\alpha'}{}^{\beta'} &= \tfrac{1}{8} \, \Lambda_{\mu\nu}(A,B) \, \Lambda_{\mu'\nu'}(A,B) \\
&\qquad \times \left(\sigma^\mu \, \widetilde{\sigma}^{\mu'} \{\tfrac{1}{2}\}\right)_{\alpha\alpha'} \, \left(\{\tfrac{1}{2}\} \, \sigma^\nu \, \widetilde{\sigma}^{\nu'}\right)^{\beta\beta'}, \\
B_{\dot\alpha}{}^{\dot\beta} \, B_{\dot\alpha'}{}^{\dot\beta'} &= \tfrac{1}{8} \, \Lambda_{\mu\nu}(A,B) \, \Lambda_{\mu'\nu'}(A,B) \\
&\qquad \times \left(\{\tfrac{1}{2}\} \, \widetilde{\sigma}^\mu \, \sigma^{\mu'}\right)_{\dot\alpha\dot\alpha'} \, \left(\widetilde{\sigma}^\nu \, \sigma^{\nu'} \, \{\tfrac{1}{2}\}\right)^{\dot\beta\dot\beta'}, \\
A_\alpha{}^\beta \, B_{\dot\alpha'}{}^{\dot\beta'} &= \tfrac{1}{2} \, \Lambda_{\mu\nu}(A,B) \, \sigma^\mu_{\alpha\dot\alpha'} \, \widetilde{\sigma}^{\nu \, \dot\beta'\dot\beta}.
\end{aligned}
\tag{A1.21}
$$

A1.2 C-G Coefficients as Isotropic Spinors

The identity

$$
\begin{aligned}
\delta_{jj'} \, \mathrm{D}^j(A)_\alpha{}^{\alpha'} &= \sum_{\beta\beta'\gamma\gamma'} C\,(j_1 j_2 j; \beta\gamma\alpha) \;\, C\,(j_1 j_2 j'; \beta'\gamma'\alpha') \\
&\qquad \times \mathrm{D}^{j_1}(A)_\beta{}^{\beta'} \, \mathrm{D}^{j_2}(A)_\gamma{}^{\gamma'},
\end{aligned}
\tag{A1.22}
$$

for $j = |j_1{-}j_2|, |j_1{-}j_2|{+}1, \ldots, j_1{+}j_2$, expresses the fact that the C-G coefficients, $C\,(j_1 j_2 j; \beta\gamma\alpha)$ in the notation of Rose [13], are matrix elements of a unitary transformation that reduces a direct product of representations into a direct sum. By using the orthogonality of the C-G coefficients,

$$
\sum_{j,\gamma} C\,(j_1 j_2 j; \alpha\beta\gamma) \; C\,(j_1 j_2 j; \alpha'\beta'\gamma) = \delta_\alpha{}^{\alpha'} \, \delta_\beta{}^{\beta'},
\tag{A1.23}
$$

one easily finds that

$$
\begin{aligned}
C\,(j_1 j_2 j; \alpha\beta\gamma) &= \sum_{\alpha'\beta'\gamma'} \mathrm{D}^{j_1}(A)^{-1}{}_{\alpha'}{}^{\alpha} \, \mathrm{D}^{j_2}(A)^{-1}{}_{\beta'}{}^{\beta} \\
&\qquad \times \mathrm{D}^{j}(A)_\gamma{}^{\gamma'} \, C\,(j_1 j_2 j; \alpha'\beta'\gamma').
\end{aligned}
\tag{A1.24}
$$

This is just the transformation law of an isotropic spinor (a spinor with the same numerical values in every Lorentz frame) with a lower undotted spin-j index, an upper undotted spin-j_1 index, and an upper undotted spin-j_2 index.[22] Because A is arbitrary in (A1.24) one can just as well replace it by B and get the transformation law of an isotropic spinor with correspondingly dotted indices. Thus one can write

$$
C\,(j_1 j_2 j; \alpha\beta\gamma) \equiv [\, j j_1 j_2 \,]_\gamma{}^{\alpha\beta} = [\, j j_1 j_2 \,]_{\dot\gamma}{}^{\dot\alpha\dot\beta}.
\tag{A1.25}
$$

[21] The general expression for A or B in terms of Λ is given by Wightman [11]. The formulas here are an easy consequence of (A1.4), (A1.10), and (A1.11).

[22] This property is familiar from the rotation group. *Cf.* A .R .Edmonds [23, p. 46] and E. P. Wigner [26, pp. 292-296].

Because the C-G coefficient is a spinor, one can raise its spin-j index, using (A1.18), to get

$$\begin{aligned} [\,jj_1j_2\,]^{\gamma\alpha\beta} &= (-1)^{j-\gamma}\, C\,(j_1j_2j;\alpha,\beta,-\gamma) \\ &= (-1)^{j+j_2-j_1}\,\sqrt{2j+1}\;(\,jj_1j_2\,)^{\gamma\alpha\beta}\,, \end{aligned} \tag{A1.26}$$

where

$$(\,jj_1j_2\,)^{\gamma\alpha\beta} = (\,jj_1j_2\,)_{\gamma\alpha\beta} = \begin{pmatrix} j_1 & j_2 & j \\ \alpha & \beta & \gamma \end{pmatrix} \tag{A1.27}$$

is the standard Wigner 3-j symbol [26, p. 290]. By carrying out the raising and lowering operations one finds that

$$[\,jj_1j_2\,]^{\gamma}{}_{\alpha\beta} = (-1)^{2j}\,[\,jj_1j_2\,]_{\gamma}{}^{\alpha\beta}. \tag{A1.28}$$

The orthogonality relations for the C-G coefficients then become

$$\begin{aligned} (-1)^{2j}\,[\,jj_1j_2\,]_{\gamma}{}^{\alpha\beta}\,[\,j'j_1j_2\,]^{\gamma'}{}_{\alpha\beta} &= \delta_{jj'}\,\delta_{\gamma}{}^{\gamma'}, \\ (-1)^{2(j_1+j_2)}\sum_j [\,jj_1j_2\,]_{\gamma}{}^{\alpha'\beta'}\,[\,jj_1j_2\,]^{\gamma}{}_{\alpha\beta} &= \delta_{\alpha}{}^{\alpha'}\,\delta_{\beta}{}^{\beta'}, \end{aligned} \tag{A1.29}$$

where the fact that $(-1)^{2j} = (-1)^{2j_1+2j_2}$ has been used. The unsightly factor $(-1)^{2j}$ can be absorbed in the definition of an "inverse" spinor,

$$\{j_1j_2j\}_{\alpha\beta}{}^{\gamma} \equiv [\,jj_1j_2\,]_{\gamma}{}^{\alpha\beta}, \tag{A1.30}$$

if desired; various formulas such as (A1.29) then acquire a neater look.[23]

A useful expression for the metric spinors (A1.18) can be calculated from explicit formulas for the C-G coefficients:

$$\begin{aligned} [\,j\,]^{\alpha\beta} &= \sqrt{2j+1}\,[\,0jj\,]_0{}^{\alpha\beta}, \\ \{j\}_{\alpha\beta} &= \sqrt{2j+1}\;\{jj0\}\,{}_{\alpha\beta}{}^{0}. \end{aligned} \tag{A1.31}$$

It is clear that Eqs. (A1.26)-(A1.31) remain valid when all indices are dotted.

A1.3 Reduction of a Spinor: Isotropic Spinors as C-G Coefficients

Just as for the rotation group, an arbitrary spinor under $\mathcal{L}_+$ can be reduced into its irreducible parts by projecting with C-G coefficients. The projection

[23] In S-matrix theory one can usually arrange it so that this factor does not occur, because the total spin is an integer.

operators are constructed by the standard method [12]. The reduction operators for a spinor with N lower undotted or lower dotted indices can be written in the form

$$[\, j\mathcal{J} : j_1 \cdots j_N \,]_\alpha{}^{\alpha_1 \cdots \alpha_N} = [\, j\mathcal{J} : j_1 \cdots j_N \,]_{\dot\alpha}{}^{\dot\alpha_1 \cdots \dot\alpha_N}, \tag{A1.32}$$

where $\mathcal{J}$ stands for the set of $N-1$ intermediate spin values that occur in the step-wise reduction, beginning at the left with j_1 and j_2 and resulting in spin j. In other words,

$$\mathcal{J} = (l_1, \ldots, l_{N-1})\,, \tag{A1.33}$$

where $l_{i+1} = l_i + j_{i+2}$, $l_0 \equiv j_1$, in the sense of vector addition of angular momenta. These spinors are defined inductively in terms of C-G spinors, (A1.25), by

$$\begin{aligned}[\, j(\mathcal{J}'j') : j_1 \cdots j_N \,]_\alpha{}^{\alpha_1 \cdots \alpha_N} \\ = [\, jj'j_N \,]_\alpha{}^{\alpha' \alpha_N} \, [\, j'\mathcal{J}' : j_1 \cdots j_{N-1} \,]_{\alpha'}{}^{\alpha_1 \cdots \alpha_{N-1}}.\end{aligned} \tag{A1.34}$$

One can, of course, obtain reduction operators for spinors with upper indices by lowering the indices of the symbols just defined. It is convenient, however, to define another symbol, differing by a phase,

$$\begin{aligned}\{\, j_1 \cdots j_N : j\mathcal{J} \,\}_{\alpha_1 \cdots \alpha_N}{}^{\alpha} &\equiv [\, j\mathcal{J} : j_1 \cdots j_N \,]_\alpha{}^{\alpha_1 \cdots \alpha_N} \\ &= \{\, j_1 \cdots j_N : j\mathcal{J} \,\}_{\dot\alpha_1 \cdots \dot\alpha_N}{}^{\dot\alpha},\end{aligned} \tag{A1.35}$$

which is constructed by precise analogy with (A1.34), using the symbol defined in (A1.30). One can then prove the orthogonality relations

$$\begin{aligned}[\, j\mathcal{J} : j_1 \cdots j_N \,]_\alpha{}^{\alpha_1 \cdots \alpha_N} \; \{\, j_1 \cdots j_N : j'\mathcal{J}' \,\}_{\alpha_1 \cdots \alpha_N}{}^{\alpha'} \\ = \delta_{JJ'} \, \delta_{\mathcal{J}\mathcal{J}'} \, \delta_\alpha{}^{\alpha'},\end{aligned} \tag{A1.36}$$

$$\begin{aligned}\sum_{j,\mathcal{J}} \{\, j_1 \cdots j_N : j\mathcal{J} \,\}_{\alpha_1 \cdots \alpha_N}{}^{\alpha} \; [\, j\mathcal{J} : j_1 \cdots j_N \,]_\alpha{}^{\alpha_1' \cdots \alpha_N'} \\ = \delta_{\alpha_1}{}^{\alpha_1'} \, \cdots \, \delta_{\alpha_N}{}^{\alpha_N'},\end{aligned} \tag{A1.37}$$

where the sum is over all $j, \mathcal{J}$ occurring in the reduction of the spins $j_1 \cdots j_N$. The proof is by induction from (A1.29), using the relation

$$(-1)^{2j} = (-1)^{2(j_1 + \cdots + j_N)} \tag{A1.38}$$

which also follows by induction.

Exactly the same equations hold for the spinors with all indices dotted.

In order to reduce an arbitrary spinor, it is clearly sufficient to consider the case where all indices are lower. The notation will be simplified by writing $(\alpha, M) \equiv \alpha_1 \cdots \alpha_M$, and $(j, M) \equiv j_1 \cdots j_M$. Then an arbitrary spinor $\eta_{(\alpha,M)(\dot{\beta},N)}$ reduces to

$$\eta_{\alpha\dot{\beta}}(jj' : \mathcal{J}\mathcal{J}') \\ = [\, j\mathcal{J} : (j,M)\,]_\alpha{}^{(\alpha,M)}\ [\, j'\mathcal{J}' : (j',N)\,]_{\dot{\beta}}{}^{(\dot{\beta},N)}\ \eta_{(\alpha,M)(\dot{\beta},N)}\,, \tag{A1.39}$$

which transforms according to the irreducible representation $\mathcal{D}^{j,j'}$. One can use the orthogonality relation (A1.37) to transform the set of irreducible spinors on the left side of (A1.39) back into the original spinor.

Because they are constructed from C-G coefficients, the reduction operators are isotropic spinors. Suppose that ξ is an arbitrary isotropic spinor satisfying the transformation law

$$\bigotimes_m \mathrm{D}^{j_m}(A)\ \bigotimes_n \mathrm{D}^{j'_n}(B)\,\xi = \xi\,. \tag{A1.40}$$

If one substitutes ξ into (A1.39), the resulting irreducible spinor must be isotropic. This implies that $j = j' = 0$, for the isotropy of $\xi_{\alpha\dot{\beta}}(jj' : \mathcal{J}\mathcal{J}')$ means that it spans a one-dimensional, invariant subspace of the irreducible representation $\mathcal{D}^{j,j'}$. Inverting (A1.39), one has the result for any isotropic spinor:

$$\xi_{(\alpha,M)(\dot{\beta},N)} \\ = \sum_{\mathcal{J},\mathcal{J}'} \{(j,M) : 0\mathcal{J}\}_{(\alpha,M)}{}^{0}\ \{(j',N) : 0\mathcal{J}'\}_{(\dot{\beta},N)}{}^{0}\ a(\mathcal{J},\mathcal{J}')\,, \tag{A1.41}$$

where the summation is over those sets $\mathcal{J}, \mathcal{J}'$ that lead to zero spins in the reduction.

The Wigner-Eckart Theorem for the rotation group is a special case of a formula that is analogous to (A1.41) [12, Chap. 14].

Equations (A1.38) and (A1.41) imply that an isotropic spinor has an even number of undotted and an even number of dotted half odd-integer spin indices.

A1.4 Isotropic Spin Tensors

The Pauli matrices, σ_μ, form the fundamental spin tensor. That this spin tensor is isotropic follows by moving the Λ in (A1.4) to the left-hand side of the equation. It is convenient for the construction of orthogonality relations to define normalized spin tensors:

$$\rho_\mu \equiv \tfrac{1}{\sqrt{2}}\,\sigma_\mu\,, \qquad \tilde{\rho}_\mu \equiv \tfrac{1}{\sqrt{2}}\,\tilde{\sigma}_\mu \tag{A1.42}$$

The ρ spinors can be regarded as a transformation from a tensor index to an equivalent dotted and undotted pair of spin-$\frac{1}{2}$ spinor indices, and *vice* *versa*. In fact, (A1.6) expresses the well-known equivalence between the $\mathcal{D}^{\frac{1}{2},\frac{1}{2}}$ and the self representations of $\mathcal{L}_+$. Any function having tensor indices can be converted to an equivalent function having only spinor indices by contracting each tensor index with that of a ρ spinor. The resulting spinor is converted back to the original function by means of the orthogonality relation (A1.5). Not all spinors can be converted into equivalent tensors, however, because not all representations of $\mathcal{L}_+$ are tensorial representations. The conversion is possible if and only if the function has an even number of half odd-integer spin indices. This property holds, for example, for isotropic spinors, from the comment following (A1.41), and for isotropic spin tensors, which can be converted to isotropic spinors. Thus isotropic spinors and spin tensors are equivalent to isotropic tensors.

In order to get a representation for an arbitrary isotropic spin tensor, one converts to an isotropic spinor, applies (A1.41), and converts back again. As a result, all isotropic spin tensors can be decomposed in terms of Pauli matrices and C-G coefficients.

It is also possible to represent an arbitrary isotropic spin tensor in terms of generalized Pauli matrices. These quantities have been constructed and some of their properties discussed elsewhere [1]. In the present notation, they are defined by

$$\begin{aligned}\rho_{(\mu,N)}(jj' : \mathcal{J}\mathcal{J}')_{\alpha\dot{\beta}} &= [\, j\mathcal{J} : (\tfrac{1}{2}, N)\,]_{\alpha}{}^{(\alpha,N)} \\ &\quad\times [\, j'\mathcal{J}' : (\tfrac{1}{2}, N)\,]_{\dot{\beta}}{}^{(\dot{\beta},N)}\ \rho_{\mu_1\,\alpha_1\dot{\beta}_1} \cdots \rho_{\mu_N\,\alpha_N\dot{\beta}_N}\,,\end{aligned} \tag{A1.43}$$

$$\begin{aligned}\tilde{\rho}_{(\mu,N)}(j'j : \mathcal{J}'\mathcal{J})^{\dot{\beta}\alpha} &= \big\{ (\tfrac{1}{2}, N) : j'\mathcal{J}' \big\}_{(\dot{\beta},N)}{}^{\dot{\beta}} \\ &\quad\times \big\{ (\tfrac{1}{2}, N) : j\mathcal{J} \big\}_{(\alpha,N)}{}^{\alpha}\ \tilde{\rho}_{\mu_1}{}^{\dot{\beta}_1\alpha_1} \cdots \tilde{\rho}_{\mu_N}{}^{\dot{\beta}_N\alpha_N}\,.\end{aligned} \tag{A1.44}$$

These spin tensors transform the irreducible parts of an arbitrary Nth rank tensor into the corresponding irreducible spinor transforming according to $\mathcal{D}^{j,j'}$. The tensor indices of these ρ spinors have the maximum symmetry of an irreducible tensor. From (A1.5), (A1.11), (A1.36), (A1.37), and (A1.42), one gets the orthogonality relations

$$\begin{aligned}\rho_{(\mu,N)}\,(jj' : \mathcal{J}\mathcal{J}')_{\alpha\dot{\beta}}\ \tilde{\rho}^{(\mu,N)}\,(s's : \mathcal{S}'\mathcal{S})^{\dot{\beta}'\alpha'} \\ = \delta_{js}\ \delta_{j's'}\ \delta_{\mathcal{J}\mathcal{S}}\ \delta_{\mathcal{J}'\mathcal{S}'}\ \delta_{\alpha}{}^{\alpha'}\ \delta_{\dot{\beta}}{}^{\dot{\beta}'},\end{aligned} \tag{A1.45}$$

$$\begin{aligned}\sum_{jj',\mathcal{J}\mathcal{J}'} \rho_{(\mu,N)}\,(jj' : \mathcal{J}\mathcal{J}')_{\alpha\dot{\beta}}\ \tilde{\rho}_{(\mu',N)}\,(j'j : \mathcal{J}'\mathcal{J})^{\dot{\beta}\alpha} \\ = g_{\mu_1\mu_1'} \cdots g_{\mu_N\mu_N'}\,.\end{aligned} \tag{A1.46}$$

It was shown elsewhere [1] that the individual terms in the sum in (A1.46) are projection operators for the reduction of a tensor into its irreducible parts.

All of the remarks in this section apply to the representation of isotropic tensors, which are isotropic spin tensors having only spin-0 indices.

A1.5 Isotropic Tensors

It is easy to show that all isotropic tensors under $\mathcal{L}_+$ are composed from the metric tensor $g_{\mu\nu}$ and the alternating symbol $\epsilon_{\mu\nu\lambda\rho}$.[24] First, one converts to an equivalent isotropic spinor with spin-$\frac{1}{2}$ indices. Next, one shows that all spinors of that type can be composed from $[\,\frac{1}{2}\,]$ metric symbols, by looking at the structure of the $[\,0\mathcal{J}: \big(\frac{1}{2}, N\big)\,]$ symbols that occur in the expansion (A1.41). Finally, one converts back to the original isotropic tensor. By using the standard identity

$$\rho_\mu\,\tilde{\rho}_\nu\,\rho_\lambda = \tfrac{1}{2}\left(g_{\mu\nu}\,\rho_\lambda - g_{\mu\lambda}\,\rho_\nu + g_{\nu\lambda}\,\rho_\mu - i\,\epsilon_{\mu\nu\lambda\sigma}\,\rho^\sigma\right)\,, \tag{A1.47}$$

induction, and (A1.5), the result follows.

[24]This well-known fact was proved for the general linear groups by Cramlet [27]. *Cf.* also P. Franklin [28].

A2 The Spin Basis in Terms of Isotropic Spin Tensors: Projection Operators on $\mathcal{T}(J:v)$

The spin basis constructed in Sec. 4 is related to one of the transformations of a $(2J+1)$-dimensional spinor into an equivalent irreducible tensor defined in (A1.43). It can be written in terms of the isotropic spin tensor

$$\rho^{\mu_1\nu_1\cdots\mu_J\nu_J}\,(J0:\mathcal{J}_0\mathcal{J}_0')_{\alpha\dot{0}}\;,$$

where the special choices $\mathcal{J}_0 = \left(1, \frac{3}{2}, \ldots, J-\frac{1}{2}\right)$ and $\mathcal{J}_0' = \left(0, \frac{1}{2}, 0, \frac{1}{2}, \ldots, 0, \frac{1}{2}\right)$ are made. A brief calculation shows that, up to a sign, this spin tensor is the same as

$$[\,J:(1,J)\,]\bigotimes_{i=1}^{J}\rho^{\mu_i\nu_i}(1,0)\,,$$

where $\rho(1,0)$ is selfdual in its tensor indices, i.e.,

$$\tfrac{i}{2}\,\epsilon_{\mu\nu\lambda\sigma}\,\rho^{\lambda\sigma}(1,0) = \rho_{\mu\nu}(1,0)\,. \tag{A2.1}$$

Then the spinors in (4.8) can be written

$$\begin{aligned}\left(-\tfrac{1}{2}\right)^{J}\,\rho^{(\mu)}(J:v_1,\ldots,v_J)&\\ &= v_{1\,\nu_1}\,\cdots\,v_{J\,\nu_J}\,[\,J:(1,J)\,]\bigotimes_{i=1}^{J}\rho^{\mu_i\nu_i}(1,0)\,.\end{aligned} \tag{A2.2}$$

Equation (4.2) results from the more general identity (A2.1). Spinors of the type $\rho(J0:\mathcal{J}_0\mathcal{J}_0')$ above, which have the minimum possible number of tensor indices, vanish upon contraction of any pair of tensor indices.[25] It is because of this general property that the expression in (4.8) or (A2.2) is traceless.

At the end of Sec. 4 it is mentioned that the construction of a spin basis has in a certain sense been reduced from a problem for $\mathcal{L}_+$ to a problem for $\mathrm{O}_+(3,\mathrm{C})$. To make this notion precise, consider the space $\mathcal{T}(N:v)$ of Nth rank tensors that vanish upon contracting any index with the four-vector v, where $v{\cdot}v \neq 0$. The subgroup of $\mathcal{L}_+$ that leaves the space $\mathcal{T}(N:v)$ invariant, denoted by $\mathcal{L}_+(v)$, is the group of transformations that leave v unchanged. It is easy to see that the group $\mathcal{L}_+(v)$ is isomorphic to $\mathrm{O}_+(3,\mathrm{C})$.

The invariant subspaces of $\mathcal{T}(N:v)$ with respect to $\mathcal{L}_+(v)$ thus correspond to the invariant subspaces of the Nth rank tensors with respect to $\mathrm{O}_+(3,\mathrm{C})$. One can generalize the $\rho(J:v)$ spinors in Sec. 4 to

$$\rho^{(\mu,N)}(J\mathcal{J}:v) = [\,J\mathcal{J}:(1,N)\,]\bigotimes_{i=1}^{N}\rho^{\mu_i}(1,v)\,, \tag{A2.3}$$

[25] *Cf.* [1], where a verification of this well-known fact is given in terms of the properties of C-G coefficients and Pauli matrices.

where J can have the values $0, 1, \ldots, N$. Then from (4.4) and (A1.36) one gets the orthogonality relation

$$(v\cdot v)^{-N}\, \rho_{(\mu,N)}\,(J\mathcal{J}:v)_\alpha\;\; \rho^{(\mu,N)}\,(J'\mathcal{J}':v)^\beta = \delta_{JJ'}\,\delta_{\mathcal{J}\mathcal{J}'}\,\delta_\alpha{}^\beta\,; \tag{A2.4}$$

and from (4.5) and (A1.37) one gets

$$\begin{aligned}(v\cdot v)^{-N} \sum_{J,\mathcal{J}} \rho^{(\mu,N)}\,(J\mathcal{J}:v)_\alpha\;\; \rho^{(\nu,N)}\,(J\mathcal{J}:v)^\alpha&\\ \equiv \sum_{J,\mathcal{J}} P^{(\mu)(\nu)}(J\mathcal{J}:N,v)&\\ = h^{\mu_1\nu_1}(v)\,\cdots\, h^{\mu_N\nu_N}(v) \equiv P^{(\mu)(\nu)}(N:v)\,.&\end{aligned} \tag{A2.5}$$

It is easily verified that $P(N:v)$ is the projection operator from the space of Nth rank tensors onto $\mathcal{T}(N:v)$, and by arguments similar to those given by the author elsewhere [1], that the $P(J\mathcal{J}:N,v)$ are projection operators for the invariant subspaces of $\mathcal{T}(N:v)$ with respect to $\mathcal{L}_+(v)$. By putting v in its rest frame one gets the projection operators for the irreducible tensors under $\mathrm{O}_+(3,\mathrm{C})$.

So far, only spinors with undotted indices have been considered. An analogous construction exists for spinors with dotted indices, and the relation beween the two constructions can be used to show that the operators $P(J\mathcal{J}:N,v)$ are tensors not only under $\mathcal{L}_+$ but also under $\mathcal{L}$. Instead of (4.1) one has

$$\tilde\rho_\mu\,(1:v)^{\dot\alpha} \equiv -[\,1\,\tfrac12\,\tfrac12\,]^{\dot\alpha}{}_{\dot\beta\dot\gamma}\,\left(\tilde\rho_\mu\, v\cdot\sigma\,[\,\tfrac12\,]\right)^{\dot\beta\dot\gamma}, \tag{A2.6}$$

or

$$\tilde\rho_\mu\,(1:\tilde v)^{\dot\alpha} = \rho^\mu\,(1:v)_\alpha\;, \tag{A2.7}$$

where $\tilde v_\mu \equiv v^\mu$. In general one has

$$\tilde\rho^{(\mu,N)}(J\mathcal{J}:v) = \left[\bigotimes_{i=1}^{N} \tilde\rho^{\mu_i}(1:v)\right]\,\{(1,N):J\mathcal{J}\}\;, \tag{A2.8}$$

and

$$\tilde\rho_{(\mu,N)}\,(J\mathcal{J}:\tilde v)^{\dot\alpha} = \rho^{(\mu,N)}\,(J\mathcal{J}:v)_\alpha\;. \tag{A2.9}$$

Equations (A2.4) and (A2.5) remain valid when ρ is replaced by $\tilde\rho$. It is easy to show that

$$[\,J:v\,]\,\rho_{(\mu,N)}(J\mathcal{J}:v) = \tilde\rho_{(\mu,N)}(J\mathcal{J}:v)\,, \tag{A2.10}$$

where $[\,J:v\,]$ is defined in (3.1). From this and (3.2) one sees that

$$\begin{aligned}\tilde\rho_{(\mu,N)}\,(J\mathcal{J}:v)^{\dot\alpha}\;\; \tilde\rho_{(\nu,N)}\,(J\mathcal{J}:v)_{\dot\alpha}&\\ = \rho_{(\mu,N)}\,(J\mathcal{J}:v)_\alpha\;\; \rho_{(\nu,N)}\,(J\mathcal{J}:v)^\alpha\,,&\end{aligned} \tag{A2.11}$$

and hence that

$$P^{(\mu)(\nu)}(J\mathcal{J}:N,v) = P_{(\mu)(\nu)}(J\mathcal{J}:N,\tilde{v})\,, \tag{A2.12}$$

where (A2.9) has been used. Manifestly, $P(J\mathcal{J}:N,v)$ is a tensor under $\mathcal{L}_+$. Equation (A2.12) shows that it is also a tensor under $\mathcal{L}$, because it transforms as a tensor under space inversion.

A3 Example of a Kinematical Singularity

First, the projection operator $S(2 : v)$ defined by (4.13) can be computed:

$$S^{(\mu)(\nu)}(2 : v) = \tfrac{1}{2}\,\left[h^{\mu_1\nu_1}(v)\,h^{\mu_2\nu_2}(v) + h^{\mu_1\nu_2}(v)\,h^{\mu_2\nu_1}(v)\right] \\ - \tfrac{1}{3}\,h^{\mu_1\mu_2}(v)\,h^{\nu_1\nu_2}(v)\,. \qquad \text{(A3.1)}$$

An example of a holomorphic, $\mathcal{L}_+$-covariant function for spin two is:

$$M(2) = {k_2}^{\mu}\,{k_2}^{\nu}\,\rho_{\mu\nu}(2 : k_3)\,. \qquad \text{(A3.2)}$$

Computing from (5.2),

$$\begin{aligned} f^{(\mu)} &= M(2)_{\alpha}\,\rho^{(\mu)}\,(2 : k_3)^{\alpha}\ {m_3}^{-4} \\ &= {S^{(\mu)}}_{(\nu)}(2 : k_3)\ {k_2}^{\nu_1}\,{k_2}^{\nu_2}\,; \end{aligned} \qquad \text{(A3.3)}$$

and using (5.16), one finds in particular that

$$\begin{aligned} a(2,0,0) &= f_{\mu\nu}\,{\widehat{k}_1}{}^{\mu}\,{\widehat{k}_1}{}^{\nu} \\ &= -\frac{\left[{m_2}^2\,{m_3}^2 - (k_2\!\cdot\! k_3)^2\right]^2}{3\,{m_3}^2\,G(K)}\,. \end{aligned} \qquad \text{(A3.4)}$$

This expression clearly has a pole for most of the zeroes of G.

A4 Proof of Lemma 5

The proof follows very closely that given by Bargmann, Hall, and Wightman for $n = r$ points and the map I. As already mentioned, the maps I_+ and I coincide for $l = 3$. Thus $\mathcal{M}_{3+}$ can be regarded as embedded in the space of 3×3 symmetric matrices, and the topology of $\mathcal{M}_{3+}$ taken to be the restriction of that of the 3×3 matrices. If the mass constraints are disregarded, then $\mathcal{M}_{3+}$ is simply replaced by the space of symmetric 3×3 matrices. The lemma will be proved without the mass constraints, except for the assumption that no vectors on the light cone occur. The lemma with mass constraints then follows by restriction of the topology.

Thus it is to be proved that for any neighborhood N of a point $Z^{(0)} = \left(z_1^{(0)}, z_2^{(0)}, z_3^{(0)}\right)$ in the space of three complex four-vectors, excluding vectors on the light cone,[26] $I_+(N)$ is a neighborhood of $I_+\left(Z^{(0)}\right)$. Because of the proof given by Bargmann, Hall, and Wightman, it is sufficient to assume that $Z^{(0)}$ is an $n \neq r$ point, with $n = r + 1$, and $r = 1$ or 2. The proof for the more easily treated $n = r$ points would also follow from the proof given here, with slight changes of wording.

The convention that $z_1^{(0)}, \ldots, z_r^{(0)}$ have nonzero Gram determinant and that $z_1^{(0)}, \ldots, z_n^{(0)}$ be linearly independent will be used, along with the notation $\mathcal{Z} \equiv I_+(Z)$, $\mathcal{Y} \equiv I_+(Y)$, etc.

A series of transformations depending only on $Z^{(0)}$ will be made to simplify the problem.

For any $r = n - 1$ point, one can write

$$z_i^{(0)} = \sum_{j=1}^{r} \alpha_{ij}\, z_j^{(0)} + \kappa_i\, \omega\,, \tag{A4.1}$$

where ω is on the light cone and orthogonal to the space spanned by $z_1, \ldots, z_r$,[27] and where $\kappa_i = 0$ for $1 \leq i \leq r$. The α_{ij} depend only on $Z^{(0)}$. Define the new variables w_i by

$$\begin{aligned} w_i &= z_i\,, && i = 1, \ldots, r \\ w_i &= z_i - \sum_{j=1}^{r} \alpha_{ij}\, z_j\,, && i > r\,. \end{aligned} \tag{A4.2}$$

This transformation has determinant one, and it gives $w_i^{(0)} = \kappa_i\, \omega$, for $i > r$. One can express the $w_i^{(0)}$ in the form

$$w_i^{(0)} = \sum_{j=1}^{n} \beta_{ij}\, w_j^{(0)}\,. \tag{A4.3}$$

[26] If $Z^{(0)}$ contains no vectors on the light cone, then there is a neighborhood of $Z^{(0)}$ with the same property.

[27] *Cf.* Sec. 6.1.

Then let

$$
\begin{aligned}
x_i &= w_i \,, & i \le n \\
x_i &= w_i - \sum_{j=1}^{n} \beta_{ij}\, w_j \,, & i > n \,.
\end{aligned}
\tag{A4.4}
$$

The β_{ij} depend only on $Z^{(0)}$. This transformation also has determinant one, and it makes $x_i^{(0)} = 0$ for $i > n$. Next one performs the transformation that orthonormalizes the first r vectors of $X^{(0)}$ and does nothing to the rest. Then by a Lorentz transformation (Lemma 1), the resulting vectors can be brought to the form

$$
\begin{aligned}
y_1^{(0)} &= (1,0,0,0)\,, \\
y_2^{(0)} &= (0,1,i,0) \equiv \omega_{12}\,, \\
y_3^{(0)} &= 0\,,
\end{aligned}
\tag{A4.5a}
$$

for $r = 1$, and

$$
\begin{aligned}
y_1^{(0)} &= (1,0,0,0)\,, \\
y_2^{(0)} &= (0,i,0,0)\,, \\
y_3^{(0)} &= (0,0,1,i) \equiv \omega_{23}\,,
\end{aligned}
\tag{A4.5b}
$$

for $r = 2$.

The net effect of all of these transformations is a nonsingular matrix A, depending only on $Z^{(0)}$,

$$
y_i = \sum_{j=1}^{3} A_{ij}\, z_j \,, \tag{A4.6}
$$

with the property

$$
A\,\mathcal{Z}^{(0)} A^{\mathrm{T}} = \mathcal{Y}^{(0)} = \left(\begin{array}{c:c} I & 0 \\ \hdashline 0 & 0 \end{array}\right), \tag{A4.7}
$$

where the identity block of the 3×3 matrix $\mathcal{Y}^{(0)}$ is an $r \times r$ matrix. The transformation A is evidently a homeomorphism, both of the space of vectors and of the space of invariants, that preserves the map I_+. Thus it is sufficient to consider neighborhoods of $Y^{(0)}$.

Let Y be in a neighborhood of $Y^{(0)}$. The final transformation to be made depends on the corresponding point $\mathcal{Y}$ which is in a neighborhood of $\mathcal{Y}^{(0)}$. It will be such as to make the first r vectors of Y orthogonal to the rest. In particular take

$$
\begin{aligned}
y_i' &= y_i\,, & i \le r\\
y_i' &= y_i - \sum_{j=1}^{r} \gamma_{ij}\, y_j\,, & i > r\,.
\end{aligned}
\tag{A4.8}
$$

where γ_{ij} is determined by the requirement that $y_i'\cdot y_j' = 0$ for $i > r$ and $j \le r$. Then

$$
\gamma_{ij} = \sum_{k=1}^{r} \mathcal{Y}_{ik}\,\left[{\mathcal{Y}_{(r)}}^{-1}\right]_{kj}\,, \quad i > r\,, \quad j \le r\,, \tag{A4.9}
$$

where $\mathcal{Y}_{(r)}$ is the upper left $r\times r$ block of $\mathcal{Y}$.

Note that $\mathcal{Y}'^{(0)} = \mathcal{Y}^{(0)}$. Also note that, because of its continuity, one can keep det $\left[\mathcal{Y}_{(r)}\right]$ bounded away from zero in a sufficiently small neighborhood of $\mathcal{Y}^{(0)}$ (or of $Y^{(0)}$), and thus guarantee the existence and boundedness of ${\mathcal{Y}_{(r)}}^{-1}$ there. Thus, if one writes, in such a neighborhood, (A4.8) in the form

$$
y_i' = \sum_{j=1}^{3} B_{ij}\,(\mathcal{Y})\; y_j\,, \tag{A4.10}
$$

then the set of $B(\mathcal{Y})$ is bounded and the set of inverses $B^{-1}(\mathcal{Y})$ exists and is bounded. The boundedness of the $B(\mathcal{Y})$ implies that a neighborhood of $\mathcal{Y}^{(0)}$ (or of $Y^{(0)}$) can be found such that the corresponding points $\mathcal{Y}'$ (or Y') lie inside of any preassigned neighborhood of $\mathcal{Y}^{(0)}$ (or of $Y^{(0)}$). This plus the existence of the bounded set of inverses $B^{-1}(\mathcal{Y})$ implies that for any neighborhood N of $Y^{(0)}$, one can find neighborhoods $\mathcal{N}_0$ of $\mathcal{Y}^{(0)}$ and N_0 of $Y^{(0)}$ such that if $\mathcal{Y}' = B(\mathcal{Y})\,\mathcal{Y}\,B(\mathcal{Y})^{\mathrm{T}}$, with $\mathcal{Y}$ in $\mathcal{N}_0$, then any Y' in N_0 satisfying $I_+(Y') = \mathcal{Y}'$ has the property that the vectors $\sum_{j=1}^{3} B^{-1}(\mathcal{Y})_{ij}\, y_j'$ form a point in N.

The above statement also holds for any subneighborhood $\mathcal{N}_1$ of $\mathcal{Y}^{(0)}$ in $\mathcal{N}_0$, and it reduces the problem to the question of whether one can find a sufficiently small $\mathcal{N}_1$ such that for any point of the form $\mathcal{Y}' = B(\mathcal{Y})\,\mathcal{Y}\,B(\mathcal{Y})^{\mathrm{T}}$, with $\mathcal{Y}$ in $\mathcal{N}_1$, there is a corresponding Y' in N_0. Because of the boundedness of $B(\mathcal{Y})$ it is sufficient to find a small neighborhood $\mathcal{N}'$ of $\mathcal{Y}'^{(0)} = \mathcal{Y}^{(0)}$ in the space of $\mathcal{Y}'$ such that there is a corresponding Y' in the fixed neighborhood N_0. To show that such an $\mathcal{N}'$ exists, write $Y' = Y^{(0)} + V$, and

$$
\begin{aligned}
\mathcal{Y}' &= \left(\begin{array}{c:c} C_1 & 0 \\ \hdashline 0 & C_2 \end{array}\right) + \left(\begin{array}{c:c} I & 0 \\ \hdashline 0 & 0 \end{array}\right),\\
&= \left(\begin{array}{c:cc} \left(y_1^{(0)} + v_1\right)^2 & \left(y_1^{(0)} + v_1\right)\cdot\left(y_2^{(0)} + v_2\right) & \left(y_1^{(0)} + v_1\right)\cdot\left(y_3^{(0)} + v_3\right) \\ \hdashline \cdots & 2\,\omega_{12}\cdot v_2 + v_2\cdot v_2 & \omega_{12}\cdot v_3 + v_2\cdot v_3 \\ \cdots & \cdots & v_3\cdot v_3 \end{array}\right),
\end{aligned}
\tag{A4.11}
$$

for $r = 1$; and

$$\mathcal{Y}' = \left(\begin{array}{c:c} \left(y_i^{(0)} + v_i\right)\cdot\left(y_j^{(0)} + v_j\right) & \left(y_i^{(0)} + v_i\right)\cdot\left(y_3^{(0)} + v_3\right) \\ \hdashline \cdots & 2\,\omega_{23}\cdot v_3 + v_3\cdot v_3 \end{array}\right) \tag{A4.12}$$

with $i, j = 1, 2$, for $r = 2$.

Taking all but the first r components of v_i, for $i \leq r$, to vanish, Bargmann, Hall, and Wightman showed that one can find a bound for C_1 such that there are always solutions for the first r components of these vectors that are as small as desired. Accordingly, we take the first r components of v_i, for $i > r$, to vanish and show that C_2 can be bounded in such a way that small solutions exist for the remaining components. These choices already guarantee that the off-diagonal blocks in (A4.11) and (A4.12) vanish.

For $r = 1$, write

$$C_2 = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}, \tag{A4.13}$$

and choose

$$v_2 = \left(0, -\frac{\alpha}{4}, -\frac{\alpha}{4i}, 0\right), \qquad v_3 = \left(0, -\frac{\beta}{2}, -\frac{\beta}{2i}, i\sqrt{\gamma}\right). \tag{A4.14}$$

If C_2 is small, then v_2 and v_3 are small, as required.

For $r = 2$, C_2 is a number, and

$$v_3 = \left(0, 0, -\frac{C_2}{4}, -\frac{C_2}{4i}\right) \tag{A4.15}$$

is a solution with the right property.

Therefore Lemma 5 is proved.

It is worth emphasizing that the same proof does not work for neighborhoods of $n \neq r$ points in spaces with $l > 3$. Upon adding the necessary extra elements to the lower right block of the matrix in (A4.11), for example, one soon discovers that it is not possible in general to find small solutions for V.

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