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Semiparametric methods for survival analysis of case-control data subject to dependent censoring: Supplementary Materials

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Appendix S1

In this report, we provide a proof of Theorem 2 and an examination of the corresponding variance estimator. We begin with a review of the notation.

A.01 Notation

$i =$ subject $(i = 1, \ldots, n)$

$T_i =$ failure time

$C_{1i} =$ independent censoring time

$C_{2i} =$ dependent censoring time

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$C_i = C_{1i} \wedge C_{2i}$

$X_i = T_i \wedge C_{1i} \wedge C_{2i}$

$Y_i(t) = I\left\{X_i \geq t\right\}$

$\Delta_{1i} = I\left(T_i \leq C_i\right)$

$\Delta_{2i} = I\left\{C_{2i} \leq C_{1i}, C_{2i} < T_i)\right\}$

$\Delta_{3i} = (1 - \Delta_{1i})(1 - \Delta_{2i})$

$N_i(t) = I\left\{X_i \leq t, \Delta_{1i} = 1\right\}$

$N_i^C(t) = I\left\{X_i \leq t, \Delta_{2i} = 1\right\}$

$\mathbf{Z}_{1i} =$ time-constant covariate vector

$\mathbf{Z}_{2i}(t) =$ time-dependent covariate vector

$\mathbf{Z}_i(t) = \left\{\mathbf{Z}_{1i}^T, \mathbf{Z}_{2i}^T(t)\right\}^T$

$\overline{\mathbf{Z}}_i(t) = \left\{\mathbf{Z}_i(u) : 0 \leq u \leq t\right\}$

$\mathbf{V}_i(t) = \left\{V_{1i}(t), \ldots, V_{qi}(t)\right\} =$ functions of $\mathbf{Z}_i(t)$

$\lambda_i(t) = \lambda_0(t)e^{\boldsymbol{\beta}^T\mathbf{Z}_i(0)}$

$\lambda_i^C(t) = \lambda_0^C(t)e^{\boldsymbol{\alpha}^T\mathbf{V}_i(t)}$

$\xi_i = I\,($ individual i is selected for the subcohort $)$

$p_k = Pr(\xi_i = 1 \mid \Delta_{ki} = 1)$, $k = 1, 2, 3$

$\mathbf{p} = (p_1, p_2, p_3)^T$

$\rho_i(\mathbf{p}) = \sum_{k=1}^{3} \Delta_{ki}\xi_i / p_k$

$dM_i(t) = dN_i(t) - Y_i(t)e^{\boldsymbol{\beta}^T\mathbf{Z}_i(0)}\lambda_0(t)dt$

$dM_i^C(t) = dN_i^C(t) - Y_i(t)e^{\boldsymbol{\alpha}^T\mathbf{V}_i(t)}\lambda_0^C(t)dt$

A.02 Matrices pertaining to Theorem 1

$$\mathbf{\Omega}(\boldsymbol{\alpha}) = E\left\{\boldsymbol{\psi}_i(\boldsymbol{\alpha},\mathbf{p})^{\otimes 2}\right\}$$

$$\boldsymbol{\psi}_i(\boldsymbol{\alpha},\mathbf{p}) = \mathbf{K}_i(\boldsymbol{\alpha},\mathbf{p}) + \mathbf{B}^C(\boldsymbol{\alpha},\mathbf{p})\mathbf{Q}_i(\mathbf{p})$$

$$\mathbf{K}_i(\boldsymbol{\alpha},\mathbf{p}) = \int_0^\tau \left\{\mathbf{V}_i(t) - \overline{\mathbf{v}}(\boldsymbol{\alpha},t)\right\}\rho_i(\mathbf{p})dM_i^C(t)$$

$$\mathbf{B}^C(\boldsymbol{\alpha},\mathbf{p}) = \int_0^\tau \left\{\frac{\mathbf{s}_c^{(1)}(\boldsymbol{\alpha},\mathbf{p},t)}{s_c^{(0)}(\boldsymbol{\alpha},\mathbf{p},t)^2}\mathbf{r}^{(0)}(\boldsymbol{\alpha},\mathbf{p},t) - \frac{\mathbf{r}^{(1)}(\boldsymbol{\alpha},\mathbf{p},t)}{s_c^{(0)}(\boldsymbol{\alpha},\mathbf{p},t)}\right\}dF^C(t) + \mathbf{d}(\boldsymbol{\alpha},\mathbf{p})$$

$$\mathbf{r}_k^{(d)}(\boldsymbol{\alpha},\mathbf{p},t) = -\frac{1}{p_k}E\left\{\Delta_{k1}Y_1(t)\mathbf{V}_1(t)^{\otimes d}e^{\boldsymbol{\alpha}^T\mathbf{V}_1(t)}\right\}, \quad d=0,1$$

$$\mathbf{r}^{(d)}(\boldsymbol{\alpha},\mathbf{p},t) = \left\{\mathbf{r}_1^{(d)}(\boldsymbol{\alpha},\mathbf{p},t) \quad \mathbf{r}_2^{(d)}(\boldsymbol{\alpha},\mathbf{p},t) \quad \mathbf{r}_3^{(d)}(\boldsymbol{\alpha},\mathbf{p},t)\right\}, \quad d=0,1,$$

where we further define

$$\mathbf{d}_k(\boldsymbol{\alpha},\mathbf{p}) = -\frac{1}{p_k}E\left[\int_0^\tau \left\{\mathbf{V}_1(t) - \overline{\mathbf{v}}(\boldsymbol{\alpha},t)\right\}\Delta_{k1}dN_1^C(t)\right]$$

$$\mathbf{d}(\boldsymbol{\alpha},\mathbf{p}) = \left\{\mathbf{d}_1(\boldsymbol{\alpha},\mathbf{p}) \quad \mathbf{d}_2(\boldsymbol{\alpha},\mathbf{p}) \quad \mathbf{d}_3(\boldsymbol{\alpha},\mathbf{p})\right\}$$

$$Q_{ki}(\mathbf{p}) = \eta_k^{-1}\Delta_{ki}(\xi_i - p_k)$$

$$\eta_k = \text{pr}(\Delta_k = 1), k = 1,2,3$$

$$\mathbf{Q}_i(\mathbf{p}) = \left\{Q_{1i}(\mathbf{p}) \quad Q_{2i}(\mathbf{p}) \quad Q_{3i}(\mathbf{p})\right\}^T,$$

with $dM_i^C(t) = dN_i^C(t) - Y_i(t)d\Lambda_i^C(t)$.

The following is a proof of Theorem 2, for the case where $\widehat{W}_{1i}$ is used in the proposed estimators. The proofs for stabilized weights, $\widehat{W}_{2i}$ and $\widehat{W}_{3i}$, proceed through steps analogous to those listed below. To begin, we list explicit definitions of the matrices referred to in the theorem.

A.02 Matrices pertaining to Theorem 2

$$
\begin{aligned}
\mathbf{A}(\boldsymbol{\beta}) &= \int_0^\tau \left\{ \frac{\mathbf{s}^{(2)}(\boldsymbol{\beta}, R, t)}{s^{(0)}(\boldsymbol{\beta}, R, t)} - \overline{\mathbf{z}}(\boldsymbol{\beta}, R, t)^{\otimes 2} \right\} dF(t) \\
\boldsymbol{\Sigma}(\boldsymbol{\beta}, R) &= E\left\{ \boldsymbol{\Theta}_i\left(\boldsymbol{\beta}, R\right)^{\otimes 2} \right\} \\
\boldsymbol{\Theta}_i\left(\boldsymbol{\beta}, R\right) &= \mathbf{O}(\boldsymbol{\beta}, R)\mathbf{Q}_i(\mathbf{p}_0) \\
&\quad + \mathbf{H}(\boldsymbol{\beta}, R)\mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1}\boldsymbol{\psi}_i(\boldsymbol{\alpha}_0, \mathbf{p}_0) \\
&\quad + \int_0^\tau \boldsymbol{\chi}(u, \tau) d\Phi_i(\boldsymbol{\alpha}_0, \mathbf{p}_0, u) \\
\mathbf{O}(\boldsymbol{\beta}, R) &= E\left[\int_0^\tau \left\{ \mathbf{Z}_i(0) - \overline{\mathbf{z}}(\boldsymbol{\beta}, R, t) \right\} \boldsymbol{\mu}_i(\mathbf{p}_0) W_{1i}(t) dM_i(t) \right] \\
\mu_{ki}(\mathbf{p}) &= \frac{d\rho_i(\mathbf{p})}{dp_k} = -\frac{\Delta_{ki}\xi_i}{p_k^2} \\
\boldsymbol{\mu}_i(\mathbf{p}) &= \left\{ \mu_{1i}(\mathbf{p}) \quad \mu_{2i}(\mathbf{p}) \quad \mu_{3i}(\mathbf{p}) \right\}^T \\
\mathbf{H}(\boldsymbol{\beta}, R) &= E\left[\int_0^\tau \left\{ \mathbf{Z}_i(0) - \overline{\mathbf{z}}(\boldsymbol{\beta}, R, t) \right\} \boldsymbol{\psi}_i^T(t) R_i(t) dM_i(t) \right] \\
\boldsymbol{\Psi}_i(t) &= \int_0^t \mathbf{V}_i(u) d\Lambda_i^C(u) \\
\boldsymbol{\chi}(t_1, t_2) &= E\left[e^{\boldsymbol{\alpha}_0^T \mathbf{V}_i(t_1)} \int_{t_1}^{t_2} \left\{ \mathbf{Z}_i(0) - \overline{\mathbf{z}}(\boldsymbol{\beta}, R, t) \right\} R_i(t) dM_i(t) \right]
\end{aligned}
$$

$$d\Phi_i(\boldsymbol{\alpha},\mathbf{p},u) = s_C^{(0)}(\boldsymbol{\alpha},u)^{-1}\left\{d\mathbf{J}(u) - \mathbf{r}^{(0)}(\boldsymbol{\alpha},\mathbf{p},u)d\Lambda_0^C(u)\right\}\mathbf{Q}_i(\mathbf{p})$$
$$-\overline{\mathbf{v}}^T(\boldsymbol{\alpha},u)d\Lambda_0^C(u)\mathbf{A}^C(\boldsymbol{\alpha})^{-1}\boldsymbol{\psi}_i(\boldsymbol{\alpha},\mathbf{p})$$
$$+s_C^{(0)}(\boldsymbol{\alpha},u)^{-1}\rho_i(\mathbf{p})dM_i^C(u)$$
$$d\mathbf{J}(u) = E\left\{\boldsymbol{\mu}_i(\mathbf{p}_0)^T dN_i^C(u)\right\},$$

A.1 $n^{-\frac{1}{2}}\boldsymbol{U}^C(\boldsymbol{\alpha}_0,\boldsymbol{p}_0)$

The estimating function for the dependent censoring model is

$$\mathbf{U}^C(\boldsymbol{\alpha}_0,\mathbf{p}_0) = \sum_{i=1}^n \int_0^\tau \left\{\mathbf{V}_i(t) - \overline{\mathbf{V}}(\boldsymbol{\alpha}_0,\mathbf{p}_0,t)\right\}\rho_i(\mathbf{p}_0)dN_i^C(t),$$

where

$$\overline{\mathbf{V}}(\boldsymbol{\alpha},\mathbf{p},t) = \frac{\mathbf{S}_C^{(1)}(\boldsymbol{\alpha},\mathbf{p},t)}{S_C^{(0)}(\boldsymbol{\alpha},\mathbf{p},t)}$$
$$\mathbf{S}_C^{(d)}(\boldsymbol{\alpha},\mathbf{p},t) = n^{-1}\sum_{i=1}^n \rho_i(\mathbf{p})Y_i(t)\mathbf{V}_i(t)^{\otimes d}e^{\boldsymbol{\alpha}^T\mathbf{V}_i(t)}.$$

Let $\mathbf{s}_C^{(d)}(t,\boldsymbol{\alpha}) = E\{Y_1(t)\mathbf{V}_1(t)^{\otimes d}e^{\boldsymbol{\alpha}^T\mathbf{V}_1(t)}\}$ and let $\overline{\mathbf{v}}(\boldsymbol{\alpha},t) = \mathbf{s}_C^{(1)}(t,\boldsymbol{\alpha})/s_C^{(0)}(t,\boldsymbol{\alpha})$. We define $dM_i^C(t) = dN_i^C(t) - Y_i(t)e^{\boldsymbol{\alpha}^T\mathbf{V}_i(t)}\lambda_0^C(t)dt$. By van der Vaart & Wellner (1996, Example 2.11.16), $H_n(t) = n^{-\frac{1}{2}}\sum_{i=1}^n \rho_i(\mathbf{p}_0)M_i^C(t)$ converges weakly to a tight Gaussian process $H(t)$ with continuous sample paths on $[0,\tau]$.

By some simple algebra, we have

$$
\begin{aligned}
n^{-\frac{1}{2}}\mathbf{U}^C(\boldsymbol{\alpha}_0,\mathbf{p}_0) &= n^{-\frac{1}{2}}\sum_{i=1}^n\int_0^\tau\{\mathbf{V}_i(t)-\overline{\mathbf{V}}(\boldsymbol{\alpha}_0,\mathbf{p}_0,t)\}\rho_i(\mathbf{p}_0)dM_i^C(t)\\
&= n^{-\frac{1}{2}}\sum_{i=1}^n\int_0^\tau\{\mathbf{V}_i(t)-\overline{\mathbf{v}}(\boldsymbol{\alpha}_0,t)\}\rho_i(\mathbf{p}_0)dM_i^C(t)\\
&\quad -n^{-\frac{1}{2}}\sum_{i=1}^n\int_0^\tau\left\{\overline{\mathbf{V}}(\boldsymbol{\alpha}_0,\mathbf{p}_0,t)-\overline{\mathbf{v}}(\boldsymbol{\alpha}_0,t)\right\}\rho_i(\mathbf{p}_0)dM_i^C(t)\\
&= n^{-\frac{1}{2}}\sum_{i=1}^n\int_0^\tau\left\{\mathbf{V}_i(t)-\overline{\mathbf{v}}(\boldsymbol{\alpha}_0,t)\right\}\rho_i(\mathbf{p}_0)dM_i^C(t)+o_p(1)\\
&= n^{-\frac{1}{2}}\sum_{i=1}^n\mathbf{K}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0)+o_p(1),
\end{aligned}
$$

with $\mathbf{K}_i(\boldsymbol{\alpha},\mathbf{p})=\int_0^\tau\{\mathbf{V}_i(t)-\overline{\mathbf{v}}(\boldsymbol{\alpha},t)\}\rho_i(\mathbf{p})dM_i^C(t)$. Note that $E\{\rho_i(\mathbf{p}_0)dM_i^C(t)\}=0$, such that $E\{\mathbf{K}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0)\}=0$, for $i=1,\cdots,n$.

A.2 $n^{1/2}(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0)$

Using a Taylor expansion, we can show that

$$
n^{-1/2}\mathbf{U}^C(\boldsymbol{\alpha}_0,\widehat{\mathbf{p}})=n^{-1/2}\mathbf{U}^C(\boldsymbol{\alpha}_0,\mathbf{p}_0)+\mathbf{B}_n^C(\boldsymbol{\alpha}_0,\mathbf{p}_*)n^{1/2}\left(\widehat{\mathbf{p}}-\mathbf{p}_0\right),
$$

where $\mathbf{p}_*$ is on the line segment between $\widehat{\mathbf{p}}$ and $\mathbf{p}_0$, and

$$
\begin{aligned}
\mathbf{B}_n^C(\boldsymbol{\alpha},\mathbf{p}) &= n^{-1}\frac{\partial}{\partial\mathbf{p}}\mathbf{U}^C(\boldsymbol{\alpha},\mathbf{p})\\
&= n^{-1}\sum_{i=1}^n\int_0^\tau\left\{\frac{\mathbf{S}_C^{(1)}(\boldsymbol{\alpha},\mathbf{p},t)}{S_C^{(0)}(\boldsymbol{\alpha},\mathbf{p},t)^2}\frac{\partial}{\partial\mathbf{p}}S_C^{(0)}(\boldsymbol{\alpha},\mathbf{p},t)\right.\\
&\quad\left.-\frac{1}{S_C^{(0)}(\boldsymbol{\alpha},\mathbf{p},t)}\frac{\partial}{\partial\mathbf{p}}\mathbf{S}_C^{(1)}(\boldsymbol{\alpha},\mathbf{p},t)\right\}\rho_i(\mathbf{p})dN_i^C(t)\\
&\quad+n^{-1}\sum_{i=1}^n\int_0^\tau\left\{\mathbf{V}_i(t)-\overline{\mathbf{V}}(\boldsymbol{\alpha},\mathbf{p},t)\right\}\frac{\partial}{\partial\mathbf{p}}\rho_i(\mathbf{p})dN_i^C(t).
\end{aligned}
$$

We define $\mathbf{R}_k^{(d)}(\boldsymbol{\alpha}, \mathbf{p}, t)$, $d = 0, 1$ and $\mathbf{D}_k(\boldsymbol{\alpha}, \mathbf{p})$ as follows,

$$\begin{aligned}\mathbf{R}_k^{(d)}(\boldsymbol{\alpha}, \mathbf{p}, t) &= \frac{\partial}{\partial p_k}\mathbf{S}_C^{(d)}(\boldsymbol{\alpha}, \mathbf{p}, t) \\ &= n^{-1}\sum_{i=1}^{n} -\frac{\Delta_{ki}\xi_i}{p_k^2} Y_i(t)\mathbf{V}_i(t)^{\otimes d} e^{\boldsymbol{\alpha}^T \mathbf{V}_i(t)}\end{aligned}$$

such that

$$\begin{aligned}\mathbf{R}_k^{(d)}(\boldsymbol{\alpha}, \mathbf{p}, t) &\longrightarrow -\frac{1}{p_k} E\{\Delta_{k1} Y_1(t)\mathbf{V}_1(t)^{\otimes d} e^{\boldsymbol{\alpha}^T \mathbf{V}_1(t)}\} \\ &\equiv \ \mathbf{r}_k^{(d)}(\boldsymbol{\alpha}, \mathbf{p}, t)\end{aligned}$$

in probability. We then have

$$\begin{aligned}\mathbf{D}_k(\boldsymbol{\alpha}, \mathbf{p}) &= n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\{\mathbf{V}_i(t) - \overline{\mathbf{V}}(\boldsymbol{\alpha}, \mathbf{p}, t)\}\frac{\partial}{\partial p_k}\rho_i(\mathbf{p}) dN_i^C(t) \\ &= -n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\{\mathbf{V}_i(t) - \overline{\mathbf{V}}(\boldsymbol{\alpha}, \mathbf{p}, t)\}\frac{\Delta_{ki}\xi_i}{p_k^2} dN_i^C(t) \\ &= -n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\{\mathbf{V}_i(t) - \overline{\mathbf{v}}(\boldsymbol{\alpha}, t)\}\frac{\Delta_{ki}\xi_i}{p_k^2} dN_i^C(t) \\ &\quad + \int_0^{\tau}\{\overline{V}(\boldsymbol{\alpha}, \mathbf{p}, t) - \overline{\mathbf{v}}(\boldsymbol{\alpha}, t)\} n^{-1}\sum_{i=1}^{n}\frac{\Delta_{ki}\xi_i}{p_k^2} dN_i^C(t)\end{aligned}$$

such that

$$\begin{aligned}\mathbf{D}_k(\boldsymbol{\alpha}, \mathbf{p}) &\longrightarrow -\frac{1}{p_k} E[\int_0^{\tau}\{\mathbf{V}_1(t) - \overline{\mathbf{v}}(\boldsymbol{\alpha}, t)\}\Delta_{k1} dN_1^C(t)] \\ &\equiv \ \mathbf{d}_k(\boldsymbol{\alpha}, \mathbf{p})\end{aligned}$$

in probability. Let $\mathbf{r}^{(d)}(\boldsymbol{\alpha}, \mathbf{p}, t) = \begin{bmatrix}\mathbf{r}_1^{(d)}(\boldsymbol{\alpha}, \mathbf{p}, t) & \mathbf{r}_2^{(d)}(\boldsymbol{\alpha}, \mathbf{p}, t) & \mathbf{r}_3^{(d)}(\boldsymbol{\alpha}, \mathbf{p}, t)\end{bmatrix}$ and let $\mathbf{d}(\boldsymbol{\alpha}, \mathbf{p}, t) = [\mathbf{d}_1(\boldsymbol{\alpha}, \mathbf{p}, t) \quad \mathbf{d}_2(\boldsymbol{\alpha}, \mathbf{p}, t) \quad \mathbf{d}_3(\boldsymbol{\alpha}, \mathbf{p}, t)]$. Then by continuous mapping,

$$\mathbf{B}_n^C(\boldsymbol{\alpha},\mathbf{p}) \longrightarrow \int_0^\tau \left\{ \frac{\mathbf{s}_c^{(1)}(\boldsymbol{\alpha},\mathbf{p},t)}{s_c^{(0)}(\boldsymbol{\alpha},\mathbf{p},t)^2}\mathbf{r}^{(0)}(\boldsymbol{\alpha},\mathbf{p},t) - \frac{1}{s_c^{(0)}(\boldsymbol{\alpha},\mathbf{p},t)}\mathbf{r}^{(1)}(\boldsymbol{\alpha},\mathbf{p},t) \right\} dF^C(t) + \mathbf{d}(\boldsymbol{\alpha},\mathbf{p}) \equiv \mathbf{B}^C(\boldsymbol{\alpha},\mathbf{p})$$

in probability, where $F^C(t) = E\{\rho_i(\mathbf{p})N_i^C(t)\}$. It is easy to show that

$$\begin{aligned} n^{1/2}(\widehat{p}_k - p_{k0}) &= n^{1/2}\left(\frac{\sum_{i=1}^n \Delta_{ki}\xi_i}{\sum_{i=1}^n \Delta_{ki}} - p_{k0}\right) \\ &= n^{1/2}\frac{n^{-1}\sum_{i=1}^n \Delta_{ki}(\xi_i - p_{k0})}{n^{-1}\sum_{i=1}^n \Delta_{ki}} \\ &= n^{-1/2}\sum_{i=1}^n \eta_k^{-1}\Delta_{ki}(\xi_i - p_{k0}) + o_p(1) \\ &= n^{-1/2}\sum_{i=1}^n Q_{ki}(p_0) + o_p(1), \end{aligned}$$

where $Q_{ki}(\mathbf{p}) = \eta_k^{-1}\Delta_{ki}(\xi_i - p_k)$, $\eta_k = \text{pr}(\Delta_k = 1), k = 1,2,3$. Let $Q_i(\mathbf{p}) = [Q_{1i}(\mathbf{p}) \quad Q_{2i}(\mathbf{p}) \quad Q_{3i}(\mathbf{p})]^T$. Note that $E\{Q_{ki}(p_0)\} = 0$, therefore,

$$\begin{aligned} n^{-1/2}\mathbf{U}^C(\boldsymbol{\alpha}_0,\widehat{\mathbf{p}}) &= n^{-1/2}\sum_{i=1}^n \left\{\mathbf{K}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0) + \mathbf{B}^C(\boldsymbol{\alpha}_0,\mathbf{p}_0)\mathbf{Q}_i(\mathbf{p}_0)\right\} + o_p(1) \\ &= n^{-1/2}\sum_{i=1}^n \boldsymbol{\psi}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0) + o_p(1), \end{aligned}$$

where $\boldsymbol{\psi}_i(\boldsymbol{\alpha},\mathbf{p}) = \mathbf{K}_i(\boldsymbol{\alpha},\mathbf{p}) + \mathbf{B}^C(\boldsymbol{\alpha},\mathbf{p})\mathbf{Q}_i(\mathbf{p})$. Since $E\{\boldsymbol{\psi}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0)\} = 0$, by the Multivariate Central Limit Theorem (MCLT),

$$n^{-1/2}\mathbf{U}^C(\boldsymbol{\alpha}_0,\widehat{\mathbf{p}}) \longrightarrow N(0,\boldsymbol{\Omega}(\boldsymbol{\alpha}_0))$$

in distribution, where $\boldsymbol{\Omega}(\boldsymbol{\alpha}_0) = E\{\boldsymbol{\psi}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0)^{\otimes 2}\}$.

We then have

$$n^{1/2}(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) = \mathbf{A}_n^C(\boldsymbol{\alpha}_*, \widehat{\mathbf{p}})^{-1} n^{-1/2} \mathbf{U}^C(\boldsymbol{\alpha}_0, \widehat{\mathbf{p}}),$$

where $\mathbf{A}_n^C(\boldsymbol{\alpha}, \mathbf{p}) = n^{-1} \sum_{i=1}^n \int_0^\tau \{\mathbf{S}_c^{(2)}(\boldsymbol{\alpha}, \mathbf{p}, t)/S_c^{(0)}(\boldsymbol{\alpha}, \mathbf{p}, t) - \overline{\mathbf{V}}(\boldsymbol{\alpha}, \mathbf{p}, t)^{\otimes 2}\}\rho_i(\mathbf{p}) dN_i^C(t)$ and $\boldsymbol{\alpha}_*$ is on the line segment between $\widehat{\boldsymbol{\alpha}}$ and $\boldsymbol{\alpha}_0$. Note the fact that $n^{-1} \sum_{i=1}^n \rho_i(\mathbf{p}) dN_i^C(t) \longrightarrow dF^C(t)$ in probability, such that $\mathbf{A}_n^C(\boldsymbol{\alpha}_*, \widehat{\mathbf{p}})$ converges in probability to $\mathbf{A}^C(\boldsymbol{\alpha}_0)$, with $\mathbf{A}^C(\boldsymbol{\alpha}) = \int_0^\tau \{\mathbf{s}_c^{(2)}(\boldsymbol{\alpha}, t)/s_c^{(0)}(\boldsymbol{\alpha}, t) - \overline{\mathbf{v}}(\boldsymbol{\alpha}, t)^{\otimes 2}\} dF^C(t)$. Therefore, by Slutsky's Theorem, $n^{1/2}(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0)$ converges in distribution to a $N(0, \mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1} \boldsymbol{\Omega}(\boldsymbol{\alpha}_0) \mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1})$ variate.

A.3 $n^{1/2}\{\widehat{\Lambda}_0^C(t) - \Lambda_0^C(t)\}$

We can decompose $n^{1/2}\{\widehat{\Lambda}_0^C(t) - \Lambda_0^C(t)\}$ as follows,

$$n^{1/2}\{\widehat{\Lambda}_0^C(t) - \Lambda_0^C(t)\} = n^{1/2}\{\widehat{\Lambda}_0^C(t; \widehat{\boldsymbol{\alpha}}, \widehat{\mathbf{p}}) - \widehat{\Lambda}_0^C(t; \widehat{\boldsymbol{\alpha}}, \mathbf{p}_0)\} \quad (1)$$

$$+ n^{1/2}\{\widehat{\Lambda}_0^C(t; \widehat{\boldsymbol{\alpha}}, \mathbf{p}_0) - \widehat{\Lambda}_0^C(t; \boldsymbol{\alpha}_0, \mathbf{p}_0)\} \quad (2)$$

$$+ n^{1/2}\{\widehat{\Lambda}_0^C(t; \boldsymbol{\alpha}_0, \mathbf{p}_0) - \Lambda_0^C(t)\}. \quad (3)$$

Applying a Taylor expansion of $\rho_i(\widehat{\mathbf{p}})/S_C^{(0)}(\widehat{\boldsymbol{\alpha}},\widehat{\mathbf{p}},t)$ around $\rho_i(\mathbf{p}_0)/S_C^{(0)}(\widehat{\boldsymbol{\alpha}},\mathbf{p}_0,t)$, we can write (1) as

$$
\begin{aligned}
(1) &= n^{1/2}\sum_{i=1}^n \int_0^t \left\{ \frac{\rho_i(\widehat{\mathbf{p}})}{S_C^{(0)}(\widehat{\boldsymbol{\alpha}},\widehat{\mathbf{p}},u)} - \frac{\rho_i(\mathbf{p}_0)}{S_C^{(0)}(\widehat{\boldsymbol{\alpha}},\mathbf{p}_0,u)} \right\} n^{-1} dN_i^C(u) \\
&= n^{-1}\sum_{i=1}^n \int_0^t \left\{ \frac{1}{S_C^{(0)}(\widehat{\boldsymbol{\alpha}},\mathbf{p}_0,u)} \frac{d\rho_i(\mathbf{p})}{d\mathbf{p}} \mid_{\mathbf{p}=\mathbf{p}_0} - \frac{\rho_i(\mathbf{p}_0)}{S_C^{(0)}(\widehat{\boldsymbol{\alpha}},\mathbf{p}_0,u)^2} \frac{\partial}{\partial \mathbf{p}} S_C^{(0)}(\widehat{\boldsymbol{\alpha}},\mathbf{p},u) \mid_{\mathbf{p}=\mathbf{p}_0} \right\} dN_i^C(u) \\
&\quad \times n^{1/2}\left(\widehat{\mathbf{p}}-\mathbf{p}_0\right) + o_p(1) \\
&= \int_0^t \left\{ \frac{n^{-1}\sum_{i=1}^n \boldsymbol{\mu}_i^T(\mathbf{p}_0) dN_i^C(u)}{s_C^{(0)}(\boldsymbol{\alpha}_0,u)} - \frac{r^{(0)}(\boldsymbol{\alpha}_0,\mathbf{p}_0,u)d\widehat{\Lambda}_0^C(u;\widehat{\boldsymbol{\alpha}},\mathbf{p}_0)}{s_C^{(0)}(\boldsymbol{\alpha}_0,u)} \right\} \\
&\quad \times n^{-1/2}\sum_{i=1}^n \mathbf{Q}_i(\mathbf{p}_0) + o_p(1) \\
&= \mathbf{L}(\boldsymbol{\alpha}_0,\mathbf{p}_0,t) n^{-1/2}\sum_{i=1}^n \mathbf{Q}_i(\mathbf{p}_0) + o_p(1),
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_{ki}(\mathbf{p}) &= \frac{d\rho_i(\mathbf{p})}{dp_k} = -\frac{\Delta_{ki}\xi_i}{p_k^2} \\
\boldsymbol{\mu}_i(\mathbf{p}) &= \left[\mu_{1i}(\mathbf{p}) \quad \mu_{2i}(\mathbf{p}) \quad \mu_{3i}(\mathbf{p})\right]^T \\
\mathbf{L}(\boldsymbol{\alpha}_0,\mathbf{p}_0,t) &= \int_0^t s_C^{(0)}(\boldsymbol{\alpha}_0,u)^{-1}\left\{ d\mathbf{J}(u) - \mathbf{r}^{(0)}(\boldsymbol{\alpha}_0,\mathbf{p}_0,u) d\Lambda_0^C(u) \right\},
\end{aligned}
$$

with $d\mathbf{J}_n(u) = n^{-1}\sum_{i=1}^n \boldsymbol{\mu}_i^T(\mathbf{p}_0) dN_i^C(u)$, which converges to $d\mathbf{J}(u)$ in probability.

Considering (2),

$$
\begin{aligned}
(2) &= n^{1/2}\sum_{i=1}^{n}\int_0^t \left\{\frac{1}{S_C^{(0)}(\widehat{\boldsymbol{\alpha}},\mathbf{p}_0,u)} - \frac{1}{S_C^{(0)}(\boldsymbol{\alpha}_0,\mathbf{p}_0,u)}\right\} n^{-1}\rho_i(\mathbf{p}_0)dN_i^C(u) \\
&= n^{-1}\left\{\sum_{i=1}^{n}\int_0^t -\frac{\mathbf{S}_C^{(1)}(\boldsymbol{\alpha}_0,\mathbf{p}_0,u)}{S_C^{(0)}(\boldsymbol{\alpha}_0,\mathbf{p}_0,u)^2}\rho_i(\mathbf{p}_0)dN_i^C(u)\right\}^T n^{1/2}(\widehat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}_0)+o_p(1) \\
&= \left\{-\int_0^t \overline{\mathbf{V}}(\boldsymbol{\alpha}_0,\mathbf{p}_0,u)d\widehat{\Lambda}_0^C(u;\boldsymbol{\alpha}_0,\mathbf{p}_0)\right\}^T \mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1}n^{-1/2}\sum_{i=1}^{n}\boldsymbol{\psi}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0)+o_p(1) \\
&= \widehat{\mathbf{h}}_C^T(t;\boldsymbol{\alpha}_0,\mathbf{p}_0)\mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1}n^{-1/2}\sum_{i=1}^{n}\boldsymbol{\psi}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0) \\
&= \mathbf{h}_C^T(t;\boldsymbol{\alpha}_0,\mathbf{p}_0)\mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1}n^{-1/2}\sum_{i=1}^{n}\boldsymbol{\psi}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0)+o_p(1),
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{\mathbf{h}}_C(t;\boldsymbol{\alpha}_0,\mathbf{p}_0) &= -\int_0^t \overline{\mathbf{V}}(\boldsymbol{\alpha}_0,\mathbf{p}_0,u)d\widehat{\Lambda}_0^C(u;\boldsymbol{\alpha}_0,\mathbf{p}_0) \\
\mathbf{h}_C(t;\boldsymbol{\alpha}_0,\mathbf{p}_0) &= -\int_0^t \overline{\mathbf{v}}(\boldsymbol{\alpha},u)d\Lambda_0^C(u).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
(3) &= n^{-1/2}\sum_{i=1}^{n}\int_0^t S_C^{(0)}(\boldsymbol{\alpha}_0,\mathbf{p}_0,u)^{-1}\rho_i(\mathbf{p}_0)dM_i^C(u) \\
&= n^{-1/2}\sum_{i=1}^{n}\int_0^t s_C^{(0)}(\boldsymbol{\alpha}_0,u)^{-1}\rho_i(\mathbf{p}_0)dM_i^C(u)+o_p(1).
\end{aligned}
$$

Combining the above results, one obtains

$$
\begin{aligned}
n^{1/2}\{\widehat{\Lambda}_0^C(t)-\Lambda_0^C(t)\} &= n^{-1/2}\sum_{i=1}^{n}\boldsymbol{\Phi}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0,t)+o_p(1) \\
&= n^{-1/2}\sum_{i=1}^{n}\int_0^t d\boldsymbol{\Phi}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0,u)+o_p(1)
\end{aligned}
$$

where

$$\Phi_i(\boldsymbol{\alpha}_0, \mathbf{p}_0, t) = \mathbf{L}(\boldsymbol{\alpha}_0, \mathbf{p}_0, t)\mathbf{Q}_i(\mathbf{p}_0) + \mathbf{h}_C^T(t; \boldsymbol{\alpha}_0, \mathbf{p}_0)\mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1}\boldsymbol{\psi}_i(\boldsymbol{\alpha}_0, \mathbf{p}_0)$$
$$+ \int_0^t s_C^{(0)}(\boldsymbol{\alpha}_0, u)^{-1}\rho_i(\mathbf{p}_0)dM_i^C(u)$$
$$d\Phi_i(\boldsymbol{\alpha}_0, \mathbf{p}_0, u) = s_C^{(0)}(\boldsymbol{\alpha}_0, u)^{-1}\left\{d\mathbf{J}(u) - \mathbf{r}^{(0)}(\boldsymbol{\alpha}_0, \mathbf{p}_0, u)d\Lambda_0^C(u)\right\}\mathbf{Q}_i(\mathbf{p}_0)$$
$$-\overline{\mathbf{v}}^T(\boldsymbol{\alpha}_0, u)d\Lambda_0^C(u)\mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1}\boldsymbol{\psi}_i(\boldsymbol{\alpha}_0, \mathbf{p}_0)$$
$$+s_C^{(0)}(\boldsymbol{\alpha}_0, u)^{-1}\rho_i(\mathbf{p}_0)dM_i^C(u).$$

Note that $E\{\mathbf{Q}_i(\mathbf{p}_0)\} = 0$, $E\{\boldsymbol{\psi}_i(\boldsymbol{\alpha}_0, \mathbf{p}_0)\} = 0$ and $E\{\rho_i(\mathbf{p}_0)dM_i^C(u)\} = 0$, such that

$E\{d\Phi_i(\boldsymbol{\alpha}_0, \mathbf{p}_0, u)\} = 0.$

A.4 $n^{1/2}\{\widehat{\Lambda}_i^C(t) - \Lambda_i^C(t)\}$

We can decompose $n^{1/2}\{\widehat{\Lambda}_i^C(t) - \Lambda_i^C(t)\}$ as follows,

$$n^{1/2}\{\widehat{\Lambda}_i^C(t) - \Lambda_i^C(t)\}$$
$$= n^{1/2}\{\int_0^t e^{\widehat{\boldsymbol{\alpha}}^T\mathbf{V}_i(u)}d\widehat{\Lambda}_0^C(u) - \int_0^t e^{\boldsymbol{\alpha}_0^T\mathbf{V}_i(u)}d\widehat{\Lambda}_0^C(u)\} \quad (4)$$
$$+n^{1/2}\{\int_0^t e^{\boldsymbol{\alpha}_0^T\mathbf{V}_i(u)}d\widehat{\Lambda}_0^C(u) - \int_0^t e^{\boldsymbol{\alpha}_0^T\mathbf{V}_i(u)}d\Lambda_0^C(u)\}. \quad (5)$$

By a Taylor expansion,

$$\begin{aligned}(4) &= n^{1/2}\int_0^t \left\{e^{\widehat{\boldsymbol{\alpha}}^T \mathbf{V}_i(u)} - e^{\boldsymbol{\alpha}_0^T \mathbf{V}_i(u)}\right\} d\widehat{\Lambda}_0^C(u) \\ &= \int_0^t \mathbf{V}_i^T(u) e^{\boldsymbol{\alpha}_0^T \mathbf{V}_i(u)} d\widehat{\Lambda}_0^C(u) n^{1/2}(\widehat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) + o_p(1) \\ &= \int_0^t \mathbf{V}_i^T(u) d\Lambda_i^C(u) \mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1} n^{-1/2} \sum_{l=1}^n \boldsymbol{\psi}_l(\boldsymbol{\alpha}_0, \mathbf{p}_0) + o_p(1).\end{aligned}$$

Now considering the second term (5),

$$\begin{aligned}(5) &= n^{1/2}\int_0^t e^{\boldsymbol{\alpha}_0^T \mathbf{V}_i(u)} d\{\widehat{\Lambda}_0^C(u) - \Lambda_0^C(u)\} \\ &= \int_0^t e^{\boldsymbol{\alpha}_0^T \mathbf{V}_i(u)} n^{-1/2} \sum_{l=1}^n d\boldsymbol{\Phi}_l(\boldsymbol{\alpha}_0, \mathbf{p}_0, u) + o_p(1).\end{aligned}$$

It follows that

$$\begin{aligned}n^{1/2}\{\widehat{\Lambda}_i^C(t) - \Lambda_i^C(t)\} &= \int_0^t \mathbf{V}_i^T(u) d\Lambda_i^C(u) \mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1} n^{-1/2} \sum_{l=1}^n \boldsymbol{\psi}_l(\boldsymbol{\alpha}_0, \mathbf{p}_0) \\ &\quad + \int_0^t e^{\boldsymbol{\alpha}_0^T \mathbf{V}_i(u)} n^{-1/2} \sum_{l=1}^n d\boldsymbol{\Phi}_l(\boldsymbol{\alpha}_0, \mathbf{p}_0, u) + o_p(1) \\ &= n^{-1/2} \sum_{l=1}^n \mathbf{G}_l(t) + o_p(1),\end{aligned}$$

where

$$\mathbf{G}_l(t) = \boldsymbol{\Psi}_i^T(t) \mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1} \boldsymbol{\psi}_l(\boldsymbol{\alpha}_0, \mathbf{p}_0) + \int_0^t e^{\boldsymbol{\alpha}_0^T \mathbf{V}_i(u)} d\boldsymbol{\Phi}_l(\boldsymbol{\alpha}_0, \mathbf{p}_0, u)$$

$$\boldsymbol{\Psi}_i(t) = \int_0^t \mathbf{V}_i(u) d\Lambda_i^C(u)$$

A.5 $n^{1/2}\{\widehat{R}_i(t) - R_i(t)\}$

Letting $R_i(t) = \rho_i(\mathbf{p})W_{1i}(t)$, we have

$$\begin{aligned}
n^{1/2}\{\widehat{R}_i(t) - R_i(t)\} &= n^{1/2}\{\rho_i(\widehat{\mathbf{p}})e^{\widehat{\Lambda}_i^C(t)} - \rho_i(\mathbf{p}_0)e^{\widehat{\Lambda}_i^C(t)}\} \\
&\quad + n^{1/2}\{\rho_i(\mathbf{p}_0)e^{\widehat{\Lambda}_i^C(t)} - \rho_i(\mathbf{p}_0)e^{\Lambda_i^C(t)}\} \\
&= \boldsymbol{\mu}_i(\mathbf{p}_0)^T e^{\Lambda_i^C(t)} n^{1/2}(\widehat{\mathbf{p}} - \mathbf{p}_0) \\
&\quad + \rho_i(\mathbf{p}_0)e^{\Lambda_i^C(t)} n^{1/2}\{\widehat{\Lambda}_i^C(t) - \Lambda_i^C(t)\} + o_p(1) \\
&= \boldsymbol{\mu}_i(\mathbf{p}_0)^T W_{1i}(t) n^{-1/2}\sum_{l=1}^n \mathbf{Q}_l(\mathbf{p}_0) + R_i(t)n^{-1/2}\sum_{l=1}^n \mathbf{G}_l(t) + o_p(1).
\end{aligned}$$

A.6 $n^{1/2}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0)$

It is easy to show that

$$n^{1/2}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0) = \mathbf{A}_n(\boldsymbol{\beta}_0)^{-1} n^{-1/2}\sum_{i=1}^n \mathbf{U}_i\{\boldsymbol{\beta}_0, \widehat{R}_i(t)\} + o_p(1),$$

where

$$\begin{aligned}
\mathbf{U}_i\{\boldsymbol{\beta}, R\} &= \int_0^\tau \left\{\mathbf{Z}_i(0) - \overline{\mathbf{Z}}(\boldsymbol{\beta}, R, t)\right\} R_i(t)dN_i(t) \\
\overline{\mathbf{Z}}(\boldsymbol{\beta}, R, t) &= \frac{\mathbf{S}^{(1)}(\boldsymbol{\beta}, R, t)}{S^{(0)}(\boldsymbol{\beta}, R, t)} \\
\mathbf{S}^{(d)}(\boldsymbol{\beta}, R, t) &= n^{-1}\sum_{i=1}^n R_i(t)Y_i(t)\mathbf{Z}_i(0)^{\otimes d} e^{\boldsymbol{\beta}^T \mathbf{Z}_i(0)}, d = 0, 1, 2 \\
\mathbf{s}^{(d)}(\boldsymbol{\beta}, R, t) &= E\left\{R_i(t)Y_i(t)\mathbf{Z}_i(0)^{\otimes d} e^{\boldsymbol{\beta}^T \mathbf{Z}_i(0)}\right\} \\
\overline{\mathbf{z}}(\boldsymbol{\beta}, R, t) &= \mathbf{s}^{(1)}(\boldsymbol{\beta}, R, t)/s^{(0)}(\boldsymbol{\beta}, R, t).
\end{aligned}$$

We then write

$$\begin{aligned}\mathbf{A}_n(\boldsymbol{\beta}) &= n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\left\{\frac{\mathbf{S}^{(2)}(\boldsymbol{\beta},R,t)}{S^{(0)}(\boldsymbol{\beta},R,t)}-\overline{\mathbf{Z}}(\boldsymbol{\beta},R,t)^{\otimes 2}\right\}R_i(t)dN_i(t)\\ &\longrightarrow \int_0^{\tau}\left\{\frac{\mathbf{s}^{(2)}(\boldsymbol{\beta},R,t)}{s^{(0)}(\boldsymbol{\beta},R,t)}-\overline{\mathbf{z}}(\boldsymbol{\beta},R,t)^{\otimes 2}\right\}dF(t)\\ &\equiv \mathbf{A}(\boldsymbol{\beta})\end{aligned}$$

in probability, with $n^{-1}\sum_{i=1}^{n}R_i(t)dN_i(t)$ converging in probability to $dF(t)$.

We can decompose $n^{-1/2}\mathbf{U}(\boldsymbol{\beta},\widehat{R})$ as follows,

$$\begin{aligned}& n^{-1/2}\mathbf{U}\left(\boldsymbol{\beta}_0,\widehat{R}\right)\\ &= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{Z}}(\boldsymbol{\beta},\widehat{R},t)\right\}\widehat{R}_i(t)dM_i(t)\\ &= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{Z}}(\boldsymbol{\beta},R,t)\right\}R_i(t)dM_i(t) \qquad (6)\\ &\quad +n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{Z}}(\boldsymbol{\beta},R,t)\right\}\left\{\widehat{R}_i(t)-R_i(t)\right\}dM_i(t) \qquad (7)\\ &\quad -n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\left\{\overline{\mathbf{Z}}(\boldsymbol{\beta},\widehat{R},t)-\overline{\mathbf{Z}}(\boldsymbol{\beta},R,t)\right\}\widehat{R}_i(t)dM_i(t) \qquad (8)\end{aligned}$$

It is easy to show that $(6) = n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{Z}}(\boldsymbol{\beta},R,t)\right\}R_i(t)dM_i(t)+o_p(1)$. The third term (8) converges in probability to 0. We can express (7) as follows,

$$
\begin{aligned}
(7) &= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{Z}}(\boldsymbol{\beta},R,t)\right\} \\
&\quad \left\{\boldsymbol{\mu}_i(\mathbf{p}_0)^T W_{1i}(t) n^{-1}\sum_{l=1}^{n}\mathbf{Q}_l(\mathbf{p}_0) + R_i(t) n^{-1}\sum_{l=1}^{n}\mathbf{G}_l(t)\right\} dM_i(t) + o_p(1) \\
&= n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{Z}}(\boldsymbol{\beta},R,t)\right\}\boldsymbol{\mu}_i(\mathbf{p}_0)^T W_{1i}(t) dM_i(t) n^{-1/2}\sum_{l=1}^{n}\mathbf{Q}_l(\mathbf{p}_0) \quad (9) \\
&\quad + n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{Z}}(\boldsymbol{\beta},R,t)\right\} R_i(t) \\
&\quad \times n^{-1}\left\{\sum_{l=1}^{n}\boldsymbol{\Psi}_i^T(t)\mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1}\boldsymbol{\psi}_l(\boldsymbol{\alpha}_0,\mathbf{p}_0)\right\} dM_i(t) \quad (10) \\
&\quad + n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{Z}}(\boldsymbol{\beta},R,t)\right\} R_i(t) \\
&\quad \times n^{-1}\sum_{l=1}^{n}\int_0^{t} e^{\boldsymbol{\alpha}_0^T\mathbf{V}_i(u)} d\boldsymbol{\Phi}_l(\boldsymbol{\alpha}_0,\mathbf{p}_0,u) dM_i(t). \quad (11)
\end{aligned}
$$

We can show that

$$
\begin{aligned}
(9) &= \widehat{\mathbf{O}}(\boldsymbol{\beta},R) n^{-1/2}\sum_{i=1}^{n}\mathbf{Q}_i(\mathbf{p}_0) \\
&= \mathbf{O}(\boldsymbol{\beta},R) n^{-1/2}\sum_{i=1}^{n}\mathbf{Q}_i(\mathbf{p}_0) + o_p(1),
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{\mathbf{O}}(\boldsymbol{\beta},R) &= n^{-1}\sum_{i=1}^{n}\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{Z}}(\boldsymbol{\beta},R,t)\right\}\boldsymbol{\mu}_i(\mathbf{p}_0)^T W_{1i}(t) dM_i(t) \\
\mathbf{O}(\boldsymbol{\beta},R) &= E\left[\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{z}}(\boldsymbol{\beta},R,t)\right\}\boldsymbol{\mu}_i(\mathbf{p}_0)^T W_{1i}(t) dM_i(t)\right].
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
(10) = n^{-1} &\left[\sum_{i=1}^{n} \int_{0}^{\tau} \left\{\mathbf{Z}_i(0) - \overline{\mathbf{Z}}(\boldsymbol{\beta}, R, t)\right\} R_i(t)\boldsymbol{\Psi}_i^T(t) dM_i(t)\right] \\
&\mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1} n^{-1/2} \sum_{l=1}^{n} \boldsymbol{\psi}_l(\boldsymbol{\alpha}_0, \mathbf{p}_0) \\
&= \widehat{\mathbf{H}}(\boldsymbol{\beta}, R)\mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1} n^{-1/2} \sum_{l=1}^{n} \boldsymbol{\psi}_l(\boldsymbol{\alpha}_0, \mathbf{p}_0) \\
&= \mathbf{H}(\boldsymbol{\beta}, R)\mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1} n^{-1/2} \sum_{l=1}^{n} \boldsymbol{\psi}_l(\boldsymbol{\alpha}_0, \mathbf{p}_0) + o_p(1),
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{\mathbf{H}}(\boldsymbol{\beta}, R) &= n^{-1} \left[\sum_{i=1}^{n} \int_{0}^{\tau} \left\{\mathbf{Z}_i(0) - \overline{\mathbf{Z}}(\boldsymbol{\beta}, R, t)\right\} \boldsymbol{\Psi}_i^T(t) R_i(t) dM_i(t)\right] \\
\mathbf{H}(\boldsymbol{\beta}, R) &= E\left[\int_{0}^{\tau} \left\{\mathbf{Z}_i(0) - \overline{\mathbf{z}}(\boldsymbol{\beta}, R, t)\right\} \boldsymbol{\Psi}_i^T(t) R_i(t) dM_i(t)\right].
\end{aligned}
$$

Changing the orders of integration and summation,

$$
\begin{aligned}
(11) &= n^{-1/2} \sum_{l=1}^{n} \int_{0}^{\tau} \left[n^{-1} \sum_{i=1}^{n} e^{\boldsymbol{\alpha}_0^T \mathbf{V}_i(u)} \int_{u}^{\tau} \left\{\mathbf{Z}_i(0) - \overline{\mathbf{Z}}(\boldsymbol{\beta}, R, t)\right\} R_i(t) dM_i(t)\right] d\boldsymbol{\Phi}_l(\boldsymbol{\alpha}_0, \mathbf{p}_0, u) \\
&= n^{-1/2} \sum_{l=1}^{n} \int_{0}^{\tau} \widehat{\boldsymbol{\chi}}(u, \tau) d\boldsymbol{\Phi}_l(\boldsymbol{\alpha}_0, \mathbf{p}_0, u) \\
&= n^{-1/2} \sum_{l=1}^{n} \int_{0}^{\tau} \boldsymbol{\chi}(u, \tau) d\boldsymbol{\Phi}_l(\boldsymbol{\alpha}_0, \mathbf{p}_0, u) + o_p(1),
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{\boldsymbol{\chi}}(t_1, t_2) &= n^{-1} \sum_{i=1}^{n} e^{\boldsymbol{\alpha}_0^T \mathbf{V}_i(t_1)} \int_{t_1}^{t_2} \left\{\mathbf{Z}_i(0) - \overline{\mathbf{Z}}(\boldsymbol{\beta}, R, t)\right\} R_i(t) dM_i(t) \\
\boldsymbol{\chi}(t_1, t_2) &= E\left[e^{\boldsymbol{\alpha}_0^T \mathbf{V}_i(t_1)} \int_{t_1}^{t_2} \left\{\mathbf{Z}_i(0) - \overline{\mathbf{z}}(\boldsymbol{\beta}, R, t)\right\} R_i(t) dM_i(t)\right].
\end{aligned}
$$

Combining the above results,

$$n^{1/2}\left(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0\right)=\mathbf{A}(\boldsymbol{\beta}_0)^{-1}n^{-1/2}\sum_{i=1}^{n}\boldsymbol{\Theta}_i\left(\boldsymbol{\beta}_0,R\right)+o_p(1),$$

where

$$\begin{aligned}\boldsymbol{\Theta}_i\left(\boldsymbol{\beta},R\right)=\;&\mathbf{O}(\boldsymbol{\beta},R)\mathbf{Q}_i(\mathbf{p}_0)\\&+\mathbf{H}(\boldsymbol{\beta},R)\mathbf{A}^C(\boldsymbol{\alpha}_0)^{-1}\boldsymbol{\psi}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0)\\&+\int_0^{\tau}\boldsymbol{\chi}(u,\tau)d\boldsymbol{\Phi}_i(\boldsymbol{\alpha}_0,\mathbf{p}_0,u).\end{aligned}$$

Note that $E\{\boldsymbol{\Theta}_i(\boldsymbol{\beta},R)\}=0$. By the MCLT and Slutsky's Theorem, $n^{1/2}(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}_0)$ converges in distribution to a $N(0,\mathbf{A}(\boldsymbol{\beta}_0)^{-1}\boldsymbol{\Sigma}(\boldsymbol{\beta},R)\mathbf{A}(\boldsymbol{\beta}_0)^{-1})$ variate, where $\boldsymbol{\Sigma}(\boldsymbol{\beta},R)=E\{\boldsymbol{\Theta}_i(\boldsymbol{\beta},R)^{\otimes 2}\}$.

A.7 Estimating var($\widehat{\boldsymbol{\beta}}$)

The variance of $\widehat{\boldsymbol{\beta}}$ can be consistently estimated by $n^{-1}\sum_{i=1}^{n}\widehat{\boldsymbol{\Theta}}_i\left(\boldsymbol{\beta},R\right)$, with $\widehat{\boldsymbol{\Theta}}_i\left(\boldsymbol{\beta},R\right)$ being obtained by substituting limiting values in $\boldsymbol{\Theta}_i\left(\boldsymbol{\beta},R\right)$ with the sample analogs. However, as shown in the Web Appendix, the computation of $\widehat{\boldsymbol{\Theta}}_i\left(\boldsymbol{\beta},R\right)$ is very complicated and difficult to implement numerically. A useful alternative is to estimate the variance of the proposed estimators by treating the weights $R_i(t)$ as known rather than estimated.

By some simple algebra, we have

$$
\begin{aligned}
n^{-1/2}\mathbf{U}\left(\boldsymbol{\beta}_0, R\right) &= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{Z}}(\boldsymbol{\beta}, R, t)\right\}R_i(t)dN_i(t)\\
&= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{Z}}(\boldsymbol{\beta}, R, t)\right\}R_i(t)dM_i(t)\\
&= n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{z}}(\boldsymbol{\beta}, R, t)\right\}R_i(t)dM_i(t) \qquad (12)\\
&\quad -n^{-1/2}\sum_{i=1}^{n}\int_0^{\tau}\left\{\overline{\mathbf{Z}}(\boldsymbol{\beta}, R, t)-\overline{\mathbf{z}}(\boldsymbol{\beta}, R, t)\right\}R_i(t)dM_i(t) \qquad (13)
\end{aligned}
$$

Note that $E\left\{W_{1i}(t)dM_i(t) \mid \mathbf{Z}_i(0)\right\}=0$, such that

$$
\begin{aligned}
E\left\{R_i(t)dM_i(t) \mid \mathbf{Z}_i(0)\right\} &= E\left\{\rho_i(\mathbf{p})W_{1i}(t)dM_i(t) \mid \mathbf{Z}_i(0)\right\}\\
&= E\left[E\left\{\rho_i(\mathbf{p})W_{1i}(t)dM_i(t) \mid \Delta_{1i}, \Delta_{2i}, \Delta_{3i}, \mathbf{Z}_i(t)\right\} \mid \mathbf{Z}_i(0)\right]\\
&= E\left\{W_{1i}(t)dM_i(t) \mid \mathbf{Z}_i(0)\right\}\\
&= 0.
\end{aligned}
$$

Therefore, (13) converges in probability to 0. It follows that

$$
n^{-1/2}\mathbf{U}\left(\boldsymbol{\beta}_0, R\right) = n^{-1/2}\sum_{i=1}^{n}\mathbf{U}_i^{\ddagger}(\boldsymbol{\beta}_0, R) + o_p(1),
$$

where $\mathbf{U}_i^{\ddagger}(\boldsymbol{\beta}_0, R)=\int_0^{\tau}\left\{\mathbf{Z}_i(0)-\overline{\mathbf{z}}(\boldsymbol{\beta}, R, t)\right\}R_i(t)dM_i(t)$. Hence, under the assumed conditions, $\{\mathbf{U}(\boldsymbol{\beta}_0, R)\}$ is asymptotically a sum of independent and identically distributed zero-mean random quantities. By the MCLT, $n^{-1/2}\mathbf{U}\left(\boldsymbol{\beta}_0, R\right)$ converges asymptotically to a $N(0, \boldsymbol{\Sigma}^{\ddagger}(\boldsymbol{\beta}_0, R))$ distribution, where $\boldsymbol{\Sigma}^{\ddagger}(\boldsymbol{\beta}, R) =$

$E\left\{\mathbf{U}_i^{\ddagger}(\boldsymbol{\beta}, R)^{\otimes 2}\right\}$. By the Functional Delta methods,

$$n^{1/2}\left(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}_0\right) = \mathbf{A}(\boldsymbol{\beta}_0)^{-1} n^{-1/2} \sum_{i=1}^{n} \mathbf{U}_i^{\ddagger}\left\{\boldsymbol{\beta}_0, R\right\} + o_p(1),$$

Therefore, the variance of $\widehat{\boldsymbol{\beta}}$ is estimated by $\widehat{\mathbf{A}}(\widehat{\boldsymbol{\beta}})^{-1}\widehat{\boldsymbol{\Sigma}}^{\ddagger}(\widehat{\boldsymbol{\beta}}, \widehat{R})\widehat{\mathbf{A}}(\widehat{\boldsymbol{\beta}})^{-1}$, where $\widehat{\mathbf{A}}(\widehat{\boldsymbol{\beta}})$ and $\widehat{\boldsymbol{\Sigma}}^{\ddagger}(\widehat{\boldsymbol{\beta}}, \widehat{R})$ are calculated by replacing limiting values with their corresponding empirical counterparts.

We now provide a proof of the large-sample distribution of the cumulative baseline hazard estimator.

A.8 Proof of Theorem 3

We can decompose $n^{1/2}\{\hat{\Lambda}_0(t) - \Lambda_0(t)\}$ as follows,

$$\begin{aligned} n^{1/2}\{\hat{\Lambda}_0(t) - \Lambda_0(t)\} =& n^{1/2}\{\hat{\Lambda}_0(t; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{p}}) - \hat{\Lambda}_0(t; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \boldsymbol{p_0})\} && (14) \\ &+ n^{1/2}\{\hat{\Lambda}_0(t; \hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \boldsymbol{p_0}) - \hat{\Lambda}_0(t; \hat{\boldsymbol{\alpha}}, \boldsymbol{\beta_0}, \boldsymbol{p_0})\} && (15) \\ &+ n^{1/2}\{\hat{\Lambda}_0(t; \hat{\boldsymbol{\alpha}}, \boldsymbol{\beta_0}, \boldsymbol{p_0}) - \hat{\Lambda}_0(t; \boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0})\} && (16) \\ &+ n^{1/2}\{\hat{\Lambda}_0(t; \boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}) - \Lambda_0(t)\} && (17) \end{aligned}$$

Applying a Taylor expansion of $\hat{R}_i(t;\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{p}})/S^{(0)}(\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}},\hat{\boldsymbol{p}},t)$ around $\hat{R}_i(t;\hat{\boldsymbol{\alpha}},\boldsymbol{p_0})/S^{(0)}(\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}},\boldsymbol{p_0},t)$, we can write (1) as

$$(1) = n^{1/2}\sum_{i=1}^{n}\int_0^t\{\frac{\hat{R}_i(u;\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{p}})}{S^{(0)}(\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}},\hat{\boldsymbol{p}},u)}-\frac{\hat{R}_i(u;\hat{\boldsymbol{\alpha}},\boldsymbol{p_0})}{S^{(0)}(\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}},\boldsymbol{p_0},u)}\}n^{-1}dN_i(u)$$

$$= n^{-1}\sum_{i=1}^{n}\int_0^t\{\frac{1}{S^{(0)}(\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}},\boldsymbol{p_0},u)}\frac{dR_i(u;\hat{\boldsymbol{\alpha}},\boldsymbol{p})}{d\boldsymbol{p}}|_{\boldsymbol{p}=\boldsymbol{p_0}}-\frac{R_i(u;\hat{\boldsymbol{\alpha}},\boldsymbol{p_0})}{S^{(0)}(\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}},\boldsymbol{p_0},u)^2}\frac{\partial}{\partial\boldsymbol{p}}S^{(0)}(\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}},\boldsymbol{p},u)|_{\boldsymbol{p}=\boldsymbol{p}}$$

$$\times n^{1/2}(\hat{\boldsymbol{p}}-\boldsymbol{p_0})+o_p(1)$$

where

$$\frac{dR_i(t;\hat{\boldsymbol{\alpha}},\boldsymbol{p})}{d\boldsymbol{p}} = \frac{d\rho_i(\boldsymbol{p})}{d\boldsymbol{p}}e^{\hat{\Lambda}_i^C(t;\hat{\boldsymbol{\alpha}},\boldsymbol{p})}+\rho_i(\boldsymbol{p})e^{\hat{\Lambda}_i^C(t;\hat{\boldsymbol{\alpha}},\boldsymbol{p})}\frac{d\hat{\Lambda}_i^C(t;\hat{\boldsymbol{\alpha}},\boldsymbol{p})}{d\boldsymbol{p}}$$

and

$$\frac{d\hat{\Lambda}_i^C(t;\hat{\boldsymbol{\alpha}},\boldsymbol{p})}{d\boldsymbol{p}} = \int_0^t e^{\hat{\boldsymbol{\alpha}}^T\boldsymbol{V}_i(u)}n^{-1}\sum_{j=1}^{n}\{\frac{\boldsymbol{\mu}_j^T(\boldsymbol{p})}{S_C^{(0)}(\hat{\boldsymbol{\alpha}},\boldsymbol{p},u)}-\frac{\rho_j(\boldsymbol{p})}{S_C^{(0)}(\hat{\boldsymbol{\alpha}},\boldsymbol{p},u)^2}\frac{\partial}{\partial\boldsymbol{p}}S_C^{(0)}(\hat{\boldsymbol{\alpha}},\boldsymbol{p},u)\}dN_j^C(u)$$

and

$$\frac{\partial}{\partial\boldsymbol{p}}S^{(0)}(\hat{\boldsymbol{\alpha}},\hat{\boldsymbol{\beta}},\boldsymbol{p},t)|_{\boldsymbol{p}=\boldsymbol{p_0}} = n^{-1}\sum_{i=1}^{n}\frac{\partial R_i(t;\hat{\boldsymbol{\alpha}},\boldsymbol{p})}{\partial\boldsymbol{p}}Y_i(t)e^{\hat{\boldsymbol{\beta}}^T\boldsymbol{Z}_i(0)}|_{\boldsymbol{p}=\boldsymbol{p_0}}\to\boldsymbol{r}_{\boldsymbol{p}}^{*(0)}(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},t)$$

Thus

$$(1) \equiv \boldsymbol{L}_1^*(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},t)n^{-1/2}\sum_{i=1}^{n}\boldsymbol{Q}_i(\boldsymbol{p_0})+o_p(1)$$

where

$$\boldsymbol{L}_{n1}^{*}(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, t) = n^{-1} \sum_{i=1}^{n} \int_0^t s^{(0)}(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, u)^{-1} \{ \boldsymbol{\mu}_i^T(\boldsymbol{p_0}) e^{\Lambda_i(u)} + \rho_i(\boldsymbol{p_0}) e^{\Lambda_i^C(u)} \int_0^u e^{\hat{\boldsymbol{\alpha}}^T \boldsymbol{V}_i(s)} s_C^{(}$$

$$\times [d\boldsymbol{J}(s) - r^{(0)}(\boldsymbol{\alpha_0}, \boldsymbol{p_0}, s) d\Lambda_0^C(s)] \} - \int_0^t s^{(0)}(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, u)^{-1} \boldsymbol{r}_{\boldsymbol{p}}^{*(0)}(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0},$$

$$\rightarrow \boldsymbol{L}_1^{*}(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, t)$$

Considering (2),

$$\begin{aligned}
(2) =& n^{1/2} \sum_{i=1}^{n} \int_0^t \{ \frac{1}{S^{(0)}(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, \boldsymbol{p_0}, u)} - \frac{1}{S^{(0)}(\hat{\boldsymbol{\alpha}}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, u)} \} n^{-1} \hat{R}_i(u; \hat{\boldsymbol{\alpha}}, \boldsymbol{p_0}) dN_i(u) \\
=& n^{-1} \{ \sum_{i=1}^{n} \int_0^t - \frac{\boldsymbol{S}^{(1)}(\hat{\boldsymbol{\alpha}}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, u)}{S^{(0)}(\hat{\boldsymbol{\alpha}}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, u)^2} \hat{R}_i(u; \hat{\boldsymbol{\alpha}}, \boldsymbol{p_0}) dN_i(u) \}^T n^{1/2} (\hat{\boldsymbol{\beta}} - \boldsymbol{\beta_0}) + o_p(1) \\
=& \{ - \int_0^t \overline{\boldsymbol{V}}^{*}(\hat{\boldsymbol{\alpha}}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, u) d\hat{\Lambda}_0(u; \hat{\boldsymbol{\alpha}}, \boldsymbol{\beta_0}, \boldsymbol{p_0}) \}^T \boldsymbol{A}(\boldsymbol{\beta_0})^{-1} n^{-1/2} \sum_{i=1}^{n} \boldsymbol{\Theta}_i(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}) + o_p(1) \\
=& \{ - \int_0^t \overline{\boldsymbol{v}}^{*}(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, u) d\Lambda_0(u) \}^T \boldsymbol{A}(\boldsymbol{\beta_0})^{-1} n^{-1/2} \sum_{i=1}^{n} \boldsymbol{\Theta}_i(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}) + o_p(1) \\
\equiv& \boldsymbol{L}_2^{*}(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, t) n^{-1/2} \sum_{i=1}^{n} \boldsymbol{\Theta}_i(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}) + o_p(1)
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{V}^{*}(\hat{\boldsymbol{\alpha}}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, t) =& \frac{\boldsymbol{S}^{(1)}(\hat{\boldsymbol{\alpha}}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, t)}{S^{(0)}(\hat{\boldsymbol{\alpha}}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, t)} \\
\boldsymbol{v}^{*}(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, t) =& \frac{\boldsymbol{s}^{(1)}(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, t)}{s^{(0)}(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, t)} \\
\boldsymbol{L}_2^{*}(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, t) =& \{ - \int_0^t \overline{\boldsymbol{v}}^{*}(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, u) d\Lambda_0(u) \}^T \boldsymbol{A}(\boldsymbol{\beta_0})^{-1}
\end{aligned}$$

Moreover,

$$
\begin{aligned}
(3) =& n^{1/2}\sum_{i=1}^{n}\int_0^t\{\frac{\hat{R}_i(u;\hat{\boldsymbol{\alpha}},\boldsymbol{p_0})}{S^{(0)}(\hat{\boldsymbol{\alpha}},\boldsymbol{\beta_0},\boldsymbol{p_0},u)}-\frac{R_i(u;\boldsymbol{\alpha_0},\boldsymbol{p_0})}{S^{(0)}(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},u)}\}n^{-1}dN_i(u)\\
=& n^{-1}\sum_{i=1}^{n}\int_0^t\{\frac{1}{S^{(0)}(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},u)}\frac{dR_i(u;\boldsymbol{\alpha},\boldsymbol{p_0})}{d\boldsymbol{\alpha}}|_{\boldsymbol{\alpha}=\boldsymbol{\alpha_0}}-\frac{R_i(u;\boldsymbol{\alpha_0},\boldsymbol{p_0})}{S^{(0)}(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},u)^2}\\
&\times\frac{\partial}{\partial\boldsymbol{\alpha}}S^{(0)}(\boldsymbol{\alpha},\boldsymbol{\beta_0},\boldsymbol{p_0},u)|_{\boldsymbol{\alpha}=\boldsymbol{\alpha_0}}\}dN_i(u)\times n^{1/2}(\boldsymbol{\alpha}-\boldsymbol{\alpha_0})+o_p(1)
\end{aligned}
$$

where

$$
\frac{dR_i(t;\boldsymbol{\alpha},\boldsymbol{p_0})}{d\boldsymbol{\alpha}}=\rho_i(\boldsymbol{p_0})e^{\Lambda_i^C(t;\boldsymbol{\alpha},\boldsymbol{p_0})}\frac{d\Lambda_i^C(t;\boldsymbol{\alpha},\boldsymbol{p_0})}{d\boldsymbol{\alpha}}
$$

and

$$
\begin{aligned}
\frac{d\Lambda_i^C(t;\boldsymbol{\alpha},\boldsymbol{p_0})}{d\boldsymbol{\alpha}}=&\int_0^t\boldsymbol{V}_i(u)^Te^{\boldsymbol{\alpha}^T\boldsymbol{V}_i(u)}n^{-1}\sum_{j=1}^{n}\frac{\rho_j(\boldsymbol{p_0})}{S_C^{(0)}(\boldsymbol{\alpha},\boldsymbol{p_0},u)}dN_j^C(u)\\
&+\int_0^t e^{\boldsymbol{\alpha}^T\boldsymbol{V}_i(u)}n^{-1}\sum_{j=1}^{n}(-\frac{\rho_j(\boldsymbol{p_0})\boldsymbol{S}_C^{(1)}(\boldsymbol{\alpha},\boldsymbol{p_0},u)}{S_C^{(0)}(\boldsymbol{\alpha},\boldsymbol{p_0},u)^2})dN_j^C(u)
\end{aligned}
$$

and

$$
\frac{\partial}{\partial\boldsymbol{\alpha}}S^{(0)}(\boldsymbol{\alpha},\boldsymbol{\beta_0},\boldsymbol{p_0},t)|_{\boldsymbol{\alpha}=\boldsymbol{\alpha_0}}=n^{-1}\sum_{i=1}^{n}\frac{\partial R_i(t;\boldsymbol{\alpha},\boldsymbol{p_0})}{\partial\boldsymbol{\alpha}}Y_i(t)e^{\boldsymbol{\beta_0}^T\boldsymbol{Z}_i(0)}|_{\boldsymbol{\alpha}=\boldsymbol{\alpha_0}}\to\boldsymbol{r}_{\boldsymbol{\alpha}}^{*(0)}(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},t)
$$

Thus,

$$
(3)\equiv\boldsymbol{L}_3^*(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},t)n^{-1/2}\sum_{i=1}^{n}\boldsymbol{\psi}_i(\boldsymbol{\alpha_0},\boldsymbol{p_0})+o_p(1)
$$

where

$$\begin{aligned}\boldsymbol{L}^*_{n3}(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},t) =&\{n^{-1}\sum_{i=1}^n\int_0^t s^{(0)}(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},u)^{-1}\{\rho_i(\boldsymbol{p_0})e^{\Lambda_i^C(u)}[\int_0^u \boldsymbol{V}_i(s)^T e^{\boldsymbol{\alpha}^T\boldsymbol{V}_i(s)} s_C^{(0)}(\boldsymbol{\alpha_0}, s\\ &- \int_0^u e^{\boldsymbol{\alpha}^T\boldsymbol{V}_i(s)} v(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},s)d\Lambda_0^C(s)]\}\\ &- \int_0^t s^{(0)}(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},u)^{-1}\boldsymbol{r}^*_{\boldsymbol{\alpha}}(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},u)d\Lambda_0(u)\} \times \boldsymbol{A}^C(\boldsymbol{\alpha_0})\\ &\to \boldsymbol{L}^*_3(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},t)\end{aligned}$$

with

$$d\boldsymbol{J}^*_n(t) = n^{-1}\sum_{i=1}^n \rho_i(\boldsymbol{p_0})dN_j^C(t)$$

which converges to $d\boldsymbol{J}^*(t)$ in probability.

The last part,

$$\begin{aligned}(4) =&n^{-1/2}\sum_{i=1}^n\int_0^t S^{(0)}(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},u)^{-1}R_i(u;\boldsymbol{\alpha_0},\boldsymbol{p_0})dM_i(u)\\ =&n^{-1/2}\sum_{i=1}^n\int_0^t s^{(0)}(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},u)^{-1}R_i(u;\boldsymbol{\alpha_0},\boldsymbol{p_0})dM_i(u) + o_p(1)\end{aligned}$$

Combining the above result, one obtains

$$n^{1/2}\{\hat{\Lambda}_0(t) - \Lambda_0(t)\} =n^{-1/2}\sum_{i=1}^n \Phi_i^*(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},t) + o_p(1)$$

where

$$\begin{aligned}\Phi_i^*(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},t) =&\boldsymbol{L}^*_1(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},t)\boldsymbol{Q}_i(\boldsymbol{p_0}) + \boldsymbol{L}^*_2(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},t)\boldsymbol{\Theta}_i(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0})\\ &+ \boldsymbol{L}^*_3(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},t)\boldsymbol{\psi}_i(\boldsymbol{\alpha_0},\boldsymbol{p_0}) + \int_0^t s^{(0)}(\boldsymbol{\alpha_0},\boldsymbol{\beta_0},\boldsymbol{p_0},u)^{-1}R_i(u;\boldsymbol{\alpha_0},\boldsymbol{p_0})dM_i(u\end{aligned}$$

Note that $E\{\boldsymbol{Q}_i(\boldsymbol{p_0})\} = 0$, $E\{\boldsymbol{\Theta}_i(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0})\} = 0$, $E\{\boldsymbol{\psi}_i(\boldsymbol{\alpha_0}, \boldsymbol{p_0})\} = 0$ and $E\{R_i(u; \boldsymbol{\alpha_0}, \boldsymbol{p_0}) dM_i(u)\} = 0$, such that $E\{d\Phi_i^*(\boldsymbol{\alpha_0}, \boldsymbol{\beta_0}, \boldsymbol{p_0}, t)\} = 0$.

TABLE 1: Supplementary analysis comparing HCC (assigned MELD of 22; reference) versus non-HCC (by lab MELD) pre-transplant mortality

		Case-Control Weighted: W_1			Case-control Weighted: W_3		
Patients	MELD	$\hat{\beta}$	SE	p	$\hat{\beta}$	SE	p
HCC	"22"	0	–	–	0	–	–
non-HCC	6-8	-1.15	0.14	< 0.0001	-1.05	0.15	< 0.0001
	9-11	-0.65	0.10	< 0.0001	-0.53	0.12	< 0.0001
	12-13	-0.60	0.10	< 0.0001	-0.44	0.12	0.0003
	14-15	**-0.20**	**0.11**	**0.056**	**-0.05**	**0.12**	**0.69**
	16-17	**0.02**	**0.10**	**0.88**	0.20	0.13	0.13
	18-19	0.28	0.14	0.04	0.54	0.16	0.0006
	20-22	0.44	0.14	0.001	0.73	0.13	< 0.0001
	23-24	0.78	0.21	0.001	1.29	0.20	< 0.0001
	25-29	1.38	0.17	< 0.0001	1.82	0.17	< 0.0001
	30-39	1.88	0.17	< 0.0001	2.30	0.18	< 0.0001
	40	2.64	0.36	< 0.0001	2.80	0.45	< 0.0001

BIBLIOGRAPHY

Flury, B. (1997). A first course in multivariate statistics. *Springer Texts in Statistics*.

Pollard, D. (1990). *Empirical processes: Theory and applications*. Institute of Mathematical Statistic.

TABLE 2: Simulation results based on 1000 replications: Mis-specified model:

Setting 2A-2B, but with (Z_{i1} deleted)

$W_i(t)$	Estimator	BIAS	ESD	ASE	CP
		Setting 2A: $p_1 = p_2 = p_3 = 0.15$			
1	$\widehat{\beta}_1$	0.096	0.136	0.139	0.901
W_1	$\widehat{\beta}_1^{W_1}$	0.124	0.144	0.146	0.862
W_2	$\widehat{\beta}_1^{W_2}$	0.095	0.137	0.141	0.910
W_3	$\widehat{\beta}_1^{W_3}$	0.082	0.135	0.139	0.929
1	$\widehat{\beta}_2$	-0.104	0.133	0.134	0.876
W_1	$\widehat{\beta}_2^{W_1}$	-0.031	0.135	0.139	0.945
W_2	$\widehat{\beta}_2^{W_2}$	-0.018	0.132	0.136	0.945
W_3	$\widehat{\beta}_2^{W_3}$	-0.013	0.132	0.136	0.946
		Setting 2B: $p_1 = p_2 = p_3 = 0.10$			
1	$\widehat{\beta}_1$	0.101	0.169	0.171	0.916
W_1	$\widehat{\beta}_1^{W_1}$	0.132	0.179	0.179	0.890
W_2	$\widehat{\beta}_1^{W_2}$	0.101	0.170	0.173	0.914
W_3	$\widehat{\beta}_1^{W_3}$	0.088	0.168	0.171	0.927
1	$\widehat{\beta}_2$	-0.102	0.167	0.164	0.912
W_1	$\widehat{\beta}_2^{W_1}$	-0.030	0.169	0.170	0.948
W_2	$\widehat{\beta}_2^{W_2}$	-0.018	0.164	0.167	0.947
W_3	$\widehat{\beta}_2^{W_3}$	-0.013	0.165	0.167	0.949

van der Vaart, A. W. & Wellner, J. A. (1996). *Weak convergence and empirical processes: With application to statistics*. New York: Springer.

Zhang, M. & Schaubel, D.E. (2011). Estimating differences in restricted mean lifetime using observational data subject to dependent censoring. *Biometrics*, 67, 740–749.

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