Generalizations of the Lerch zeta function

by

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Contents

Abstract

The Lerch zeta function is a three-variable generalization of the Riemann zeta function and the Hurwitz zeta function. In this thesis, we study generalizations and analogues of the Lerch zeta function. Our approaches proceed along three directions: local, global and classical.

In the local study, we construct a new family of local zeta integrals, called *local Lerch-Tate zeta integrals*. These local zeta integrals have analytic continuation and functional equation which resembles the functional quation of the Lerch zeta function. Our local zeta integrals generalize Tate's local zeta integrals.

In the global investigation, we introduce a family of global zeta integrals over global number field and a family of global zeta integrals over global function field; these families are called *global Lerch-Tate zeta integrals*. These global zeta integrals converge absolutely on a right half-plane; they have meromorphic continuation and functional equation. Our global zeta integrals generalize Tate's global zeta integrals. In the special case when the number field is the field of rational numbers, specializations of these global zeta integrals give the symmetrized Lerch zeta functions.

In the classical approach, we study a generalized Lerch zeta function with four variables. We compute its Fourier expansion and establish its analytic continuation.

Chapter 1

Introduction

1.1 The Lerch zeta function

This thesis studies generalizations and analogues of the Lerch zeta function. The Lerch zeta function is given by the Dirichlet series

$$\zeta(s, a, c) = \sum_{n=0}^{\infty} \frac{1}{(n+c)^s} \, \mathbf{e}(na)$$

where $\mathbf{e}(z) = e^{2\pi i z}$ for a complex number z. If a is complex with non-negative imaginary part and c is real but not a non-negative integer, $\zeta(s, a, c)$ converges absolutely for $\Re(s) > 1$.

Special cases of the Lerch zeta function include the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \zeta(s, 0, 1),$$

the Hurwitz zeta function

$$\zeta(s, c) = \sum_{n=0}^{\infty} (n+c)^{-s} = \zeta(s, 0, c),$$

and the periodic zeta function [23, Section 25.13]

$$F(a, s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \, \mathbf{e}(na) = \mathbf{e}(-a) \, \zeta(s, a, 1).$$

First investigations on the Lerch zeta function can be found in papers of Lipschitz [13, 14] and Lerch [12]. Weil gave a beautiful exposition of the analytic continuation and the functional equation of the Lerch zeta function in his classic book [42]. The book of Laurinčikas and Garunkštis [11] carefully studies analytic properties of the Lerch zeta function.

1.2 Properties of the Lerch zeta function

Lerch [12] showed that $\zeta(s,a,c)$ extends to a meromorphic function for $s \in \mathbb{C}$ by virtue of the integral representation

$$\zeta(s,a,c) = \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{z^{s-1}e^{cz}}{1-e^{z+2\pi ia}}\,\mathrm{dz}$$

where C is the contour running from $-\infty$ below the negative real axis to the origin, encircling around the origin in positive orientation, and tracing back to ∞ above the negative real axis. Lerch [12] proved the three-term functional equation

$$\zeta(1-s,a,c) = \frac{\Gamma(s)}{(2\pi)^s}\big\{e^{\frac{\pi is}{2}}e^{-2\pi iac}\zeta(s,-c,a) + e^{-\frac{\pi is}{2}}e^{2\pi ic(1-a)}\zeta(s,c,1-a)\big\}, \qquad (1.1)$$

which researchers call Lerch's transformation formula.

Now suppose that a and c are real variables with $0 < a < 1$ and $0 < c < 1$. Fundamental properties of the Lerch zeta function $\zeta(s,a,c)$ include the following:

(i) *Functional equation:* Lerch's transformation formula (1.1) admits a more symmetric form. Define

$$\begin{aligned} L^+(s,a,c) &= \zeta(s,a,c) + e^{-2\pi ia}\zeta(s,1-a,1-c) \\ L^-(s,a,c) &= \zeta(s,a,c) - e^{-2\pi ia}\zeta(s,1-a,1-c). \end{aligned}$$

The symmetrized Lerch zeta functions $L^{\pm}(s,a,c)$ can be expressed, for $\mathrm{Re}(s) > 1$, as Dirichlet series

$$\begin{aligned} L^+(s,a,c) &= \sum_{n=-\infty}^{\infty} \frac{1}{|n+c|^s}\,\mathbf{e}(na) \\ L^-(s,a,c) &= \sum_{n=-\infty}^{\infty} \frac{\mathrm{sgn}(n+c)}{|n+c|^s}\,\mathbf{e}(na). \end{aligned}$$

On completing $L^{\pm}(s,a,c)$ with

$$\hat{L}^{+}(s,a,c) = \pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\,L^{+}(s,a,c)$$
$$\hat{L}^{-}(s,a,c) = \pi^{-\frac{s+1}{2}}\,\Gamma\left(\frac{s+1}{2}\right)\,L^{-}(s,a,c),$$

it is shown that $\hat{L}^{\pm}(s,a,c)$ extends to entire functions of s and admits functional equations

$$\hat{L}^{+}(s,a,c) = \mathbf{e}(-ac)\,\hat{L}^{+}(1-s,1-c,a) \tag{1.2}$$
$$\hat{L}^{-}(s,a,c) = i\mathbf{e}(-ac)\,\hat{L}^{-}(1-s,1-c,a). \tag{1.3}$$

The functional equations (1.2) and (1.3) were proved by Weil [42, p. 57] and Lagarias and Li [7, Theorem 2.1].

(ii) *Differential-difference equations:* The Lerch zeta function satisfies, for all s by analytic continuation, the differential-difference equations

$$\frac{1}{2\pi i}\,\partial_a\zeta(s,a,c) = \zeta(s-1,a,c) - c\,\zeta(s,a,c) \tag{1.4}$$
$$\partial_c\zeta(s,a,c) = -s\,\zeta(s+1,a,c). \tag{1.5}$$

From (1.4) and (1.5), one derives a second order linear partial differential equation satisfied by $\zeta(s,a,c)$, namely

$$\left(\frac{1}{2\pi i}\,\partial_a\partial_c + c\,\partial_c + s\right)\zeta(s,a,c) = 0. \tag{1.6}$$

The differential equations (1.4)–(1.6) were shown for $\zeta(s,a,c)$ by Apostol [1, p. 163] and extended to an analytic continuation of $\zeta(s,a,c)$ by Lagarias and Li [8, Theorem 5.1].

The zeta functions $L^{\pm}(s,a,c)$ satisfy the periodicity relations

$$L^{\pm}(s,a+1,c) = L^{\pm}(s,a,c) \tag{1.7}$$
$$L^{\pm}(s,a,c+1) = \mathbf{e}(-a)\,L^{\pm}(s,a,c). \tag{1.8}$$

Beside (i) and (ii), the Lerch zeta function exhibits several salient features which are interesting and worthwhile for further investigation.

(iii) *Multivaluedness:* Viewed as a function of three complex variables, $\zeta(s,a,c)$ exhibits multi-valued nature. Lagarias and Li [8] showed that $\zeta(s,a,c)$ can be defined as a multi-valued function on the complex manifold

$$\{(s,a,c) \in \mathbb{C} \times (\mathbb{C} - \mathbb{Z}) \times (\mathbb{C} - \mathbb{Z})\}$$

and that the multi-valued function descends to a single-valued function on the maximal abelian cover of this complex manifold.

(iv) Special values of $\zeta(s,a,c)$ and its variants have interesting arithmetic properties, as shown by Lagarias and Li [9] and Shimura [31, 32].

(v) The Lerch zeta function $\zeta(s,a,c)$ can not be expressed as an Euler product in general.

1.3 Motivation and goal of the thesis

Tate reformulated classical number-theoretic zeta functions, for instance the Riemann zeta function, Dirichlet L-functions, and Hecke L-functions, as zeta integrals. Modern number theory transforms Tate's perspective into the theory of automorphic L-functions. The automorphic L-functions have Euler products which provide them with vast connections with arithmetic geometry and representation theory.

Because of the translation twist by the variable c and the additive character twist by the variable a, the Lerch zeta function $\zeta(s,a,c)$ does not have an Euler product in general. It is natural to ask whether there is an automorphic framework for which the Lerch zeta function fits in. This is too broad a question to answer, but let us raise the following concrete questions:

(1) Can the Lerch zeta function be realized as a zeta integral on the adeles?

(2) Is there a generalization of the Lerch zeta function for global number field?

(3) Does the Lerch zeta function have an analogue over global function field?

(4) If Question 1 has a positive answer, is there an analogous zeta integral over local fields?

This thesis constructs new generalizations and analogues of the Lerch zeta function. We give an affirmative answer to each of these questions. In Section 1.5 we describe and summarize new results obtained.

1.4 A recollection on Tate's Thesis

Let us briefly review Tate's Thesis, preparing appropriate background to describe the results of the thesis on generalizations and analogues of the Lerch zeta function. The best references are the original Tate's Thesis [33], which is reprinted in [5], and the reinterpretation of Weil [41] in the language of distributions (see also [6, 40]).

Suppose that G is a locally compact abelian group. A *quasi-character* (resp. *character*) of G is a continuous homomorphism $\chi : G \to \mathbb{C}^\times$ (resp. $\chi : G \to \mathbb{T}$). We denote by $X(G)$ (resp. $\widehat{G}$) the group of all quasi-characters (resp. characters) of G. The group $\widehat{G}$ endowed with the compact-open topology is called the *Pontryagin dual* of G.

Tate's Thesis [33] realizes the spaces $X(G)$ of those G arising from number-theoretic contexts as complex varieties; it is on these varieties that the zeta integrals will live. In number-theoretic considerations, we encounter situations where G admits a norm map $|\cdot| : G \to \mathbb{R}_+$ with compact kernel G^1 and with image $H \subset \mathbb{R}_+$. We assume that either

(a) $H = \mathbb{R}_+$ or

(b) H is an infinite discrete subgroup of the form $q^{\mathbb{Z}}$ for some real number $q > 1$.

There holds an exact sequence

$$1 \to G^1 \to G \xrightarrow{|\cdot|} H \to 1.$$

Interesting situations include, but not exclusively, instances with

- $G = F^\times$ where F is an archimedean local field
- G is the idele class group of a global number field K
- $G = F^\times$ where F is a nonarchimedean local field
- G is the idele class group of a global function field K;

the first two instances yield case (a), whereas the latter two yield case (b).

In case (a), every quasi-character of H is of the form $t \mapsto t^s$ for some $s \in \mathbb{C}$ and hence $X(H) = \mathbb{C}$. In case (b), every quasi-character of H is uniquely determined by its value on q, whence one has a natural isomorphism $X(H) \xrightarrow{\sim} \mathbb{C}^\times, \chi \mapsto \chi(q)$. Alternatively, in case (b), the inclusion $H \subset \mathbb{R}_+$ induces the surjection $X(\mathbb{R}_+) \to X(H)$, which in turn realizes $X(H)$ as the quotient $\mathbb{C}/\frac{2\pi i}{\log(q)}\mathbb{Z}$ because $X(\mathbb{R}_+) = \mathbb{C}$. In either case, $X(H)$ is naturally endowed with a structure of *one-dimensional connected complex variety.*

It follows from our assumptions on G and H that there is a natural exact sequence (of abstract groups)

$$1 \to X(H) \to X(G) \to \widehat{G^1} \to 1,$$

with $\widehat{G^1}$ being discrete (because G^1 is compact). This exact sequence realizes $X(G)$ as a disjoint union of copies of the complex algebraic variety $X(H)$. One therefore endows $X(G)$ with a structure of *one-dimensional complex variety.* The connected complex variety $X(H)$ is the connected component at the identity of $X(G)$, called the *neutral component* of $X(G)$. The discrete group $\widehat{G^1}$ is the *component group* of $X(G)$; we write $\widehat{G^1} = \pi_0(X(G))$.

Our assumptions on G and H imply that every quasi-character of H, which is either $\mathbb{R}_+$ or $q^{\mathbb{Z}}$, is of the form $\chi(t) = t^s$, where $s \in \mathbb{C}$ in case (a) and where $s \in \mathbb{C}/\frac{2\pi i}{\log(q)}\mathbb{Z}$ in case (b). It allows one to define the *real part of the quasi-character* $\chi \in X(H)$ to be $\mathrm{Re}(\chi) := \mathrm{Re}(s)$. Moreover, the notion of real part extends to a quasi-character of G as follows. If $\chi \in X(G)$, the quasi-character $|\chi|$ of G defined by $|\chi|(x) := |\chi(x)|$ is trivial on G^1 and hence descends to a quasi-character $|\chi|_H \in X(H)$. Then one defines the *real part of the quasi-character* $\chi \in X(G)$ to be $\mathrm{Re}(\chi) := \mathrm{Re}(|\chi|_H)$.

Tate's local theory

Suppose that F is a nonarchimedean local field. Hence F is either a finite extension of $\mathbb{Q}_p$ or the field of formal Laurent series $\mathbb{F}_q((t))$. Let $\mathcal{O}_F$ be its ring of integers and π a uniformizer. For comparison with the global theory, let $X_F = X(F^\times)$ denote the group of quasi-characters of $F^\times$.

A quasi-character of $F^\times$ is said to be *unramified* if it has trivial restriction to $\mathcal{O}_F^\times$; such a quasi-character is determined by its value $\chi(\pi) \in \mathbb{C}^\times$. It is clear that the neutral component of X_F consists of unramified quasi-characters. The component group $\pi_0(X_F)$ is identified with the (discrete) Pontryagin dual $\widehat{\mathcal{O}_F^\times}$. Each connected component of X_F is labeled by a unique character of $\mathcal{O}_F^\times$, namely the restriction to $\mathcal{O}_F^\times$ of any of the element in the same connected component. Furthermore, each connected component of X_F can be (non-canonically) identified with the connected variety $\mathbb{C}^\times$.

A complex-valued function on a locally compact Hausdorff and totally disconnected group G is called a *Schwartz-Bruhat function* if it is locally constant and compactly supported; the space of all Schwartz-Bruhat functions on G is denoted by

$\mathcal{S}(G)$.

For each Schwartz-Bruhat function $f \in \mathcal{S}(F)$, Tate defined the local zeta integral

$$Z(f,\chi) = \int_{F^\times} f(x)\chi(x)\ \mathrm{d}^\times \mathrm{x}, \tag{1.9}$$

where $\mathrm{d}^\times\mathrm{x}$ is a Haar measure on $F^\times$. The assignment $\chi \mapsto Z(f,\chi)$ defines a function on X_F. At first $Z(f,\cdot)$ is absolutely convergent and defines an analytic function for $\mathrm{Re}(\chi) > 0$. Then $Z(f,\cdot)$ meromorphically extends to a function, also denoted by $Z(f,\cdot)$, on X_F. The function $Z(f,\cdot)$ is holomorphic away from the neutral component; on the neutral component, it is meromorphic with at most a simple pole at the trivial character $\chi_{\mathrm{triv}} \in X_F$ whose residue equals $\frac{f(0)}{1-\chi(\pi)}$. In particular, if $f \in \mathcal{S}(F^\times)$, then $Z(f,\cdot)$ is a holomorphic function on X_F.

Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive character. For a Schwartz-Bruhat function $f \in \mathcal{S}(F)$, its Fourier transform with respect to ψ and a Haar measure $\mathrm{d}_+\mathrm{x}$ on F is given by

$$\hat{f}(y) = \int_F f(x)\psi(xy)\,\mathrm{d}_+\mathrm{x}.$$

One chooses $\mathrm{d}_+\mathrm{x}$ to be the unique Haar measure on F for which the Fourier inversion $\hat{\hat{f}}(x) = f(-x)$ holds. Tate showed that $Z(f,\chi)$ satisfies the functional equation

$$Z(\hat{f},\chi') = \gamma(\chi,\psi,\mathrm{d}_+\mathrm{x})\ Z(f,\chi)$$

where $\chi\chi' = |\cdot|$ and $\gamma(\chi,\psi,\mathrm{d}_+\mathrm{x})$ is the local gamma factor independent of f. The local gamma factor encodes important arithmetic information.

Tate's global theory

Let K be a number field; let $\mathbb{A}_K$ be its ring of adeles and $\mathbb{A}_K^\times$ its group of ideles. Let ν denote a generic place of K; let $\mathfrak{p}$ denote a generic finite place of K. Write $K_\mathbb{R} = K \otimes_\mathbb{Q} \mathbb{R}$ for the Minkowski space of K. If r_1 (resp. r_2) is the number of real embeddings (resp. (equivalence classes of) complex embeddings) from K into $\mathbb{C}$, then $K_\mathbb{R}$ is a vector space of real dimension $r_1 + 2r_2$.

Let $\mathbb{A}_K$ be the group of adeles and $\mathbb{A}_K^\times$ its multiplicative group. By definition, these are the restricted direct products $\mathbb{A}_K = K_\mathbb{R} \times \prod'_\mathfrak{p} K_\mathfrak{p}$ and $\mathbb{A}_K^\times = K_\mathbb{R}^\times \times \prod'_\mathfrak{p} K_\mathfrak{p}^\times$. For an idele $x \in \mathbb{A}_K^\times$, let $|x|$ denote the standard idele norm, so the product formula $|x| = \prod_\nu |x|_\nu = 1$ holds true for every $x \in K^\times$. Recall the standard normalization: if ν is a real place, $|\cdot|_\nu$ is the usual absolute value; if ν is a complex place, $|\cdot|_\nu$ is the square of the usual absolute value; if $\mathfrak{p}$ is nonarchimedean, $|x|_\mathfrak{p} = q^{-\mathrm{val}_\mathfrak{p}(\mathrm{x})}$ where

$q = |\kappa(\mathfrak{p})|$ is the cardinality of the residue field and $\mathrm{val}_{\mathfrak{p}}(\mathrm{x})$ is the valuation which takes value one on any uniformizer. Write $\mathbb{A}_K^1$ for the group of ideles of norm one. Write $C_K = \mathbb{A}_K^\times / K^\times$ for the idele class group and $C_K^1 = \mathbb{A}_K^1/K^\times$ for the norm one idele class group.

For a global number field K, one takes $G = C_K$, so $G^1 = C_K^1$ and $H = \mathbb{R}_+$. Set $X_K = X(G)$. The exact sequence

$$1 \to G^1 \to G \xrightarrow{|\cdot|} \mathbb{R}_+ \to 1,$$

where $|\cdot|$ denotes the standard norm, splits non-canonically. The splittings $G = G^1 \times \mathbb{R}_+$ correspond bijectively to sections of the norm map $|\cdot| : G \to \mathbb{R}_+$. Later on we will define a continuous homomorphism $\tau : \mathbb{R}_+ \to G$ to be a *norm section* if $|\cdot| \circ \tau = \mathrm{id}_{\mathbb{R}_+}$. Each splitting $G = G^1 \times \mathbb{R}_+$ induces a decomposition $X_K = \widehat{C_K^1} \times \mathbb{C}$; the component group $\widehat{C_K^1}$ has discrete topology. For the rational number field, $C_{\mathbb{Q}}^1$ is isomorphic to the profinite group $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p^\times$, thus the (torsion) Pontryagin dual $\widehat{C_{\mathbb{Q}}^1}$ is identified with the set of all Dirichlet characters. If K is a generic number field, $\widehat{C_K^1}$ is not necessarily torsion; it consists of (finite order or infinite order) Hecke Grössencharacters.

In parallel with the local theory, Tate defined for every Schwartz-Bruhat function $f \in \mathcal{S}(\mathbb{A}_K)$ the global zeta integral

$$Z(f, \chi) = \int_{\mathbb{A}_K^\times} f(x)\chi(x)\,\mathrm{d}^\times\mathrm{x} \tag{1.10}$$

where $\mathrm{d}^\times\mathrm{x}$ is a Haar measure on $\mathbb{A}_K^\times$. The assignment $\chi \mapsto Z(f, \chi)$ defines a function on X_K. The global zeta integral $Z(f, \chi)$ converges absolutely for $\mathrm{Re}(\chi) > 1$.

Let $\psi : \mathbb{A}_K \to \mathbb{C}^\times$ be a nontrivial additive character which is trivial on K. For a Schwartz-Bruhat function $f \in \mathcal{S}(\mathbb{A}_K)$, its Fourier transform with respect to ψ and a Haar measure $\mathrm{d}_+\mathrm{x}$ on $\mathbb{A}_K$ is given by

$$\hat{f}(y) = \int_{\mathbb{A}_K} f(x)\psi(xy)\,\mathrm{d}_+\mathrm{x}.$$

One chooses $\mathrm{d}_+\mathrm{x}$ to be the unique Haar measure on $\mathbb{A}_K$ for which the Fourier inversion $\hat{\hat{f}}(x) = f(-x)$ holds. Tate showed that the global zeta integral $Z(f, \chi)$ admits meromorphic continuation to the whole variety X_K and satisfies the functional equation

$$Z(\hat{f}, \chi') = Z(f, \chi)$$

where $\chi\chi' = |\cdot|$.

1.5 Statements of main results

In Chapter 2 we introduce a new family of local zeta integrals, called *local Lerch-Tate zeta integrals*. These local zeta integrals have analytic continuation and functional equation which resembles the functional quation of the Lerch zeta function. Our local zeta integrals generalize Tate's local zeta integrals. In Chapter 3 we study a family of global zeta integrals over a number field; called *global Lerch-Tate zeta integrals over a number field*. These global zeta integrals converge absolutely on a right half-plane; they have meromorphic continuation and functional equation. Our global zeta integrals generalize Tate's global zeta integrals. In the special case when the number field is the field of rational numbers, our global zeta integrals give the Lerch zeta function. In Chapter 4 we show that an analogue for function field of curve over finite field, namely *global Lerch-Tate zeta integrals over a function field*, also has interesting analytic properties. In Chapter 5 we introduce and study a generalized Lerch zeta function.

We now give a more detailed description of the main results of the thesis.

In Chapter 2 we construct *local Lerch-Tate zeta integrals* over a nonarchimedean local field F as follows. Define

$$Z_{\tau,a,c}(f,\chi) = \int_{F^\times} \psi\left(\frac{ax}{\tau(|x|)}\right) f\big(x + c\,\tau(|x|)\big)\,\chi(x)\ \mathrm{d}^\times\mathrm{x}$$

where χ is a quasi-character of $F^\times$, (τ, a, c) is a local Lerch tuple, f is a Schwartz-Bruhat function on F. For the definition of local Lerch tuple, see Section 2.4; here $a, c \in F$. Let X_F be the variety of all quasi-characters of $F^\times$. We show the Main Local Theorem which establishes analytic properties of local Lerch-Tate zeta integrals.

Theorem 1.5.1. (Main Local Theorem) *Let F be a nonarchimedean local field. Let (τ, a, c) be a local Lerch tuple and f be a Schwartz-Bruhat function on F; set $\pi = \tau(q^{-1})$. Let ψ be an additive character of level l of F. Take for F the self-dual measure* $\mathrm{d}_+\mathrm{x}$ *with respect to ψ; take for $F^\times$ any Haar measure* $\mathrm{d}^\times\mathrm{x}$.

(i) *(Absolute convergence and analytic continuation) On each connected component of X_F, the zeta integral $Z_{\tau,a,c}(f,\chi)$ converges absolutely for* $0 < \mathrm{Re}(\chi) < 1$ *to a rational function (of the variable $z = \chi(\pi)$).*

(ii) *(Poles and residues) Let χ_0 be a character on the unit group.*

If $c \notin \mathcal{O}^\times$, the restriction of $Z_{\tau,a,c}(f,\chi)$ to the connected component X_{F,χ_0} has at most a simple pole at the unique quasi-character $\chi_{\tau,0} \in X_{F,\chi_0}$ which takes value 1 on π. The residue of $Z_{\tau,a,c}(f,\cdot)$ at $\chi_{\tau,0}$ is

$$\mathrm{Res}(Z_{\tau,a,c}(f,\chi) : \chi = \chi_{\tau,0}) = -f(0)\,\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)\,G(\psi_a,\chi_0).$$

If $c \in \mathcal{O}^\times$, the restriction of $Z_{\tau,a,c}(f,\chi)$ to the connected component X_{F,χ_0} has at most two simple poles at the unique quasi-character $\chi_{\tau,0} \in X_{F,\chi_0}$ which takes value 1 on π and at the unique quasi-character $\chi_{\tau,-1} \in X_{F,\chi_0}$ which takes value q^{-1} on π. The residue of $Z_{\tau,a,c}(f,\cdot)$ at $\chi_{\tau,0}$ is

$$\mathrm{Res}(Z_{\tau,a,c}(f,\chi) : \chi = \chi_{\tau,0}) = -f(0)\,\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)\,G(\psi_a,\chi_0).$$

The residue of $Z_{\tau,a,c}(f,\cdot)$ at $\chi_{\tau,-1}$ is

$$\mathrm{Res}(Z_{\tau,a,c}(f,\chi) : \chi = \chi_{\tau,-1}) = \frac{h}{q}\psi(-ac)\chi_0(-c)\int_F f(x)\,\mathrm{d}_+\mathrm{x}$$

where $h = \frac{\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)}{\mathrm{d}_+\mathrm{x}(\mathcal{O}^\times)}$.

(iii) *(Functional equation) Let χ_0 be a character of level m on the unit group. Let $Z_{\pi,a,c}(f,\chi_0,z)$ denote the restriction of $Z_{\tau,a,c}(f,\cdot)$ to the connected component X_{F,χ_0}. In other words, if $\chi|_{\mathcal{O}^\times} = \chi_0$ and $\chi(\pi) = z$, then $Z_{\tau,a,c}(f,\chi) = Z_{\pi,a,c}(f,\chi_0,z)$. There holds the symmetry:*

- *if χ is unramified, then*

$$\begin{aligned}&\frac{Z_{\pi,a,c}(f,\chi_{0,\mathrm{nr}},z)}{\psi(-ac)\,\mathrm{d}_+\mathrm{x}\,(\mathcal{O}^\times)}\\ &= \sum_{r\geq l} z^r\, Z_{\pi,-c\pi^r,a\pi^{-r}}\left(\hat{f},\chi_{0,\mathrm{nr}},\frac{1}{qz}\right) + \frac{z^{l-1}}{1-q}\,Z_{\pi,-c\pi^{l-1},a\pi^{1-l}}\left(\hat{f},\chi_{0,\mathrm{nr}},\frac{1}{qz}\right);\end{aligned}$$

- *if χ is ramified of level $m \geq 1$, then*

$$\frac{Z_{\pi,a,c}(f,\chi_0,z)}{\chi_0(-1)\,\psi(-ac)\,\mathrm{d}_+\mathrm{x}\,(\mathcal{O}^\times)} = z^{l-m}\,G(\psi_{\pi^{l-m}},\chi_0)\;Z_{\pi,-c\pi^{l-m},a\pi^{m-l}}\left(\hat{f},\chi_0^{-1},\frac{1}{qz}\right).$$

In Chapter 3 we construct *global Lerch-Tate zeta integrals* over a number field K

as follows. Define

$$Z_{\tau,a,c}(f,\chi) = \int_{\mathbb{A}_K^\times} \psi\left(\frac{ax}{\tau(|x|)}\right) f\left(x + c\,\tau(|x|)\right)\, \chi(x)\ \mathrm{d}^\times\mathrm{x}.$$

where χ is a quasi-character of the idele class group $\mathbb{A}_K^\times/K^\times$, (τ, a, c) is a global Lerch tuple, and f is a Schwartz-Bruhat function on $\mathbb{A}_K$. For the definition of global Lerch tuple, see Section 3.4; here $a, c \in \mathbb{A}_K$. Let X_K be the variety of all quasi-characters of $\mathbb{A}_K^\times/K^\times$. We show the Main Global Theorem over Number Field which establishes analytic properties of global number field Lerch-Tate zeta integrals. For each Schwartz-Bruhat function f on $\mathbb{A}_K$, the allowed set of values $(a, c) \in \mathbb{A}_K^2$ excludes a "singular set", which is specified by a "compatibility" condition defined in Section 3.

Theorem 1.5.2. (Main Global Theorem over a number field) *Let K be a number field. Let (τ, a, c) be a global Lerch tuple and ψ be an additive character of $\mathbb{A}_K$ with trivial restriction to K. Let f be a Schwartz-Bruhat function on $\mathbb{A}_K$ and $\hat{f}$ be the Fourier transform of f with respect to ψ and* $\mathrm{d}_+\mathrm{x}$. *Suppose that c is compatible with f and that a is compatible with $\hat{f}$.*

(i) *(Absolute convergence) On each connected component of X_K, the zeta integral $Z_{\tau,a,c}(f,\chi)$ converges absolutely for* $\mathrm{Re}(\chi) > 1$.

(ii) *(Analytic continuation) The zeta integral $Z_{\tau,a,c}(f,\chi)$ extends to a meromorphic function on X_K.*

(iii) *(Poles and residues) The meromorphic function $Z_{\tau,a,c}(f,\chi)$ is holomorphic outside of the unramified component. On the unramified component, $Z_{\tau,a,c}(f,\chi)$ has at most simple poles at $\chi = |\cdot|$ and $\chi = \chi_{\mathrm{triv}}$. Observe that each of the two integrals*

$$\int_1^\infty \hat{f}\,\left(a\,\tau(t)\right)\chi'(\tau(t))\ \mathrm{d}^\times\mathrm{t} \quad \textit{and} \quad \int_0^1 f\left(c\,\tau(t)\right)\,\chi(\tau(t))\ \mathrm{d}^\times\mathrm{t}$$

converges absolutely for $\mathrm{Re}(\chi) > 1$ *and has a meromorphic continuation to $\chi \in X_K$. Then the function*

$$Z_{\tau,a,c}(f,\chi) - \psi(-ac)\,v_K \int_1^\infty \hat{f}\,\left(a\,\tau(t)\right)\chi'(\tau(t))\ \mathrm{d}^\times\mathrm{t} + v_K \int_0^1 f\left(c\,\tau(t)\right)\,\chi(\tau(t))\ \mathrm{d}^\times\mathrm{t}$$

extends to a holomorphic function on the unramified component of X_K.

(iv) *(Functional equation) Observe that the integral*

$$S_1(f,\chi) = \int_0^1 f\left(c\,\tau(t)\right)\,\chi(\tau(t))\ \mathrm{d}^\times\mathrm{t}$$

converges absolutely for $\mathrm{Re}(\chi) > 0$ *and extends to a meromorphic function of* $\chi \in X_K$*; we denote this meromorphic function also by* $S_1(f,\chi)$*. Observe also that the integral*

$$S_2(f,\chi) = \int_1^\infty f\left(c\,\tau(t)\right)\,\chi(\tau(t))\ \mathrm{d}^\times\mathrm{t}$$

converges absolutely for $\mathrm{Re}(\chi) < 0$ *and extends to a meromorphic function of* $\chi \in X_K$*; we denote this meromorphic function also by* $S_2(f,\chi)$*.*

Set

$$\widetilde{Z}_{\tau,a,c}(f,\chi) = \begin{cases} Z_{\tau,a,c}(f,\chi) + \upsilon_K\left(S_1(f,\chi) + S_2(f,\chi)\right) & \text{if } \chi \text{ is unramified} \\ Z_{\tau,a,c}(f,\chi) & \text{if } \chi \text{ is ramified.} \end{cases}$$

One has

$$\widetilde{Z}_{\tau,a,c}(f,\chi) = \psi(-ac)\ \widetilde{Z}_{\tau,-c,a}(\hat{f},\chi').$$

Remarks 1.5.3. (i) We show in Section 3.8 that, when $K = \mathbb{Q}$, the global Lerch-Tate zeta integral can be specialized to give the Lerch zeta function.

(ii) Since the Lerch zeta function can not be expressed as an Euler product, it would be interesting if there is some relation between the global and local Lerch-Tate zeta integrals.

In Chapter 4 we fix a smooth projective geometrically connected curve C over a finite field $k = \mathbb{F}_q$ and let $K = k(C)$ be the function field of C. This chapter constructs *global Lerch-Tate zeta integrals* over the function field K as follows. Define

$$Z_{\tau,a,c}(f,\chi) = \int_{\mathbb{A}_K^\times} \psi\left(\frac{ax}{\tau(|x|)}\right)\ f\left(x + c\,\tau(|x|)\right)\,\chi(x)\ \mathrm{d}^\times\mathrm{x}.$$

where χ is a quasi-character of the idele class group $\mathbb{A}_K^\times/K^\times$, (τ,a,c) is a global Lerch tuple, and f is a Schwartz-Bruhat function on $\mathbb{A}_K$. For the definition of global Lerch tuple, see Section 4.2; here $a, c \in \mathbb{A}_K$. Let X_K be the variety of all quasi-characters of $\mathbb{A}_K^\times/K^\times$. We show the Main Global Theorem over Function Field which establishes analytic properties of global function field Lerch-Tate zeta integrals.

Theorem 1.5.4. (Main Global Theorem over a function field) *Let* (τ, a, c) *be a global Lerch tuple and* ψ *be an additive character of* $\mathbb{A}_K$ *with trivial restriction to* K*. Let* f *be a Schwartz-Bruhat function on* $\mathbb{A}_K$ *and* $\hat{f}$ *be the Fourier transform of* f *with respect to* ψ *and* $\mathrm{d}_+\mathrm{x}$*. Suppose that* c *is compatible with* f *and that* a *is compatible with* $\hat{f}$*.*

(i) *(Absolute convergence) On each connected component of* X_K*, the zeta integral* $Z_{\tau,a,c}(f, \chi)$ *converges absolutely for* $\mathrm{Re}(\chi) > 1$*.*

(ii) *(Analytic continuation) The zeta integral* $Z_{\tau,a,c}(f, \chi)$ *extends to a meromorphic function on* X_K*.*

(iii) *(Poles and residues) The meromorphic function* $Z_{\tau,a,c}(f, \chi)$ *is holomorphic outside of the unramified component. On the unramified component,* $Z_{\tau,a,c}(f, \chi)$ *has at most simple poles at* $\chi = |\cdot|$ *and* $\chi = \chi_{\mathrm{triv}}$*. Observe that each of the two sums*

$$\sum_{n>0} \hat{f}\ (a\,\tau(q^n))\,\chi'(\tau(q^n)) \quad \textit{and} \quad \sum_{n<0} f\,(c\,\tau(q^n))\ \chi(\tau(q^n))$$

converges absolutely for $\mathrm{Re}(\chi) > 1$ *and has a meromorphic continuation to* $\chi \in X_K$*. Then the function*

$$Z_{\tau,a,c}(f, \chi) - \psi(-ac)\,v_K \sum_{n>0} \hat{f}\ (a\,\tau(q^n))\,\chi'(\tau(q^n)) + v_K \sum_{n<0} f\,(c\,\tau(q^n))\ \chi(\tau(q^n))$$

extends to a holomorphic function on the unramified component of X_K*.*

(iv) *(Functional equation) Observe that the integral*

$$S_1(f, \chi) = \sum_{m=-\infty}^{-1} f\,(c\,\tau(q^m))\ \chi(\tau(q^m))$$

converges absolutely for $\mathrm{Re}(\chi) > 0$ *and extends to a meromorphic function of* $\chi \in X_K$*; we denote this meromorphic function also by* $S_1(f, \chi)$*. Observe also that the integral*

$$S_2(f, \chi) = \sum_{m=0}^{\infty} f\,(c\,\tau(q^m))\ \chi(\tau(q^m))$$

converges absolutely for $\mathrm{Re}(\chi) < 0$ *and extends to a meromorphic function of*

$\chi \in X_K$; we denote this meromorphic function also by $S_2(f,\chi)$. Set

$$\widetilde{Z}_{\tau,a,c}(f,\chi) = \begin{cases} Z_{\tau,a,c}(f,\chi) + \upsilon_K\,(S_1(f,\chi) + S_2(f,\chi)) & \text{if } \chi \text{ is unramified} \\ Z_{\tau,a,c}(f,\chi) & \text{if } \chi \text{ is ramified.} \end{cases}$$

One has

$$\widetilde{Z}_{\tau,a,c}(f,\chi) = \psi(-ac)\ \widetilde{Z}_{\tau,-c,a}(\hat{f},\chi').$$

In Chapter 5 we study a generalized Lerch zeta function given by the Dirichlet series

$$Z(x,y;\alpha,\beta) = {\sum_{n=-\infty}^{\infty}}' \frac{1}{(n+x)^{\alpha}(n+\overline{x})^{\beta}}\ \mathbf{e}(ny)$$

where x, y, α and β are complex variables, $\mathbf{e}(z) = e^{2\pi i z}$, and the primed sum excludes the term $n = -x$ when x is an integer. Here we fix the log branch $\log z = \log|z| + i \arg z$ with $-\pi \leq \arg z < \pi$ and define $z^{\gamma} = e^{\gamma \log z}$ for two complex numbers z and γ.

We establish analytic continuation of $Z(x,y;\alpha,\beta)$ by means of Fourier development. More precisely, we define the upsilon function

$$\upsilon(x;\alpha,\beta) = \int_0^{\infty} e^{-t}(t+x)^{\alpha-1}t^{\beta-1}dt$$

which is absolutely convergent and defines an analytic function for $x \in \mathbb{C} - \mathbb{R}_{\leq 0}$ and $\alpha,\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta) > 0$. We prove that $\upsilon(x;\alpha,\beta)$ extends to an analytic function for $x \in \mathbb{C} - \mathbb{R}_{\leq 0}$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C} - \mathbb{Z}_{\leq 0}$. We show the following

Theorem 1.5.5. *Suppose that $x = x' + ix'' \in \mathbb{H}$ and that $y \in \mathbb{R}$. Suppose further that $\alpha,\beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha+\beta) > 1$. Then*

$$Z(x,y;\alpha,\beta) = \mathbf{e}(-x'y) \sum_{n=-\infty}^{\infty} c_n\ \mathbf{e}(nx') \tag{1.11}$$

where the Fourier coefficients $c_n = c_n(x'',y;\alpha,\beta)$ are given by

$$c_n = 2\pi\, e^{\frac{1}{2}\pi i(\beta-\alpha)}\,(2x'')^{1-\alpha-\beta} \begin{cases} \frac{e^{-2\pi(n-y)x''}}{\Gamma(\alpha)\,\Gamma(\beta)}\ \upsilon(4\pi(n-y)x'';\alpha,\beta) & \text{if } n > y \\ \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\,\Gamma(\beta)} & \text{if } n = y \\ \frac{e^{-2\pi(y-n)x''}}{\Gamma(\alpha)\,\Gamma(\beta)}\ \upsilon(4\pi(y-n)x'';\beta,\alpha) & \text{if } n < y. \end{cases} \tag{1.12}$$

From the above theorem, we deduce the analytic continuation of $\zeta(x, y; \alpha, \beta)$:

Theorem 1.5.6. (i) *For fixed $x \in \mathbb{H}$ and fixed $y \in \mathbb{R}-\mathbb{Z}$, the function $Z(x, y; \alpha, \beta)$ extends to an analytic function for $\alpha \in \mathbb{C}, \beta \in \mathbb{C}$.*

(ii) *For fixed $x \in \mathbb{H}$ and fixed $y \in \mathbb{Z}$, the function*

$$\tilde{Z}(x, y; \alpha, \beta) = Z(x, y; \alpha, \beta) - 2\pi\, e^{\frac{1}{2}\pi i(\beta-\alpha)}\, (2x'')^{1-\alpha-\beta}\, \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}$$

extends to an analytic function for $\alpha \in \mathbb{C}, \beta \in \mathbb{C}$.

The generalized Lerch zeta function $Z(x, y; \alpha, \beta)$ is related to the Lerch zeta function via certain limiting behaviors. Suppose that $\mathrm{Re}(s) > 1$, that $x \in \mathbb{H}$ satisfies $0 < \mathrm{Re}(x) < 1$, and that $y \in \mathbb{R}$. Write $x = x' + ix''$ with $x', x'' \in \mathbb{R}$, so $0 < x' < 1$. Among other relations, we show that

$$\lim_{x''\to 0} Z(x' + ix'', y; s, 0) = \zeta(s, y, x') + \mathbf{e}(-y)\, e^{-\pi i s}\, \zeta(s, -y, 1 - x').$$

Chapter 2

Local zeta integrals

2.1 Overview of the chapter

In this chapter we construct a family of local zeta integrals over nonarchimedean local field and prove its analytic properties. The organization of the chapter is as follows.

- Section 2.2 gives preliminary concepts which are necessary for the $\mathfrak{p}$-adic analysis of local zeta integral.
- Section 2.4 defines norm section, Lerch tuple, and local Lerch-Tate zeta integrals.
- The analytic properties of local Lerch-Tate zeta integrals are formulated in Section 2.5. Theorem 2.5.1 asserts that these zeta integrals are rational functions; Theorem 2.5.2 asserts that they have functional equations which are similar to that of the Lerch zeta function.
- Section 2.6 computes certain Gauss sums and Fourier transform.
- Section 2.7 is devoted to the proofs of Theorems 2.5.1 and 2.5.2.
- Section 2.8 works out an example of local Lerch-Tate zeta integrals.

2.2 Notations and basic definitions

Suppose that F is a nonarchimedean local field. Hence F is either a finite extension of $\mathbb{Q}_p$ or the field $\mathbb{F}_q((\pi))$ of formal Laurent series over a finite field. We write $\mathcal{O} = \mathcal{O}_F$ for its ring of integers and $\kappa = \kappa_F$ for its residue field; put $q = \operatorname{card} \kappa$.

The space X_F of all quasi-characters of $F^\times$ has a natural structure of a complex algebraic variety. Furthermore, considered as a complex analytic variety, X_F is a disjoint union of connected components, each of which can be non-canonically identified with $\mathbb{C}^\times$. All quasi-characters of $F^\times$ which have the same restriction to the unit group $\mathcal{O}^\times$ belong to the same connected component. Let us set some notations to specify the connected components of X_F more conveniently.

Definition 2.2.1. If $\chi_0 : \mathcal{O}^\times \to \mathbb{C}^\times$ is a character of the unit group, set

$$X_{F,\chi_0} = \{\chi \in X_F : \chi|_{\mathcal{O}^\times} = \chi_0\}. \tag{2.1}$$

In other words, X_{F,χ_0} is the connected component of X_F labeled by $\chi_0 \in \widehat{\mathcal{O}^\times}$. In particular, if $\chi_{0,\mathrm{nr}}$ is the trivial character on the unit group, then $X_{F,\chi_{0,\mathrm{nr}}}$ is the neutral component consisting of unramified quasi-characters of $F^\times$; we write $X_{F,\mathrm{nr}}$ for this neutral component.

Much of the arithmetic content of an (additive or multiplicative) character is contained in the following notion called *level of a character*. Throughout this chapter, we use ψ to denote an additive character and χ to denote a multiplicative character. Let $\mathrm{val} : F \to \mathbb{Z} \cup \{\infty\}$ denote the normalized valuation on F for which every uniformizer π has valuation one. In other words

$$\mathrm{val}(a) = \begin{cases} \mathrm{ord}_\pi(a) & \text{if } a = u\pi^{\mathrm{ord}_\pi(a)}, u \in \mathcal{O}^\times, \mathrm{ord}_\pi(a) \in \mathbb{Z} \\ +\infty & \text{if } a = 0. \end{cases}$$

Definition 2.2.2. Suppose that $\psi : F \to \mathbb{C}^\times$ is an additive character. Define the *level* of ψ to be the smallest integer l such that ψ has trivial restriction to $\mathfrak{p}^l$. In particular, if ψ has level l and $\psi_a(x) = \psi(ax)$ $(a \in F^\times)$, then the level of ψ_a is $l - \mathrm{val(a)}$.

Definition 2.2.3. (i) Suppose that $\chi_0 : \mathcal{O}^\times \to \mathbb{C}^\times$ is a character. χ_0 is said to be *unramified* if it is trivial; otherwise χ_0 is said to be *ramified*. If χ_0 is ramified, define the *level* of χ_0 is the smallest positive integer m such that χ_0 has trivial restriction to $1 + \mathfrak{p}^m$. If χ_0 is unramified, define the *level* of χ_0 to be 0.

(ii) A quasi-character $\chi : F^\times \to \mathbb{C}^\times$ is said to be *unramified* (resp. *ramified*) if its restriction to $\mathcal{O}^\times$ is unramified (resp. ramified). Define the *level* of χ to be the level of its restriction to $\mathcal{O}^\times$.

We recall the standard notions of Fourier transform on local field and of self-dual measure.

Definition 2.2.4. Suppose that $\psi : F \to \mathbb{C}^\times$ is a nontrivial character and that $\mathrm{d}_+\mathrm{x}$ is a Haar measure on F. If f is a Schwartz-Bruhat function on F, define the Fourier transform of f with respect to ψ and $\mathrm{d}_+\mathrm{x}$ to be the function

$$\hat{f}(y) = \int_F f(x)\,\psi(xy)\ \mathrm{d}_+\mathrm{x} \qquad (y \in F). \tag{2.2}$$

The Fourier transform $\hat{f}$ is also a Schwartz-Bruhat function on F.

Definition 2.2.5. For every nontrivial character ψ of F, there exists a unique Haar measure $\mathrm{d}_+\mathrm{x}$ on F such that, for all Schwartz-Bruhat functions f on F, one has $\hat{\hat{f}}(x) = f(-x)$. This Haar measure $\mathrm{d}_+\mathrm{x}$ is called the *self-dual measure with respect to* ψ.

Example 2.2.6. Suppose that F is a finite extension of $\mathbb{Q}_p$. If $x \in \mathbb{Q}_p$, write $\{x\}_p$ for the p-adic fractional part of x; in particular $x - \{x\}_p \in \mathbb{Z}_p$ for every $x \in \mathbb{Q}_p$. Denote by $\mathrm{tr}_{F/\mathbb{Q}_p}$ the trace homomorphism from F to $\mathbb{Q}_p$. Let $\mathfrak{D}$ denote the different of $F/\mathbb{Q}_p$, i.e. the smallest integral ideal of $\mathcal{O}$ for which $\mathrm{tr}_{F/\mathbb{Q}_p}(\mathfrak{D}^{-1}\mathcal{O}) \subset \mathbb{Z}_p$. Take for $\psi : F \to \mathbb{C}^\times$ the character

$$\psi(x) = \mathbf{e}\left(\left\{\mathrm{tr}_{F/\mathbb{Q}_p}(x)\right\}_p\right) \quad (x \in F).$$

Then the self-dual measure with respect to ψ is the measure for which the ring of integers has volume $(\mathrm{N}\mathfrak{D})^{-\frac{1}{2}}$.

We now introduce Gauss sums which play an important role in the study of local zeta integrals.

Definition 2.2.7. Suppose that $\chi_0 : \mathcal{O}^\times \to \mathbb{C}^\times$ is a character and that f is a complex-valued locally constant function on F. If n is a positive integrer, define the *Gauss sums* $G(f, \chi_0)$ and $G_n(f, \chi_0)$ to be

$$G(f, \chi_0) = \frac{1}{\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)} \int_{\mathcal{O}^\times} f(x)\,\chi_0(x)\ \mathrm{d}^\times\mathrm{x} \tag{2.3}$$

$$G_n(f, \chi_0) = \frac{1}{\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)} \int_{1+\mathfrak{p}^n} f(x)\,\chi_0(x)\ \mathrm{d}^\times\mathrm{x} \tag{2.4}$$

where $\mathrm{d}^\times\mathrm{x}$ is any Haar measure on $F^\times$. Note that the values of $G(f, \chi_0)$ and $G_n(f, \chi_0)$ are independent of choice of Haar measure on $F^\times$.

Remark 2.2.8. Suppose that $\psi : F \to \mathbb{C}^\times$ is a character of level l and $\chi_0 : \mathcal{O}^\times \to \mathbb{C}^\times$

is a character of the same level $l \geq 1$. Then ψ and χ_0 induce finite characters

$$\overline{\psi} : \mathcal{O}/\mathfrak{p}^l \to \mathbb{C}^\times, \qquad \overline{\chi_0} : \left(\mathcal{O}/\mathfrak{p}^l\right)^\times \to \mathbb{C}^\times.$$

Let n be a positive integer such that $n \leq l$. Consider the *finite Gauss sums*

$$G_{\mathfrak{p}^l}(\overline{\psi}, \overline{\chi_0}) := \sum_{\alpha \in (\mathcal{O}/\mathfrak{p}^l)^\times} \overline{\psi}(\alpha)\overline{\chi_0}(\alpha). \tag{2.5}$$

$$G_{\mathfrak{p}^l,n}(\overline{\psi}, \overline{\chi_0}) := \sum_{\substack{\alpha \in (\mathcal{O}/\mathfrak{p}^l)^\times \\ \alpha \equiv 1 \pmod{\mathfrak{p}^n}}} \overline{\psi}(\alpha)\overline{\chi_0}(\alpha). \tag{2.6}$$

It follows from (2.3) and (2.5) that

$$G(\psi, \chi_0) = \frac{1}{q^l - q^{l-1}}\ G_{\mathfrak{p}^l}(\overline{\psi}, \overline{\chi_0}). \tag{2.7}$$

Similarly, by (2.4) and (2.6),

$$G_n(\psi, \chi_0) = \frac{1}{q^l - q^{l-1}}\ G_{\mathfrak{p}^l,n}(\overline{\psi}, \overline{\chi_0}). \tag{2.8}$$

The identities (2.7) and (2.8) justify why the integrals (2.3) and (2.4) are called Gauss sums.

2.3 Local norm sections

Recall $\mathrm{val} : \mathrm{F} \to \mathbb{Z} \cup \{\infty\}$ is the normalized valuation which takes unit value on any uniformizer. The continuous homomorphism

$$|\cdot| : F^\times \to \mathbb{R}_+, \quad |x| = q^{-\mathrm{val(x)}}$$

is called the *normalized norm* of F. Write N for its image, so $N = q^{\mathbb{Z}} \subset \mathbb{R}_+$.

Definition 2.3.1. A group homomorphism $\tau : N \to F^\times$ is called a *norm section* if $|\cdot| \circ \tau = \mathrm{id_N}$.

Lemma 2.3.2. *The assignment $\tau \mapsto \tau(q^{-1})$ defines a bijection between the set of norm sections and the set of uniformizers of F.*

Proof. Since $N = q^{\mathbb{Z}}$ is cyclic, every norm section τ is determined by its value on q^{-1}. It is apparent that $\pi = \tau(q^{-1})$ is a uniformizer of F. The lemma is shown. □

2.4 Local Lerch-Tate zeta integrals

Definition 2.4.1. A *local Lerch tuple* is a triple (τ, a, c) consisting of a norm section τ and a pair $(a, c) \in F^2$.

Definition 2.4.2. *(Local Lerch-Tate zeta integral)* Suppose that (τ, a, c) is a local Lerch tuple. If f is a Schwartz-Bruhat function on F and χ is a quasi-character of $F^\times$, define the local Lerch-Tate zeta integral

$$Z_{\tau,a,c}(f, \chi) = \int_{F^\times} \psi\left(\frac{ax}{\tau(|x|)}\right) f\big(x + c\,\tau(|x|)\big)\,\chi(x)\ \mathrm{d}^\times \mathrm{x}. \tag{2.9}$$

Remark 2.4.3. On setting $a = c = 0$, one recovers Tate's local zeta integral (1.9). Namely

$$Z_{\tau,0,0}(f, \chi) = Z(f, \chi) = \int_{F^\times} f(x)\chi(x)\,\mathrm{d}^\times x.$$

This special caes explains the name *local Lerch-Tate zeta integrals.*

Remark 2.4.4. One gains a better understanding of the local zeta integral $Z_{\tau,a,c}(f, \chi)$ with the following observation: every quasi-character of $F^\times$ is determined by its restriction to the unit group and its value on a uniformizer.

More exactly, suppose that a norm section τ is given and that $\pi = \tau(q^{-1})$. If χ is a quasi-character of $F^\times$, put $\chi_0 = \chi|_{\mathcal{O}^\times}$ and $z = \chi(\pi) \in \mathbb{C}^\times$. Then χ_0 determines the connected component of X_F that χ belongs to, namely the connected component X_{F,χ_0} (Definition 2.2.1). One can view X_{F,χ_0} as the complex algebraic variety $\mathbb{C}^\times$ by identifying a quasi-character χ with its value z on π. One can also define a complex variable $s \in \mathbb{C}/\frac{2\pi i}{\log q}\mathbb{Z}$ such that $z = \chi(\pi) = q^{-s}$. This definition of s depends on the choice of the uniformizer π, but $\mathrm{Re}(s)$ is independent of this choice.

Then the restriction of $Z_{\tau,a,c}(f, \cdot)$ to X_{F,χ_0} becomes a function of z; write $Z_{\pi,a,c}(f, \chi_0, z)$ for this function. By (2.9) we have

$$Z_{\pi,a,c}(f, \chi_0, z) = \sum_{n=-\infty}^{\infty} \int_{\mathcal{O}^\times} \psi(au)\, f\left(\pi^n(u + c)\right)\, \chi_0(u)\, z^n\ \mathrm{d}^\times \mathrm{u}.$$

We shall use $Z_{\tau,a,c}(f, \cdot)$ to refer to a function on the whole complex variety X_F and use $Z_{\pi,a,c}(f, \chi_0, \cdot)$ to mean a function on the connected component X_{F,χ_0}. One only needs to remember that, if the norm section τ corresponds to the uniformizer π and if χ is a quasi-character on $F^\times$ with restriction χ_0 to the unit group and with value z on π, then $Z_{\tau,a,c}(f, \chi) = Z_{\pi,a,c}(f, \chi_0, z)$.

Remark 2.4.5. We shall see that the convergence of the integral (2.9) is delicate.

Suppose that $f = 1_{\mathcal{O}_F}$. For $c = -1$, the equation

$$x_0 + c\,\tau(|x_0|) = 0$$

with $x_0 = \pi^m$ holds for all $m \in \mathbb{Z}$. For $m \leq -1$ and

$$x := x_0 + y \in D(x_0, 1) = \{x : |x - x_0| < 1\}$$

one has

$$f(x + c\,\tau(|x|)) = f(y) = 1.$$

For a suitable choice of a, the term $\psi\left(\frac{ax}{\tau(|x|)}\right)$ is non-constant on $D(x_0, 1)$ for all $m \geq 1$. In this case the integrand

$$\psi\left(\frac{ax}{\tau(|x|)}\right) f\big(x + c\,\tau(|x|)\big)$$

is not compactly supported on F, so it is not a Schwartz-Bruhat function on F. The convergence of the integral (2.9) will be restricted to the annulus

$$\frac{1}{q} < |z| < 1, \text{ resp. } 0 < \mathrm{Re}(s) < 1.$$

2.5 The Main Local Theorem

In this section we formulate the analytic properties of the local Lerch-Tate zeta integrals as two theorems: Theorem 2.5.1 and Theorem 2.5.2. Their combination gives Theorem 1.5.1 which we call the Main Local Theorem.

Theorem 2.5.1. *Let F be a nonarchimedean local field. Let (τ, a, c) be a local Lerch tuple and f be a Schwartz-Bruhat function on F; set $\pi = \tau(q^{-1})$. Let ψ be an additive character of level l of F. Take for F the self-dual measure $\mathrm{d}_+\mathrm{x}$ with respect to ψ; take for $F^\times$ any Haar measure $\mathrm{d}^\times\mathrm{x}$.*

(i) *(Absolute convergence and analytic continuation) On each connected component of X_F, the zeta integral $Z_{\tau,a,c}(f, \chi)$ converges absolutely for $0 < \mathrm{Re}(\chi) < 1$ to a rational function (of the variable $z = \chi(\pi)$).*

(ii) *(Poles and residues) Let χ_0 be a character on the unit group.*

If $c \notin \mathcal{O}^\times$, the restriction of $Z_{\tau,a,c}(f, \chi)$ to the connected component X_{F,χ_0} has at most a simple pole at the unique quasi-character $\chi_{\tau,0} \in X_{F,\chi_0}$ which takes *value* 1 *on* π. *The residue of* $Z_{\tau,a,c}(f,\cdot)$ *at* $\chi_{\tau,0}$ *is*

$$\operatorname{Res}(Z_{\tau,a,c}(f,\chi) : \chi = \chi_{\tau,0}) = -f(0)\, \mathrm{d}^{\times}\mathrm{x}(\mathcal{O}^{\times})\, G(\psi_a, \chi_0).$$

If $c \in \mathcal{O}^{\times}$, *the restriction of* $Z_{\tau,a,c}(f,\chi)$ *to the connected component* X_{F,χ_0} *has at most two simple poles at the unique quasi-character* $\chi_{\tau,0} \in X_{F,\chi_0}$ *which takes value* 1 *on* π *and at the unique quasi-character* $\chi_{\tau,-1} \in X_{F,\chi_0}$ *which takes value* q^{-1} *on* π. *The residue of* $Z_{\tau,a,c}(f,\cdot)$ *at* $\chi_{\tau,0}$ *is*

$$\operatorname{Res}(Z_{\tau,a,c}(f,\chi) : \chi = \chi_{\tau,0}) = -f(0)\, \mathrm{d}^{\times}\mathrm{x}(\mathcal{O}^{\times})\, G(\psi_a, \chi_0).$$

The residue of $Z_{\tau,a,c}(f,\cdot)$ *at* $\chi_{\tau,-1}$ *is*

$$\operatorname{Res}(Z_{\tau,a,c}(f,\chi) : \chi = \chi_{\tau,-1}) = \frac{h}{q}\psi(-ac)\chi_0(-c) \int_F f(x)\, \mathrm{d}_+\mathrm{x}$$

where $h = \frac{\mathrm{d}^{\times}\mathrm{x}(\mathcal{O}^{\times})}{\mathrm{d}_+\mathrm{x}(\mathcal{O}^{\times})}$.

Observe that Tate's local zeta integral

$$Z_{\tau,0,0}(f,\chi) = \int_{F^{\times}} f(x)\chi(x)\, \mathrm{d}^{\times}\mathrm{x}$$

has at most a simple pole at $z = 1$. The local Lerch-Tate zeta integrals are more complicated and may have poles at $z = 1$ and $z = \frac{1}{q}$.

Theorem 2.5.2. *Let* F *be a nonarchimedean local field. Let* (τ, a, c) *be a local Lerch tuple and* f *be a Schwartz-Bruhat function on* F; *set* $\pi = \tau(q^{-1})$. *Let* ψ *be an additive character of level* l *of* F. *Take for* F *the self-dual measure* $\mathrm{d}_+\mathrm{x}$ *with respect to* ψ; *take for* $F^{\times}$ *any Haar measure* $\mathrm{d}^{\times}\mathrm{x}$.

Let χ_0 *be a character of level* m *on the unit group. Let* $Z_{\pi,a,c}(f,\chi_0,z)$ *denote the restriction of* $Z_{\tau,a,c}(f,\cdot)$ *to the connected component* X_{F,χ_0}. *In other words, if* $\chi|_{\mathcal{O}^{\times}} = \chi_0$ *and* $\chi(\pi) = z$, *then* $Z_{\tau,a,c}(f,\chi) = Z_{\pi,a,c}(f,\chi_0,z)$. *There holds the functional equations:*

- *if* χ *is unramified, then*

$$\frac{Z_{\pi,a,c}(f,\chi_{0,\mathrm{nr}},z)}{\psi(-ac)\, \mathrm{d}_+\mathrm{x}\,(\mathcal{O}^{\times})} = \sum_{r \geq l} z^r\, Z_{\pi,-c\pi^r,a\pi^{-r}}\left(\hat{f}, \chi_{0,\mathrm{nr}}, \frac{1}{qz}\right) + \frac{z^{l-1}}{1-q}\, Z_{\pi,-c\pi^{l-1},a\pi^{1-l}}\left(\hat{f}, \chi_{0,\mathrm{nr}}, \frac{1}{qz}\right);$$

- *if χ is ramified of level $m \geq 1$, then*

$$\frac{Z_{\pi,a,c}(f,\chi_0,z)}{\chi_0(-1)\,\psi(-ac)\,\mathrm{d}_+\mathrm{x}\,(\mathcal{O}^\times)} = z^{l-m}\, G(\psi_{\pi^{l-m}},\chi_0)\ Z_{\pi,-c\pi^{l-m},a\pi^{m-l}}\left(\hat{f},\chi_0^{-1},\frac{1}{qz}\right).$$

Observe that the local functional equation for ramified χ has the schematic form

$$Z_{\pi,a,c}(f,\chi_0,z) = \psi(-ac)\,\chi_0(-1)\,T(\chi,\psi)\,z^{l-m}\,Z_{\pi,-c',a'}\left(\hat{f},\chi_0^{-1},\frac{1}{qz}\right)$$

where $T(\chi,\psi)$ depends on χ and ψ and

$$c' = c\pi^{l-m}, \quad a' = a\pi^{m-l}$$

so that $ac = a'c'$. The local functional equation for unramified χ is more complicated. It has an extra term with an infinite sum over powers of z. It is a interesting question to find out a conceptual explanation of this sum. In both cases, the functional equation has a prefactor $\psi(-ac)$ which resembles the functional equation of the Lerch zeta function.

2.6 Gauss sums and Fourier transform

To prepare for the proof of Theorem 2.5.1, in this section we prove several lemmas on $\mathfrak{p}$-adic integration. Lemmas 2.6.1 and 2.6.2 compute Gauss sums of an additive character and a multiplicative character. Lemma 2.6.3 and Corollary 2.6.4 demonstrate a duality of Gauss sum via Fourier transform. Finally, Lemma 2.6.6 records a Fourier transform calculation.

Lemma 2.6.1. *Suppose that $\psi : F \to \mathbb{C}^\times$ is a character of level l and that $\chi_0 : \mathcal{O}^\times \to \mathbb{C}^\times$ is a character of level m.*

(i) *If χ_0 is unramified, then*

$$G(\psi,\chi_0) = \begin{cases} 1 & (\textit{if } \ l \leq 0) \\ \frac{1}{1-q} & (\textit{if } \ l = 1) \\ 0 & (\textit{if } \ l > 1). \end{cases}$$

(ii) *If χ_0 is ramified, then*

$$G(\psi,\chi_0)=\begin{cases}0 & (\textit{if}\ \ l\neq m)\\ \frac{1}{q^l-q^{l-1}}\ G_{\mathfrak{p}^l}(\overline{\psi},\overline{\chi_0}) & (\textit{if}\ \ l=m)\end{cases}$$

where, in the case $l=m$, ψ and χ_0 induce finite characters

$$\overline{\psi}:\mathcal{O}/\mathfrak{p}^l\to\mathbb{C}^\times,\qquad \overline{\chi_0}:\left(\mathcal{O}/\mathfrak{p}^l\right)^\times\to\mathbb{C}^\times$$

and $G_{\mathfrak{p}^l}(\overline{\psi},\overline{\chi_0})$ is given by (2.5).

Proof. (i) Assume χ_0 is unramified. Take for $F^\times$ a Haar measure $\mathrm{d}^\times\mathrm{x}$; it follows that

$$G(\psi,\chi_0)=\frac{1}{\mathrm{d}^\times\mathrm{x}\,(\mathcal{O}^\times)}\int_{\mathcal{O}^\times}\psi(x)\chi_0(x)\ \mathrm{d}^\times\mathrm{x}=\frac{1}{\mathrm{d}^\times\mathrm{x}\,(\mathcal{O}^\times)}\int_{\mathcal{O}^\times}\psi(x)\ \mathrm{d}^\times\mathrm{x}.$$

If $l\leq 0$, then ψ is trivial on $\mathcal{O}$, thereby $G(\psi,\chi_0)=1$. If $l=1$, then ψ induces a nontrivial character $\overline{\psi}:\mathcal{O}/\mathfrak{p}\to\mathbb{C}^\times$ and hence

$$G(\psi,\chi_0)=\frac{1}{q-1}\sum_{a\in\mathcal{O}/\mathfrak{p},a\neq 0}\overline{\psi}(a)=\frac{1}{1-q}.$$

If $l>1$, then ψ induces a nontrivial character $\overline{\psi}:\mathcal{O}/\mathfrak{p}^l\to\mathbb{C}^\times$ and a nontrivial character $\overline{\psi}_l:\mathfrak{p}^{l-1}/\mathfrak{p}^l\to\mathbb{C}^\times$. Let $\{a_j:0\leq j<q\}\subset\mathcal{O}$ be the set of Teichmüller representatives of the residue field κ; rearrange the elements if necessary so that $a_0=0$. Let π be an arbitrary uniformizer. On writing $a\equiv a_{j_0}+\pi a_{j_1}+\cdots+\pi^{l-1}a_{j_{l-1}}$ mod $\mathfrak{p}^l$, we find that

$$\begin{aligned}G(\psi,\chi_0)&=\frac{\mathrm{d}^\times\mathrm{x}\,(1+\mathfrak{p}^l)}{\mathrm{d}^\times\mathrm{x}\,(\mathcal{O}^\times)}\sum_{a\in\mathcal{O}/\mathfrak{p}^l,a\neq 0}\overline{\psi}(a)\\&=\frac{1}{q^l-q^{l-1}}\sum_{j_0=1}^{q-1}\psi(a_{j_0})\sum_{j_1=0}^{q-1}\psi(\pi a_{j_1})\cdots\sum_{j_{l-1}=0}^{q-1}\psi(\pi^{l-1}a_{j_{l-1}}).\end{aligned}$$

But in the last sum, the elements $\{\pi^{l-1}a_{j_{l-1}}:0\leq j_{l-1}<q\}$ form a complete set of representatives of $\mathfrak{p}^{l-1}/\mathfrak{p}^l$. Hence, by the nontriviality of $\overline{\psi}_l$,

$$\sum_{j_{l-1}=0}^{q-1}\psi(\pi^{l-1}a_{j_{l-1}})=\sum_{a\in\mathfrak{p}^{l-1}/\mathfrak{p}^l}\overline{\psi}_l(a)=0.$$

Thus $G(\psi, \chi_0) = 0$. This completes the proof of (i).

(ii) Assume χ_0 is ramified of level $m \geq 1$. This means χ_0 has trivial restriction to $1 + \mathfrak{p}^m$ but not to $1 + \mathfrak{p}^{m-1}$. We consider three cases:

(a) Case $l < m$. Then χ_0 is nontrivial on $\mathcal{O}^\times \cap (1 + \mathfrak{p}^l)$. If $l \leq 0$, it follows from the nontriviality of χ_0 on the compact group $\mathcal{O}^\times$ that

$$G(\psi, \chi_0) = \frac{1}{\mathrm{d}^\times \mathrm{x}(\mathcal{O}^\times)} \int_{\mathcal{O}^\times} \chi_0(x) \, \mathrm{d}^\times \mathrm{x} = 0.$$

If $l > 0$, on decomposing the integral $\int_{\mathcal{O}^\times}$ modulo $\mathfrak{p}^l$,

$$G(\psi, \chi_0) = \frac{1}{\mathrm{d}^\times \mathrm{x}(\mathcal{O}^\times)} \sum_{a \, (\mathrm{mod}^\times \mathfrak{p}^l)} \psi(a) \chi_0(a) \int_{1+\mathfrak{p}^l} \chi_0(u) \, \mathrm{d}^\times \mathrm{u},$$

which vanishes because of the nontriviality of χ_0 on the compact group $1 + \mathfrak{p}^l$. Thus, $G(\psi, \chi_0) = 0$ provided that $l < m$ and that $m \geq 1$.

(b) Case $l > m$. Then χ_0 is trivial on $1 + \mathfrak{p}^{l-1}$. Let $\{a_j : 0 \leq j < q\} \subset \mathcal{O}$ be the set of Teichmüller representatives of the residue field κ; rearrange the elements if necessary so that $a_0 = 0$. Let π be an arbitrary uniformizer. On decomposing the integral $\int_{\mathcal{O}^\times}$ modulo $\mathfrak{p}^l$, we deduce that

$$G(\psi, \chi_0) = \frac{\mathrm{d}^\times \mathrm{x} \, (1 + \mathfrak{p}^l)}{\mathrm{d}^\times \mathrm{x}(\mathcal{O}^\times)} \sum_{a \, (\mathrm{mod}^\times \mathfrak{p}^l)} \psi(a) \, \chi_0(a).$$

On writing $a = b + \pi^{l-1} a_j$ with $b \, (\mathrm{mod}^\times \, \mathfrak{p}^{l-1})$ and $0 \leq j < q$, we find that

$$\begin{aligned} G(\psi, \chi_0) &= (q^l - q^{l-1}) \sum_{b \, (\mathrm{mod}^\times \mathfrak{p}^{l-1})} \sum_{j=0}^{q-1} \psi(b + \pi^{l-1} a_j) \chi_0(b + \pi^{l-1} a_j) \\ &= (q^l - q^{l-1}) \sum_{b \, (\mathrm{mod}^\times \mathfrak{p}^{l-1})} \psi(b) \chi_0(b) \sum_{j=0}^{q-1} \psi(\pi^{l-1} a_j). \end{aligned}$$

But the last sum $\sum_{j=0}^{q-1}$ vanishes because ψ is nontrivial on $\mathfrak{p}^{l-1}$. Thus, $G(\psi, \chi_0) = 0$ provided that $l > m \geq 1$.

(c) Case $l = m$. Then ψ and χ_0 induce finite characters

$$\overline{\psi} : \mathcal{O}/\mathfrak{p}^l \to \mathbb{C}^\times, \qquad \overline{\chi_0} : \left(\mathcal{O}/\mathfrak{p}^l\right)^\times \to \mathbb{C}^\times,$$

the latter via the natural isomorphism $\mathcal{O}^\times/1+\mathfrak{p}^l \cong \left(\mathcal{O}/\mathfrak{p}^l\right)^\times$. By (2.7),

$$G(\psi,\chi_0) = \frac{\mathrm{d}^\times\mathrm{x}\,(1+\mathfrak{p}^l)}{\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)}\ G_{\mathfrak{p}^l}(\overline{\psi},\overline{\chi_0}) = \frac{1}{q^l-q^{l-1}}\ G_{\mathfrak{p}^l}(\overline{\psi},\overline{\chi_0}).$$

This completes the proof of (ii).

□

Lemma 2.6.2. *Suppose that $\psi : F \to \mathbb{C}^\times$ is a character of level l and that $\chi_0 : \mathcal{O}^\times \to \mathbb{C}^\times$ is a character of level m. Suppose that n is a positive integer.*

(i) *If $m \le n$, then*

$$G_n(\psi,\chi_0) = \begin{cases} \frac{\psi(1)}{q^n-q^{n-1}} & (\textit{if } \ l \le n) \\ 0 & (\textit{if } \ l > n). \end{cases}$$

(ii) *If $m > n$, then*

$$G_n(\psi,\chi_0) = \begin{cases} 0 & (\textit{if } \ l \ne m) \\ \frac{1}{q^l-q^{l-1}}\ G_{\mathfrak{p}^l,n}(\overline{\psi},\overline{\chi_0}) & (\textit{if } \ l = m) \end{cases}$$

where, in the case $l = m$, ψ and χ_0 induce finite characters

$$\overline{\psi} : \mathcal{O}/\mathfrak{p}^l \to \mathbb{C}^\times, \qquad \overline{\chi_0} : \left(\mathcal{O}/\mathfrak{p}^l\right)^\times \to \mathbb{C}^\times$$

and $G_{\mathfrak{p}^l,n}(\overline{\psi},\overline{\chi_0})$ is given by (2.6).

Proof. (i) Assume $m \le n$. Take for F a Haar measure $\mathrm{d}_+\mathrm{x}$ and for $F^\times$ the Haar measure $\mathrm{d}^\times\mathrm{x} = \frac{\mathrm{d}_+\mathrm{x}}{|x|}$; it follows that

$$G(\psi,\chi_0) = \frac{1}{\mathrm{d}^\times\mathrm{x}\,(\mathcal{O}^\times)} \int_{1+\mathfrak{p}^n} \psi(x)\ \mathrm{d}^\times\mathrm{x} = \frac{\psi(1)}{\mathrm{d}_+\mathrm{x}\,(\mathcal{O}^\times)} \int_{\mathfrak{p}^n} \psi(x)\ \mathrm{d}_+\mathrm{x}.$$

If $l \le n$, then ψ is trivial on $\mathfrak{p}^n$, thereby $G(\psi,\chi_0) = \frac{\mathrm{d}_+\mathrm{x}\,(\mathfrak{p}^n)}{\mathrm{d}_+\mathrm{x}\,(\mathcal{O}^\times)}\,\psi(1) = \frac{\psi(1)}{q^n-q^{n-1}}$. If $l > n$, then ψ induces a nontrivial character on the compact group $\mathfrak{p}^n$ and hence $G(\psi,\chi_0) = 0$.

(ii) Assume $m > n$. Hence χ_0 has nontrivial restriction to $1+\mathfrak{p}^n$. Take for F a Haar measure $\mathrm{d}_+\mathrm{x}$ and for $F^\times$ a Haar measure $\mathrm{d}^\times\mathrm{x} = \frac{\mathrm{d}_+\mathrm{x}}{|x|}$. We consider three cases:

(a) <u>Case $l < m$.</u> Then χ_0 is nontrivial on $(1+\mathfrak{p}^n)\cap(1+\mathfrak{p}^l)$. If $l \le n$, it follows from the nontriviality of χ_0 on the compact group $1+\mathfrak{p}^n$ that

$$G(\psi,\chi_0) = \frac{\psi(1)}{\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)} \int_{1+\mathfrak{p}^n} \chi_0(x)\ \mathrm{d}^\times\mathrm{x} = 0.$$

If $n < l < m$, on decomposing the integral $\int_{\mathcal{O}^\times}$ modulo $\mathfrak{p}^l$,

$$G(\psi,\chi_0) = \frac{1}{\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)} \sum_{\substack{a \,(\mathrm{mod}^\times \mathfrak{p}^l) \\ a\equiv 1 \,(\mathrm{mod}\ \mathfrak{p}^n)}} \psi(a)\chi_0(a) \int_{1+\mathfrak{p}^l} \chi_0(u)\,\mathrm{d}^\times\mathrm{u},$$

which vanishes because of the nontriviality of χ_0 on the compact group $1+\mathfrak{p}^l$. Thus, $G(\psi,\chi_0) = 0$ provided that $m > n$ and that $m > l$.

(b) <u>Case $l > m$</u>. Then χ_0 is trivial on $1+\mathfrak{p}^{l-1}$. Let $\{a_j : 0 \le j < q\} \subset \mathcal{O}$ be the set of Teichmüller representatives of the residue field κ; rearrange the elements if necessary so that $a_0 = 0$. Let π be an arbitrary uniformizer. On decomposing the integral $\int_{\mathcal{O}^\times}$ modulo $\mathfrak{p}^l$, we deduce that

$$G(\psi,\chi_0) = \frac{\mathrm{d}^\times\mathrm{x}\,(1+\mathfrak{p}^l)}{\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)} \sum_{\substack{a \,(\mathrm{mod}^\times \mathfrak{p}^l) \\ a\equiv 1 \,(\mathrm{mod}\ \mathfrak{p}^n)}} \psi(a)\,\chi_0(a).$$

On writing $a = b + \pi^{l-1}a_j$ with $b\,(\mathrm{mod}^\times\ \mathfrak{p}^{l-1})$ and $0 \le j < q$, we find that

$$\begin{aligned} G(\psi,\chi_0) &= (q^l - q^{l-1}) \sum_{\substack{b \,(\mathrm{mod}^\times \mathfrak{p}^l) \\ b\equiv 1 \,(\mathrm{mod}\ \mathfrak{p}^n)}} \sum_{j=0}^{q-1} \psi(b+\pi^{l-1}a_j)\chi_0(b+\pi^{l-1}a_j) \\ &= (q^l - q^{l-1}) \sum_{\substack{b \,(\mathrm{mod}^\times \mathfrak{p}^l) \\ b\equiv 1 \,(\mathrm{mod}\ \mathfrak{p}^n)}} \psi(b)\chi_0(b) \sum_{j=0}^{q-1} \psi(\pi^{l-1}a_j). \end{aligned}$$

But the last sum $\sum_{j=0}^{q-1}$ vanishes because ψ is nontrivial on $\mathfrak{p}^{l-1}$. Thus, $G(\psi,\chi_0) = 0$ provided that $l > m > n$.

(c) <u>Case $l = m$</u>. Then ψ and χ_0 induce finite characters

$$\overline{\psi} : \mathcal{O}/\mathfrak{p}^l \to \mathbb{C}^\times, \qquad \overline{\chi_0} : \left(\mathcal{O}/\mathfrak{p}^l\right)^\times \to \mathbb{C}^\times,$$

the latter via the natural isomorphism $\mathcal{O}^\times/1+\mathfrak{p}^l \cong \left(\mathcal{O}/\mathfrak{p}^l\right)^\times$. By (2.8),

$$G(\psi, \chi_0) = \frac{1}{q^l - q^{l-1}}\; G_{\mathfrak{p}^l,n}(\overline{\psi}, \overline{\chi_0}).$$

This completes the proof of (ii).

$\square$

Lemma 2.6.3. *Suppose that $\psi : F \to \mathbb{C}^\times$ is a character of level l and that $\chi_0 : \mathcal{O}^\times \to \mathbb{C}^\times$ is a character of level m. Take for F a Haar measure $\mathrm{d}_+\mathrm{x}$. Let f be a Schwartz-Bruhat function on F and write $\hat{f}$ for the Fourier transform of f with respect to ψ and $\mathrm{d}_+\mathrm{x}$.*

(i) *If χ_0 is unramified, then*

$$G(\hat{f}, \chi_0) = \int_F f_1(x)\;\mathrm{d}_+\mathrm{x}$$

where

$$f_1(x) = \begin{cases} f(x) & (\textit{if}\;\; \mathrm{val(x)} \geq \mathrm{l}) \\ \frac{f(x)}{1-q} & (\textit{if}\;\; \mathrm{val(x)} = \mathrm{l}-1) \\ 0 & (\textit{if}\;\; \mathrm{val(x)} < \mathrm{l}-1). \end{cases}$$

(ii) *If χ_0 is ramified and π is a uniformizer, then*

$$G(\hat{f}, \chi_0) = q^{m-l}\,\mathrm{d}_+\mathrm{x}(\mathcal{O}^\times)\, G(\psi_{\pi^{l-m}}, \chi_0)\, G(f_{\pi^{l-m}}, \chi_0^{-1})$$

where $f_{\pi^{l-m}}$ is the function given by $f_{\pi^{l-m}}(x) = f(\pi^{l-m}x)$.

Proof. The idea is to rewrite $G(\hat{f}, \chi_0)$ using Fubini's theorem and then appeal to Lemma 2.6.1.

For every $x \in F$, let ψ_x denote the character $\psi_x(y) = \psi(xy)$. If $x \in F^\times$, the character ψ_x has level $l - \mathrm{val(x)}$. Since the Gauss sum does not depend on choice of Haar measure on $F^\times$, take for $F^\times$ the normalized Haar measure $\mathrm{d}^\times\mathrm{x}$ such that the unit group has volume one. By Fubini's theorem,

$$\begin{aligned} G(\hat{f},\chi_0) &= \int_{\mathcal{O}^\times} \hat{f}(u)\,\chi_0(u)\ \mathrm{d}^\times \mathrm{u} \\ &= \int_{\mathcal{O}^\times}\int_F f(x)\,\psi(xu)\,\chi_0(u)\ \mathrm{d}_+\mathrm{x}\ \mathrm{d}^\times \mathrm{u} \\ &= \int_F\int_{\mathcal{O}^\times} f(x)\,\psi(xu)\,\chi_0(u)\ \mathrm{d}^\times \mathrm{u}\ \mathrm{d}_+\mathrm{x} \\ &= \int_F f(x)\,G(\psi_x,\chi_0)\ \mathrm{d}_+\mathrm{x}. \end{aligned}$$

We are in a position to show (i). The integral over F can be decomposed according to the valuation of x. By the unramified assumption on χ_0 and Lemma 2.6.1 (i), if $\mathrm{val(x)} \geq \mathrm{l}$ (resp. $\mathrm{val(x)} = \mathrm{l}-1$, resp. $\mathrm{val(x)} < \mathrm{l}-1$), then $G(\psi_x,\chi_0)$ takes value 1 (resp. $\frac{1}{1-q}$, resp. 0). It follows that

$$G(\hat{f},\chi_0) = \int_F f_1(x)\ \mathrm{d}_+\mathrm{x}$$

where

$$f_1(x) = \begin{cases} f(x) & (\text{if}\ \ \mathrm{val(x)} \geq \mathrm{l}) \\ \frac{f(x)}{1-q} & (\text{if}\ \ \mathrm{val(x)} = \mathrm{l}-1) \\ 0 & (\text{if}\ \ \mathrm{val(x)} < \mathrm{l}-1). \end{cases}$$

This proves (i).

We now show (ii). Assume χ_0 is ramified of level m. By Lemma 2.6.1 (ii), $G(\psi_x,\chi_0) = 0$ unless the levels of ψ_x and χ_0 coincide, in which case $\mathrm{val(x)} = \mathrm{l}-\mathrm{m}$. The change of variable $x = \pi^{l-m}x'$ translates this condition into $x' \in \mathcal{O}^\times$. Since $\mathrm{d}_+\mathrm{x} = q^{m-l}\ \mathrm{d}_+\mathrm{x}'$, by Lemma 2.6.1 (ii),

$$\begin{aligned} G(\hat{f},\chi_0) &= \frac{1}{q^m-q^{m-1}}\,q^{m-l}\int_F f(\pi^{l-m}x')\,G_{\mathfrak{p}^m}(\overline{\psi_{\pi^{l-m}x'}},\overline{\chi_0})\,1_{\mathcal{O}^\times}(x')\ \mathrm{d}_+\mathrm{x}' \\ &= \frac{1}{q^l-q^{l-1}}\int_{\mathcal{O}^\times} f(\pi^{l-m}x')\,G_{\mathfrak{p}^m}(\overline{\psi_{\pi^{l-m}x'}},\overline{\chi_0})\ \mathrm{d}_+\mathrm{x}'. \end{aligned}$$

But it follows from (2.5) that

$$\begin{aligned} G_{\mathfrak{p}^m}(\overline{\psi_{\pi^{l-m}x'}}, \overline{\chi_0}) &= \sum_{a \in \mathcal{O}/\mathfrak{p}^m, a\neq 0} \overline{\psi_{\pi^{l-m}x'}}(a)\overline{\chi_0}(a) \\ &= \chi_0^{-1}(x') \sum_{a \in \mathcal{O}/\mathfrak{p}^m, a\neq 0} \overline{\psi_{\pi^{l-m}}}(ax')\overline{\chi_0}(ax') \\ &= \chi_0^{-1}(x')\, G_{\mathfrak{p}^m}(\overline{\psi_{\pi^{l-m}}}, \overline{\chi_0}). \end{aligned}$$

Therefore

$$\begin{aligned} G(\hat{f}, \chi_0) &= \frac{1}{q^l - q^{l-1}}\, G_{\mathfrak{p}^m}(\overline{\psi_{\pi^{l-m}}}, \overline{\chi_0}) \int_{\mathcal{O}^\times} f(\pi^{l-m}x')\, \chi_0^{-1}(x')\ \mathrm{d_+x'} \\ &= q^{m-l}\, G(\psi_{\pi^{l-m}}, \chi_0) \int_{\mathcal{O}^\times} f(\pi^{l-m}x')\, \chi_0^{-1}(x')\ \mathrm{d_+x'}. \end{aligned}$$

This proves (ii). $\square$

Corollary 2.6.4. *Suppose that $\psi : F \to \mathbb{C}^\times$ is a nontrivial character of level l and that $\chi_0 : \mathcal{O}^\times \to \mathbb{C}^\times$ is a character of level m. Take for F the self-dual measure $\mathrm{d_+x}$ with respect to ψ. Let f be a Schwartz-Bruhat function on F and write $\hat{f}$ for the Fourier transform of f with respect to ψ and $\mathrm{d_+x}$.*

(i) *If χ_0 is unramified, then*

$$G(f, \chi_0) = \int_F \left(\hat{f}\right)_1 (y)\ \mathrm{d_+y}$$

where

$$\left(\hat{f}\right)_1 (x) = \begin{cases} \hat{f}(x) & (\textit{if}\ \ \mathrm{val(x) \geq l}) \\ \frac{\hat{f}(x)}{1-q} & (\textit{if}\ \ \mathrm{val(x) = l-1}) \\ 0 & (\textit{if}\ \ \mathrm{val(x) < l-1}). \end{cases}$$

(ii) *If χ_0 is ramified and π is a uniformizer, then*

$$G(f, \chi_0) = \chi_0(-1)\, q^{m-l}\, \mathrm{d_+x}(\mathcal{O}^\times)\, G(\psi_{\pi^{l-m}}, \chi_0)\, G(\hat{f}_{\pi^{l-m}}, \chi_0^{-1})$$

where $\hat{f}_{\pi^{l-m}}$ is the function given by $\hat{f}_{\pi^{l-m}}(x) = \hat{f}(\pi^{l-m}x)$.

Proof. Let f^τ denote the function on F given by $f^\tau(x) = f(-x)$. Since the Gauss sum does not depend on choice of Haar measure on $F^\times$, take for $F^\times$ the normalized

Haar measure $\mathrm{d}^\times\mathrm{x}$ such that the unit group has volume one. Therefore

$$\begin{aligned} G(f^\tau,\chi_0) &= \int_{\mathcal{O}^\times} f(-x)\,\chi_0(x)\ \mathrm{d}^\times\mathrm{x} \\ &= \chi_0(-1)\int_{\mathcal{O}^\times} f(-x)\,\chi_0(-x)\ \mathrm{d}^\times\mathrm{x} \\ &= \chi_0(-1)\,G(f,\chi_0). \end{aligned}$$

In particular, since $\mathrm{d}_+\mathrm{x}$ is normalized to be self-dual with respect to ψ (Definition 2.2.5), the equality $\hat{\hat{f}} = f^\tau$ holds true, whence $G(f,\chi_0) = \chi_0(-1)\,G(\hat{\hat{f}},\chi_0)$. Parts (i) and (ii) are now immediate consequences of Lemma 2.6.3, on replacing f by its Fourier transform. □

Remark 2.6.5. On choosing appropriate f, for example $f = 1 + \mathfrak{p}^m$ for $m \geq 1$ in Lemma 2.6.4 (ii) and on combining with Lemma 2.6.3, one obtains another proof of the classical fact about the absolute value of the Gauss sum.

Lemma 2.6.6. *Suppose that $\psi : F \to \mathbb{C}^\times$ is a character and that* $\mathrm{d}_+\mathrm{x}$ *is a Haar measure on F. Suppose further that π is a uniformizer and that $(a,c) \in F^2$. If f is a Schwartz-Bruhat function on F, then the Fourier transforms of f and of the function*

$$g(x) = \psi(ax) f\left(\pi^n(x+c)\right)$$

are related by

$$\hat{g}(x) = q^n\,\psi\left(-c(x+a)\right)\,\hat{f}\left(\frac{x+a}{\pi^n}\right),$$

where the Fourier transforms are taken with respect to ψ and $\mathrm{d}_+\mathrm{x}$.

Proof. The Fourier transform $\hat{g}$ can be computed in three steps. First, if $f_1(x) = f(\pi^n x)$, then

$$\hat{f}_1(y) = \int_F f(\pi^n x)\,\psi(xy)\ \mathrm{d}_+\mathrm{x} = q^n\,\hat{f}\left(\frac{y}{\pi^n}\right).$$

Secondly, if $f_2(x) = f_1(x+c) = f\left(\pi^n(x+c)\right)$, then

$$\hat{f}_2(y) = \int_F f_1(x+c)\,\psi(xy)\ \mathrm{d}_+\mathrm{x} = \psi(-cy)\hat{f}_1(y).$$

Thirdly, since $g(x) = \psi(ax)\, f_2(x) = \psi(ax)\, f\,(\pi^n(x+c))$, it follows that

$$\hat{g}(y) = \int_F \psi(ax)\, f_2(x)\, \psi(xy)\ \mathrm{d}_+\mathrm{x} = \hat{f}_2(a+y).$$

Thus $\hat{g}(x) = q^n\, \psi\,(-c(x+a))\ \hat{f}\left(\frac{x+a}{\pi^n}\right)$. □

2.7 Proof of the Main Local Theorem

We are now in a position to establish the analytic properties of local Lerch-Tate zeta integrals.

Proof of Theorem 2.5.1. Let us briefly recall some notations. A norm section τ corresponds to a uniformizer π, namely $\pi = \tau(q^{-1})$. If χ is a quasi-character of $F^\times$, put $\chi_0 = \chi|_{\mathcal{O}^\times}$ and $z = \chi(\pi) \in \mathbb{C}^\times$. Then $\chi \in X_{F,\chi_0}$. On identifying a quasi-character in the connected component X_{F,χ_0} with its value on π, we view the restriction of $Z_{\tau,a,c}(f,\cdot)$ to X_{F,χ_0} as the function

$$z \mapsto Z_{\pi,a,c}(f,\chi_0,z) = \sum_{n=-\infty}^{\infty} \int_{\mathcal{O}^\times} \psi(au)\, f\,(\pi^n(u+c))\ \chi_0(u)\, z^n\ \mathrm{d}^\times\mathrm{u}.$$

Since f is a Schwartz-Bruhat function on F, there is a sufficiently large integer M such that f is supported on $\mathfrak{p}^{-M}$ and that f is constant on every coset $x + \mathfrak{p}^M$ $(x \in F)$.

We consider two cases.

(a) Assume $c \notin \mathcal{O}^\times$. Put $r = \min(\mathrm{val(c)}, 0)$ and write $Z_{\pi,a,c}(f,\chi_0,z)$ as

$$\left(\sum_{n<-M-r} + \sum_{-M-r\le n<M-r} + \sum_{n\ge M-r}\right) \int_{\mathcal{O}^\times} \psi(au)\, f\,(\pi^n(u+c))\ \chi_0(u)\, z^n\ \mathrm{d}^\times\mathrm{u}.$$

The first sum vanishes because $\pi^n(u+c)$ does not belong to $\mathfrak{p}^{-M}$ and hence does not belong to the support of f. The second finite sum is clearly an element of $\mathbb{C}[z,z^{-1}]$. The third sum equals

$$\sum_{n\ge M-r} \int_{\mathcal{O}^\times} \psi(au)\, f(0)\, \chi_0(u)\, z^n\ \mathrm{d}^\times\mathrm{u} = \int_{\mathcal{O}^\times} \psi(au)\, f(0)\, \chi_0(u)\ \mathrm{d}^\times\mathrm{u} \sum_{n\ge M-r} z^n$$

which converges absolutely for $|z| < 1$ to a rational function of z. Hence $Z_{\pi,a,c}(f,\chi_0,z)$, viewed as a rational function of z, has at most a pole of finite order at $z=0$ and a simple pole at $z=1$; the residue at $z=1$ equals

$$-\int_{\mathcal{O}^\times} \psi(au)\, f(0)\, \chi_0(u)\ \mathrm{d}^\times\mathrm{u} = -f(0)\, \mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)\, G(\psi_a, \chi_0).$$

Thus the restriction of $Z_{\tau,a,c}(f,\cdot)$ to the connected component X_{F,χ_0} has at most a simple pole at the unique quasi-character $\chi_{\tau,0} \in X_{F,\chi_0}$ which takes unit value on π. The residue of $Z_{\tau,a,c}(f,\cdot)$ at $\chi_{\tau,0}$ is $-f(0)\,\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)\, G(\psi_a,\chi_0)$.

(b) Assume $c \in \mathcal{O}^\times$. Write $Z_{\pi,a,c}(f,\chi_0,z)$ as

$$\left(\sum_{n<-M} + \sum_{-M\le n<M} + \sum_{n\ge M}\right) \int_{\mathcal{O}^\times} \psi(au)\, f\left(\pi^n(u+c)\right)\, \chi_0(u)\, z^n\ \mathrm{d}^\times\mathrm{u}.$$

The second finite sum is clearly a function in $\mathbb{C}[z,z^{-1}]$. The third sum equals

$$\sum_{n\ge M} \int_{\mathcal{O}^\times} \psi(au)\, f(0)\, \chi_0(u)\, z^n\ \mathrm{d}^\times\mathrm{u} = \int_{\mathcal{O}^\times} \psi(au)\, f(0)\, \chi_0(u)\ \mathrm{d}^\times\mathrm{u} \sum_{n\ge M} z^n,$$

which converges absolutely for $|z|<1$ to a rational function of z. However, the first sum

$$T := \sum_{n<-M} \int_{\mathcal{O}^\times} \psi(au)\, f\left(\pi^n(u+c)\right)\, \chi_0(u)\, z^n\ \mathrm{d}^\times\mathrm{u}$$

does not vanish as in the preceding case.

Now suppose that $n<-M$ and put $u=-cw$ with $w\in\mathcal{O}^\times$. Then

$$T = \chi_0(-c)\cdot \sum_{n<-M} \int_{\mathcal{O}^\times} \psi(-acw)\, f\left(\pi^n c(1-w)\right)\, \chi_0(w)\, z^n\ \mathrm{d}^\times\mathrm{w}.$$

Since f is supported on $\mathfrak{p}^{-M}$, it follows that $f\left(\pi^n c(1-w)\right) = 0$ unless $w \in 1+\mathfrak{p}^{-M-n}$. Now suppose further that $w\in 1+\mathfrak{p}^{-M-n}$. By the local constancy of the characters ψ and χ_0, apart from finitely many integers n, the equalities $\psi(-acw) = \psi(-ac)$ and $\chi_0(1+\mathfrak{p}^{-M-n}) = 1$ hold true. The finitely many integers n that are excluded, say $-M'<n<-M$, contributes to T a function in $\mathbb{C}[z,z^{-1}]$. On setting $w = 1-c^{-1}\pi^{-n}t$ with $t\in\mathfrak{p}^{-M}$, we find that the remaining integers n contribute to T the quantity

$$\frac{\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)}{\mathrm{d}_+\mathrm{x}(\mathcal{O}^\times)} \cdot \chi_0(-c)\, \psi(-ac) \cdot \int_{\mathfrak{p}^{-M}} f(t)\ \mathrm{d}_+\mathrm{t} \cdot \sum_{n\le -M'} (qz)^n \tag{2.10}$$

where $\mathrm{d}_+\mathrm{t}$ is any Haar measure on F. Thus T converges absolutely for $|z| > \frac{1}{q}$ to a rational function of z.

The argument given above also shows that $Z_{\pi,a,c}(f, \chi_0, z)$, viewed as a rational function of z, has at most a pole of finite order at $z = 0$, a simple pole at $z = 1$ and a simple pole at $z = \frac{1}{q}$. The residue at $z = 1$ is

$$\int_{\mathcal{O}^\times} \psi(au)\, f(0)\, \chi_0(u)\ \mathrm{d}^\times \mathrm{u} = -f(0)\, \mathrm{d}^\times \mathrm{u}(\mathcal{O}^\times)\, G(\psi_a, \chi_0).$$

The residue at $z = \frac{1}{q}$ is, on putting $h = \frac{\mathrm{d}^\times \mathrm{x}(\mathcal{O}^\times)}{\mathrm{d}_+ \mathrm{x}(\mathcal{O}^\times)}$,

$$\frac{h}{q}\, \chi_0(-c)\, \psi(-ac) \int_{\mathfrak{p}^{-M}} f(t)\ \mathrm{d}_+\mathrm{t} = \frac{h}{q}\, \chi_0(-c)\, \psi(-ac) \int_F f(t)\ \mathrm{d}_+\mathrm{t}.$$

Thus the restriction of $Z_{\tau,a,c}(f, \cdot)$ to the connected component X_{F,χ_0} has at most two simple poles at the unique quasi-character $\chi_{\tau,0} \in X_{F,\chi_0}$ which takes unit value on π and at the unique quasi-character $\chi_{\tau,-1} \in X_{F,\chi_0}$ which takes value q^{-1} on π. The residue of $Z_{\tau,a,c}(f, \cdot)$ at $\chi_{\tau,0}$ is $-f(0)\, G(\psi_a, \chi_0)$; the residue of $Z_{\tau,a,c}(f, \cdot)$ at $\chi_{\tau,-1}$ is, by (2.10),

$$\frac{h}{q}\, \chi_0(-c)\, \psi(-ac) \int_F f(t)\ \mathrm{d}_+\mathrm{t}.$$

□

Proof Theorem 2.5.2. We consider two cases.

(a) Assume χ is unramified. By Remark 2.4.4, the restriction of $Z_{\tau,a,c}(f, \cdot)$ to the unramified component $X_{F,\mathrm{nr}}$ is the function $Z_{\pi,a,c}(f, \chi_{0,\mathrm{nr}}, z)$. It follows that

$$Z_{\pi,a,c}(f, \chi_{0,\mathrm{nr}}, z) = \sum_{n=-\infty}^{\infty} \int_{\mathcal{O}^\times} \psi(au)\, f\left(\pi^n(u + c)\right)\ z^n\ \mathrm{d}^\times \mathrm{u}.$$

Put $g_n(x) = \psi(ax)\, f\left(\pi^n(x + c)\right)$, so that

$$Z_{\pi,a,c}(f, \chi_{0,\mathrm{nr}}, z) = \mathrm{d}^\times \mathrm{x}\, (\mathcal{O}^\times) \sum_{n=-\infty}^{\infty} G(g_n, \chi_{0,\mathrm{nr}})\, z^n.$$

By Lemma 2.6.6, the Fourier transform of g_n is

$$\hat{g}_n(x) = q^n\, \psi\left(-c(x+a)\right)\, \hat{f}\left(\frac{x+a}{\pi^n}\right),$$

Corollary 2.6.4 (i) then gives

$$\begin{aligned} G(g_n, \chi_{0,\mathrm{nr}}) &= \int_F (\hat{g}_n)_1(x)\; \mathrm{d}_+\mathrm{x} \\ &= \psi(-ac)\left(\int_{\mathfrak{p}^l} \mathrm{d}_+\mathrm{x} + \frac{1}{1-q}\int_{\mathfrak{p}^{l-1}-\mathfrak{p}^l} \mathrm{d}_+\mathrm{x}\right)\psi(-cx)\,\hat{f}\left(\frac{x+a}{\pi^n}\right) q^n. \end{aligned}$$

Put

$$\begin{aligned} Z_1 &= \mathrm{d}^\times\mathrm{x}\,(\mathcal{O}^\times) \sum_{n=-\infty}^{\infty} \int_{\mathfrak{p}^l} \psi(-cx)\,\hat{f}\left(\frac{x+a}{\pi^n}\right)\,(qz)^n\;\mathrm{d}_+\mathrm{x} \\ Z_2 &= \mathrm{d}^\times\mathrm{x}\,(\mathcal{O}^\times) \sum_{n=-\infty}^{\infty} \int_{\mathfrak{p}^{l-1}-\mathfrak{p}^l} \psi(-cx)\,\hat{f}\left(\frac{x+a}{\pi^n}\right)\,(qz)^n\;\mathrm{d}_+\mathrm{x}\,, \end{aligned}$$

so that

$$Z_{\pi,a,c}(f, \chi_{0,\mathrm{nr}}, z) = \psi(-ac)\,\left(Z_1 + \frac{Z_2}{1-q}\right). \tag{2.11}$$

We proceed to compute Z_1 and Z_2. On the one hand

$$\begin{aligned} \frac{Z_1}{\mathrm{d}^\times\mathrm{x}\,(\mathcal{O}^\times)} &= \sum_{n=-\infty}^{\infty} \int_{\mathfrak{p}^l} \psi(-cx)\,\hat{f}\left(\frac{x+a}{\pi^n}\right)\,(qz)^n\;\mathrm{d}_+\mathrm{x} \\ &= \sum_{n=-\infty}^{\infty}\sum_{r\geq l} z^r \int_{\mathcal{O}^\times} \psi(-c\pi^r y)\,\hat{f}\left(\frac{\pi^r y+a}{\pi^n}\right)\,(qz)^{n-r}\;\mathrm{d}_+\mathrm{y} \\ &= \sum_{r\geq l} z^r \sum_{n=-\infty}^{\infty} \int_{\mathcal{O}^\times} \psi(-c\pi^r y)\,\hat{f}\left(\frac{\pi^r y+a}{\pi^n}\right)\,(qz)^{n-r}\;\mathrm{d}_+\mathrm{y} \\ &= \sum_{r\geq l} z^r \sum_{n'=-\infty}^{\infty} \int_{\mathcal{O}^\times} \psi(-c\pi^r y)\hat{f}\left(\pi^{n'}(y+a\pi^{-r})\right)\,(qz)^{-n'}\;\mathrm{d}_+\mathrm{y}. \end{aligned}$$

Since $\int_{\mathcal{O}^\times}\varphi(x)\mathrm{d}_+\mathrm{x} = \frac{\mathrm{d}_+\mathrm{x}(\mathcal{O}^\times)}{\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)}\int_{\mathcal{O}^\times}\varphi(x)\;\mathrm{d}^\times\mathrm{x}$, it follows that

$$Z_1 = \mathrm{d}_+\mathrm{x}(\mathcal{O}^\times)\,\sum_{r\geq l} z^r\, Z_{\pi,-c\pi^r,a\pi^{-r}}\left(\hat{f}, \chi_{0,\mathrm{nr}}, 1/(qz)\right). \tag{2.12}$$

On the other hand

$$
\begin{aligned}
Z_2 &= \sum_{n=-\infty}^{\infty} \int_{\mathfrak{p}^{l-1}-\mathfrak{p}^l} \psi(-cx)\, \hat{f}\left(\frac{x+a}{\pi^n}\right) (qz)^n \ \mathrm{d}_+\mathrm{x}\,, \\
&= z^{l-1} \sum_{n=-\infty}^{\infty} \int_{\mathcal{O}^\times} \psi(-c\pi^{l-1}y)\, \hat{f}\left(\frac{\pi^{l-1}y+a}{\pi^n}\right) (qz)^{n-l+1} \ \mathrm{d}_+\mathrm{y}\,, \\
&= z^{l-1} \sum_{n'=-\infty}^{\infty} \int_{\mathcal{O}^\times} \psi(-c\pi^{l-1}y)\, \hat{f}\left(\pi^{n'}(\pi^{l-1}y+a\pi^{1-l})\right) (qz)^{-n'} \ \mathrm{d}_+\mathrm{y}\,.
\end{aligned}
$$

Since $\int_{\mathcal{O}^\times} \varphi(x)\mathrm{d}_+\mathrm{x} = \frac{\mathrm{d}_+\mathrm{x}(\mathcal{O}^\times)}{\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)} \int_{\mathcal{O}^\times} \varphi(x)\ \mathrm{d}^\times\mathrm{x}$, it follows that

$$
Z_2 = \mathrm{d}_+\mathrm{x}(\mathcal{O}^\times)\, z^{l-1}\, Z_{\pi,-c\pi^{l-1},a\pi^{1-l}}\left(\hat{f}, \chi_{0,\mathrm{nr}}, 1/(qz)\right). \tag{2.13}
$$

Finally, (2.11) together with (2.12) and (2.13) yield the functional equation

$$
\begin{aligned}
&\frac{Z_{\pi,a,c}(f,\chi_{0,\mathrm{nr}},z)}{\psi(-ac)\,\mathrm{d}_+\mathrm{x}\,(\mathcal{O}^\times)} \\
&= \sum_{r\geq l} z^r\, Z_{\pi,-c\pi^r,a\pi^{-r}}\left(\hat{f},\chi_{0,\mathrm{nr}},\frac{1}{qz}\right) + \frac{z^{l-1}}{1-q}\, Z_{\pi,-c\pi^{l-1},a\pi^{1-l}}\left(\hat{f},\chi_{0,\mathrm{nr}},\frac{1}{qz}\right),
\end{aligned}
$$

provided that χ is unramified.

(b) <u>Assume χ is ramified of level $m \geq 1$.</u> By Remark 2.4.4, the restriction of $Z_{\tau,a,c}(f,\cdot)$ to the connected component X_{F,χ_0} is the function $Z_{\pi,a,c}(f,\chi_0,z)$. It follows that

$$
Z_{\pi,a,c}(f,\chi_0,z) = \sum_{n=-\infty}^{\infty} \int_{\mathcal{O}^\times} \psi(au)\, f\left(\pi^n(u+c)\right)\, \chi_0(u)\, z^n \ \mathrm{d}^\times\mathrm{u}.
$$

Put $g_n(x) = \psi(ax)\, f\left(\pi^n(x+c)\right)$, so that

$$
Z_{\pi,a,c}(f,\chi_0,z) = \sum_{n=-\infty}^{\infty} G(g_n,\chi_0)\, z^n.
$$

By Lemma 2.6.6, the Fourier transform of g_n is

$$
\hat{g}_n(x) = q^n\, \psi\left(-c(x+a)\right)\, \hat{f}\left(\frac{x+a}{\pi^n}\right),
$$

Corollary 2.6.4 (ii) then gives

$$G(g_n, \chi_0) = \chi_0(-1)\, q^{m-l}\, G(\psi_{\pi^{l-m}x}, \chi_0) \int_{\mathcal{O}^\times} \hat{g}_n(\pi^{l-m}x)\, \chi_0^{-1}(x)\ \mathrm{d}_+\mathrm{x}.$$

Put

$$Z = \mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times) \sum_{n=-\infty}^{\infty} \int_{\mathcal{O}^\times} \psi(-c\pi^{l-m}x)\, \hat{f}\left(\frac{\pi^{l-m}x+a}{\pi^n}\right)\, (qz)^n\, \chi_0^{-1}(x)\ \mathrm{d}_+\mathrm{x}$$

so that

$$Z_{\pi,a,c}(f, \chi_0, z) = \psi(-ac)\, \chi_0(-1)\, q^{m-l}\, G(\psi_{\pi^{l-m}}, \chi_0)\, Z. \tag{2.14}$$

It remains to compute Z. We have

$$\begin{aligned}
\frac{Z}{\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)} &= \sum_{n=-\infty}^{\infty} \int_{\mathcal{O}^\times} \psi(-c\pi^{l-m}x)\, \hat{f}\left(\frac{\pi^{l-m}x+a}{\pi^n}\right)\, (qz)^n\, \chi_0^{-1}(x)\ \mathrm{d}_+\mathrm{x} \\
&= \sum_{n=-\infty}^{\infty} \int_{\mathcal{O}^\times} \psi(-c\pi^{l-m}x)\, \hat{f}\left(\frac{x+a\pi^{m-l}}{\pi^{n+m-l}}\right)\, (qz)^n\, \chi_0^{-1}(x)\ \mathrm{d}_+\mathrm{x} \\
&= \sum_{n'=-\infty}^{\infty} \int_{\mathcal{O}^\times} \psi(-c\pi^{l-m}x)\, \hat{f}\left(\pi^{n'}(x+a\pi^{m-l})\right)(qz)^{l-m-n'} \chi_0^{-1}(x)\, \mathrm{d}_+\mathrm{x} \\
&= (qz)^{l-m}\, Z_{\pi,-c\pi^{l-m},a\pi^{m-l}}\left(\hat{f}, \chi_0^{-1}, 1/(qz)\right).
\end{aligned}$$

Since $\int_{\mathcal{O}^\times} \varphi(x)\mathrm{d}_+\mathrm{x} = \frac{\mathrm{d}_+\mathrm{x}(\mathcal{O}^\times)}{\mathrm{d}^\times\mathrm{x}(\mathcal{O}^\times)} \int_{\mathcal{O}^\times} \varphi(x)\ \mathrm{d}^\times\mathrm{x}$, it follows that

$$Z = (qz)^{l-m}\, \mathrm{d}_+\mathrm{x}(\mathcal{O}^\times)\ Z_{\pi,-c\pi^{l-m},a\pi^{m-l}}\left(\hat{f}, \chi_0^{-1}, 1/(qz)\right). \tag{2.15}$$

Finally, (2.14) and (2.15) yield

$$\frac{Z_{\pi,a,c}(f, \chi_0, z)}{\chi_0(-1)\, \psi(-ac)\mathrm{d}_+\mathrm{x}(\mathcal{O}^\times)} = z^{l-m}\, G(\psi_{\pi^{l-m}}, \chi_0)\ Z_{\pi,-c\pi^{l-m},a\pi^{m-l}}\left(\hat{f}, \chi_0^{-1}, 1/(qz)\right).$$

The proof is complete.

□

2.8 Example

In this section we work out an example of the local zeta integrals.

Let now $F = \mathbb{Q}_p$. Suppose that $f_0 = 1_{\mathbb{Z}_p}$ and that the norm section $\tau : p^{\mathbb{Z}} \to F^\times$ is given by $\tau(p^{-1}) = p$. Suppose further that

$$\psi(x) = \mathbf{e}_p(x) := \mathbf{e}(\{x\}_p) \quad (x \in \mathbb{Q}_p)$$

where $\{x\}_p$ denotes the p-adic fractional part of x. If $x = \sum_{m=-N}^{N} a_m p^{-m}$ where N is a positive integer, then by definition $\{x\}_p = \sum_{m=1}^{N} a_{-m} p^{-m}$. Take for $\mathbb{Q}_p^\times$ the normalized measure $\mathrm{d}^\times\mathrm{x}$ such that the unit group has volume one. If $(a, c) \in \mathbb{Q}_p^2$ and χ is a quasi-character of $\mathbb{Q}_p^\times$, consider

$$Z_{\tau,a,c}(f_0, \chi) = \int_{\mathbb{Q}_p^\times} \psi\left(\frac{ax}{p^{\mathrm{val(x)}}}\right) f_0(x + c\, p^{\mathrm{val(x)}})\, \chi(x)\ \mathrm{d}^\times\mathrm{x}. \tag{2.16}$$

If χ_0 is a character on $\mathbb{Z}_p^\times$, then the restriction of $Z_{\tau,a,c}(f, \cdot)$ to the connected component $X_{\mathbb{Q}_p,\chi_0}$ of $X_{\mathbb{Q}_p}$ is the function $Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, \cdot)$ of the variable $z = \chi(p)$. More exactly, if $\chi|_{\mathbb{Z}_p^\times} = \chi_0$ and if $z = \chi(p)$, then $Z_{\tau,a,c}(f_0, \chi) = Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z)$, where

$$Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \sum_{n=-\infty}^{\infty} \int_{\mathbb{Z}_p^\times} \mathbf{e}_p(au)\, 1_{\mathbb{Z}_p}(p^n(u + c))\, \chi_0(u) z^n\ \mathrm{d}^\times\mathrm{x}. \tag{2.17}$$

It follows that $Z_{\tau,a,c}(f_0, \chi) = Z_{\tau,a+v,c}(f_0, \chi)$ for every $v \in \mathbb{Z}_p$. Let $I(\chi_0)$ be the function which takes value 1 if χ_0 is unramified and 0 otherwise. Propositions 2.8.1 and 2.8.2 give explicit formulas for the local zeta integral $Z_{\tau,a,c}(f_0, \chi)$.

Proposition 2.8.1. *Suppose that $a \in \mathbb{Z}_p$.*

(i) *If $c \in p\mathbb{Z}_p$, then $Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \frac{I(\chi_0)}{1-z}$.*

(ii) *If $c \notin \mathbb{Z}_p$, then $Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \frac{z^{-\mathrm{val(c)}}\, I(\chi_0)}{1-z}$.*

(iii) *If $c \in \mathbb{Z}_p^\times$ and χ_0 is unramified, then*

$$Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \frac{1}{1-z} + \frac{1}{(1-p^{-1})(pz-1)}.$$

If $c \in \mathbb{Z}_p^\times$ and χ_0 is ramified of level m, then

$$Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \frac{\chi_0(-c)}{(pz)^{m-1}} \cdot \frac{1}{(1-p^{-1})(pz-1)}.$$

Proposition 2.8.2. *Suppose that $a \in \frac{1}{p^l}\mathbb{Z}_p - \frac{1}{p^{l-1}}\mathbb{Z}_p$ where $l \in \mathbb{Z}_{\geq 1}$.*

(i) *If* $c \in p\mathbb{Z}_p$, *then* $Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \frac{G(\psi_a,\chi_0)}{1-z}$.

(ii) *If* $c \notin \mathbb{Z}_p$, *then* $Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \frac{z^{-\mathrm{val(c)}}\, G(\psi_a,\chi_0)}{1-z}$.

(iii) *If* $c \in \mathbb{Z}_p^\times$ *and* $l = m$, *then*

$$Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \frac{G(\psi_a, \chi_0)}{1-z} + \frac{\chi_0(-c)\psi(1)}{(pz)^{m-1}} \cdot \frac{1}{(1-p^{-1})(pz-1)} + \frac{\chi_0(-c)}{p^m - p^{m-1}} \sum_{n=1}^{m-1} z^{-n}\, G_{p^m\mathbb{Z}_p,n}(\overline{\psi_{-ac}}, \overline{\chi_0}).$$

If $c \in \mathbb{Z}_p^\times$ *and* $l \neq m$, *then*

$$Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \frac{G(\psi_a, \chi_0)}{1-z} + \frac{\chi_0(-c)\psi(1)}{(pz)^{m-1}} \cdot \frac{1}{(1-p^{-1})(pz-1)}.$$

Proof of Proposition 2.8.1. If $a \in \mathbb{Z}_p$, then $\mathbf{e}_p\left(\frac{ax}{p^{\mathrm{val(x)}}}\right) = 1$ for every $x \in \mathbb{Q}_p^\times$. It follows from (2.17) that

$$Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \int_{\mathbb{Q}_p^\times} 1_{\mathbb{Z}_p}(x + c\, p^{\mathrm{val(x)}})\, \chi(x)\ \mathrm{d}^\times\mathrm{x}.$$

First assume that $c \in p\mathbb{Z}_p$. Then $1_{\mathbb{Z}_p}(x + c\, p^{\mathrm{val(x)}}) = 0$ precisely when $x \in \mathbb{Z}_p$ and hence

$$Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \int_{\mathbb{Z}_p\setminus\{0\}} \chi(x)\ \mathrm{d}^\times\mathrm{x} = \int_{\mathbb{Z}_p^\times} \chi_0(x)\ \mathrm{d}^\times\mathrm{x}\ \sum_{n=0}^{\infty} z^n = \frac{I(\chi_0)}{1-z}.$$

Next assume that $c \notin \mathbb{Z}_p$. Then $1_{\mathbb{Z}_p}(x + c\, p^{\mathrm{val(x)}}) = 0$ unless $x \in p^{-\mathrm{val(c)}}\mathbb{Z}_p$. Hence

$$\begin{aligned} Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) &= \int_{p^{-\mathrm{val(c)}}\mathbb{Z}_p\setminus\{0\}} \chi(x)\ \mathrm{d}^\times\mathrm{x} \\ &= \int_{\mathbb{Z}_p^\times} \chi_0(x)\ \mathrm{d}^\times\mathrm{x} \sum_{m=-\mathrm{val(c)}}^{\infty} z^m = \frac{z^{-\mathrm{val(c)}}\, I(\chi_0)}{1-z}. \end{aligned}$$

Finally assume that $c \in \mathbb{Z}_p^\times$. Write $\int_{\mathbb{Q}_p^\times} = \int_{\mathbb{Z}_p\setminus\{0\}} + \sum_{n=1}^{\infty} \int_{p^{-n}\mathbb{Z}_p^\times}$. The first integral is

$$\int_{\mathbb{Z}_p\setminus\{0\}} \chi(x)\ \mathrm{d}^\times\mathrm{x} = \frac{I(\chi_0)}{1-z},$$

whereas the second term equals

$$\sum_{n=1}^{\infty} z^{-n} \int_{\mathbb{Z}_p^\times} 1_{p^n\mathbb{Z}_p}(u+c)\,\chi_0(u)\ \mathrm{d}^\times \mathrm{u} = \chi_0(-c) \sum_{n=1}^{\infty} z^{-n} \int_{\mathbb{Z}_p^\times} 1_{1+p^n\mathbb{Z}_p}(v)\,\chi_0(v)\ \mathrm{d}^\times \mathrm{v},$$

by a change of variable $v = -uc$. If χ_0 is unramified, then

$$\int_{\mathbb{Z}_p^\times} 1_{1+p^n\mathbb{Z}_p}(v)\,\chi_0(v)\ \mathrm{d}^\times \mathrm{v} = \frac{1}{p^n - p^{n-1}}$$

and hence the second term is

$$\sum_{n=1}^{\infty} z^{-n} \frac{1}{p^n - p^{n-1}} = \frac{1}{(1-p^{-1})(pz-1)}.$$

If χ_0 is ramified of level m, the integral $\int_{\mathbb{Z}_p^\times} 1_{1+p^n\mathbb{Z}_p}(v)\,\chi_0(v)\ \mathrm{d}^\times \mathrm{v}$ is zero unless $n \geq m$, in which case it is $\frac{1}{p^n - p^{n-1}}$; hence the second term is

$$\chi_0(-c) \sum_{n=m}^{\infty} z^{-n} \frac{1}{p^n - p^{n-1}} = \frac{\chi_0(-c)}{(pz)^{m-1}} \cdot \frac{1}{(1-p^{-1})(pz-1)}.$$

This completes the proof. $\square$

Proof of Proposition 2.8.2. It follows from (2.17) that

$$Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \sum_{n=-\infty}^{\infty} \int_{\mathbb{Z}_p^\times} \psi(au) 1_{\mathbb{Z}_p}(p^n(u+c))\,\chi_0(u)\, z^n\ \mathrm{d}^\times \mathrm{u}.$$

First assume that $c \in p\mathbb{Z}_p$. Then $1_{\mathbb{Z}_p}(p^n(u+c)) = 0$ precisely when $n \geq 0$ and hence

$$Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p^\times} \psi(au)\,\chi_0(u)\, z^n\ \mathrm{d}^\times \mathrm{u} = \frac{G(\psi_a, \chi_0)}{1-z}.$$

Next assume that $c \notin \mathbb{Z}_p$. Then $1_{\mathbb{Z}_p}(p^n(u+c)) = 0$ unless $n \geq -\mathrm{val(c)}$. Hence

$$Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \sum_{n=-\mathrm{val(c)}}^{\infty} \int_{\mathbb{Z}_p^\times} \psi(au)\,\chi_0(u)\, z^n\ \mathrm{d}^\times \mathrm{u} = \frac{z^{-\mathrm{val(c)}}\, G(\psi_a, \chi_0)}{1-z}.$$

Finally assume that $c \in \mathbb{Z}_p^\times$. A change of variable $u = -cv$ yields

$$Z_{p,a,c}(1_{\mathbb{Z}_p}, \chi_0, z) = \chi_0(-c) \sum_{n=-\infty}^{\infty} \int_{\mathbb{Z}_p^\times} \psi(-acv) 1_{1+p^{-n}\mathbb{Z}_p}(v)\, \chi_0(v)\, z^n \ \mathrm{d}^\times \mathrm{v}.$$

Write $\int_{\mathbb{Q}_p^\times} = \int_{\mathbb{Z}_p \setminus \{0\}} + \sum_{n=1}^\infty \int_{p^{-n}\mathbb{Z}_p^\times}$. The first integral is

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p^\times} \psi(-acv)\, \chi_0(-cv)\, z^n \ \mathrm{d}^\times \mathrm{v} = \frac{G(\psi_a, \chi_0)}{1-z},$$

whereas the second term equals

$$\chi_0(-c) \sum_{n=1}^{\infty} z^{-n} \int_{1+p^n\mathbb{Z}_p} \psi(-acv)\, \chi_0(v) \ \mathrm{d}^\times \mathrm{u} = \chi_0(-c) \sum_{n=1}^{\infty} z^{-n} G_n(\psi_{-ac}, \chi_0).$$

Suppose first that $l = m$. By Lemma 2.6.2, the $n \geq m$ contribute to the second term the quantity

$$\sum_{n \geq m} \frac{\psi(1)}{p^n - p^{n-1}} z^{-n} = \frac{\psi(1)}{(pz)^{m-1}} \cdot \frac{1}{(1-p^{-1})(pz-1)},$$

whereas the $n < m$ contribute to the second term the quantity

$$\frac{\chi_0(-c)}{p^m - p^{m-1}} \sum_{n=1}^{m-1} G_{p^m\mathbb{Z}_p, n}(\overline{\psi_{-ac}}, \overline{\chi_0}).$$

Thus the second term now equals

$$\frac{\psi(1)}{(pz)^{m-1}} \cdot \frac{1}{(1-p^{-1})(pz-1)} + \frac{\chi_0(-c)}{p^m - p^{m-1}} \sum_{n=1}^{m-1} G_{p^m\mathbb{Z}_p, n}(\overline{\psi_{-ac}}, \overline{\chi_0}).$$

Next suppose that $l \neq m$. By Lemma 2.6.2, only the $n \geq m$ contribute to the second term, and thus the second term equals

$$\sum_{n \geq m} \frac{\psi(1)}{p^n - p^{n-1}} z^{-n} = \frac{\psi(1)}{(pz)^{m-1}} \cdot \frac{1}{(1-p^{-1})(pz-1)}.$$

This completes the proof. $\square$

Chapter 3

Global zeta integrals over number fields

3.1 Overview of the chapter

The goal of this chapter is to give an adelic generalization of the Lerch zeta function. This is accomplished in Sections 3.4 and 3.5. Here is an overview of the chapter.

- In Section 3.2 we describe the variety of quasi-characters for a number field and the connected components of this variety.
- In Section 3.4 we construct a family of global zeta integrals over a number field. Therein we define norm section, Lerch tuple and global Lerch-Tate zeta integrals.
- The analytic properties of global Lerch-Tate zeta integrals are formulated in Section 3.5 as Theorem 3.5.1, Theorem 3.5.2 and Theorem 3.5.3.
- Section 3.6 contains the proof of Theorem 3.5.1.
- Section 3.7 is devoted to the proofs of Theorems 3.5.2 and 3.5.3.

3.2 Notations and basic definitions

Suppose that K is an algebraic number field. Write $C_K = \mathbb{A}_K^\times/K^\times$ for the idele class group and $C_K^1 = \mathbb{A}_K^1/K^\times$ for the norm one idele class group. The space X_K of all quasi-characters of C_K has a natural structure of a complex variety. Furthermore, X_K is a disjoint union of connected components, each of which can be non-canonically identified with $\mathbb{C}$. All quasi-characters of the idele class group which have the same restriction to C_K^1 belong to the same connected component. Let us set some notations to specify the connected components of X_K more conveniently.

Definition 3.2.1. If χ_0 is a character of C_K^1, set

$$X_{K,\chi_0} = \{\chi \in X_K : \chi|_{C_K^1} = \chi_0\}. \tag{3.1}$$

In other words, X_{K,χ_0} is the connected component of X_K labeled by $\chi_0 \in \widehat{C_K^1}$. In particular, if $\chi_{0,\mathrm{nr}}$ is the trivial character on C_K^1, then $X_{K,\chi_{0,\mathrm{nr}}}$ is the neutral component consisting of unramified quasi-characters of C_K; we write $X_{K,\mathrm{nr}}$ for this neutral component.

Remark 3.2.2. There are non-canonical isomorphisms from $\mathbb{C}$ onto X_{K,χ_0}. If $\tilde{\chi_0} : C_K \to \mathbb{C}^\times$ is an extension of χ_0, then the map which sends a complex number s to the quasi-character $\chi(x) = \tilde{\chi_0}(x)\,|x|^s$ defines an isomorphism from $\mathbb{C}$ onto X_{K,χ_0}.

We recall the standard notions of adelic Fourier transform and of self-dual measure.

Definition 3.2.3. Suppose that $\psi : \mathbb{A}_K \to \mathbb{C}^\times$ is a nontrivial character and that $\mathrm{d}_+\mathrm{x}$ is a Haar measure on $\mathbb{A}_K$. If f is a Schwartz-Bruhat function on $\mathbb{A}_K$, define the Fourier transform of f with respect to ψ and $\mathrm{d}_+\mathrm{x}$ to be the function

$$\hat{f}(y) = \int_{\mathbb{A}_K} f(x)\,\psi(xy)\;\mathrm{d}_+\mathrm{x} \qquad (y \in \mathbb{A}_K). \tag{3.2}$$

The Fourier transform $\hat{f}$ is also a Schwartz-Bruhat function on $\mathbb{A}_K$.

Definition 3.2.4. For every nontrivial character ψ of $\mathbb{A}_K$, there exists a unique Haar measure $\mathrm{d}_+\mathrm{x}$ on $\mathbb{A}_K$ such that, for all Schwartz-Bruhat functions f on $\mathbb{A}_K$, one has $\hat{\hat{f}}(x) = f(-x)$. This Haar measure $\mathrm{d}_+\mathrm{x}$ is called the *self-dual measure with respect to* ψ.

3.3 Global norm sections

Let $|\cdot| : \mathbb{A}_K^\times \to \mathbb{R}_+$ denote the standard norm map of $\mathbb{A}_K^\times$. Then $|\cdot|$ is trivial on $K^\times$ and induces a continuous homomorphism, also denoted by $|\cdot|$, from C_K to $\mathbb{R}_+$.

Definition 3.3.1. A continuous group homomorphism $\tau : \mathbb{R}_+ \to \mathbb{A}_K^\times$ is called a *norm section* if $|\cdot| \circ \tau = \mathrm{id}_{\mathbb{R}_+}$.

Remark 3.3.2. Since $\mathbb{R}_+$ is connected and $K_\mathfrak{p}^\times$ is totally disconnected for every finite place $\mathfrak{p}$, every continuous homomorphism $\mathbb{R}_+ \to K_\mathfrak{p}^\times$ is trivial. Therefore the image of τ lands entirely in $K_\mathbb{R}^\times$.

The following lemma describes the totality of norm sections over a generic number field.

Lemma 3.3.3. *Suppose that K is a number field of degree n and that r_1 (resp. r_2) is the number of real embeddings (resp. complex embeddings). Then the set of all norm sections is in bijection with*

$$\mathcal{T} = \left\{ ((a_\mu), (z_\nu)) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} : \sum_{\mu=1}^{r_1} a_\mu + 2 \sum_{\nu=1}^{r_2} \mathrm{Re}(z_\nu) = 1 \right\}.$$

More precisely, each norm section

$$\tau : \mathbb{R}_+ \to \mathbb{A}_K^\times$$

is of the form $\tau(t) = (\tau_\infty(t), 1)$ where $\tau_\infty(t) \in K_\mathbb{R}^\times$ is given by

$$\tau_\infty(t) = \left((t^{a_\mu})_{1 \le \mu \le r_1}, (t^{z_\nu})_{1 \le \nu \le r_2}\right).$$

for a unique element $((a_\mu), (z_\nu)) \in \mathcal{T}$.

Proof. Observe that every continuous homomorphism $\phi : \mathbb{R}_+ \to G$, where G is a topological group which is uniquely divisible, is determined by its value on an arbitrary element $t_0 \in \mathbb{R}_+$ with $t_0 \neq 1$. This is clear on noting that the image of the map

$$\mathbb{Q} \to \mathbb{R}_+, \ r \mapsto t_0^r$$

is dense in $\mathbb{R}_+$. In particular, if $G = \mathbb{R}_+$, this observation implies that every continuous homomorphism $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is of the form $\phi(t) = t^a$ for a uniquely determined real number a. Consequentially, every continuous homomorphism $\phi : \mathbb{R}_+ \to \mathbb{R}^\times$ is of the form $\phi(t) = t^a$ for a unique $a \in \mathbb{R}$. On the other hand, it is a standard fact that every quasi-character $\phi : \mathbb{R}_+ \to \mathbb{C}^\times$ is of the form $\phi(t) = t^z$ for a unique complex number z.

Suppose that $\tau : \mathbb{R}_+ \to \mathbb{A}_K^\times$ is a norm section. By Remark 3.3.2, τ is necessarily of the form

$$\tau(t) = (\tau_\infty(t), 1)$$

where $\tau_\infty : \mathbb{R}_+ \to K_\mathbb{R}^\times$ satisfies $|\cdot| \circ \tau_\infty = \mathrm{id}_{\mathbb{R}_+}$.

If r_1 (resp. r_2) is the number of real embeddings (resp. complex embeddings), then $K_\mathbb{R}^\times = (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2}$. For every real embedding, the composition of $\tau_\infty : \mathbb{R}_+ \to K_\mathbb{R}^\times$ with the corresponding projection from $K_\mathbb{R}^\times$ onto $\mathbb{R}^\times$ gives rise to a continuous homomorphism $\mathbb{R}_+ \to \mathbb{R}^\times$ which is necessarily of the form $t \mapsto t^{a_\mu}$ for $a_\mu \in \mathbb{R}$. Similarly, for every complex embedding, the composition of $\tau_\infty : \mathbb{R}_+ \to K_\mathbb{R}^\times$ with the corresponding projection from $K_\mathbb{R}^\times$ onto $\mathbb{C}^\times$ gives rise to a quasi-character $\mathbb{R}_+ \to \mathbb{C}^\times$ which is necessarily of the form $t \mapsto t^{z_\nu}$ for $z_\nu \in \mathbb{C}$. The condition for τ to be a norm section is tantamount to

$$\sum_{\mu=1}^{r_1} a_\mu + 2\sum_{\nu=1}^{r_2} \mathrm{Re}(z_\nu) = 1.$$

This completes the proof. $\square$

Examples 3.3.4. 1. If K is a totally real number field of degree n, then

$$\mathcal{T} = \left\{ (a_\mu) \in \mathbb{R}^n : \sum_{\mu=1}^{n} a_\mu = 1 \right\}.$$

2. If K is a totally imaginary number field of degree $2n$, then

$$\mathcal{T} = \left\{ (z_\nu) \in \mathbb{C}^n : \sum_{\nu=1}^{n} \mathrm{Re}(z_\nu) = \frac{1}{2} \right\}.$$

Definition 3.3.5. . Suppose that $((a_\mu), (z_\nu))$ is an element of

$$\mathcal{T} = \left\{ ((a_\mu), (z_\nu)) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} : \sum_{\mu=1}^{r_1} a_\mu + 2\sum_{\nu=1}^{r_2} \mathrm{Re}(z_\nu) = 1 \right\}.$$

Suppose further that $\tau_\infty : \mathbb{R}_+ \to K_\mathbb{R}^\times$ is given by

$$\tau_\infty(t) = ((t^{a_\mu})_{1 \le \mu \le r_1}, (t^{z_\nu})_{1 \le \nu \le r_2})$$

and that $\tau : \mathbb{R}_+ \to \mathbb{A}_K^\times$ is the norm section

$$\tau(t) = (\tau_\infty(t), 1).$$

We call τ a *positive norm section* if $a_\mu > 0$ for every $1 \le \mu \le r_1$ and $\mathrm{Re}(z_\nu) > 0$ for every $1 \le \nu \le r_2$,

3.4 Global Lerch-Tate zeta integrals over a number field

Definition 3.4.1. A *global Lerch tuple* is a triple (τ, a, c) consisting of a positive norm section τ and a pair of adeles $(a, c) \in \mathbb{A}_K^2$.

Definition 3.4.2. *(Global Lerch-Tate zeta integral)* Suppose that (τ, a, c) is a global Lerch tuple. If f is a Schwartz-Bruhat function on $\mathbb{A}_K$ and χ is a quasi-character of C_K, define the global Lerch-Tate zeta integral

$$Z_{\tau,a,c}(f,\chi) = \int_{\mathbb{A}_K^\times} \psi\left(\frac{ax}{\tau(|x|)}\right) f\left(x + c\,\tau(|x|)\right) \chi(x)\ \mathrm{d}^\times \mathrm{x}. \tag{3.3}$$

Remark 3.4.3. On setting $a = c = 0$, one recovers Tate's global zeta integral (1.10). Namely

$$Z_{\tau,0,0}(f,\chi) = Z(f,\chi) = \int_{\mathbb{A}_K^\times} f(x)\chi(x)\,\mathrm{d}^\times x.$$

This special case explains the name *global Lerch-Tate zeta integrals*.

Remark 3.4.4. One gains a better understanding of the global zeta integral $Z_{\tau,a,c}(f,\chi)$ with the following two observations. First, a norm section τ induces a splitting $\mathbb{A}_K^\times = \mathbb{A}_K^1 \times \tau(\mathbb{R}_+)$. Secondly, every quasi-character of $\mathbb{A}_K^\times$ is determined by its restrictions to $\mathbb{A}_K^1$ and to $\tau(\mathbb{R}_+)$.

Suppose that a norm section τ is given. Then every $x \in \mathbb{A}_K^\times$ can be written uniquely as $x = \tau(t)\,y$ where $t = |x| \in \mathbb{R}_+$ and $y = \frac{x}{\tau(|x|)} \in \mathbb{A}_K^1$. If χ is a quasi-character of $\mathbb{A}_K^\times$, write χ_0 for its restriction to $\mathbb{A}_K^1$. By Definition 3.2.1, χ belongs to the connected component X_{K,χ_0} of X_K. There is an isomorphism $\mathbb{C} \to X_{K,\chi_0}$ which sends a complex number s to the quasi-character $\chi(x) = \chi_0\left(\frac{x}{\tau(|x|)}\right)|x|^s$. Via this isomorphism, the restriction of $Z_{\tau,a,c}(f,\cdot)$ to X_{K,χ_0} becomes a function of s; write $Z_{\tau,a,c}(f,\chi_0,s)$ for this function. By (4.3) we have

$$Z_{\tau,a,c}(f,\chi_0,s) = \int_{\mathbb{R}_+}\int_{\mathbb{A}_K^1} \psi(ay)\, f\left((y+c)\,\tau(t)\right)\,\chi_0(y)\,t^s\ \ \mathrm{d}^\times \mathrm{y}\ \mathrm{d}^\times \mathrm{t}.$$

Remark 3.4.5. It is non-trivial to show the convergence of the global Lerch-Tate zeta integrals. The function

$$\psi\left(\frac{ax}{\tau(|x|)}\right) f\left(x + c\,\tau(|x|)\right)$$

is not a Schwartz-Bruhat function on $\mathbb{A}_K$ in general. To show absolute convergence we assume:

- the norm section τ is a *positive norm section*;
- for each adelic Schwartz-Bruhat function f, a certain range of parameters $(a,c) \in \mathbb{A}_K^2$ is excluded; the remaining range of parameters $(a,c) \in \mathbb{A}_K^2$ satisfies the conditions that c is compatible with f.

The following definition is a technical condition that we need in order to show the analytic properties of global zeta integrals. It will be used to exclude, for each Schwartz-Bruhat function f on $\mathbb{A}_K$, a singular set of (a, c) in $\mathbb{A}_K^2$ to ensure that the global zeta integrals are absolutely convergent and satisfy functional equation.

Definition 3.4.6. (i) Suppose that $\alpha = \otimes_{\mathfrak{p}}\alpha_{\mathfrak{p}} \otimes \alpha_\infty \in \mathbb{A}_K$. The *singular support* of $\alpha_{\mathfrak{p}}$ is the smallest fractional ideal $\mathfrak{a}_{\mathfrak{p}} \subset K_{\mathfrak{p}}$ such that $\mathcal{O}_{\mathfrak{p}} \subset \mathfrak{a}_{\mathfrak{p}}$ and that $\alpha_{\mathfrak{p}} \in \mathfrak{a}_{\mathfrak{p}}$. The *singular support* of α is the smallest fractional ideal $\mathfrak{a} \subset K$ such that $\mathcal{O} \subset \mathfrak{a}$ and that $\alpha_{\mathfrak{p}} \in \mathfrak{a}\mathcal{O}_{\mathfrak{p}}$ for every $\mathfrak{p}$.

(ii) Suppose that $f = \otimes_{\mathfrak{p}} f_{\mathfrak{p}} \otimes f_\infty$ is a factorizable Schwartz-Bruhat function on $\mathbb{A}_K$. The *singular support* of $f_{\mathfrak{p}}$ is the smallest fractional ideal $\mathfrak{a}_{\mathfrak{p}} \subset K_{\mathfrak{p}}$ such that $\mathcal{O}_{\mathfrak{p}} \subset \mathfrak{a}_{\mathfrak{p}}$ and that $\mathrm{Supp}(f_{\mathfrak{p}}) \subset \mathfrak{a}_{\mathfrak{p}}$. The *singular support* of f is the smallest fractional ideal $\mathfrak{a} \subset K$ such that $\mathcal{O} \subset \mathfrak{a}$ and that $\mathrm{Supp}(f_{\mathfrak{p}}) \subset \mathfrak{a}\mathcal{O}_{\mathfrak{p}}$ for every $\mathfrak{p}$.

(iii) Suppose that f is a Schwartz-Bruhat function on $\mathbb{A}_K$. Suppose further that $f = \sum_{i=1}^n f_i$ where each f_i is a factorizable Schwartz-Bruhat function on $\mathbb{A}_K$. The *singular support of f with respect to the decomposition* $f = \sum_{i=1}^n f_i$ is the fractional ideal $\mathfrak{a} = \sum_{i=1}^n \mathfrak{a}_i$ where each $\mathfrak{a}_i$ is the singular support of f_i. The *pole* of f is the smallest fractional ideal of K among all singular supports of f with respect to all decompositions $f = \sum_{i=1}^n f_i$.

(iv) Suppose that $\alpha = \otimes_{\mathfrak{p}}\alpha_{\mathfrak{p}} \otimes \alpha_\infty \in \mathbb{A}_K$ and that f is a Schwartz-Bruhat function on $\mathbb{A}_K$. Write $d = [K : \mathbb{Q}]$ and let $\mathfrak{a} \subset K$ (resp. $\mathfrak{b} \subset K$) be the singular support of α (resp. f). We say α and f are *compatible* if $\frac{|\alpha_\infty|}{\mathrm{N}(\mathfrak{a}+\mathfrak{b})} \notin \mathbb{Z} \setminus \{0\}$. Otherwise we say α and f are *incompatible.*

Examples 3.4.7. Suppose that $K = \mathbb{Q}$.

(i) If $\alpha = \otimes_\nu \alpha_\nu \in \mathbb{A}_{\mathbb{Q}}$ with $\alpha_\infty \in \mathbb{R}$, $\alpha_2 = \frac{1}{2}, \alpha_3 = \frac{1}{9}$ and $\alpha_p \in \mathbb{Z}_p$ for every prime $p \geq 5$, then the singular support of α_2 is $\frac{1}{2}\mathbb{Z}_2$, of α_3 is $\frac{1}{9}\mathbb{Z}_3$ and of α_p for $p \geq 5$ is $\mathbb{Z}_p$. Therefore the singular support of α is $\frac{1}{18}\mathbb{Z}$.

(ii) If $f = (1_{\frac{1}{8}\mathbb{Z}_2} \otimes_{p>2} 1_{\mathbb{Z}_p}) \otimes f_\infty$ with $f_\infty \in \mathcal{S}(\mathbb{R})$, then the singular support of f_2 is $\frac{1}{8}\mathbb{Z}_2$ and of f_p, where $p > 2$, is $\mathbb{Z}_p$.

(iii) If α and f are as in (i) and (ii), then α and f are compatible if and only if $|\alpha_\infty|$ is different from a nonzero integer multiple of $\frac{1}{72}$ $(= \frac{1}{8\times 9})$.

3.5 The Main Global Theorem over a number field

In this section, we formulate the analytic properties of the global Lerch-Tate zeta integrals as 3 theorems: Theorem 3.5.1, Theorem 3.5.2 and Theorem 3.5.3. Their combination gives Theorem 1.5.2 which we call the Main Global Theorem over a number field.

Let K be a number field. Take for $\mathbb{A}_K$ the self-dual measure $\mathrm{d}_+\mathrm{x}$ with respect to ψ and take for $\mathbb{A}_K^\times$ any Haar measure $\mathrm{d}^\times\mathrm{x}$. Put $v_K = \mathrm{d}^\times\dot{\mathrm{x}}\,(C_K^1)$ where $\mathrm{d}^\times\dot{\mathrm{x}}$ is induced by $\mathrm{d}^\times\mathrm{x}$. Write $\mathrm{d}^\times\mathrm{t} = \frac{\mathrm{d}_+\mathrm{t}}{t}$ where $\mathrm{d}_+\mathrm{t}$ denotes the Lebesgue measure on $\mathbb{R}$.

For a Schwartz-Bruhat function f on $\mathbb{A}_K$, let $\hat{f}$ denote the Fourier transform of f with respect to ψ and $\mathrm{d}_+\mathrm{x}$. For a quasi-character χ on C_K, let χ' denote the quasi-character given by $\chi\chi' = |\cdot|$.

Theorem 3.5.1. (Absolute convergence) *Let (τ, a, c) be a global Lerch tuple and ψ be a nontrivial additive character of $\mathbb{A}_K$ with trivial restriction to K. Let f be a Schwartz-Bruhat function on $\mathbb{A}_K$ and $\hat{f}$ be the Fourier transform of f with respect to ψ and $\mathrm{d}_+\mathrm{x}$. Suppose that c is compatible with f (in the sense of Definition 3.4.6* (iv)*). Then the global zeta integral $Z_{\tau,a,c}(f,\chi)$ is absolutely convergent for* $\mathrm{Re}(\chi) > 1$ *and defines a holomorphic function on this right half-plane of X_K.*

Note that the absolute convergence theorem imposes a compatibility condition on f and the variable $c \in \mathbb{A}_K$.

Theorem 3.5.2. (Analytic continuation) *Let (τ, a, c) be a global Lerch tuple and ψ be a nontrivial additive character of $\mathbb{A}_K$ with trivial restriction to K. Let f be a Schwartz-Bruhat function on $\mathbb{A}_K$ and $\hat{f}$ be the Fourier transform of f with respect to ψ and $\mathrm{d}_+\mathrm{x}$. Suppose that c is compatible with f and that a is compatible with $\hat{f}$ (in the sense of Definition 3.4.6* (iv)*).*

(i) *The zeta integral $Z_{\tau,a,c}(f,\chi)$ extends to a meromorphic function on X_K.*

(ii) *The meromorphic function $Z_{\tau,a,c}(f,\chi)$ is holomorphic outside of the unramified component. On the unramified component, $Z_{\tau,a,c}(f,\chi)$ has at most simple poles at $\chi = |\cdot|$ and $\chi = \chi_{\mathrm{triv}}$. Observe that each of the two integrals*

$$\int_1^\infty \hat{f}\,(a\,\tau(t))\,\chi'(\tau(t))\,\mathrm{d}^\times\mathrm{t} \quad \textit{and} \quad \int_0^1 f\,(c\,\tau(t))\,\chi(\tau(t))\,\mathrm{d}^\times\mathrm{t}$$

converges absolutely for $\mathrm{Re}(\chi) > 1$ *and has a meromorphic continuation to*

$\chi \in X_K$. *Then the function*

$$Z_{\tau,a,c}(f,\chi) - \psi(-ac)\, v_K \int_1^\infty \hat{f}\,(a\,\tau(t))\,\chi'(\tau(t))\;\mathrm{d}^\times\mathrm{t} + v_K \int_0^1 f\,(c\,\tau(t))\;\chi(\tau(t))\;\mathrm{d}^\times\mathrm{t}$$

extends to a holomorphic function on the unramified component of X_K.

Note that the analytic continuation theorem imposes the compatibility condition on $a \in \mathbb{A}_K$ with $\hat{f}$ and the the compatibility condition on $c \in \mathbb{A}_K$ with f.

Theorem 3.5.3. (Functional equation) *Let* (τ, a, c) *be a global Lerch tuple and* ψ *be a nontrivial additive character of* $\mathbb{A}_K$ *with trivial restriction to* K. *Let* f *be a Schwartz-Bruhat function on* $\mathbb{A}_K$ *and* $\hat{f}$ *be the Fourier transform of* f *with respect to* ψ *and* $\mathrm{d}_+\mathrm{x}$. *Suppose that* c *is compatible with* f *and that* a *is compatible with* $\hat{f}$ *(in the sense of Definition 3.4.6* (iv)*).*

(i) *The integral*

$$S_1(f,\chi) = \int_0^1 f\,(c\,\tau(t))\;\chi(\tau(t))\;\mathrm{d}^\times\mathrm{t}$$

converges absolutely for $\mathrm{Re}(\chi) > 0$ *and extends to a meromorphic function of* $\chi \in X_K$*; we denote this meromorphic function also by* $S_1(f,\chi)$. *The integral*

$$S_2(f,\chi) = \int_1^\infty f\,(c\,\tau(t))\;\chi(\tau(t))\;\mathrm{d}^\times\mathrm{t}$$

converges absolutely for $\mathrm{Re}(\chi) < 0$ *and extends to a meromorphic function of* $\chi \in X_K$*; we denote this meromorphic function also by* $S_2(f,\chi)$.

(ii) *Set*

$$\widetilde{Z}_{\tau,a,c}(f,\chi) = \begin{cases} Z_{\tau,a,c}(f,\chi) + v_K\left(S_1(f,\chi) + S_2(f,\chi)\right) & \text{if } \chi \text{ is unramified} \\ Z_{\tau,a,c}(f,\chi) & \text{if } \chi \text{ is ramified.} \end{cases}$$

One has

$$\widetilde{Z}_{\tau,a,c}(f,\chi) = \psi(-ac)\;\widetilde{Z}_{\tau,-c,a}(\hat{f},\chi'). \tag{3.4}$$

In Section 3.8, we show that the symmetrized Lerch zeta functions $L^{\pm}(s,a,c)$ in Chapter 1 can be obtained from a suitable $Z_{\tau,a,c}(f,\chi)$ for $\mathrm{Re}(\chi) > 1$. Theorem 3.5.2 and Theorem 3.5.3 give analytic continuation and functional equation respectively.

3.6 Absolute convergence

In this section we prove the absolute convergence of the global zeta integral $Z_{\tau,a,c}(f,\chi)$. We briefly recall some notations. Let (τ,a,c) be a global Lerch tuple and f be a Schwartz-Bruhat function on $\mathbb{A}_K$. The norm section τ induces a splitting $\mathbb{A}_K^\times = \mathbb{A}_K^1 \times \tau(\mathbb{R}_+)$. If χ is a quasi-character of C_K with restriction χ_0 to C_K^1, then χ belongs to the connected component X_{K,χ_0} of X_K. There is an isomorphism from $\mathbb{C}$ onto X_{K,χ_0} which sends a complex number s to the quasi-character $\chi(x) = \chi_0\left(\frac{x}{\tau(|x|)}\right)|x|^s$. Via this isomorphism, we view the restriction of $Z_{\tau,a,c}(f,\cdot)$ to X_{K,χ_0} as the function

$$s \mapsto Z_{\tau,a,c}(f,\chi_0,s) = \int_{\mathbb{R}_+}\int_{\mathbb{A}_K^1} \psi(ay)\, f\left((y+c)\,\tau(t)\right)\, \chi_0(y)\, t^s \;\mathrm{d}^\times \mathrm{y}\; \mathrm{d}^\times \mathrm{t}. \tag{3.5}$$

To show the absolute convergence of $Z_{\tau,a,c}(f,\chi_0,s)$, assume without loss of generality that f is factorizable. Write $f = \otimes_{\mathfrak{p}} f_{\mathfrak{p}} \otimes f_\infty$ where $f_{\mathfrak{p}} \in \mathcal{S}(K_{\mathfrak{p}})$ and $f_\infty \in \mathcal{S}(K_{\mathbb{R}})$. Let N_0 be a sufficiently large integer so that $\mathrm{Supp}(f_{\mathfrak{p}}) \subset \mathfrak{p}^{-N_0}\mathcal{O}_{\mathfrak{p}}$ and that $c_{\mathfrak{p}} \in \mathfrak{p}^{-N_0}\mathcal{O}_{\mathfrak{p}}$ for every $\mathfrak{p}$.

Let us start with a preparatory lemma.

Lemma 3.6.1. *Suppose that $f = \otimes_{\mathfrak{p}} f_{\mathfrak{p}} \otimes f_\infty$ is a Schwartz-Bruhat function on $\mathbb{A}_K$ and that $c \in \mathbb{A}_K$. Suppose that $C > 0$ is an arbitrary positive real. Then there exists a finite set S of places of K which includes all infinite places and which satisfies three properties:*

(a) *for every $\mathfrak{p} \notin S$, one has $f_{\mathfrak{p}} = 1_{\mathcal{O}_{\mathfrak{p}}}$;*

(b) *for every $\mathfrak{p} \notin S$, one has $c_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$;*

(c) *if $y \in \mathbb{A}_K^1$ with $y_{\mathfrak{p}} + c_{\mathfrak{p}} \in \mathrm{Supp}(f_{\mathfrak{p}})$ for every $\mathfrak{p}$ and $y_{\mathfrak{p}'} \in \mathfrak{p}'\mathcal{O}_{\mathfrak{p}'}$ for some $\mathfrak{p}' \notin S$, then $|y_\infty| > C$.*

Proof. Since f is a Schwartz-Bruhat function and $c \in \mathbb{A}_K$, there is a finite set S of places which includes all infinite places and satisfies (a) and (b). Then the set S can be enlarged, if necessary, to another finite set S' so that, if $\mathfrak{p}' \notin S'$, then $\mathrm{N}\mathfrak{p}' > C\ \prod_{\mathfrak{p}\in S}(\mathrm{N}\mathfrak{p})^{N_0}$. It is clear that the set S' necessarily satisfies (a) and (b).

Suppose now that $y \in \mathbb{A}_K^1$ with $y_{\mathfrak{p}} + c_{\mathfrak{p}} \in \mathrm{Supp}(f_{\mathfrak{p}})$ for every $\mathfrak{p}$ and $y_{\mathfrak{p}'} \in \mathfrak{p}'\mathcal{O}_{\mathfrak{p}'}$ for some $\mathfrak{p}' \notin S'$. Hence $|y_{\mathfrak{p}'}| \leq \frac{1}{\mathrm{N}\mathfrak{p}'}$. By definition N_0 is a sufficiently large integer so that $\mathrm{Supp}(f_{\mathfrak{p}}) \subset \mathfrak{p}^{-N_0}\mathcal{O}_{\mathfrak{p}}$ and that $c_{\mathfrak{p}} \in \mathfrak{p}^{-N_0}\mathcal{O}_{\mathfrak{p}}$ for every $\mathfrak{p}$. In particular, at every place $\mathfrak{p} \in S$, $y_{\mathfrak{p}} \in \mathfrak{p}^{-N_0}\mathcal{O}_{\mathfrak{p}}$ and hence $|y_{\mathfrak{p}}| \leq (\mathrm{N}\mathfrak{p})^{N_0}$; whereas at every place $\mathfrak{p} \notin S$, $y_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$

and hence $|y_\mathfrak{p}| \leq 1$. It follows that

$$\begin{aligned} |y_\infty| &= \frac{1}{\prod_\mathfrak{p} |y_\mathfrak{p}|_\mathfrak{p}} \geq \frac{1}{|y_{\mathfrak{p}'}| \ \prod_{\mathfrak{p}\in S} |y_\mathfrak{p}|} \geq \frac{\mathrm{N}\mathfrak{p}'}{\prod_{\mathfrak{p}\in S} |y_\mathfrak{p}|} \\ &\geq \frac{\mathrm{N}\mathfrak{p}'}{\prod_{\mathfrak{p}\in S} (\mathrm{N}\mathfrak{p})^{N_0}} > C. \end{aligned}$$

Therefore the set S' satisfies the property (c), besides the properties (a) and (b) that S' inherits from S. The existence of a finite set of places of K with the three properties is thus verified. $\square$

We are in a position to prove the absolute convergence of global Lerch-Tate zeta integrals.

Proof of Theorem 3.5.1. Let $C > 0$ be a constant which depends on $c \in \mathbb{A}_K$ and which is to be chosen. Let S be a finite set of places of K which contains all infinite places and which satisfies the three properties in Lemma 3.6.1. In (3.5), we decompose the integral over $\mathbb{A}_K^1$ into the integrals over the two subsets

$$\begin{aligned} H_1 &= \{y \in \mathbb{A}_K^1 : y_{\mathfrak{p}'} \in \mathfrak{p}'\mathcal{O}_{\mathfrak{p}'} \text{ for some } \mathfrak{p}' \notin S\} \\ H_2 &= \{y \in \mathbb{A}_K^1 : y_\mathfrak{p} \in \mathcal{O}_\mathfrak{p}^\times \text{ for all } \mathfrak{p} \notin S\}. \end{aligned}$$

By definition, H_1 is a closed subset of $\mathbb{A}_K^1$ and H_2 is an open subset of $\mathbb{A}_K^1$. We will show that the integral over H_1 converges absolutely for $\sigma = \mathrm{Re}(s) > 1$ and the integral over H_2 converges absolutely for $\sigma > 0$.

Let us set some notations for the proof. Suppose that K has r_1 real embeddings and r_2 complex embeddings; put $R = r_1 + r_2$. Write

$$S = \{\text{infinite places}\} \cup \{\mathfrak{p}_j : 1 \leq j \leq r\} = S_\infty \cup \{\mathfrak{p}_j : 1 \leq j \leq r\}.$$

and put $q_j = \mathrm{N}\mathfrak{p}_j$ for $1 \leq j \leq r$. Write

$$S_\infty = \{\nu_j : 1 \leq j \leq R\}$$

for the set of archimedean places of K. We therefore have

$$S = S_\infty \cup \{\mathfrak{p}_j : 1 \leq j \leq r\}.$$

If $y = \otimes_\nu y_\nu \in \mathbb{A}_K^\times$ is an idele, write $y_\infty = \otimes_{1\leq j\leq R} y_{\nu_j}$ for its archimedean component and $y_{\mathrm{fin}} = \otimes_{\mathfrak{p}} y_{\mathfrak{p}}$ for its nonarchimedean component. Put $|y|_{\nu_j} = |y_\infty|_{\nu_j} := |y_{\nu_j}|_{\nu_j}$. Recall that the norms at real places are the usual absolute value and at complex places are the square of the usual absolute value. For a positive real A, put

$$K^\times_{\mathbb{R},|\cdot|=A} = \{y \in K^\times_{\mathbb{R}} : |y| = A\}.$$

For a place ν of K and a positive constant A, the sets $K^\times_{\nu,|\cdot|>A}$ and $K^\times_{\nu,|\cdot|\leq A}$ have their obvious meanings. For an archimedean place ν, set

$$U_\nu = \{x \in K^\times_\nu : |x|_\nu = 1\};$$

for instance $U_\nu = \{\pm 1\}$ if ν is real and $U_\nu = \mathbb{S}^1$ if ν is complex. We also need some notations related to the norm section $\tau : \mathbb{R}_+ \to \mathbb{A}^\times_K$. Let τ_∞ be the composition

$$\mathbb{R}_+ \xrightarrow{\tau} \mathbb{A}^\times_K \xrightarrow{\mathrm{proj}} K^\times_{\mathbb{R}}.$$

For $1 \leq j \leq R$ let τ_j be the projection

$$\mathbb{R}_+ \xrightarrow{\tau_\infty} K^\times_{\mathbb{R}} \xrightarrow{\mathrm{proj}} K^\times_{\nu_j}.$$

For $1 \leq j \leq R$ let $b_j \in \mathbb{R}$ be such that $|\tau_j(t)|_{\nu_j} = t^{b_j}$. The condition that the norm section is positive is equivalent to $b_j > 0$ for every $1 \leq j \leq R$.

Consider the integral over H_2, namely

$$\int_{\mathbb{R}_+} \int_{H_2} \psi(ay)\, f\left((y+c)\,\tau(t)\right)\, \chi_0(y)\, t^s \;\, \mathrm{d}^\times\mathrm{y}\; \mathrm{d}^\times\mathrm{t}. \tag{3.6}$$

Our goal is to show that this integral converges absolutely and uniformly on compacta for $\sigma > 0$. We do this by carrying out a series of reductions. Note first that ψ and χ_0 are unitary characters and that we may assume without loss of generality that each component of f takes value in $\mathbb{R}_{\geq 0}$. The absolute and uniform convergence of (3.6) is reduced to showing that, for arbitrary positive reals $0 < \delta < A$, the integral

$$\int_{\mathbb{R}_+} \int_{H_2} f\left((y+c)\,\tau(t)\right)\, t^\sigma \;\, \mathrm{d}^\times\mathrm{y}\; \mathrm{d}^\times\mathrm{t} \tag{3.7}$$

is uniformly bounded for $\delta < \sigma < A$.

By the properties (a) and (b) of S given by Lemma 3.6.1, $f_{\mathfrak{p}} = 1_{\mathcal{O}_{\mathfrak{p}}}$ and $c_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$ for every $\mathfrak{p} \notin S$. Recall that N_0 is chosen as a sufficiently large integer so that

$\mathrm{Supp}(f_{\mathfrak{p}}) \subset \mathfrak{p}^{-N_0}\mathcal{O}_{\mathfrak{p}}$ and that $c_{\mathfrak{p}} \in \mathfrak{p}^{-N_0}\mathcal{O}_{\mathfrak{p}}$ for every $\mathfrak{p}$. It follows that

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{H_2} f\left((y+c)\,\tau(t)\right)\, t^{\sigma}\ \mathrm{d}^{\times}\mathrm{y}\ \mathrm{d}^{\times}\mathrm{t} \\
&\ll \int_{\mathbb{R}_+}\Big\{\int_{\left(\prod_{\mathfrak{p}\in S} K_{\mathfrak{p}}^{\times}\times K_{\mathbb{R}}^{\times}\right)\cap\mathbb{A}_K^1} f_{\infty}\left((y_{\infty}+c_{\infty})\,\tau(t)\right)\prod_{\mathfrak{p}\in S} f_{\mathfrak{p}}(y_{\mathfrak{p}}+c_{\mathfrak{p}})\otimes_{\mathfrak{p}\in\mathrm{S}}\mathrm{d}^{\times}\mathrm{y}_{\mathfrak{p}}\otimes\mathrm{d}^{\times}\mathrm{y}_{\infty}\Big\} t^{\sigma}\mathrm{d}^{\times}\mathrm{t} \\
&\ll \int_{\mathbb{R}_+}\left\{\int_{\left(\prod_{\mathfrak{p}\in S}\left(K_{\mathfrak{p}}^{\times}\cap\mathfrak{p}^{-N_0}\mathcal{O}_{\mathfrak{p}}\right)\times K_{\mathbb{R}}^{\times}\right)\cap\mathbb{A}_K^1} f_{\infty}\left((y_{\infty}+c_{\infty})\,\tau(t)\right)\ \otimes_{\mathfrak{p}\in\mathrm{S}}\mathrm{d}^{\times}\mathrm{y}_{\mathfrak{p}}\otimes\mathrm{d}^{\times}\mathrm{y}_{\infty}\right\} t^{\sigma}\ \mathrm{d}^{\times}\mathrm{t} \\
&\ll \int_{\mathbb{R}_+}\left\{\sum_{\substack{(n_1,\ldots,n_r)\in\mathbb{Z}^r\\ n_j\geq -N_0}}\int_{K^{\times}_{\mathbb{R},|\cdot|=\prod_{j=1}^{r}\mathrm{N}\mathfrak{p}_j^{n_j}}} f_{\infty}\left((y_{\infty}+c_{\infty})\,\tau(t)\right)\mathrm{d}^{\times}\mathrm{y}_{\infty}\right\} t^{\sigma}\ \mathrm{d}^{\times}\mathrm{t}.
\end{aligned}$$

The uniform boundedness of (3.7) now reduces to the uniform boundedness of

$$\sum_{\substack{(n_1,\ldots,n_r)\in\mathbb{Z}^r\\ n_j\geq -N_0}}\int_{\mathbb{R}_+}\int_{K^{\times}_{\mathbb{R},|\cdot|=\prod_{j=1}^{r}\mathrm{N}\mathfrak{p}_j^{n_j}}} f_{\infty}\left((y_{\infty}+c_{\infty})\,\tau(t)\right)\, t^{\sigma}\ \mathrm{d}^{\times}\mathrm{y}_{\infty}\,\mathrm{d}^{\times}\mathrm{t}\qquad(\delta<\sigma<A). \tag{3.8}$$

To simplify notation, assume that $r = 1$; the proof of convergence of (3.8) for general r is the same. When $r = 1$, we have $S = \{\nu_j : 1 \leq j \leq R\} \cup \{\mathfrak{p}\}$ and the integral (3.8) equals

$$\sum_{n\geq -N_0}\int_{\mathbb{R}_+}\int_{K^{\times}_{\mathbb{R},|\cdot|=\mathrm{N}\mathfrak{p}^n}} f_{\infty}\left((y_{\infty}+c_{\infty})\,\tau(t)\right)\, t^{\sigma}\ \mathrm{d}^{\times}\mathrm{y}_{\infty}\ \mathrm{d}^{\times}\mathrm{t}\qquad(\delta<\sigma<A). \tag{3.9}$$

It follows from the compatibility assumption of c and f, which is prescribed in Definition 3.4.6 (iv), that

$$|c_{\infty}| \neq \mathrm{N}\mathfrak{p}^n \quad \text{for every} \quad n \geq -N_0.$$

In the integral (3.9), one has $|y_{\infty}| = \mathrm{N}\mathfrak{p}^n$. When y_{∞} belongs to the hypersurface $K^{\times}_{\mathbb{R},|\cdot|=\mathrm{N}\mathfrak{p}^n}$, there is at least one place $\nu \in S_{\infty}$ such that

$$|y_{\infty}+c_{\infty}|_{\nu} \gg |y_{\infty}|_{\nu} \tag{3.10}$$

Here the notation $f \gg g$ means there is a positive constant C such that $f(x) \geq$

$C|g(x)|$. Without loss of generality assume $\nu = \nu_1$. We then rewrite (3.9) as

$$\sum_{n \geq -N_0} \int_{\mathbb{R}_+} \int_{K^\times_{\mathbb{R},|\cdot|=\mathrm{N}\mathfrak{p}^n}} \mathrm{d}^\times \mathrm{y}_\infty \, \mathrm{d}^\times \mathrm{t} = \sum_{n \geq -N_0} \int_{\mathbb{R}_+} \int_{U_1 \times \prod_{j=2}^R K^\times_{\nu_j}} \mathrm{d}^\times \mathrm{u}_1 \mathrm{d}^\times \mathrm{y}_2 \dots \mathrm{d}^\times \mathrm{y}_\mathrm{R} \, \mathrm{d}^\times \mathrm{t}. \tag{3.11}$$

The integrand of the second member of (3.11) is

$$f_\infty \left(\left(\frac{\mathrm{N}\mathfrak{p}^n}{|y|_{\nu_2} \dots |y|_{\nu_R}} u_1 + c_1 \right) \tau_1(t), (y_2 + c_2)\tau_2(t), \dots, (y_R + c_R)\tau_R(t) \right). \tag{3.12}$$

The inequality (3.10) now becomes

$$\left| \frac{\mathrm{N}\mathfrak{p}^n}{|y|_{\nu_2} \dots |y|_{\nu_R}} u_1 + c_1 \right|_{\nu_1} \gg \frac{\mathrm{N}\mathfrak{p}^n}{|y|_{\nu_2} \dots |y|_{\nu_R}}. \tag{3.13}$$

We are now reduced to showing the uniform boundedness of the second member of (3.11) for $\delta < \sigma < A$, given that (3.13) holds. For this purpose we make use of the fact that f_∞ is of rapid decay; in particular it decays rapidly with respect to each coordinate. For each $2 \leq j \leq R$, write $K^\times_{\nu_j} = D_j \cup E_j$ where

$$\begin{aligned} D_j &= \{x \in K^\times_{\nu_j} : |x|_{\nu_j} \leq 2|c|_{\nu_j} + 1\} \\ E_j &= \{x \in K^\times_{\nu_j} : |x|_{\nu_j} > 2|c|_{\nu_j} + 1\}. \end{aligned}$$

By definition, if $y_j \in E_j$, then $|y + c|_{\nu_j} \gg |y|_{\nu_j}$. In the integral (3.11), we make two decompositions:

- decompose $\int_{\mathbb{R}_+} = \int_0^1 + \int_1^\infty$;
- decompose the integral over $\prod_{j=2}^R K^\times_{\nu_j}$ into the integral over "tubes" where each factor is either D_j or E_j. If $y_j \in D_j$, then there is a pole at 0 which is taken care by the rapid decay of f_∞ with respect to the first coordinate. If $y_j \in E_j$, then the convergence at infinity is taken care by the rapid decay of f_∞ with respect to the coordinate y_j.

More precisely, first look at $\int_0^1$. Let $\delta' > 0$ be arbitrary and suppose that $y_j \in D_j$ for $2 \leq j \leq T$ and that $y_j \in E_j$ for $T + 1 \leq j \leq R$. Then (3.12) is

$$\ll \left(\frac{N\mathfrak{p}^n}{|y|_{\nu_2} \dots |y|_{\nu_R}} t^{b_1} \right)^{-\delta'} \prod_{j=T+1}^R \left(|y|_{\nu_j} t^{b_j} \right)^{-2\delta'}.$$

Then the contribution to (3.11) is

$$\ll \sum_{n \geq -N_0} (N\mathfrak{p})^{-n\delta'} \int_0^1 t^{\sigma - \delta' b_1 - 2\delta' \sum_{j=T+1}^R b_j} \frac{\mathrm{dt}}{t} \cdot$$
$$\int_{U_1 \times \prod_{j=2}^T D_j \times \prod_{j=T+1}^R E_j} \prod_{j=2}^T |y|_{\nu_j}^{\delta'} \prod_{j=T+1}^R |y|_{\nu_j}^{-\delta'} \mathrm{d}^\times \mathrm{u}_1 \mathrm{d}^\times \mathrm{y}_2 \dots \mathrm{d}^\times \mathrm{y}_\mathrm{R},$$

which is finite given that $\sigma > \delta' \left(b_1 + 2 \sum_{j=T+1}^R b_j \right)$.

Next look at $\int_1^\infty$. Let $A' > 0$ be arbitrary and suppose that $y_j \in D_j$ for $2 \leq j \leq T$ and that $y_j \in E_j$ for $T+1 \leq j \leq R$. Then (3.12) is

$$\ll \left(\frac{N\mathfrak{p}^n}{|y|_{\nu_2} \dots |y|_{\nu_R}} t^{b_1} \right)^{-A'} \prod_{j=T+1}^R \left(|y|_{\nu_j} t^{b_j} \right)^{-A' - \epsilon}.$$

Then the contribution to (3.11) is

$$\ll \sum_{n \geq -N_0} (N\mathfrak{p})^{-nA'} \int_1^\infty t^{\sigma - A' b_1 - (A'+\epsilon) \sum_{j=T+1}^R b_j} \frac{\mathrm{dt}}{t} \cdot$$
$$\int_{U_1 \times \prod_{j=2}^T D_j \times \prod_{j=T+1}^R E_j} \prod_{j=2}^T |y|_{\nu_j}^{A'} \prod_{j=T+1}^R |y|_{\nu_j}^{-\epsilon} \mathrm{d}^\times \mathrm{u}_1 \mathrm{d}^\times \mathrm{y}_2 \dots \mathrm{d}^\times \mathrm{y}_\mathrm{R},$$

which is finite given that $\sigma < A' b_1 + (A' + \epsilon) \sum_{j=T+1}^R b_j$. Here we crucially use that fact that all b_j are positive reals, since all combinations of the b_j may occur in general.

Finally, by making δ' sufficiently small and A' sufficiently large, we can ensure that the condition $\delta < \sigma < A$ implies the two requirements on σ. This proves that absolute convergence of (3.6) for $\sigma > 0$; the convergence is uniform in strips $\delta < \sigma < A$ for arbitrary $0 < \delta < A$.

Now consider the integral over H_1, namely

$$\int_{\mathbb{R}_+} \int_{H_1} \psi(ay) \, f\left((y+c)\, \tau(t) \right) \, \chi_0(y) \, t^s \; \mathrm{d}^\times \mathrm{y} \; \mathrm{d}^\times \mathrm{t}. \tag{3.14}$$

Our goal is to show that this integral converges absolutely and uniformly on compacta for $\sigma > 1$. We now set the constant $C > 0$ at the beginning of the proof to be

$$C = 2^R \prod_{j=1}^R \max\left(|c|_{\nu_j}, 1 \right).$$

By definition, if $y \in H_1$ then $y_{\mathfrak{p}'} \in \mathfrak{p}'\mathcal{O}_{\mathfrak{p}'}$ for some $\mathfrak{p}' \notin S$. For every $\mathfrak{p} \notin S$, the properties (a) and (b) of S in Lemma 3.6.1 imply that $f_{\mathfrak{p}} = 1_{\mathcal{O}_{\mathfrak{p}}}$ and that $c_{\mathfrak{p}} \in \mathcal{O}_{\mathfrak{p}}$. If, furthermore, $t \in \mathbb{R}_+$ and $(y + c)\,\tau(t) \in \mathrm{Supp}(f)$, then y verifies the condition (c) of Lemma 3.6.1 and hence $|y_\infty| > C$. By our choice of C, when $y \in H_1$, there is a place $\nu \in S_\infty$ such that

$$|y_\infty|_\nu > 2|c_\infty|_\nu,$$

which in turn implies that

$$|y_\infty + c_\infty|_\nu \gg |y_\infty|_\nu. \tag{3.15}$$

Proceeding exactly the same as above, we deduce that (3.14) converges absolutely for $\sigma > 1$; the convergence is uniform in strips $1 + \delta < \sigma < A$ for arbitrary $0 < \delta < A - 1$. Again the positivity assumption on the norm section, namely $b_j > 0$ for every $1 \le j \le R$ is used here.

On combining the absolute convergence of (3.14) and (3.6), we conclude the absolute convergence of $Z_{\tau,a,c}(f, \chi_0, s)$ for $\mathrm{Re}(s) > 1$, provided that c and f is compatible. This completes the proof of Theorem 3.5.1. □

3.7 Analytic continuation and functional equation

In this section we prove the analytic continuation and the functional equation of the global zeta integral $Z_{\tau,a,c}(f, \chi)$. In Section 3.6, the global zeta integral $Z_{\tau,a,c}(f, \chi)$ is shown to converge absolutely for $\mathrm{Re}(\chi) > 1$. To establish analytic continuation and functional equation, we follow the adelic analysis framework which was first introduced in Tate's Thesis. The origin of Tate's approach traced back to Riemann's second proof of the analytic continuation and functional equation of the Riemann's zeta function.

Definition 3.7.1. Suppose that $F \subset \mathbb{A}_K$ is a fundamental domain for the additive action of K on $\mathbb{A}_K$. Let $\mathcal{Z}(\mathbb{A}_K)$ denote the class of functions $f : \mathbb{A}_K \to \mathbb{C}$ which satisfy the following conditions:

(i) $f(x)$ and $\hat{f}(x)$ are continuous and are in $L^1(\mathbb{A}_K)$;

(ii) $\sum_{\xi \in K} f(x + \xi)$ and $\sum_{\xi \in K} \hat{f}(x + \xi)$ are absolutely and uniformly convergent for $x \in F$;

Note that any Schwartz-Bruhat function $f \in \mathcal{S}(\mathbb{A}_K)$ is also in $\mathcal{Z}(\mathbb{A}_K)$. The main tool that we need in proving analytic continuation and functional equation is the following result of Tate.

Proposition 3.7.2. (Tate's adelic Poisson summation formula [5, p. 333]) *If $f \in \mathcal{Z}(\mathbb{A}_K)$ and $x \in \mathbb{A}_K^\times$, then*

$$\sum_{\xi \in K} f(\xi x) = \frac{1}{|x|} \sum_{\xi \in K} \hat{f}(\xi / x).$$

The following preparatory lemma computes a Fourier transform.

Lemma 3.7.3. *Suppose that $\psi : \mathbb{A}_K / K \to \mathbb{C}^\times$ is a character and that $\mathrm{d}_+\mathrm{x}$ is a Haar measure on $\mathbb{A}_K$. Suppose further that $\tau : \mathbb{R}_+ \to \mathbb{A}_K^\times$ is a norm section and that $(a, c) \in \mathbb{A}_K^2$. If f is a Schwartz-Bruhat function on $\mathbb{A}_K$ and $t \in \mathbb{R}_+$, then the Fourier transforms of f and of the function*

$$g_t(x) = \psi(ax)\, f\left((x + c)\, \tau(t)\right)$$

are related by

$$\hat{g}_t(x) = \frac{1}{t}\, \psi\left(-c(x + a)\right)\, \hat{f}\left(\frac{x + a}{\tau(t)}\right),$$

where the Fourier transforms are taken with respect to ψ and $\mathrm{d}_+\mathrm{x}$.

Proof. The Fourier transform $\hat{g}$ can be computed in three steps. First, if $f_1(x) = f(x\, \tau(t))$, then

$$\hat{f}_1(y) = \int_{\mathbb{A}_K} f\left(x\, \tau(t)\right)\, \psi(xy)\ \mathrm{d}_+\mathrm{x} = \frac{1}{t}\, \hat{f}\left(\frac{y}{\tau(t)}\right).$$

Secondly, if $f_2(x) = f_1(x + c) = f\left((x + c)\, \tau(t)\right)$, then

$$\hat{f}_2(y) = \int_{\mathbb{A}_K} f_1(x + c)\, \psi(xy)\ \mathrm{d}_+\mathrm{x} = \psi(-cy) \hat{f}_1(y).$$

Thirdly, since $g_t(x) = \psi(ax)\, f_2(x) = \psi(ax)\, f\left((x + c)\, \tau(t)\right)$, it follows that

$$\hat{g}_t(y) = \int_{\mathbb{A}_K} \psi(ax)\, f_2(x)\, \psi(xy)\ \mathrm{d}_+\mathrm{x} = \hat{f}_2(a + y).$$

Thus $\hat{g}_t(x) = \frac{1}{t}\, \psi\left(-c(x + a)\right)\, \hat{f}\left(\frac{x+a}{\tau(t)}\right)$. $\square$

We are in a position to establish the analytic continuation of $Z_{\tau,a,c}(f, \chi)$ and compute its poles and residues.

Proof of Theorem 3.5.2. Let χ_0 be a character of C_K^1. By Remark 3.4.4, we view the restriction of the zeta integral $Z_{\tau,a,c}(f, \cdot)$ to the connected component X_{K,χ_0} of X_K as the function

$$s \mapsto Z_{\tau,a,c}(f,\chi_0,s) = \int_{\mathbb{R}_+}\int_{\mathbb{A}_K^1} \psi(ay)\, f\left((y+c)\,\tau(t)\right)\, \chi_0(y)\, t^s \;\; \mathrm{d}^\times \mathrm{y}\; \mathrm{d}^\times \mathrm{t}. \tag{3.16}$$

Following Riemann, let us rewrite (3.16) as $\int_{\mathbb{R}_+} = \int_1^\infty + \int_0^1$. Set

$$Z^+_{\tau,a,c}(f,\chi_0,s) = \int_1^\infty \int_{\mathbb{A}_K^1} \psi(ay)\, f\left((y+c)\,\tau(t)\right)\, \chi_0(y)\, t^s \;\; \mathrm{d}^\times \mathrm{y}\; \mathrm{d}^\times \mathrm{t} \tag{3.17}$$

$$Z^-_{\tau,a,c}(f,\chi_0,s) = \int_0^1 \int_{\mathbb{A}_K^1} \psi(ay)\, f\left((y+c)\,\tau(t)\right)\, \chi_0(y)\, t^s \;\; \mathrm{d}^\times \mathrm{y}\; \mathrm{d}^\times \mathrm{t}. \tag{3.18}$$

On the one hand, the zeta integral $Z^+_{\tau,a,c}(f,\chi_0,s)$ converges absolutely and defines an entire function of s. This can be shown similarly to the proof of Theorem 3.5.1. On the other hand

$$\begin{aligned} Z^-_{\tau,a,c}(f,\chi_0,s) &= \int_0^1 \left\{ \int_{\mathbb{A}_K^1} \psi(ay)\, f\left((y+c)\,\tau(t)\right)\, \chi_0(y)\;\; \mathrm{d}^\times \mathrm{y} \right\}\; t^s \mathrm{d}^\times \mathrm{t} \\ &= \int_0^1 \left\{ \int_{\mathbb{A}_K^1/K^\times} \sum_{\xi\in K^\times} \psi(a\xi y)\, f\left((\xi y+c)\,\tau(t)\right)\, \chi_0(y)\;\; \mathrm{d}^\times \dot{\mathrm{y}} \right\}\; t^s \mathrm{d}^\times \mathrm{t} \\ &= \int_0^1 \left\{ \int_{\mathbb{A}_K^1/K^\times} \sum_{\xi\in K^\times} g_t(\xi y)\, \chi_0(y)\;\; \mathrm{d}^\times \dot{\mathrm{y}} \right\}\; t^s \mathrm{d}^\times \mathrm{t} \end{aligned}$$

where $g_t(y) = \psi(ay) f\left((y+c)\,\tau(t)\right)$. Put

$$\begin{aligned} M &= \int_0^1 \left\{ \int_{\mathbb{A}_K^1/K^\times} \sum_{\xi\in K^\times} \hat{g}_t(\xi/y)\, \chi_0(y)\;\; \mathrm{d}^\times \dot{\mathrm{y}} \right\}\; t^s \mathrm{d}^\times \mathrm{t} \\ U &= \int_0^1 \left\{ \int_{\mathbb{A}_K^1/K^\times} \hat{g}_t(0)\, \chi_0(y)\;\; \mathrm{d}^\times \dot{\mathrm{y}} \right\}\; t^s \mathrm{d}^\times \mathrm{t} \\ V &= \int_0^1 \left\{ \int_{\mathbb{A}_K^1/K^\times} g_t(0)\, \chi_0(y)\;\; \mathrm{d}^\times \dot{\mathrm{y}} \right\}\; t^s \mathrm{d}^\times \mathrm{t}. \end{aligned}$$

It follows from Proposition 3.7.2 that

$$Z^-_{\tau,a,c}(f,\chi_0,s) = M + U - V. \tag{3.19}$$

We proceed to compute M, U and V. First, by Proposition 3.7.3 we have

$$\begin{aligned}\hat{g}_t(\xi/y) &= \frac{1}{t}\,\psi\left(-c(a+\xi/y)\right)\,\hat{f}\left(\frac{a+\xi/y}{\tau(t)}\right)\\ &= \frac{1}{t}\,\psi(-ac)\,\psi\left(-c\xi y^{-1}\right)\,\hat{f}\left(\frac{a+\xi y^{-1}}{\tau(t)}\right).\end{aligned}$$

It follows that

$$\begin{aligned}M &= \psi(-ac)\int_0^1\left\{\int_{\mathbb{A}_K^1/K^\times}\sum_{\xi\in K^\times}\psi\left(-c\xi y^{-1}\right)\,\hat{f}\left(\frac{a+\xi y^{-1}}{\tau(t)}\right)\,\chi_0(y)\;\mathrm{d}^\times\dot{\mathrm{y}}\right\}\,t^{s-1}\mathrm{d}^\times\mathrm{t}\\ &= \psi(-ac)\int_0^1\left\{\int_{\mathbb{A}_K^1/K^\times}\sum_{\xi\in K^\times}\psi\left(-c\xi y_1\right)\,\hat{f}\left(\frac{a+\xi y_1}{\tau(t)}\right)\,\chi_0^{-1}(y_1)\;\mathrm{d}^\times\dot{\mathrm{y}}_1\right\}\,t^{s-1}\mathrm{d}^\times\mathrm{t}\\ &= \psi(-ac)\int_1^\infty\left\{\int_{\mathbb{A}_K^1/K^\times}\sum_{\xi\in K^\times}\psi\left(-c\xi y_1\right)\hat{f}\left(\left(a+\xi y_1\right)\tau(t)\right)\chi_0^{-1}(y_1)\,\mathrm{d}^\times\dot{\mathrm{y}}_1\right\}t^{1-s}\mathrm{d}^\times\mathrm{t}\\ &= \psi(-ac)\;Z^+_{\tau,-c,a}(\hat{f},\chi_0^{-1},1-s). \qquad (3.20)\end{aligned}$$

Thus M extends to an entire function of s.

Secondly, observe that both U and V vanish unless χ is unramified. Put $v_K = \int_{\mathbb{A}_K^1/K^\times} 1\mathrm{d}\dot{\mathrm{y}}$. If χ is unramified, by Proposition 3.7.3 we have

$$U = \psi(-ac)\,v_K\int_0^1\hat{f}\left(a/\tau(t)\right)t^{s-1}\mathrm{d}^\times\mathrm{t} \qquad (3.21)$$

$$V = v_K\int_0^1 f\left(c\,\tau(t)\right)t^{s}\mathrm{d}^\times\mathrm{t}. \qquad (3.22)$$

Since the infinity components f_∞ and $\hat{f}_\infty$ are Schwartz function, it is clear that both U and V define holomorphic functions for $\sigma > 1$ and have meromorphic continuation to $s \in \mathbb{C}$.

On combining (3.19), (3.20), (3.21) and (3.22), we deduce that

$$Z_{\tau,a,c}(f,\chi_0,s) = Z^+_{\tau,a,c}(f,\chi_0,s) + \psi(-ac)\;Z^+_{\tau,-c,a}(\hat{f},\chi_0^{-1},1-s) + U - V.$$

Moreover, both $Z^+_{\tau,a,c}(f,\chi_0,s)$ and $Z^+_{\tau,-c,a}(\hat{f},\chi_0^{-1},1-s)$ are entire functions of s, whereas both U and V extend to a meromorphic function of $s \in \mathbb{C}$. This shows the meromorphic continuation of $Z_{\tau,a,c}(f,\chi_0,s)$. The preceeding argument also shows that, if χ is unramified, then $U = V = 0$ and hence $Z_{\tau,a,c}(f,\chi_0,s)$ extends to an entire function of s. If χ is ramified, then $Z_{\tau,a,c}(f, \chi_0, s) - U + V$ extends to an entire function of s. The proof of (ii) and (iii) is complete. $\square$

We now show the functional equation of $Z_{\tau,a,c}(f, \chi_0, s)$.

Proof of Theorem 3.5.3. Observe first that, on putting $f^\tau(x) = f(-x)$, we see that

$$Z^+_{\tau,-a,-c}(f^\tau, \chi_0, s) = Z^+_{\tau,a,c}(f, \chi_0, s). \tag{3.23}$$

In fact, by (3.17) and by the triviality of χ_0 on $K^\times$,

$$\begin{aligned}
Z^+_{\tau,-a,-c}(f^\tau, \chi_0, s) &= \int_1^\infty \int_{\mathbb{A}_K^1} \psi(-ay)\, f^\tau\left((y-c)\,\tau(t)\right)\, \chi_0(y)\, t^s \ \mathrm{d}^\times \mathrm{y}\ \mathrm{d}^\times \mathrm{t} \\
&= \int_1^\infty \int_{\mathbb{A}_K^1} \psi(-ay)\, f\left((c-y)\,\tau(t)\right)\, \chi_0(y)\, t^s \ \mathrm{d}^\times \mathrm{y}\ \mathrm{d}^\times \mathrm{t} \\
&= \int_1^\infty \int_{\mathbb{A}_K^1} \psi(ay_1)\, f\left((c+y_1)\,\tau(t)\right)\, \chi_0(y_1)\, t^s \ \mathrm{d}^\times \mathrm{y}_1\ \mathrm{d}^\times \mathrm{t} \\
&= Z^+_{\tau,a,c}(f, \chi_0, s).
\end{aligned}$$

We consider two cases:

(a) Assume χ is unramified. It follows from (3.19), (3.20), (3.21) and (3.22) that

$$\begin{aligned}
Z_{\tau,a,c}(f, \chi_0, s) = {}& Z^+_{\tau,a,c}(f, \chi_0, s) + \psi(-ac)\ Z^+_{\tau,-c,a}(\hat{f}, \chi_0^{-1}, 1-s) \\
&+ \psi(-ac)\, v_K \int_0^1 \hat{f}\left(a/\tau(t)\right)\, t^{s-1}\ \mathrm{d}^\times \mathrm{t} - v_K \int_0^1 f\left(c\,\tau(t)\right)\, t^s\ \mathrm{d}^\times \mathrm{t}.
\end{aligned} \tag{3.24}$$

This expression of $Z_{\tau,a,c}(f, \chi_0, s)$ exhibits the symmetry

$$\begin{gathered}
f \leftrightarrow \hat{f} \\
\chi = (\chi_0, s) \leftrightarrow \chi' = (\chi_0^{-1}, 1-s) \\
(a, c) \leftrightarrow (-c, a).
\end{gathered}$$

More precisely, (3.24) together with (3.23) and the fact that $\hat{\hat{f}} = f^\tau$ yield

$$
\begin{aligned}
&Z_{\tau,-c,a}(\hat{f},\chi_0^{-1},1-s)\\
&= Z^+_{\tau,-c,a}(\hat{f},\chi_0^{-1},1-s) + \psi(ac)\ Z^+_{\tau,-a,-c}(f^\tau,\chi_0,s)\\
&\quad + \psi(ac)\, v_K \int_0^1 \hat{f}\,(-c/\tau(t))\ t^{s-1}\ \mathrm{d}^\times\mathrm{t} - v_K \int_0^1 f\,(a\,\tau(t))\ t^s\ \mathrm{d}^\times\mathrm{t}\\
&= Z^+_{\tau,-c,a}(\hat{f},\chi_0^{-1},1-s) + \psi(ac)\ Z^+_{\tau,a,c}(f,\chi_0,s)\\
&\quad + \psi(ac)\, v_K \int_1^\infty \hat{f}\,(-c\,\tau(t))\ t^{1-s}\ \mathrm{d}^\times\mathrm{t} - v_K \int_1^\infty f\,(a/\tau(t))\ t^{-s}\ \mathrm{d}^\times\mathrm{t}.
\end{aligned}
\tag{3.25}
$$

It follows from (3.24) and (3.25) that

$$
\begin{aligned}
&\frac{1}{v_K}\Big(Z_{\tau,a,c}(f,\chi_0,s) - \psi(-ac)\ Z_{\tau,-c,a}(\hat{f},\chi_0^{-1},1-s)\Big)\\
&= \psi(-ac) \int_0^\infty \hat{f}\,(a/\tau(t))\ t^{s-1}\ \mathrm{d}^\times\mathrm{t} - \int_0^\infty f\,(c\,\tau(t))\,t^s\ \mathrm{d}^\times\mathrm{t}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&Z_{\tau,a,c}(f,\chi_0,s) + v_K \int_0^\infty f\,(c\,\tau(t))\,t^s\ \mathrm{d}^\times\mathrm{t}\\
&= \psi(-ac)\left(Z_{\tau,-c,a}(\hat{f},\chi_0^{-1},1-s) + v_K \int_0^\infty \hat{f}\,(a/\tau(t))\,t^{s-1}\ \mathrm{d}^\times\mathrm{t}\right).
\end{aligned}
$$

By definition, the first member is $\widetilde{Z}_{\tau,a,c}(f,\chi)$, and second member is $\widetilde{Z}_{\tau,-c,a}(\hat{f},\chi')$. This proves the functional equation (3.4) when χ is unramified.

(b) Assume χ is ramified. It follows from (3.19), (3.20), and the vanishing of both U and V that

$$
Z_{\tau,a,c}(f,\chi_0,s) = Z^+_{\tau,a,c}(f,\chi_0,s) + \psi(-ac)\ Z^+_{\tau,-c,a}(\hat{f},\chi_0^{-1},1-s). \tag{3.26}
$$

This expression of $Z_{\tau,a,c}(f,\chi_0,s)$ exhibits the symmetry

$$
\begin{gathered}
f \leftrightarrow \hat{f}\\
\chi = (\chi_0,s) \leftrightarrow \chi' = (\chi_0^{-1},1-s)\\
(a,c) \leftrightarrow (-c,a).
\end{gathered}
$$

More precisely, (3.26) together with (3.23) and the fact that $\hat{\hat{f}} = f^\tau$ yield

$$\begin{aligned} & Z_{\tau,-c,a}(\hat{f}, \chi_0^{-1}, 1-s) \\ & = Z^+_{\tau,-c,a}(\hat{f}, \chi_0^{-1}, 1-s) + \psi(ac)\ Z^+_{\tau,-a,-c}(f^\tau, \chi_0, s) \\ & = Z^+_{\tau,-c,a}(\hat{f}, \chi_0^{-1}, 1-s) + \psi(ac)\ Z^+_{\tau,a,c}(f, \chi_0, s). \end{aligned} \tag{3.27}$$

Therefore, by (3.26) and (3.27),

$$Z_{\tau,a,c}(f, \chi_0, s) = \psi(-ac)\ Z_{\tau,-c,a}(\hat{f}, \chi_0^{-1}, 1-s).$$

This proves the functional equation when χ is ramified. The proof of the functional equation of global zeta integrals is now completed.

□

3.8 Example: the case $K = \mathbb{Q}$

The purpose of this section is to show that global Lerch-Tate zeta integrals for $\mathbb{Q}$ can be specialized to give Lerch zeta function.

Observe that there is a unique norm section $\tau : \mathbb{R}_+ \to \mathbb{A}_\mathbb{Q}^\times$ because each norm section $\mathbb{R}_+ \to \mathbb{A}_\mathbb{Q}^\times$ has image in the archimedean component and $\mathbb{Q}$ has only one infinite place.

We choose adelic Haar measures as follows:

- the Haar measure $\mathrm{d_+x_p}$ on $\mathbb{Q}_p$ provides $\mathbb{Z}_p$ with unit volume;
- the Haar measure $\mathrm{d^\times x_p}$ on $\mathbb{Q}_p^\times$ provides $\mathbb{Z}_p^\times$ with unit volume;
- the measure $\mathrm{d_+x_\infty}$ on $\mathbb{R}$ is the Lebesgue measure;
- on $\mathbb{R}^\times$ take $\mathrm{d^\times x_\infty} = \frac{\mathrm{d_+x_\infty}}{|x_\infty|}$;
- for $\mathbb{A}_\mathbb{Q}$ set $\mathrm{d_+x} = \prod_\nu \mathrm{d_+x_\nu}$;
- for $\mathbb{A}_\mathbb{Q}^\times$ set $\mathrm{d^\times x} = \prod_\nu \mathrm{d^\times x_\nu}$. Since the inclusion of $\hat{\mathbb{Z}}^\times$ in $\mathbb{A}_\mathbb{Q}^\times$ induces an isomorphism $\hat{\mathbb{Z}}^\times \xrightarrow{\sim} C_\mathbb{Q}^1$, it follows that

$$v_\mathbb{Q} = \mathrm{d^\times x}\,(C_\mathbb{Q}^1) = 1.$$

We choose the additive character $\psi : \mathbb{A}_\mathbb{Q} \to \mathbb{C}^\times$ as follows:

- set $\psi_p : \mathbb{Q}_p \to \mathbb{C}^\times$ by $\psi_p(x_p) = \mathbf{e}\left(\{x_p\}_p\right)$;
- set $\psi_\infty : \mathbb{R} \to \mathbb{C}^\times$ by $\psi_\infty(x_\infty) = \mathbf{e}(-x_\infty)$;
- define $\psi : \mathbb{A}_\mathbb{Q} \to \mathbb{C}^\times$ by $\psi(x) = \prod_\nu \psi_\nu(x_\nu)$ for $x = \otimes_\nu x_\nu \in \mathbb{A}_\mathbb{Q}$.

We give the symmetrized Lerch zeta functions as examples of global Lerch-Tate zeta integrals:

Proposition 3.8.1. *Suppose that $a = \otimes_\nu a_\nu \in \mathbb{A}_\mathbb{Q}$ and $c = \otimes_\nu c_\nu \in \mathbb{A}_\mathbb{Q}$ satisfy*

(a) *$a_p = c_p = 0$ for every p;*

(b) *$a_\infty \in \mathbb{R} - \mathbb{Z}$ and $c_\infty \in \mathbb{R} - \mathbb{Z}$.*

One has:

(i) *if $f_0 = \otimes_p 1_{\mathbb{Z}_p} \otimes e^{-\pi x_\infty^2}$ and $\chi_s = |\cdot|^s$, then for* $\mathrm{Re}(s) > 1$

$$Z_{\tau,a,c}(f_0, \chi_s) = \frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{|m + c_\infty|^s}\, \mathbf{e}(-m a_\infty); \tag{3.28}$$

(ii) *if $f_1 = \otimes_{p<\infty} 1_{\mathbb{Z}_p} \otimes x_\infty e^{-\pi x_\infty^2}$ and $\chi_s = |\cdot|^s$, then for* $\mathrm{Re}(s) > 1$

$$Z_{\tau,a,c}(f_1, \chi_s) = \frac{1}{2}\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right) \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{(m + c_\infty)}{|m + c_\infty|^{s+1}}\, \mathbf{e}(-m a_\infty). \tag{3.29}$$

Remark 3.8.2. In Proposition 3.8.1, the hypothesis (b) encodes compatibility condition for the test functions f_0 and f_1. The values $c \in \mathbb{A}_K$ with $c_\infty \in \mathbb{Z} \setminus \{0\}$ are incompatible with f_0 and f_1; the values $a \in \mathbb{A}_K$ with $a_\infty \in \mathbb{Z} \setminus \{0\}$ are incompatible with $\hat{f}_0$ and $\hat{f}_1$. However, the values $c \in \mathbb{A}_K$ with $c_\infty = 0$ are compatible with f_0 and f_1; the values $a \in \mathbb{A}_K$ with $a_\infty = 0$ are compatible with $\hat{f}_0$ and $\hat{f}_1$.

Proof of Proposition 3.8.1. A fundamental domain for the action of $\mathbb{Q}^\times$ on $\mathbb{A}_\mathbb{Q}^1$ is given by $\{1\} \times \hat{\mathbb{Z}}^\times = \{1\} \times \prod_p \mathbb{Z}_p^\times \subset \mathbb{A}_\mathbb{Q}^\times$. If $f_0 = \otimes_p 1_{\mathbb{Z}_p} \otimes e^{-\pi x_\infty^2}$ and $\chi_s = |\cdot|^s$, then

$$\begin{aligned} Z_{\tau,a,c}(f_0, \chi_s) &= \int_{\mathbb{R}_+} \int_{\mathbb{A}_\mathbb{Q}^1} \psi(ay)\, f_0\left((y+c)\,\tau(t)\right)\, \chi_0(y)\, t^s \;\, \mathrm{d}^\times \mathrm{y}\; \mathrm{d}^\times \mathrm{t} \\ &= \int_{\mathbb{R}_+} \left\{ \sum_{\beta\in\mathbb{Q}^\times} \int_{\{1\}\times\hat{\mathbb{Z}}^\times} \psi(a\beta y)\, f_0\left((\beta y + c)\,\tau(t)\right)\, \chi_0(y)\;\, \mathrm{d}^\times \mathrm{y} \right\} t^s\; \mathrm{d}^\times \mathrm{t}. \end{aligned}$$

Since $f_0\left((\beta y+c)\,\tau(t)\right)=e^{-\pi(\beta+c_\infty)^2t^2}\prod_p 1_{\mathbb{Z}_p}(\beta y_p)$, it follows that $f_0\left((\beta y+c)\,\tau(t)\right)=0$ unless β is an integer. If β is an integer and $y\in\{1\}\times\hat{\mathbb{Z}}^\times$, then by assumptions

$$\psi(a\beta y)=\mathbf{e}(-\beta a_\infty)$$
$$f_0\left((\beta y+c)\,\tau(t)\right)=e^{-\pi(\beta+c_\infty)^2t^2}.$$

Therefore

$$\begin{aligned}
Z_{\tau,a,c}(f_0,\chi_s)&=\int_{\mathbb{R}_+}\sum_{m\in\mathbb{Z}\setminus\{0\}}\mathbf{e}(-ma_\infty)\,e^{-\pi(m+c_\infty)^2t^2}\,t^s\ \mathrm{d}^\times\mathrm{t}\\
&=\frac{1}{2}\sum_{m\in\mathbb{Z}\setminus\{0\}}\mathbf{e}(-ma_\infty)\int_0^\infty e^{-u}\left(\frac{u}{\pi\,|m+c_\infty|^2}\right)^{s/2}\mathrm{d}^\times\mathrm{u}\\
&=\frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{1}{|m+c_\infty|^s}\,\mathbf{e}(-ma_\infty).
\end{aligned}$$

This proves (i).

On the other hand, if $f_1=\otimes_p 1_{\mathbb{Z}_p}\otimes x_\infty\,e^{-\pi x_\infty^2}$ and $\chi_s=|\cdot|^s$, then

$$\begin{aligned}
Z_{\tau,a,c}(f_1,\chi_s)&=\int_{\mathbb{R}_+}\int_{\mathbb{A}^1_{\mathbb{Q}}}\psi(ay)\,f_1\left((y+c)\,\tau(t)\right)\,\chi_0(y)\,t^s\ \ \mathrm{d}^\times\mathrm{y}\ \mathrm{d}^\times\mathrm{t}\\
&=\int_{\mathbb{R}_+}\left\{\sum_{\beta\in\mathbb{Q}^\times}\int_{\{1\}\times\hat{\mathbb{Z}}^\times}\psi(a\beta y)\,f_1\left((\beta y+c)\,\tau(t)\right)\,\chi_0(y)\ \ \mathrm{d}^\times\mathrm{y}\right\}t^s\ \mathrm{d}^\times\mathrm{t}.
\end{aligned}$$

Since $f_1\left((\beta y+c)\,\tau(t)\right)=(\beta+c_\infty)\,t\,e^{-\pi(\beta+c_\infty)^2t^2}\prod_p 1_{\mathbb{Z}_p}(\beta y_p)$, it follows that

$$f_1\left((\beta y+c)\,\tau(t)\right)=0$$

unless β is an integer. If β is an integer and $y\in\{1\}\times\hat{\mathbb{Z}}^\times$, then by assumptions

$$\psi(a\beta y)=\mathbf{e}(-\beta a_\infty)$$
$$f_1\left((\beta y+c)\,\tau(t)\right)=(\beta+c_\infty)\,t\,e^{-\pi(\beta+c_\infty)^2t^2}.$$

Therefore

$$\begin{aligned}
Z_{\tau,a,c}(f_1,\chi_s) &= \int_{\mathbb{R}_+} \sum_{m\in\mathbb{Z}\setminus\{0\}} \mathbf{e}(-ma_\infty)\,(m+c_\infty)\,e^{-\pi(m+c_\infty)^2t^2}\,t^{s+1}\,\mathrm{d}^\times\mathrm{t} \\
&= \frac{1}{2}\sum_{m\in\mathbb{Z}\setminus\{0\}} \mathbf{e}(-ma_\infty)\,(m+c_\infty)\int_0^\infty e^{-u}\left(\frac{u}{\pi\,|m+c_\infty|^2}\right)^{\frac{s+1}{2}}\mathrm{d}^\times\mathrm{u} \\
&= \frac{1}{2}\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{m+c_\infty}{|m+c_\infty|^{s+1}}\,\mathbf{e}(-ma_\infty).
\end{aligned}$$

This proves (ii). □

Remark 3.8.3. In the notation of Section 1.2, we have

$$\begin{aligned}
Z_{\tau,a,c}(f_0,\chi_s) &= \frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\left(L^+(s,-a_\infty,c_\infty)-|c_\infty|^{-s}\right) \\
Z_{\tau,a,c}(f_1,\chi_s) &= \frac{1}{2}\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\left(L^-(s,-a_\infty,c_\infty)-\mathrm{sgn}(c_\infty)\,|c_\infty|^{-s}\right).
\end{aligned}$$

The functional equations of global zeta integrals translate to the functional equations of the symmetrized Lerch zeta functions. In fact, on setting $u = c_\infty^2 t^2$,

$$\begin{aligned}
\int_0^\infty f_0(c\,\tau(t))\,\chi_s(\tau(t))\,\mathrm{d}^\times\mathrm{t} &= \int_0^\infty e^{-\pi c_\infty^2 t^2}\,t^s\,\mathrm{d}^\times\mathrm{t} \\
&= \int_0^\infty e^{-\pi u}\,u^{\frac{s}{2}}\,|c_\infty|^{-s}\,\mathrm{d}^\times\mathrm{t} \\
&= \frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\,|c_\infty|^{-s}.
\end{aligned}$$

Similarly, on setting $u = c_\infty^2 t^2$,

$$\int_0^\infty f_1(c\,\tau(t))\,\chi_s(\tau(t))\,\mathrm{d}^\times\mathrm{t} = \frac{1}{2}\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\,\frac{c_\infty}{|c_\infty|^s}.$$

Since $v_\mathbb{Q} = 1$ and $\chi_s = |\cdot|^s$ is unramified, it follows that

$$\begin{aligned}
\tilde{Z}_{\tau,a,c}(f_0,\chi_s) &= Z_{\tau,a,c}(f_0,\chi_s) + v_\mathbb{Q}\int_0^\infty f_0(c\,\tau(t))\,\chi_s(\tau(t))\,\mathrm{d}^\times\mathrm{t} \\
&= \frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\,L^+(s,-a_\infty,c_\infty)
\end{aligned}$$

and that

$$\begin{aligned}\tilde{Z}_{\tau,a,c}(f_1, \chi_s) &= Z_{\tau,a,c}(f_1, \chi_s) + v_{\mathbb{Q}} \int_0^\infty f_1(c\,\tau(t))\, \chi_s(\tau(t))\, \mathrm{d}^\times \mathrm{t} \\ &= \frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\, L^-(s, -a_\infty, c_\infty).\end{aligned}$$

Theorems 3.5.2 and 3.5.3 apply to give the analytic continuation and functional equation for $L^+(s, -a_\infty, c_\infty)$ and $L^-(s, -a_\infty, c_\infty)$.

Chapter 4

Global zeta integrals over function fields

4.1 Overview of the chapter

The current chapter aims to provide a function field analogue of the Lerch zeta function. This analogue is produced in Sections 4.2 and 4.5.

- Section 4.2 applies the ideas of Chapter 3 to construct global Lerch-Tate zeta integrals over a function field of a curve over finite field.
- The analytic properties of global function field Lerch-Tate zeta integrals are formulated in Section 4.5 as Theorem 4.5.1, Theorem 4.5.2 and Theorem 4.5.3.
- Section 4.6 is devoted to the proofs of Theorems 4.5.1, 4.5.2 and 4.5.3.

4.2 Notations and basic definitions

Fix a smooth projective geometrically connected curve C over a finite field $k = \mathbb{F}_q$. Let $K = k(C)$ be the function field of C and $|C|$ the set of closed points of C. For each $v \in |C|$, let $\mathcal{O}_v$ denote the completed local ring at ν with fraction field K_v and residue field k_v of cardinality $q_v = q^{d_v}$. Let $\mathrm{Jac}\,(C)$ be the Jacobian variety of C, defined over k and parametrizing line bundles of degree 0.

Write $\mathbb{A}_K$ for the ring of adeles and $\mathbb{A}_K^\times$ for the ring of ideles. Write $C_K = \mathbb{A}_K^\times/K^\times$ for the idele class group and $C_K^1 = \mathbb{A}_K^1/K^\times$ for the norm one idele class group. Let $|\cdot| : \mathbb{A}_K^\times \to \mathbb{R}_+$ denote the standard norm map of $\mathbb{A}_K^\times$. Then $|\cdot|$ is trivial on $K^\times$ and induces a continuous homomorphism, also denoted by $|\cdot|$, from C_K to $\mathbb{R}_+$. One knows that the image of $|\cdot|$ is $\mathrm{N} = q^{\mathbb{Z}}$ (see [40, Chapter VII]).

The space X_K of all quasi-characters of C_K has a natural structure of a complex variety. Furthermore, considered as a complex analytic variety, X_K is a disjoint union of connected components, each of which can be non-canonically identified with

$\mathbb{C} \,/\, \frac{2\pi i}{\log q}\,\mathbb{Z}$. All quasi-characters of C_K which have the same restriction to C_K^1 belong to the same connected component. We set some notations to specify the connected components of X_K more conveniently.

Definition 4.2.1. If χ_0 is a character of C_K^1, set

$$X_{K,\chi_0} = \{\chi \in X_K : \chi|_{C_K^1} = \chi_0\}. \tag{4.1}$$

In particular, if $\chi_{0,\mathrm{nr}}$ is the trivial character on C_K^1, then $X_{K,\chi_{0,\mathrm{nr}}}$ is the neutral component consisting of those quasi-characters of C_K which have trivial restriction to C_K^1; we write $X_{K,\mathrm{nr}}$ for this neutral component.

Remark 4.2.2. There are non-canonical isomorphisms from $\mathbb{C} \,/\, \frac{2\pi i}{\log q}\,\mathbb{Z}$ onto X_{K,χ_0}. If $\tilde{\chi_0} : C_K \to \mathbb{C}^\times$ is an extension of χ_0, then the map which sends a complex number s to the quasi-character $\chi(x) = \tilde{\chi_0}(x)\,|x|^s$ induces an isomorphism from $\mathbb{C} \,/\, \frac{2\pi i}{\log q}\,\mathbb{Z}$ onto X_{K,χ_0}.

The standard notions of adelic Fourier transform and of self-dual measure for function field are the same as that for number field.

Definition 4.2.3. Suppose that $\psi : \mathbb{A}_K \to \mathbb{C}^\times$ is a nontrivial character and that $\mathrm{d}_+\mathrm{x}$ is a Haar measure on $\mathbb{A}_K$. If f is a Schwartz-Bruhat function on $\mathbb{A}_K$, define the Fourier transform of f with respect to ψ and $\mathrm{d}_+\mathrm{x}$ to be the function

$$\hat{f}(y) = \int_{\mathbb{A}_K} f(x)\,\psi(xy)\;\mathrm{d}_+\mathrm{x} \qquad (y \in \mathbb{A}_K). \tag{4.2}$$

The Fourier transform $\hat{f}$ is also a Schwartz-Bruhat function on $\mathbb{A}_K$.

Definition 4.2.4. For every nontrivial character ψ of $\mathbb{A}_K$, there exists a unique Haar measure $\mathrm{d}_+\mathrm{x}$ on $\mathbb{A}_K$ such that, for all Schwartz-Bruhat functions f on $\mathbb{A}_K$, one has $\hat{\hat{f}}(x) = f(-x)$. This Haar measure $\mathrm{d}_+\mathrm{x}$ is called the *self-dual measure with respect to* ψ.

4.3 Global norm sections

Let $|\cdot| : \mathbb{A}_K^\times \to \mathrm{N} = q^{\mathbb{Z}} \subset \mathbb{R}_+$ denote the standard norm map of $\mathbb{A}_K^\times$. Then $|\cdot|$ is trivial on $K^\times$ and induces a continuous homomorphism, also denoted by $|\cdot|$, from C_K to N.

We proceed to give preliminary notions of norm section and global Lerch tuple. They are main ingredients in the definition of global zeta integrals over a function field.

Definition 4.3.1. A group homomorphism $\tau : \mathrm{N} \to \mathbb{A}_K^\times$ is called a *norm section* if $|\cdot| \circ \tau = \mathrm{id}_{\mathrm{N}}$.

Remark 4.3.2. Since $\mathrm{N} = q^{\mathbb{Z}}$ is cyclic, every norm section τ gives rise to a divisor of degree one on the curve C, namely the divisor of the idele $\tau(q^{-1}) \in \mathbb{A}_K^\times$.

Definition 4.3.3. A norm section $\tau : \mathrm{N} \to \mathbb{A}_K^\times$ is said to be *positive* if the support of the divisor of the idele $\tau(q^{-1})$ consists of a singular point. Equivalently, τ is positive if there is an $\mathbb{F}_q$-rational point, say $\infty \in |C|$, such that $\tau(q^{-1}) = \otimes_{P \in |C|} x_P$ with $x_P \in \mathcal{O}_P^\times$ for every $P \in |C| \setminus \{\infty\}$.

Remark 4.3.4. There exists a positive norm section $\tau : \mathrm{N} \to \mathbb{A}_K^\times$ if and only if C has an $\mathbb{F}_q$-rational point.

Lemma 4.3.5. *A choice of a positive norm section then corresponds to a choice of an $\mathbb{F}_q$-rational point, say ∞, a choice of a uniformizing parameter at that point, say $\pi_\infty \in K_\infty$, and a choice of unit $u_P \in \mathcal{O}_P^\times$ for every $P \in |C| \setminus \{\infty\}$. We say that τ is supported at ∞ and is given by $\tau(q^{-1}) = \otimes_{P \in |C| \setminus \{\infty\}} x_P \otimes \pi_\infty$.*

Proof. By definition $\tau(q^{-1}) = \otimes_{P \in |C|} x_P$ has $x_P = u_P \in \mathcal{O}_P^\times$ for every $P \in |C| \setminus \{\infty\}$. For τ to be a norm section, $x_\infty \in K_\infty$ must be a uniformizer. The lemma is shown. $\square$

4.4 Global Lerch-Tate zeta integrals over a function field

Definition 4.4.1. A *global Lerch tuple* is a triple (τ, a, c) consisting of a positive norm section τ and a pair of adeles $(a, c) \in \mathbb{A}_K^2$.

Definition 4.4.2. *(Global Lerch-Tate zeta integral)* Suppose that (τ, a, c) is a global Lerch tuple. If f is a Schwartz-Bruhat function on $\mathbb{A}_K$ and χ is a quasi-character of C_K, define the global Lerch-Tate zeta integral

$$Z_{\tau,a,c}(f, \chi) = \int_{\mathbb{A}_K^\times} \psi\left(\frac{ax}{\tau(|x|)}\right) f\left(x + c\,\tau(|x|)\right) \chi(x) \, \mathrm{d}^\times \mathrm{x}. \tag{4.3}$$

Remark 4.4.3. One gains a better understanding of the global zeta integral $Z_{\tau,a,c}(f, \chi)$ with the following two observations. First, a norm section τ induces a splitting $\mathbb{A}_K^\times = \mathbb{A}_K^1 \times q^{\mathbb{Z}}$. Secondly, every quasi-character of $\mathbb{A}_K^\times$ is determined by its restrictions to $\mathbb{A}_K^1$ and its value on $\tau(q^{-1})$.

Suppose that a norm section τ is given. Then every $x \in \mathbb{A}_K^\times$ can be written uniquely as $x = t^m y$ where $t = \tau(q^{-1})$, $m \in \mathbb{Z}$ and $y \in \mathbb{A}_K^1$. If χ is a quasi-character of

C_K, write χ_0 for its restriction to C_K^1. By Definition 4.2.1, χ belongs to the connected component X_{K,χ_0} of X_K. There is an isomorphism $\mathbb{C} \;/\; \frac{2\pi i}{\log q}\,\mathbb{Z} \xrightarrow{\sim} X_{K,\chi_0}$ which sends a complex number s modulo $\frac{2\pi i}{\log q}\,\mathbb{Z}$ to the quasi-character $\chi(x) = \chi_0\left(\frac{x}{\tau(|x|)}\right)|x|^s = \chi_0(y)\,q^{-ms}$. Via this isomorphism, the restriction of $Z_{\tau,a,c}(f,\cdot)$ to X_{K,χ_0} becomes a function of s; write $Z_{\tau,a,c}(f,\chi_0,s)$ for this function. By (4.3) we have

$$Z_{\tau,a,c}(f,\chi_0,s) = \sum_{m=-\infty}^{\infty} \int_{\mathbb{A}_K^1} \psi(ay)\, f\left((y+c)\,t^m\right)\, \chi_0(y)\, q^{-ms}\ \mathrm{d}^\times \mathrm{y}.$$

The following definition is a technical condition that we need in order to show the analytic properties of global zeta integrals. It will be used to exclude, for each Schwartz-Bruhat function f on $\mathbb{A}_K$, a singular set of (a,c) in $\mathbb{A}_K^2$ to ensure that the global zeta integrals are absolutely convergent and satisfy functional equation.

Definition 4.4.4. (i) Suppose that $\alpha = \otimes_{P\in|C|}\alpha_P \in \mathbb{A}_K$. The *singular degree* of α_P is the biggest integer n_P such that $\mathcal{O}_P \subset \mathfrak{m}_P^{n_P}$ and that $\alpha_P \in \mathfrak{m}_P^{n_P}$. The *singular degree* of α is the sum $\sum_{P\in|C|} n_P$ where each n_P is the singular degree of α_P.

(ii) Suppose that $f = \otimes_{P\in|C|} f_P$ is a factorizable Schwartz-Bruhat function on $\mathbb{A}_K$. The *singular degre* of f_P is the biggest integer n_P such that $\mathcal{O}_P \subset \mathfrak{m}_P^{n_P}$ and that $\mathrm{Supp}(f_P) \subset \mathfrak{m}_P^{n_P}$. The *singular degree* of f is the sum $\sum_{P\in|C|} n_P$ where each n_P is the singular degree of f_P.

(iii) Suppose that f is a Schwartz-Bruhat function on $\mathbb{A}_K$. Suppose further that $f = \sum_{j=1}^r f_j$ where each f_j is a factorizable Schwartz-Bruhat function on $\mathbb{A}_K$. The *singular degree of f with respect to the decomposition* $f = \sum_{j=1}^r f_j$ is the sum $\sum_{j=1}^r n_j$ where each n_j is the singular degree of f_j. The *singular degree* of f is the biggest singular degree of f with respect to all the decompositions $f = \sum_{j=1}^r f_j$ into factorizable functions.

(iv) Suppose that $\alpha \in \mathbb{A}_K$ and that f is a Schwartz-Bruhat function on $\mathbb{A}_K$. Let n_α (resp. n_f) be the singular degree of α (resp. f); put $n_{\alpha,f} = \max(n_\alpha, n_f)$. Suppose further that C has an $\mathbb{F}_q$-rational point ∞ and let $\mathrm{val}_\infty : K_\infty \to \mathbb{Z}\cup\{\infty\}$ be the normalized valuation which takes value one on any uniformizing paramter. We say α and f are *compatible at the point* ∞ if $\mathrm{val}_\infty(c_\infty) > |n_{\alpha,f}|$.

Note that all of the singular degrees in Definition 4.4.4 are non-positive integers.

4.5 The Main Global Theorem over a function field

In this section, we formulate the analytic properties of the global Lerch-Tate zeta integrals over a function field as 3 theorems: Theorem 4.5.1, Theorem 4.5.2 and Theorem 4.5.3. Their combination gives Theorem 1.5.4 which we call the Main Global Theorem over a function field.

Let C be a smooth geometrically connected projective curve over a finite field $k = \mathbb{F}_q$; suppose that C has an $\mathbb{F}_q$-rational point ∞. Let $K = K(C)$ be the function field of C. Take for $\mathbb{A}_K$ the self-dual measure $\mathrm{d}_+\mathrm{x}$ with respect to ψ and take for $\mathbb{A}_K^\times$ any Haar measure $\mathrm{d}^\times \mathrm{x}$. Put $v_K = \mathrm{d}^\times \dot{\mathrm{x}}\,(C_K^1)$ where $\mathrm{d}^\times \dot{\mathrm{x}}$ is induced by $\mathrm{d}^\times \mathrm{x}$.

For a Schwartz-Bruhat function f on $\mathbb{A}_K$, let $\hat{f}$ denote the Fourier transform of f with respect to ψ and $\mathrm{d}_+\mathrm{x}$. For a quasi-character χ on C_K, let χ' denote the quasi-character given by $\chi\chi' = |\cdot|$.

Theorem 4.5.1. (Absolute convergence) *Let (τ, a, c) be a global Lerch tuple and ψ be an additive character of $\mathbb{A}_K$ with trivial restriction to K. Let f be a Schwartz-Bruhat function on $\mathbb{A}_K$ and $\hat{f}$ be the Fourier transform of f with respect to ψ and* $\mathrm{d}_+\mathrm{x}$. *Suppose that τ is supported at the given $\mathbb{F}_q$-rational point ∞ of C. Suppose further that c is compatible with f at ∞. Then on each connected component of X_K, the zeta integral $Z_{\tau,a,c}(f, \chi)$ converges absolutely for* $\mathrm{Re}(\chi) > 1$.

Note that the absolute convergence theorem imposes a compatibility condition on f and the variable $c \in \mathbb{A}_K$.

Theorem 4.5.2. (Analytic continuation) *Let (τ, a, c) be a global Lerch tuple and ψ be an additive character of $\mathbb{A}_K$ with trivial restriction to K. Let f be a Schwartz-Bruhat function on $\mathbb{A}_K$ and $\hat{f}$ be the Fourier transform of f with respect to ψ and* $\mathrm{d}_+\mathrm{x}$. *Suppose that τ is supported at the given $\mathbb{F}_q$-rational point ∞ of C. Suppose further that c is compatible with f at ∞ and that a is compatible with $\hat{f}$ at ∞.*

(i) *The zeta integral $Z_{\tau,a,c}(f, \chi)$ extends to a meromorphic function on X_K.*

(ii) *The meromorphic function $Z_{\tau,a,c}(f, \chi)$ is holomorphic outside of the unramified component. On the unramified component, $Z_{\tau,a,c}(f, \chi)$ has at most simple poles at $\chi = |\cdot|$ and $\chi = \chi_{\mathrm{triv}}$. Observe that each of the two sums*

$$\sum_{n>0} \hat{f}\,\left(a\,\tau(q^n)\right) \chi'(\tau(q^n)) \quad \textit{and} \quad \sum_{n<0} f\left(c\,\tau(q^n)\right)\, \chi(\tau(q^n))$$

converges for $\mathrm{Re}(\chi) > 0$ *and has a meromorphic continuation to* $\chi \in X_K$*. Then the function*

$$Z_{\tau,a,c}(f,\chi) - \psi(-ac)\, v_K \sum_{n>0} \hat{f}\ (a\,\tau(q^n))\,\chi'(\tau(q^n)) + v_K \sum_{n<0} f\,(c\,\tau(q^n))\ \chi(\tau(q^n))$$

extends to a holomorphic function on the unramified component of X_K*.*

Note that the analytic continuation theorem imposes the compatibility condition on $a \in \mathbb{A}_K$ with $\hat{f}$ and the the compatibility condition on $c \in \mathbb{A}_K$ with f.

Theorem 4.5.3. (Functional equation) *Let* (τ, a, c) *be a global Lerch tuple and* ψ *be an additive character of* $\mathbb{A}_K$ *with trivial restriction to* K*. Let* f *be a Schwartz-Bruhat function on* $\mathbb{A}_K$ *and* $\hat{f}$ *be the Fourier transform of* f *with respect to* ψ *and* $\mathrm{d}_+\mathrm{x}$*. Suppose that* τ *is supported at the given* $\mathbb{F}_q$*-rational point* ∞ *of* C*. Suppose further that* c *is compatible with* f *at* ∞ *and that* a *is compatible with* $\hat{f}$ *at* ∞*.*

(i) *The integral*

$$S_1(f,\chi) = \sum_{m=-\infty}^{-1} f\,(c\,\tau(q^m))\ \chi(\tau(q^m))$$

converges absolutely for $\mathrm{Re}(\chi) > 0$ *and extends to a meromorphic function of* $\chi \in X_K$*; we denote this meromorphic function also by* $S_1(f,\chi)$*. The integral*

$$S_2(f,\chi) = \sum_{m=0}^{\infty} f\,(c\,\tau(q^m))\ \chi(\tau(q^m))$$

converges absolutely for $\mathrm{Re}(\chi) < 0$ *and extends to a meromorphic function of* $\chi \in X_K$*; we denote this meromorphic function also by* $S_2(f,\chi)$*.*

(ii) *Set*

$$\widetilde{Z}_{\tau,a,c}(f,\chi) = \begin{cases} Z_{\tau,a,c}(f,\chi) + v_K\,(S_1(f,\chi) + S_2(f,\chi)) & \text{if } \chi \text{ is unramified} \\ Z_{\tau,a,c}(f,\chi) & \text{if } \chi \text{ is ramified.} \end{cases}$$

One has

$$\widetilde{Z}_{\tau,a,c}(f,\chi) = \psi(-ac)\ \widetilde{Z}_{\tau,-c,a}(\hat{f},\chi').$$

4.6 Analytic properties of global zeta integrals

In this section we prove the analytic properties the global zeta integral $Z_{\tau,a,c}(f,\chi)$. We briefly recall some notations. Let (τ, a, c) be a global Lerch tuple and f be a

Schwartz-Bruhat function on $\mathbb{A}_K$. The positive norm section τ is given by

$$\tau(q^{-1}) = \otimes_{P\in|C|\setminus\{\infty\}} u_P \otimes \pi_\infty$$

where $\pi_\infty \in K_\infty$ is a uniformizing parameter at the given $\mathbb{F}_q$-rational point $\infty \in |C|$ and $u_P \in \mathcal{O}_P^\times$ for every $P \in |C| \setminus \{\infty\}$. Note also that τ induces a splitting $\mathbb{A}_K^\times = \mathbb{A}_K^1 \times q^{\mathbb{Z}}$. If χ is a quasi-character of C_K with restriction χ_0 to C_K^1, then χ belongs to the connected component X_{K,χ_0} of X_K. There is an isomorphism from $\mathbb{C} \,/\, \frac{2\pi i}{\log q}\mathbb{Z} \xrightarrow{\sim} X_{K,\chi_0}$ which sends a complex number s modulo $\frac{2\pi i}{\log q}\mathbb{Z}$ to the quasi-character $\chi(x) = \chi_0\left(\frac{x}{\tau(|x|)}\right)|x|^s$. Via this isomorphism, we view the restriction of $Z_{\tau,a,c}(f,\cdot)$ to X_{K,χ_0} as the function

$$s \mapsto Z_{\tau,a,c}(f,\chi_0,s) = \sum_{m=-\infty}^{\infty} \int_{\mathbb{A}_K^1} \psi(ay)\, f\left((y+c)\,\tau(q^{-m})\right)\, \chi_0(y)\, q^{-ms}\ \mathrm{d}^\times \mathrm{y}. \qquad (4.4)$$

Let N_0 be a sufficiently large integer so that $\mathrm{Supp}(f_\nu) \subset \mathfrak{p}_\nu^{-N_0}\mathcal{O}_\nu$ and that $c_\nu \in \mathfrak{p}_\nu^{-N_0}\mathcal{O}_\nu$ for every place ν.

The following preparatory lemma is an analogue of Lemma 3.6.1.

Lemma 4.6.1. *Suppose that $f = \otimes_{P\in|C|} \otimes f_P$ is a Schwartz-Bruhat function on $\mathbb{A}_K$ and that $c \in \mathbb{A}_K$. Suppose further that $D > 0$ is an arbitrary positive real. Then there exists a finite set S of places of K which includes all places in D and which satisfies three properties:*

(a) *for every $\nu \notin S$, one has $f_\nu = 1_{\mathcal{O}_\nu}$;*

(b) *for every $\nu \notin S$, one has $c_\nu \in \mathcal{O}_\nu$;*

(c) *if $y \in \mathbb{A}_K^1$ with $y_\nu + c_\nu \in \mathrm{Supp}(f_\nu)$ for every ν and $y_{\nu'} \in \mathfrak{p}_{\nu'}\mathcal{O}_{\nu'}$ for some $\nu' \notin S$, then*

$$|y_\infty| > D.$$

Proof. Since f is a Schwartz-Bruhat function and $c \in \mathbb{A}_K$, there is a finite set S of places which includes all places in D and satisfies (a) and (b). Then the set S can be enlarged, if necessary, to another finite set S' so that, if $\nu' \notin S'$, then $\mathrm{N}\nu' > D\ \prod_{\nu\in S-\{\infty\}}(\mathrm{N}\nu)^{N_0}$. This is possible because there are only finitely many points on the curve C with bounded degrees. It is clear that the set S' necessarily satisfies (a) and (b).

Suppose now that $y \in \mathbb{A}_K^1$ with $y_\nu + c_\nu \in \mathrm{Supp}(f_\nu)$ for every $\nu \notin D$ and $y_{\nu'} \in \mathfrak{p}_{\nu'}\mathcal{O}_{\nu'}$ for some $\nu' \notin S'$. Hence $|y_{\nu'}| \leq \frac{1}{\mathrm{N}\nu'}$. By definition N_0 is a sufficiently large integer so that $\operatorname{Supp}(f_\nu) \subset \mathfrak{p}_\nu^{-N_0}\mathcal{O}_\nu$ and that $c_\nu \in \mathfrak{p}_\nu^{-N_0}\mathcal{O}_\nu$ for every ν. In particular, at every place $\nu \in S$, $y_\nu \in \mathfrak{p}_\nu^{-N_0}\mathcal{O}_\nu$ and hence $|y_\nu| \le (\mathrm{N}\nu)^{N_0}$; whereas at every place $\nu \notin S$, $y_\nu \in \mathcal{O}_\nu$ and hence $|y_\nu| \le 1$. It follows that

$$\begin{aligned} |y_\infty| &= \frac{1}{\prod_{\nu\neq\infty}|y_\nu|} \ge \frac{1}{|y_{\nu'}|\ \prod_{\nu\in S-\{\infty\}}|y_\nu|} \ge \frac{\mathrm{N}\nu'}{\prod_{\nu\in S-\{\infty\}}|y_\nu|} \\ &\ge \frac{\mathrm{N}\nu'}{\prod_{\nu\in S-\{\infty\}}(\mathrm{N}\nu)^{N_0}} > D. \end{aligned}$$

Therefore the set S' satisfies the property (c), besides the properties (a) and (b) that S' inherits from S. The existence of a finite set of places of K with the three properties is thus verified. $\square$

We are in a position to prove the absolute convergence of $Z_{\tau,a,c}(f,\chi)$.

Proof of Theorem 4.5.1. Let $D > 0$ be a constant which depends on $c \in \mathbb{A}_K$ and which is to be chosen. Let S be a finite set of closed points on C which contains the $\mathbb{F}_q$-rational point ∞ and which satisfies the three properties in Lemma 4.6.1. In (4.4), we decompose the integral over $\mathbb{A}_K^1$ into the integrals over the two subsets

$$\begin{aligned} H_1 &= \{y \in \mathbb{A}_K^1 : y_{P'} \in \mathfrak{m}_{P'} \text{ for some } P' \notin S\} \\ H_2 &= \{y \in \mathbb{A}_K^1 : y_P \in \mathcal{O}_P^\times \text{ for all } P \notin S\}. \end{aligned}$$

By definition, H_1 is a closed subset of $\mathbb{A}_K^1$ and H_2 is an open subset of $\mathbb{A}_K^1$.

Let us set some notations for the proof. Write

$$S = \{\infty\} \cup \{P_j : 1 \le j \le r\}.$$

Let d_j denote the degree of the point P_j, so $q_j = q^{d_j}$ is the size of the residue field $\kappa_{P_j} = \mathcal{O}_{P_j}/\mathfrak{m}_{P_j}$. If $y = \otimes_{P\in|C|}y_P \in \mathbb{A}_K^\times$, write $y_{\mathrm{fin}} = \otimes_{P\in|C|\setminus\{\infty\}}y_P$. Put $|y|_{P_j} = |y_{P_j}|_{P_j}$. For a point $P \in |C|$ and a positive real A, put

$$K^\times_{P,|\cdot|=A} = \{y \in K_P^\times : |y| = A\}.$$

Similarly, the sets $K^\times_{P,|\cdot|>A}$ and $K^\times_{P,|\cdot|\le A}$ have their obvious meanings. Recall also that $\mathrm{N} = q^{\mathbb{Z}} \subset \mathbb{R}_+$ is the image of the norm map $|\cdot| : \mathbb{A}_K^\times \to \mathbb{R}_+$ and that $\tau : \mathrm{N} \to \mathbb{A}_K^\times$ is given by

$$\tau(q^{-1}) = \otimes_{P\in|C|\setminus\{\infty\}}u_P \otimes \pi_\infty \in \mathbb{A}_K^\times$$

with $u_P \in \mathcal{O}_P^\times$ for every $P \in |C| \setminus \{\infty\}$.

Consider the integral over H_2, namely

$$\sum_{m=-\infty}^{\infty} \int_{H_2} \psi(ay)\, f\left((y+c)\,\tau(q^{-m})\right)\; \chi_0(y)\, q^{-ms}\;\; \mathrm{d}^{\times}\mathrm{y}. \tag{4.5}$$

Our goal is to show that this integral converges absolutely and uniformly on compacta for $\sigma > 0$. We do this by carrying out a series of reductions. Note first that ψ and χ_0 are unitary characters and that we may assume without loss of generality that each component of f takes value in $\mathbb{R}_{\geq 0}$. The absolute convergence of (4.5) is reduced to that of the integral

$$\sum_{m=-\infty}^{\infty} \int_{H_2} f\left((y+c)\,\pi_{\infty}^{m}\right)\, q^{-m\sigma}\;\; \mathrm{d}^{\times}\mathrm{y} \tag{4.6}$$

for $\sigma > 0$.

By the properties (a) and (b) of S given by Lemma 4.6.1, $f_P = 1_{\mathcal{O}_P}$ and $c_P \in \mathcal{O}_P$ for every $P \notin S$. Recall that N_0 is chosen as a sufficiently large integer so that $\mathrm{Supp}(f_P) \subset \mathfrak{m}_P^{-N_0}$ and that $c_P \in \mathfrak{m}_P^{-N_0}$ for every P. It follows that

$$\begin{aligned}
&\sum_{m=-\infty}^{\infty} \int_{H_2} f\left((y+c)\,\tau(q^{-m})\right)\, q^{-m\sigma}\;\; \mathrm{d}^{\times}\mathrm{y} \\
&\ll \sum_{m=-\infty}^{\infty} \Big\{ \int_{(\prod_{Q\in S} K_Q^{\times})\cap \mathbb{A}_K^1} f_{\infty}\left((y_{\infty}+c_{\infty})\,\pi_{\infty}^{m}\right) \prod_{P\in S} f_P((y_P+c_P)u_P^m) \otimes_{\mathrm{Q}\in\mathrm{S}} \mathrm{d}^{\times}\mathrm{y}_{\mathrm{Q}} \Big\} q^{-m\sigma} \\
&\ll \sum_{m=-\infty}^{\infty} \left\{ \int_{\left(\prod_{P\in S\setminus\{\infty\}} \left(K_P^{\times}\cap \mathfrak{m}_P^{-N_0}\right)\times K_{\infty}^{\times}\right)\cap \mathbb{A}_K^1} f_{\infty}\left((y_{\infty}+c_{\infty})\,\pi_{\infty}^{m}\right) \;\; \otimes_{\mathrm{Q}\in\mathrm{S}} \mathrm{d}^{\times}\mathrm{y}_{\mathrm{Q}} \right\} q^{-m\sigma} \\
&\ll \sum_{m=-\infty}^{\infty} \left\{ \sum_{\substack{(n_1,\ldots,n_r)\in\mathbb{Z}^r \\ n_j \geq -N_0}} \int_{K^{\times}_{\infty,|\cdot|=\prod_{j=1}^{r} q_j^{n_j}}} f_{\infty}\left((y_{\infty}+c_{\infty})\,\pi_{\infty}^{m}\right) \mathrm{d}^{\times}\mathrm{y}_{\infty} \right\} q^{-m\sigma}.
\end{aligned}$$

The uniform boundedness of (4.6) now reduces to the uniform boundedness of

$$\sum_{\substack{(n_1,\ldots,n_r)\in\mathbb{Z}^r \\ n_j \geq -N_0}} \sum_{m=-\infty}^{\infty} q^{-m\sigma} \int_{K^{\times}_{\infty,|\cdot|=\prod_{j=1}^{r} q_j^{n_j}}} f_{\infty}\left((y_{\infty}+c_{\infty})\,\pi_{\infty}^{m}\right) \mathrm{d}^{\times}\mathrm{y}_{\infty} \qquad (\delta < \sigma < A). \tag{4.7}$$

To simplify notation, assume that $r = 1$; the proof of convergence of (4.7) for general r is the same. When $r = 1$, we have $S = \{\infty, P_1\}$ and the integral (4.7) equals

$$\sum_{n\geq -N_0}\sum_{m=-\infty}^{\infty} q^{-m\sigma}\int_{K^\times_{\infty,|\cdot|=q_1^n}} f_\infty\left((y_\infty+c_\infty)\,\pi_\infty^m\right)\,\mathrm{d}^\times \mathrm{y}_\infty \qquad (\delta<\sigma<A). \tag{4.8}$$

By the compatibility assumption of c and f, which is prescribed in Definition 4.4.4 (iv), we have

$$|c_\infty| < q_1^n \quad \text{for every} \quad n\geq -N_0.$$

Since $|y_\infty| = q_1^n$ in (4.8), it follows that

$$|y_\infty + c_\infty| = |y_\infty|. \tag{4.9}$$

Since f_∞ is a Schwartz-Bruhat function on K_∞, it is apparent that (4.8) converges absolutely for $\sigma > 0$. This proves the absolute convergence of (4.5) for $\sigma > 0$.

Now consider the integral over H_1, namely

$$\sum_{m=-\infty}^{\infty}\int_{H_1}\psi(ay)\,f\left((y+c)\,\tau(q^{-m})\right)\,\chi_0(y)\,q^{-ms}\,\mathrm{d}^\times \mathrm{y}. \tag{4.10}$$

Our goal is to show that this integral converges absolutely for $\sigma > 1$. We now set the constant $D > 0$ at the beginning of the proof to be

$$D = \max\left(|c_\infty|, 1\right).$$

By definition, if $y \in H_1$ then $y_{P'} \in \mathfrak{m}'_P$ for some $P' \notin S$. For every $P \notin S$, the properties (a) and (b) of S in Lemma 4.6.1 imply that $f_P = 1_{\mathcal{O}_P}$ and that $c_P \in \mathcal{O}_P$. If, furthermore, $y_P + c_P \in \mathrm{Supp}(f_P)$, then y verifies the condition (c) of Lemma 4.6.1 and hence $|y_\infty| > D$. Hence $|y_\infty| > |c_\infty|$, and so

$$|y_\infty + c_\infty| = |y_\infty|. \tag{4.11}$$

Proceeding exactly the same as above, we deduce that (4.10) converges absolutely for $\sigma > 1$.

On combining the absolute convergence of (4.10) and (4.5), we conclude the absolute convergence of $Z_{\tau,a,c}(f, \chi_0, s)$ for $\mathrm{Re}(s) > 1$, provided that c and f are compatible at ∞. This completes the proof of Theorem 4.5.3.

$\square$

The proofs of Theorems 4.5.2 and 4.5.3 are the same as that of Theorems 3.5.2 and 3.5.3 respectively.

4.7 Example: the case $K = \mathbb{F}_q(T)$

The purpose of this section is to show that global function field Lerch-Tate zeta integrals can be specialized to give an analogue of Lerch zeta function for the rational function field $\mathbb{F}_q(T)$.

Let C denote the projective line $\mathbb{P}^1_{\mathbb{F}_q}$ over the finite field $\mathbb{F}_q$. Fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ and let $\overline{C} = C \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q$. Write $K = K(C)$ for the function field of C, namely the rational function field $\mathbb{F}_q(T)$; so $K(\overline{C}) = \overline{\mathbb{F}}_q(T)$ is the function field of $\overline{C}$.

Every closed point $P \in |C|$ corresponds either to an irreducible monic polynomial in $\mathbb{F}_q[T]$ or to the degree valuation for which $\frac{1}{T}$ is a uniformizer. We say in the former case that P is a finite closed point, and in the latter case that $P = \infty$ is the closed point at infinity. Let K_P denote the completion of K at P, $\mathcal{O}_P$ its ring of integers and $\mathfrak{m}_P$ the maximal ideal of $\mathcal{O}_P$. The residue field $\kappa_P = \mathcal{O}_P/\mathfrak{m}_P$ is a finite extension of $\mathbb{F}_q$. If t_P is a uniformizing parameter at P, then $K_P = \kappa_P((t_P))$, $\mathcal{O}_P = \kappa_P[[t_P]]$ and $\mathfrak{m}_P = (t_P)$. In particular, on putting $\pi_\infty = \frac{1}{T}$, we have $K_\infty = \mathbb{F}_q((\pi_\infty))$, $\mathcal{O}_\infty = \mathbb{F}_q[[\pi_\infty]]$, $\mathfrak{m}_\infty = (\pi_\infty)$ and $\kappa_\infty = \mathbb{F}_q$. Write $|x|_\infty = q^{-\mathrm{val}_\infty(x)}$ $(x \in K_\infty)$ for the normalized absolute value on K_∞ for which $|\pi_\infty|_\infty = \frac{1}{q}$.

Now consider our projective line $C = \mathbb{P}^1_{\mathbb{F}_q}$. Let ψ_q be a nontrivial additive character $\psi_q : \mathbb{F}_q \to \mathbb{C}^\times$. Let ω denote the differential form $\omega = \mathrm{dT}$ on $\mathbb{P}^1_{\mathbb{F}_q}$. For each closed point $P \in |C|$, let $\mathrm{Res}_P : \Omega^1_{K(C)_P/\kappa_P} \to \kappa_P$ denote the κ_P-linear residue map at P. Define the local additive character $\psi_P : K_P \to \mathbb{C}^\times$ by

$$\psi_P(x_P) = \psi_q\left(\mathrm{Tr}_{\kappa_P/\mathbb{F}_q} \mathrm{Res}_P\,(x_P\omega)\right) \quad (x_P \in K_P).$$

Define the global additive character $\psi : \mathbb{A}_K \to \mathbb{C}^\times$ by $\psi = \otimes_{P\in|C|}\psi_P$. By definition

$$\psi(\otimes_{P\in|C|}x_P) = \prod_{P\in|C|} \psi_P(x_P) = \prod_{P\in|C|} \psi_q\left(\mathrm{Tr}_{\kappa_P/\mathbb{F}_q} \mathrm{Res}_P\,(x_P\omega)\right)$$

The following lemma follows immediately upon unwinding the definition of residue.

Lemma 4.7.1. (i) *ψ has trivial restriction to K.*

(ii) *If $f \in \mathbb{F}_q[T]$ and $P \in |C| \setminus \{\infty\}$, then $\psi_P(f) = 1$.*

(iii) *If $f = \sum_{\nu\gg-\infty} a_\nu \pi_\infty^\nu \in K_\infty$, then $\psi_\infty(f) = \psi_q^{-1}(a_1)$.*

We now define an analogue of the Lerch zeta function for the projective line $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{F}_q}$.

Definition 4.7.2. Define the Lerch zeta function over $\mathbb{P}^1$ to be Dirichlet series

$$\zeta_{\mathbb{P}^1}(s, a_\infty, c_\infty) = \sum_{\beta \in \mathbb{F}_q[T]} \frac{\psi_\infty(\beta a_\infty)}{|\beta + c_\infty|_\infty^s}$$

where $a_\infty \in K_\infty$, $c_\infty \in K_\infty - \mathcal{O}_\infty$, and $s \in \mathbb{C}$.

Lemma 4.7.3. $\zeta_{\mathbb{P}^1}(s, a_\infty, c_\infty)$ *converges uniformly absolutely on compact subsets of* $\Re(s) > 1$ *and defines a holomorphic function on this right half-plane.*

Proof. Put $\sigma = \Re(s)$, so $\sigma > 1$. By assumption there exists a positive integer N such that

$$c_\infty = \sum_{n=-N}^{\infty} c_n \pi_\infty^n \quad (c_n \in \mathbb{F}_q, c_{-N} \neq 0).$$

Therefore, for every polynomial $\beta \in \mathbb{F}_q[T]$ of degree $m > N$, we have

$$|\beta + c_\infty|_\infty^s = q^{ms}.$$

Hence

$$\begin{aligned}
\sum_{\substack{\beta \in \mathbb{F}_q[T] \\ \deg \beta \geq N+1}} \frac{|\psi_\infty(\beta a_\infty)|}{|\beta + c_\infty|_\infty^\sigma} &\leq \sum_{m=N+1}^{\infty} (\text{number of polynomials of degree } m)\ q^{-m\sigma} \\
&\leq \sum_{m=N+1}^{\infty} (q-1) q^m\ q^{-m\sigma} \\
&= (q-1) \sum_{m=N+1}^{\infty} q^{m(1-\sigma)}
\end{aligned}$$

which is finite. The proof is complete. □

We proceed to show that $\zeta_{\mathbb{P}^1}(s, a_\infty, c_\infty)$ arises from global function field Lerch-Tate zeta integrals. We take for $\tau : q^{\mathbb{Z}} \to \mathbb{A}_K^\times$ the norm section for which $\tau(q^{-1}) = (\pi_\infty, (1))$. We normalize adelic Haar measures as follows:

- let $\mathrm{d}^\times \mathrm{x_P}$ be the Haar measure on $K_P^\times$ for which $\mathcal{O}_P^\times$ has unit volume;
- take for $\mathbb{A}_K^\times$ the product measure $\mathrm{d}^\times \mathrm{x} = \prod_{P \in |C|} \mathrm{d}^\times \mathrm{x_P}$;
- take for $\mathbb{A}_K$ the self-dual measure with respect to ψ.

Proposition 4.7.4. *Suppose that* $a = \otimes_{P\in|C|} a_P \in \mathbb{A}_K$ *and* $c = \otimes_{P\in|C|} c_P \in \mathbb{A}_K$ *satisfy*

(a) $a_P = c_P = 0$ *for every* $P \in |C| \setminus \{\infty\}$*;*

(b) $a_\infty, c_\infty \in K_\infty$ *and* $\mathrm{val}_\infty(c_\infty) > 0$.

Suppose further that $\chi_s = |\cdot|^s$ *and that* $f = \otimes_{P\in|C|} f_P \in S(\mathbb{A}_K)$ *is given by*

(a) $f_P = 1_{\mathcal{O}_P}$ *for every* $P \in |C| \setminus \{\infty\}$*;*

(b) $f_\infty = 1_{\mathfrak{m}_\infty}$.

Then for $\mathrm{Re}(s) > 1$

$$Z_{\tau,a,c}(f, \chi_s) = \zeta_{\mathbb{P}^1}(s, a_\infty, c_\infty). \tag{4.12}$$

Proof. The proof starts with the observation that a fundamental domain for the action of $K^\times$ on $\mathbb{A}_K^1$ is given by the following subset of $\mathbb{A}_K^1$

$$E := (1 + \mathfrak{m}_\infty) \times \prod_{P\in|C|\setminus\{\infty\}} \mathcal{O}_P^\times.$$

It follows from the hypotheses that

$$\begin{aligned} Z_{\tau,a,c}(f, \chi_s) &= \sum_{m=-\infty}^{\infty} \int_{\mathbb{A}_K^1} \psi(ay)\, f\left((y + c)\,\tau(q^{-m})\right)\, q^{-ms} \;\, \mathrm{d}^\times \mathrm{y} \\ &= \sum_{m=-\infty}^{\infty} q^{-ms} \int_{C_K^1} \left\{ \sum_{\beta\in K^\times} \psi(a\beta y)\, f\left((\beta y + c)\,\tau(q^{-m})\right) \right\} \;\mathrm{d}^\times \dot{\mathrm{y}} \\ &= \sum_{m=-\infty}^{\infty} q^{-ms} \int_{E} \left\{ \sum_{\beta\in K^\times} \psi(a\beta y)\, f\left((\beta y + c)\,\tau(q^{-m})\right) \right\} \;\mathrm{d}^\times \mathrm{y}. \end{aligned}$$

Since

$$f\left((\beta y + c)\,\tau(q^{-m})\right) = 1_{\mathfrak{m}_\infty}\left((\beta y_\infty + c_\infty)\pi_\infty^m\right) \prod_{P\in|C|\setminus\{\infty\}} 1_{\mathcal{O}_P}(\beta y_p),$$

it follows that $f\left((\beta y + c)\,\tau(t)\right) = 0$ unless $\beta \in \mathbb{F}_q[T]$. If $\beta \in \mathbb{F}_q[T]$ and $y = \otimes_{P\in|C|} y_P \in E$, then

$$\begin{aligned} \psi(a\beta y) &= \psi_\infty(\beta a_\infty y_\infty) \\ f\left((\beta y + c)\,\tau(q^{-m})\right) &= 1_{\mathfrak{m}_\infty}\left((\beta y_\infty + c_\infty)\,\pi_\infty^m\right). \end{aligned}$$

Therefore

$$Z_{\tau,a,c}(f,\chi_s) = \sum_{m=-\infty}^{\infty} q^{-ms} \sum_{\beta\in\mathbb{F}_q[T]\setminus\{0\}} \int_{1+\mathfrak{m}_\infty} \psi_\infty(\beta a_\infty y_\infty)\, 1_{\mathfrak{m}_\infty}\left((\beta y_\infty + c_\infty)\,\pi_\infty^m\right)\ \mathrm{d}^\times \mathrm{y}_\infty.$$

An easy calculation then gives

$$Z_{\tau,a,c}(f,\chi_s) = \frac{1}{(q-1)(1-q^{-s})}\,\zeta_{\mathbb{P}^1}(s,a_\infty,c_\infty).$$

□

Chapter 5

A generalized Lerch zeta function

5.1 Overview of the chapter

In this chapter we study a generalized Lerch zeta function given by the following

Definition. Suppose that x, y, α and β are complex variables. Set

$$Z(x, y; \alpha, \beta) = \sum_{n=-\infty}^{\infty}{}' \frac{1}{(n+x)^{\alpha}(n+\overline{x})^{\beta}} \, \mathbf{e}(ny) \tag{5.1}$$

where $\mathbf{e}(z) = e^{2\pi i z}$ and the primed sum excludes the term $n = -x$ when x is an integer. Here we fix the log branch $\log z = \log |z| + i \arg z$ with $-\pi \leq \arg z < \pi$ and define $z^{\gamma} = e^{\gamma \log z}$ for two complex numbers z and γ.

The series representation (5.1) is a Fourier expansion of $Z(x, y; \alpha, \beta)$ in terms of y. On letting x to be a complex variable, the function $\frac{1}{(n+x)^{\alpha}(n+\overline{x})^{\beta}}$ becomes multivalued in x. We will, however, restrict to the case $x \in \mathbb{H}$ for the purpose of getting a single-valued function. Our goal is to show analytic continuation of $Z(x, y; \alpha, \beta)$ in $(\alpha, \beta) \in \mathbb{C}^2$ by means of Fourier development in terms of the real part of x. The chapter outline is as follows.

- Section 5.2 shows the absolute convergence of $Z(x, y; \alpha, \beta)$ for real y and $\mathrm{Re}(\alpha + \beta) > 1$.

- In Sections 5.3–5.5, several confluent hypergeometric functions are introduced and studied in detail. We define and introduce the upsilon hypergeometric function. We give a tabulation of their analytic properties, stated in a quite general form.

- Proposition 5.6.1 in Section 5.6 shows that the upsilon hypergeometric function is the Fourier transform of a product of two complex powers.

- On putting $x = x' + ix''$, Section 5.7 gives the Fourier expansion of $\zeta(x, y; \alpha, \beta)$ in terms of x' and establishes its analytic continuation in (α, β).

- Section 5.8 relates $Z(x, y; \alpha, \beta)$ to the Lerch zeta function.

Throughout, let $\mathbb{C}' = \mathbb{C} - \mathbb{R}_{\leq 0}$ denote the cut complex plane and $\mathbb{C}^{\bullet} = \mathbb{C} - \mathbb{Z}_{\leq 0}$ denote the complex plane punctured at the non-negative integers. Let

$$\begin{aligned} \mathbb{H} = \mathbb{H}^{+} &= \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\} \\ \mathbb{H}^{-} &= \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\} \\ \mathbb{H}^{\pm} &= \mathbb{H}^{+} \cup \mathbb{H}^{-} \\ \mathbb{C}^{+} &= \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\} \\ \mathbb{C}^{-} &= \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}. \end{aligned}$$

5.2 Absolute convergence of the series

We are treating complex powers in the definition of $Z(x, y; \alpha, \beta)$. We define

$$z^{\alpha} = \exp(\alpha \, \log(z))$$

where log is the principal branch

$$\log(z) = \log|z| + i \arg(z)$$

specifying $-\pi \leq \arg(z) < \pi$. We will sometimes need to allow negative real values for z, in which case $\arg(z) = -\pi$ and hence $z^{\alpha} = |z|^{\alpha} e^{-\pi i \alpha}$.

Lemma 5.2.1. *Suppose that $w, v \in \mathbb{C}$ satisfy $-\pi \leq \arg(w) + \arg(v) < \pi$. Then $(wv)^{\alpha} = w^{\alpha} v^{\alpha}$ for every $\alpha \in \mathbb{C}$.*

Proof. Recall that we fix the log branch $\log z = \log|z| + i \arg z$ with $-\pi \leq \arg z < \pi$ and define $z^{\alpha} = \exp(\alpha \log z)$. The hypothesis $-\pi \leq \arg(w) + \arg(v) < \pi$ then implies that $\arg(wv) = \arg(w) + \arg(v)$. Thus $(wv)^{\alpha} = w^{\alpha} v^{\alpha}$ for every $\alpha \in \mathbb{C}$. $\square$

Remark 5.2.2. Lemma 5.2.1 gives for $x \in \mathbb{H}^{\pm}$ the identity

$$\frac{1}{(n + x)^{\alpha} (n + \overline{x})^{\beta}} = \frac{1}{|n + x|^{2\beta} (n + x)^{\alpha - \beta}}.$$

The following lemma shows that the series representation (5.1) of $Z(x, y; \alpha, \beta)$ converges for $y \in \mathbb{R}$.

Lemma 5.2.3. *Suppose that $x \in \mathbb{H}$ and $y \in \mathbb{R}$. Suppose further that $\alpha, \beta \in \mathbb{C}$ satisfy* $\mathrm{Re}(\alpha+\beta) > 1$. *Then on the region $(\alpha, \beta) \in \mathbb{C}^2$ with* $\mathrm{Re}(\alpha+\beta) > 1$, *the series*

$$Z(x, y; \alpha, \beta) = \sideset{}{'}\sum_{n=-\infty}^{\infty} \frac{1}{(n+x)^{\alpha}(n+\overline{x})^{\beta}}\, \mathbf{e}(ny)$$

converges absolutely and defines a function which is real analytic in x, y and which is holomorphic in α, β.

Proof. Put $x = x' + ix''$. If the hypotheses are satisfied, then for every integer n

$$\left| \frac{1}{(n+x)^{\alpha}(n+\overline{x})^{\beta}}\, \mathbf{e}(ny) \right| \ll \frac{1}{|n+x''|^{\mathrm{Re}(\alpha+\beta)}}.$$

Let $\delta > 0$ be arbitrary. Hence the summation which defines $\zeta(x, y; \alpha, \beta)$ is bounded absolutely and uniformly for $\mathrm{Re}(\alpha+\beta) > 1+\delta$ by

$$\sum_{n=-\infty}^{\infty} \frac{1}{|n+x''|^{1+\delta}},$$

which is finite because $\mathrm{Re}(\alpha+\beta) > 1$. The proof is complete. $\square$

Remarks 5.2.4.

1. On setting $\alpha = s$ and $\beta = s - a$ with $a \in \{0, 1\}$, one derives a variant of the Lerch zeta function, namely

$$Z(x, y; s, s+a) = \sideset{}{'}\sum_{n=-\infty}^{\infty} \frac{(n+\overline{x})^{a}}{|n+x|^{2s}}\, \mathbf{e}(ny)$$

which was studied by Weil [42, Chapter VII], who called it $S_a(x, -y, s)$. This function exhibits the real analyticity in the "Hurwitz variable" x and the complex analyticity in the "Lerch variable" y.

2. Suppose that $s \in \mathbb{C}$. On setting $\alpha = s$ and $\beta = 0$, one considers an analytic continuation of the Lerch zeta function, namely

$$Z(x, y; s, 0) = \sideset{}{'}\sum_{n=-\infty}^{\infty} \frac{1}{(n+x)^{s}}\, \mathbf{e}(ny) = \zeta(s, y, x)$$

which was the main object of investigation of Lagarias and Li in [8]. This function exhibits the complex analyticity in both the "Hurwitz variable" x and the "Lerch variable" y. We refer the reader to [8] for interesting results on the multivalued nature of this function.

5.3 Confluent hypergeometric function: Shimura's tau function

We start with a confluent hypergeometric function studied by Shimura, namely the *tau function* $\tau(x;\alpha,\beta)$. Properties of the tau function were proven and used extensively in [27, 28, 29, 30]. See also [31, 32] for more recent papers of Shimura related to the Lerch zeta function. Precedents of the tau function appeared in Maass' classic works [15, 16, 17, 18, 19]. Since it is not easy to extract the results, we collect the analytic properties proven by Shimura and state them in a slightly more general form.

Suppose that $x \in \mathbb{C}$ with $\mathrm{Re}(x) > 0$. Suppose that $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\beta) > 0$. Then the integral

$$\tau(x;\alpha,\beta) = \int_0^\infty e^{-xt}(1+t)^{\alpha-1}t^{\beta-1}dt \tag{5.2}$$

is uniformly convergent on compacta and defines an analytic function of x, α, β. Lemma 5.3.1 and Proposition 5.3.2 below are due to Shimura.

Recall that $\mathbb{C}' = \mathbb{C} - \mathbb{R}_{\leq 0}$ and that $\mathbb{C}^\bullet = \mathbb{C} - \mathbb{Z}_{\leq 0}$.

Lemma 5.3.1. *The function $\tau(x;\alpha,\beta)$ extends to an analytic function for $x \in \mathbb{C}'$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}^\bullet$. In this extended domain, the function $\tau(x;\alpha,\beta)$ satisfies the following properties:*

(i) $x^\beta \tau(x;\alpha,\beta) = \int_0^\infty e^{-t}\left(1+\frac{t}{x}\right)^{\alpha-1} t^{\beta-1}dt$ *provided that* $\mathrm{Re}(\beta) > 0$*;*

(ii) $x\tau(x;\alpha+1,\beta+1) = \beta\tau(x;\alpha+1,\beta) + \alpha\tau(x;\alpha,\beta+1)$*;*

(iii) $\tau(x;\alpha+1,\beta) = \tau(x;\alpha,\beta) + \tau(x;\alpha,\beta+1)$*;*

(iv) $\frac{x^{1-\alpha}}{\Gamma(1-\alpha)}\tau(x;1-\beta,1-\alpha) = \frac{x^\beta}{\Gamma(\beta)}\tau(x;\alpha,\beta)$*, both sides being analytic for* $x \in \mathbb{C}', \alpha \in \mathbb{C}, \beta \in \mathbb{C}$*;*

(v) $\partial_x \tau(x;\alpha,\beta) = -\tau(x;\alpha,\beta+1)$*.*

Proof. The analytic continuation consists of two steps. First, suppose that $x \in \mathbb{R}_+$, that $\alpha \in \mathbb{C}$, and that $\mathrm{Re}(\beta) > 0$. Then a change of variable $t \mapsto \frac{t}{x}$ in (5.2) gives the identity

$$\tau(x;\alpha,\beta) = x^{-\beta}\int_0^\infty e^{-t}\left(1+\frac{t}{x}\right)^{\alpha-1} t^{\beta-1}\,\mathrm{dt},$$

the second member of which is analytic for $x \in \mathbb{C}', \alpha \in \mathbb{C}, \mathrm{Re}(\beta) > 0$. The analyticity follows immediately from the observation that the term $\left(1+\frac{t}{x}\right)^{\alpha-1}$ is bounded as $t \to 0$ and is polynomially bounded as $t \to \infty$, provided that x is in a compact subset of $\mathbb{C}'$, that $\alpha \in \mathbb{C}$, and that $\mathrm{Re}(\beta) > 0$. Secondly, suppose that $x \in \mathbb{R}_+$, that $\alpha \in \mathbb{C}$, and that $\mathrm{Re}(\beta) > 0$. Then an integration by parts of the quantity $\frac{\mathrm{d}}{\mathrm{dt}}\left\{e^{-xt}(1+t)^{\alpha}t^{\beta}\right\}$ yields

$$x\tau(x;\alpha+1,\beta+1) = \beta\tau(x;\alpha+1,\beta) + \alpha\tau(x;\alpha,\beta+1).$$

This identity implies that $\tau(x;\alpha,\beta)$ can be analytically continued to the extended domain $x \in \mathbb{C}', \alpha \in \mathbb{C}, \beta \in \mathbb{C}^{\bullet}$. The singularities at non-negative integers β are at worst simple poles; in other words the function $(x,\alpha,\beta) \mapsto \frac{1}{\Gamma(\beta)}\tau(x;\alpha,\beta)$ is analytic for $x \in \mathbb{C}', \alpha \in \mathbb{C}, \beta \in \mathbb{C}$. It now follows by analytic continuation that (i) and (ii) hold true in the asserted domains.

Next, apply (5.2) for $\tau(x;\alpha+1,\beta)$ and write $(1+t)^{\alpha} = (1+t)^{\alpha-1}(1+t)$. This yields (iii), first for $\mathrm{Re}(x) > 0, \alpha \in \mathbb{C}, \mathrm{Re}(\beta) > 0$ and then for the extended domain by analytic continuation. Also from (5.2), differentiation with respect to x under integral sign is permissible when $\mathrm{Re}(x) > 0, \alpha \in \mathbb{C}, \mathrm{Re}(\beta) > 0$. Hence (v) is shown, originally for $\mathrm{Re}(x) > 0, \alpha \in \mathbb{C}, \mathrm{Re}(\beta) > 0$ and hence for the extended domain $x \in \mathbb{C}', \alpha \in \mathbb{C}, \beta \in \mathbb{C}^{\bullet}$ by analytic continuation.

Finally, by analytic continuation it suffices to prove (iv) for $x \in \mathbb{R}_+, 0 < \mathrm{Re}(\alpha) < 1, 0 < \mathrm{Re}(\beta) < 1$. In this region

$$\begin{aligned}
\Gamma(\beta)\tau(x;1-\beta,1-\alpha) &= \int_0^{\infty}\int_0^{\infty} e^{-u}u^{\beta}e^{-xt}(1+t)^{-\beta}t^{1-\alpha}\frac{\mathrm{du}}{u}\frac{\mathrm{dt}}{t} \\
&= \int_0^{\infty}\int_0^{\infty} e^{-v(1+t)}v^{\beta}e^{-xt}t^{1-\alpha}\frac{\mathrm{dv}}{v}\frac{\mathrm{dt}}{t} \\
&= \int_0^{\infty}\int_0^{\infty} e^{-v}v^{\beta}e^{-t(v+x)}t^{1-\alpha}\frac{\mathrm{dt}}{t}\frac{\mathrm{dv}}{v} \\
&= \int_0^{\infty}\int_0^{\infty} e^{-v}v^{\beta}(v+x)^{\alpha-1}e^{-w}w^{1-\alpha}\frac{\mathrm{dw}}{w}\frac{\mathrm{dv}}{v} \\
&= \Gamma(1-\alpha)\int_0^{\infty} e^{-v}v^{\beta}(v+x)^{\alpha-1}\frac{\mathrm{dv}}{v} \\
&= \Gamma(1-\alpha)x^{\alpha+\beta-1}\tau(x;\alpha,\beta).
\end{aligned}$$

Thus (iv) follows. The proof is complete. □

Recall that $\mathbb{C}' = \mathbb{C} - \mathbb{R}_{\leq 0}$ and that $\mathbb{C}^{\bullet} = \mathbb{C} - \mathbb{Z}_{\leq 0}$.

Proposition 5.3.2. *Suppose that $x \in \mathbb{C}'$, that $\alpha \in \mathbb{C}$, and that $\beta \in \mathbb{C}^{\bullet}$.*

(i) *The function* $\tau(x;\alpha,\beta)$ *satisfies the differential equation*

$$x\partial_x^2\tau(x;\alpha,\beta)+(\alpha+\beta-x)\partial_x\tau(x;\alpha,\beta)-\beta\tau(x;\alpha,\beta)=0. \tag{5.3}$$

(ii) *Suppose further that* $0<\delta<\frac{\pi}{2}$. *If* $\mathrm{Re}(\beta)>0$ *and* $\mathrm{Re}(\alpha+\beta)>1$, *then*

$$\tau(x;\alpha,\beta)\sim\Gamma(\alpha+\beta-1)x^{1-\alpha-\beta}$$

as $x\to 0$ *in the cone* $|\arg(x)|<\frac{\pi}{2}-\delta$.

(iii) *Suppose further that* $0<\delta<\frac{\pi}{2}$. *If* $\mathrm{Re}(\beta)>0$, *then*

$$\tau(x;\alpha,\beta)\sim\Gamma(\beta)x^{-\beta}$$

as $x\to\infty$ *in the cone* $|\arg(x)|<\frac{\pi}{2}-\delta$.

Proof. We first show (i). Lemma 5.3.1 (v) implies that $\partial_x\tau(x;\alpha,\beta)=-\tau(x;\alpha,\beta+1)$ and that $\partial_x^2\tau(x;\alpha,\beta)=-\tau(x;\alpha,\beta+2)$. Therefore, by Lemma 5.3.1 (ii) and (iii),

$$\begin{aligned}
&x\partial_x^2\tau(x;\alpha,\beta)+(\alpha+\beta-x)\partial_x\tau(x;\alpha,\beta)-\beta\tau(x;\alpha,\beta)\\
&=x\tau(x;\alpha,\beta+2)-(\alpha+\beta-x)\tau(x;\alpha,\beta+1)-\beta\tau(x;\alpha,\beta)\\
&=-(\alpha+\beta)\tau(x;\alpha,\beta+1)+(\alpha-1)\tau(x;\alpha-1,\beta+2)\\
&\qquad+(\beta+1)\tau(x;\alpha,\beta+1)+(\alpha-1)\tau(x;\alpha-1,\beta+1)\\
&=(\alpha-1)\Big\{\tau(x;\alpha-1,\beta+2)-\tau(x;\alpha,\beta+1)+\tau(x;\alpha-1,\beta+1)\Big\}=0.
\end{aligned}$$

Alternatively, one asserts that the differential operator

$$D:=x\partial_x^2+(\alpha+\beta-x)\partial_x-\beta$$

applied to the integrand of the integral representation (5.2), namely to

$$e^{-xt}(1+t)^{\alpha-1}t^{\beta-1},$$

yields an exact form

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big\{-e^{-xt}(1+t)^{\alpha}t^{\beta}\Big\}.$$

Moreover, any primitive of this exact form takes the same value at the boundaries 0 and ∞. Hence it remains to verify the assertion. On putting $f(x) = e^{-xt}(1+t)^{\alpha-1}t^{\beta-1}$,

$$\begin{aligned}\partial_x f(x) &= -tf(x)\\ \partial_x^2 f(x) &= t^2 f(x),\end{aligned}$$

and so $(Df)(x) = f(x)(xt^2 - t(\alpha+\beta-x) - \beta) = \frac{\mathrm{d}}{\mathrm{dt}}\Big\{ - e^{-xt}(t+1)^{\alpha}t^{\beta}\Big\}$. This verifies the assertion.

We now show (ii). When $\mathrm{Re}(\beta) > 0, \mathrm{Re}(\alpha+\beta) > 1$ and $x \to 0$ we have, by Lemma 5.3.1 (i) and the dominated convergence theorem,

$$\tau(x;\alpha,\beta) = x^{1-\alpha-\beta}\int_0^\infty e^{-t}(x+t)^{\alpha-1}t^{\beta-1}\mathrm{dt} \sim \Gamma(\alpha+\beta-1)x^{1-\alpha-\beta}.$$

We next show (iii). When $\mathrm{Re}(\beta) > 0$ and $x \to \infty$ we have, by Lemma 5.3.1 (i) and the dominated convergence theorem,

$$\tau(x;\alpha,\beta) = x^{-\beta}\int_0^\infty e^{-t}\left(1+\frac{t}{x}\right)^{\alpha-1} t^{\beta-1}\mathrm{dt} \sim \Gamma(\beta)x^{-\beta}.$$

This completes the proof. □

5.4 Confluent hypergeometric function: the upsilon function

In this paragraph we introduce and study the analytic properties of a confluent hypergeometric function, which we call the *upsilon function* $\upsilon(x;\alpha,\beta)$. The upsilon function is closely related to Shimura's tau function and gives the Fourier transform of complex powers (Proposition 5.6.1). In Theorem 5.7.2 below we use the upsilon function and its analytic properties in obtaining the Fourier expansion and hence the analytic continuation of the generalized Lerch zeta function $Z(x,y;\alpha,\beta)$.

Suppose that $x \in \mathbb{C}'$. Suppose further that $\alpha,\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta) > 0$. Then the integral

$$\upsilon(x;\alpha,\beta) = \int_0^\infty e^{-t}(t+x)^{\alpha-1}t^{\beta-1}dt \tag{5.4}$$

is uniformly convergent on compacta and defines an analytic function of x,α,β. This is evident on noting that $(t+x)^{\alpha-1}$ is bounded as $t \to 0$ and that $(t+x)^{\alpha-1}$ is polynomially bounded as $t \to \infty$. The relation of $\upsilon(x;\alpha,\beta)$ and the function $\tau(x;\alpha,\beta)$ in the previous paragraph, by Lemma 5.3.1 (i) and (5.4), is

$$\tau(x;\alpha,\beta) = x^{1-\alpha-\beta}\,\upsilon(x;\alpha,\beta). \tag{5.5}$$

Recall that $\mathbb{C}' = \mathbb{C} - \mathbb{R}_{\leq 0}$ and that $\mathbb{C}^\bullet = \mathbb{C} - \mathbb{Z}_{\leq 0}$.

Lemma 5.4.1. *The function $\upsilon(x;\alpha,\beta)$ extends to an analytic function for $x \in \mathbb{C}'$, $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}^\bullet$. In this extended domain, the function $\upsilon(x;\alpha,\beta)$ satisfies the following properties:*

(i) $\upsilon(x;\alpha,\beta) = x^{\alpha-1}\int_0^\infty e^{-t}\left(1+\frac{t}{x}\right)^{\alpha-1} t^{\beta-1}dt$ *provided that* $\mathrm{Re}(\beta) > 0$*;*

(ii) $\upsilon(x;\alpha+1,\beta+1) = \alpha\upsilon(x;\alpha,\beta+1) + \beta\upsilon(x;\alpha+1,\beta)$*;*

(iii) $\upsilon(x;\alpha+1,\beta) = \upsilon(x;\alpha,\beta+1) + x\upsilon(x;\alpha,\beta)$*;*

(iv) $\frac{x^{\alpha-1}}{\Gamma(1-\alpha)}\upsilon(x;1-\beta,1-\alpha) = \frac{x^{-\beta}}{\Gamma(\beta)}\upsilon(x;\alpha,\beta)$*, both sides being analytic for* $x \in \mathbb{C}', \alpha \in \mathbb{C}, \beta \in \mathbb{C}$*;*

(v) $\partial_x\upsilon(x;\alpha,\beta) = (\alpha-1)\upsilon(x;\alpha-1,\beta)$.

Proof. We first show the analytic continuation and (ii). Suppose that $x \in \mathbb{C}'$, that $\alpha \in \mathbb{C}$, and that $\mathrm{Re}(\beta) > 0$. An integration by parts of the quantity $\frac{\mathrm{d}}{\mathrm{dt}}\left\{e^{-t}(t+x)^\alpha t^\beta\right\}$ yields

$$\upsilon(x;\alpha+1,\beta+1) = \alpha\upsilon(x;\alpha,\beta+1) + \beta\upsilon(x;\alpha+1,\beta).$$

This identity implies that $\upsilon(x;\alpha,\beta)$ can be analytically continued to the extended domain $x \in \mathbb{C}', \alpha \in \mathbb{C}, \beta \in \mathbb{C}^\bullet$. The singularities at non-negative integers β are at worst simple poles: the function $(x,\alpha,\beta) \mapsto \frac{1}{\Gamma(\beta)}\upsilon(x;\alpha,\beta)$ is analytic for $x \in \mathbb{C}', \alpha \in \mathbb{C}, \beta \in \mathbb{C}$. Thus (ii) follows.

Now suppose that $x \in \mathbb{R}_+$, that $\alpha \in \mathbb{C}$, and that $\mathrm{Re}(\beta) > 0$. Then a change of variable $t \mapsto \frac{t}{x}$ in (5.4) gives the identity

$$\upsilon(x;\alpha,\beta) = x^{\alpha-1}\int_0^\infty e^{-t}\left(1+\frac{t}{x}\right)^{\alpha-1} t^{\beta-1}\,\mathrm{dt},$$

the second member of which is analytic for $x \in \mathbb{C}', \alpha \in \mathbb{C}, \mathrm{Re}(\beta) > 0$. The analyticity follows immediately from the observation that the term $\left(1+\frac{t}{x}\right)^{\alpha-1}$ is bounded as $t \to 0$ and is polynomially bounded as $t \to \infty$, provided that x is in a compact subset of $\mathbb{C}'$, that $\alpha \in \mathbb{C}$, and that $\mathrm{Re}(\beta) > 0$. By virtue of analytic continuation (i) holds true in the asserted domain.

Next, apply (5.4) for $\upsilon(x;\alpha+1,\beta)$ and write $(1+t)^\alpha = (1+t)^{\alpha-1}(1+t)$. This yields (iii), first for $\mathrm{Re}(x) > 0, \alpha \in \mathbb{C}, \mathrm{Re}(\beta) > 0$ and then for the extended domain by analytic continuation. Also from (5.4), differentiation with respect to x under integral sign is permissible when $\mathrm{Re}(x) > 0, \alpha \in \mathbb{C}, \mathrm{Re}(\beta) > 0$. Hence (v) is shown, originally for $\mathrm{Re}(x) > 0, \alpha \in \mathbb{C}, \mathrm{Re}(\beta) > 0$ and hence for the extended domain $x \in \mathbb{C}', \alpha \in \mathbb{C}, \beta \in \mathbb{C}^\bullet$ by analytic continuation.

Finally, by analytic continuation it suffices to prove (iv) for $x \in \mathbb{R}_+, 0 < \mathrm{Re}(\alpha) < 1, 0 < \mathrm{Re}(\beta) < 1$. In this region

$$\begin{aligned}
\Gamma(\beta)\upsilon(x; 1-\beta, 1-\alpha) &= \int_0^\infty \int_0^\infty e^{-u} u^\beta e^{-t} (t+x)^{-\beta} t^{1-\alpha} \frac{\mathrm{du}}{u} \frac{\mathrm{dt}}{t} \\
&= \int_0^\infty \int_0^\infty e^{-v(t+x)} v^\beta e^{-t} t^{1-\alpha} \frac{\mathrm{dv}}{v} \frac{\mathrm{dt}}{t} \\
&= \int_0^\infty \int_0^\infty e^{-vx} v^\beta e^{-t(v+1)} t^{1-\alpha} \frac{\mathrm{dt}}{t} \frac{\mathrm{dv}}{v} \\
&= \int_0^\infty \int_0^\infty e^{-vx} v^\beta (v+1)^{\alpha-1} e^{-w} w^{1-\alpha} \frac{\mathrm{dw}}{w} \frac{\mathrm{dv}}{v} \\
&= \Gamma(1-\alpha) \int_0^\infty e^{-vx} v^\beta (v+1)^{\alpha-1} \frac{\mathrm{dv}}{v} \\
&= \Gamma(1-\alpha) x^{1-\alpha-\beta} \upsilon(x; \alpha, \beta).
\end{aligned}$$

Thus (iv) follows. The proof is complete. □

Remark 5.4.2. For fixed $x \in \mathbb{C}'$ and $\alpha \in \mathbb{C}$, the function $\upsilon(x; \alpha, \beta)$ is a meromorphic function of $\beta \in \mathbb{C}$ with simple poles at negative integers. For fixed $x \in \mathbb{C}'$, Lemma 5.4.1 (iv) implies that the function $\frac{1}{\Gamma(\beta)}\upsilon(x; \alpha, \beta)$ is a holomorphic function of $(\alpha, \beta) \in \mathbb{C}^2$.

Recall that $\mathbb{C}' = \mathbb{C} - \mathbb{R}_{\leq 0}$ and that $\mathbb{C}^\bullet = \mathbb{C} - \mathbb{Z}_{\leq 0}$.

Proposition 5.4.3. *Suppose that $x \in \mathbb{C}'$, that $\alpha \in \mathbb{C}$, and that $\beta \in \mathbb{C}^\bullet$. Suppose further that $0 < \delta < \frac{\pi}{2}$.*

(i) *The function $\upsilon(x; \alpha, \beta)$ satisfies the differential equation*

$$x\partial_x^2 \upsilon(x; \alpha, \beta) + (2 - \alpha - \beta - x)\partial_x \upsilon(x; \alpha, \beta) + (\alpha - 1)\upsilon(x; \alpha, \beta) = 0. \qquad (5.6)$$

(ii) *If $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\alpha + \beta) > 1$, then*

$$\upsilon(x; \alpha, \beta) \sim \Gamma(\alpha + \beta - 1)$$

as $x \to 0$ in the cone $|\arg(x)| < \frac{\pi}{2} - \delta$.

(iii) *If $\mathrm{Re}(\beta) > 0$, then*

$$\upsilon(x; \alpha, \beta) \sim \Gamma(\beta) x^{\alpha - 1}$$

as $x \to \infty$ in the cone $|\arg(x)| < \frac{\pi}{2} - \delta$.

(iv) *Let K be a compact subset of $\mathbb{C} \times \mathbb{C}^\bullet$. As $x \to \infty$ in the cone $|\arg(x)| < \frac{\pi}{2} - \delta$ and uniformly for $(\alpha, \beta) \in K$, the function $\upsilon(x; \alpha, \beta)$ is polynomially bounded. More precisely, there exist positive constants A and B such that*

$$|\upsilon(x; \alpha, \beta)| \leq (A + |x|)^B \quad \left(|\arg(x)| < \frac{\pi}{2} - \delta, (\alpha, \beta) \in K\right).$$

Proof. We first show (i). Lemma 5.4.1 (v) implies that $\partial_x \upsilon(x; \alpha, \beta) = (\alpha - 1)\upsilon(x; \alpha - 1, \beta)$ and that $\partial_x^2 \upsilon(x; \alpha, \beta) = (\alpha - 1)(\alpha - 2)\upsilon(x; \alpha - 2, \beta)$. Therefore, by Lemma 5.4.1 (ii) and (iii),

$$\begin{aligned}
&x\partial_x^2 \upsilon(x; \alpha, \beta) + (2 - \alpha - \beta - x)\partial_x \upsilon(x; \alpha, \beta) + (\alpha - 1)\upsilon(x; \alpha, \beta) \\
&= (\alpha - 1)\Big\{x(\alpha - 2)\upsilon(x; \alpha - 2, \beta) + (2 - \alpha - \beta - x)\upsilon(x; \alpha - 1, \beta) + \upsilon(x; \alpha, \beta)\Big\} \\
&= (\alpha - 1)\Big\{ - x(\beta - 1)\upsilon(x; \alpha - 1, \beta - 1) + (1 - \alpha)\upsilon(x; \alpha - 1, \beta) \\
&\qquad\qquad + (1 - \beta)\upsilon(x; \alpha - 1, \beta) + \upsilon(x; \alpha, \beta)\Big\} \\
&= (\alpha - 1)\Big\{(1 - \beta)\upsilon(x; \alpha, \beta - 1) - (\alpha - 1)\upsilon(x; \alpha - 1, \beta) + \upsilon(x; \alpha, \beta)\Big\} = 0.
\end{aligned}$$

Alternatively, one asserts that the differential operator

$$D := x\partial_x^2 + (2 - \alpha - \beta - x)\partial_x + (\alpha - 1)$$

applied to the integrand of the integral representation (5.4), namely to

$$e^{-t}(t + x)^{\alpha - 1} t^{\beta - 1},$$

yields an exact form

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big\{(1 - \alpha)e^{-t}(t + x)^{\alpha - 2} t^{\beta}\Big\}.$$

Moreover, any primitive of this exact form takes the same value at the boundaries 0 and ∞. Hence it remains to verify the assertion. On putting $f(x) = e^{-t}(t+x)^{\alpha-1}t^{\beta-1}$,

$$\begin{aligned}
\partial_x f(x) &= f(x)\ \frac{\alpha - 1}{t + x} \\
\partial_x^2 f(x) &= f(x)\ \frac{(\alpha - 1)(\alpha - 2)}{(t + x)^2},
\end{aligned}$$

and so

$$
\begin{aligned}
(Df)(x) &= (\alpha-1)f(x)\left\{\frac{x(\alpha-2)}{(t+x)^2}+\frac{2-\alpha-\beta-x}{t+x}+1\right\}\\
&= (\alpha-1)f(x)\,\frac{t^2+tx-\beta(t+x)-(\alpha-2)t}{(t+x)^2}\\
&= \frac{\mathrm{d}}{\mathrm{dt}}\Big\{(1-\alpha)e^{-t}(t+x)^{\alpha-2}t^{\beta}\Big\}.
\end{aligned}
$$

The assertion is verifed.

We next show (ii). When $\mathrm{Re}(\beta)>0, \mathrm{Re}(\alpha+\beta)>1$ and $x\to 0$ we have, by Lemma 5.4.1 (i) and the dominated convergence theorem,

$$
\upsilon(x;\alpha,\beta)=\int_0^{\infty} e^{-t}(x+t)^{\alpha-1}t^{\beta-1}\,\mathrm{dt}\sim\Gamma(\alpha+\beta-1).
$$

We now show (iii). When $\mathrm{Re}(\beta)>0$ and $x\to\infty$ we have, by Lemma 5.4.1 (i) and the dominated convergence theorem,

$$
\upsilon(x;\alpha,\beta)=x^{\alpha-1}\int_0^{\infty} e^{-t}\left(1+\frac{t}{x}\right)^{\alpha-1}t^{\beta-1}\,\mathrm{dt}\sim\Gamma(\beta)x^{\alpha-1}.
$$

Finally we show (iv). In view of Lemma 5.4.1 (ii), we may assume without loss of generality that $\mathrm{Re}(\beta)>0$. In this case, (iv) is clear from the integral representation (5.4). This completes the proof. □

5.5 Confluent hypergeometric function: Whittaker's function

Both the tau function $\tau(x;\alpha,\beta)$ and the upsilon function $\upsilon(x;\alpha,\beta)$ studied in previous paragraphs are closely related to a Whittaker's function. For intuition as well as to shed light on their origin, in this paragraph we tabulate the analytic properties of a genuine Whittaker's function, namely the function $W(x;\alpha,\beta)$. It is a solution to a confluent hypergeometric second-order differential equation and has rapid decay at infinity.

Set

$$
\begin{aligned}
W(x;\alpha,\beta) &= \frac{1}{\Gamma(\beta)}e^{-\frac{1}{2}x}x^{\beta}\tau(x;\alpha,\beta) && (5.7)\\
&= \frac{1}{\Gamma(\beta)}e^{-\frac{1}{2}x}x^{1-\alpha}\,\upsilon(x;\alpha,\beta). && (5.8)
\end{aligned}
$$

which is called a *Whittaker confluent hypergeometric function.* It follows from Lemma

5.3.1 (i) that the Whittaker function $W(x;\alpha,\beta)$ is an analytic function of $x \in \mathbb{C}', \alpha \in \mathbb{C}, \beta \in \mathbb{C}$.

Lemma 5.5.1. *Suppose that $x \in \mathbb{C}'$ and that $\alpha, \beta \in \mathbb{C}$. Then*

(i) $W(x;\alpha,\beta) = \frac{1}{\Gamma(\beta)} e^{-\frac{1}{2}x} \int_0^\infty e^{-t} \left(1+\frac{t}{x}\right)^{\alpha-1} t^{\beta-1} dt$ *provided that* $\mathrm{Re}(\beta) > 0$;

(ii) $W(x;\alpha+1,\beta+1) = W(x;\alpha+1,\beta) + \frac{\alpha}{x} W(x;\alpha,\beta+1)$;

(iii) $W(x;\alpha+1,\beta) = W(x;\alpha,\beta) + \frac{\beta}{x} W(x;\alpha,\beta+1)$;

(iv) $W(x;1-\beta,1-\alpha) = W(x;\alpha,\beta)$;

(v) $\partial_x W(x;\alpha,\beta) = \left(\frac{\beta}{x} - \frac{1}{2}\right) W(x;\alpha,\beta) - \frac{1}{x} W(x;\alpha,\beta+1)$.

Proof. Each part is a simple translation of the corresponding part in Lemma 5.4.1. □

Proposition 5.5.2. *Suppose that $x \in \mathbb{C}'$, that $\alpha \in \mathbb{C}$, and that $\beta \in \mathbb{C}$.*

(i) *The function $W(x;\alpha,\beta)$ satisfies the generalized Whittaker's differential equation*

$$\left\{ x^2\partial_x^2 + (\alpha-\beta)x\partial_x + \left[(\alpha-\beta)\left(\frac{1}{2}x - \beta\right) + \left(\beta(1-\beta) - \frac{1}{4}x^2\right)\right]\right\} W = 0. \tag{5.9}$$

(ii) *Suppose further that $0 < \delta < \frac{\pi}{2}$. If $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\alpha+\beta) > 1$, then*

$$W(x;\alpha,\beta) \sim \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\beta)} x^{1-\alpha}$$

as $x \to 0$ in the cone $|\arg(x)| < \frac{\pi}{2} - \delta$.

(iii) *Suppose further that $0 < \delta < \frac{\pi}{2}$. If $\mathrm{Re}(\beta) > 0$, then*

$$W(x;\alpha,\beta) \sim e^{-\frac{1}{2}x}$$

as $x \to \infty$ in the cone $|\arg(x)| < \frac{\pi}{2} - \delta$.

Proof. First we show (i). For $\mathrm{Re}(x) > 0, \alpha \in \mathbb{C}, \mathrm{Re}(\beta) > 0$, the function $W(x;\alpha,\beta)$ admits an integral representation

$$W(x;\alpha,\beta) = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\frac{1}{2}x} x^\beta e^{-xt} (1+t)^{\alpha-1} t^{\beta-1} \mathrm{dt}.$$

To show (5.9), it suffices to assert that the differential operator

$$D := \partial_x^2 + \frac{\alpha-\beta}{x}\,\partial_x - \left\{\frac{\alpha-\beta}{x}\,\Big(\frac{\beta}{x}-\frac{1}{2}\Big) + \Big(\frac{1}{4}-\frac{\beta(1-\beta)}{x^2}\Big)\right\}$$

applied to the integrand of the above integral representation, namely to

$$e^{-\frac{1}{2}x}x^{\beta}e^{-xt}(1+t)^{\alpha-1}t^{\beta-1},$$

yields an exact form

$$\frac{\mathrm{d}}{\mathrm{dt}}\Big\{-e^{-\frac{1}{2}x}x^{\beta-1}e^{-xt}(1+t)^{\alpha}t^{\beta}\Big\}.$$

Then, since any primitive of this exact form takes the same value at the boundaries 0 and ∞, the differential equation (5.9) follows.

To verify the assertion, one puts $f(x)=e^{-\frac{1}{2}x}x^{\beta}e^{-xt}(1+t)^{\alpha-1}t^{\beta-1}$ and deduces that

$$\begin{aligned}
\partial_x f(x) &= f(x)\left(\frac{\beta}{x}-\frac{1}{2}-t\right)\\
\partial_x^2 f(x) &= f(x)\left(t(t+1)+\frac{\beta(\beta-1)}{x^2}-\frac{\beta}{x}-\frac{2\beta t}{x}+\frac{1}{4}\right),
\end{aligned}$$

whence

$$\begin{aligned}
(Df)(x) &= f(x)\Big\{t(t+1)+\frac{\beta(\beta-1)}{x^2}-\frac{\beta}{x}-\frac{2\beta t}{x}+\frac{1}{4}+\frac{(\alpha-\beta)\beta}{x^2}+\frac{\beta-\alpha}{2x}\\
&\qquad +\frac{(\beta-\alpha)t}{x}-\frac{\alpha-\beta}{x}\left(\frac{\beta}{x}-\frac{1}{2}\right)-\frac{1}{4}+\frac{\beta(1-\beta)}{x^2}\Big\}\\
&= f(x)\Big\{t(t+1)-\frac{\alpha t}{x}-\frac{\beta(1+t)}{x}\Big\}\\
&= \frac{\mathrm{d}}{\mathrm{dt}}\Big\{-e^{-\frac{1}{2}x}x^{\beta-1}e^{-xt}(1+t)^{\alpha}t^{\beta}\Big\}.
\end{aligned}$$

This proves the assertion.

We now show (ii). When $\mathrm{Re}(\beta)>0, \mathrm{Re}(\alpha+\beta)>1$ and $x\to 0$ we have, by Lemma 5.5.1 (i) and the dominated convergence theorem,

$$W(x;\alpha,\beta)=\frac{e^{-\frac{1}{2}x}}{\Gamma(\beta)}x^{1-\alpha}\int_0^{\infty} e^{-t}(x+t)^{\alpha-1}t^{\beta-1}\,\mathrm{dt}\sim\frac{\Gamma(\alpha+\beta-1)}{\Gamma(\beta)}x^{1-\alpha}.$$

We next show (iii). When $\mathrm{Re}(\beta)>0$ and $x\to\infty$ we have, by Lemma 5.5.1 (i) and the dominated convergence theorem,

$$W(x;\alpha,\beta) = \frac{e^{-\frac{1}{2}x}}{\Gamma(\beta)} \int_0^\infty e^{-t}\left(1+\frac{t}{x}\right)^{\alpha-1} t^{\beta-1}\,\mathrm{dt} \sim e^{-\frac{1}{2}x}.$$

This completes the proof. □

5.6 A Fourier transform calculation

The confluent hypergeometric functions are useful because they are Fourier transforms of complex powers.

Proposition 5.6.1. *Suppose that* $x_1, x_2 \in \mathbb{R}$ *with* $x_2 > 0$. *Suppose that* $\xi \in \mathbb{R}$. *Suppose further that* $\alpha, \beta \in \mathbb{C}$ *with* $\mathrm{Re}(\alpha+\beta) > 1$. *Then*

$$\int_{-\infty}^\infty \frac{\mathbf{e}(-\xi x_1)}{(x_2 - ix_1)^\alpha (x_2+ix_1)^\beta} \mathrm{dx}_1 = 2\pi\ (2x_2)^{1-\alpha-\beta} \begin{cases} \frac{e^{-2\pi\xi x_2}}{\Gamma(\alpha)\Gamma(\beta)}\,\upsilon(4\pi\xi x_2;\alpha,\beta) & \text{if } \xi > 0 \\ \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)} & \text{if } \xi = 0 \\ \frac{e^{-2\pi|\xi|x_2}}{\Gamma(\alpha)\Gamma(\beta)}\,\upsilon(4\pi|\xi|x_2;\beta,\alpha) & \text{if } \xi < 0. \end{cases} \tag{5.10}$$

Proof. This computation is due to Maass [18, pp. 208–209]. It is a standard fact that

$$\int_0^\infty e^{-zt} t^s \frac{\mathrm{dt}}{t} = \frac{\Gamma(s)}{z^s}$$

for $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(s) > 0$. First assume that $\mathrm{Re}(\alpha) > 0$, that $\mathrm{Re}(\beta) > 0$, and that $\mathrm{Re}(\alpha+\beta) > 1$. On writing

$$\begin{aligned} \frac{1}{(x_2 - ix_1)^\alpha} &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(x_2-ix_1)t} t^{\alpha-1}\,\mathrm{dt} \\ \frac{1}{(x_2 + ix_1)^\beta} &= \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-(x_2+ix_1)u} u^{\beta-1}\,\mathrm{du}, \end{aligned}$$

we find that

$$\frac{1}{(x_2 - ix_1)^\alpha (x_2+ix_1)^\beta} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^\infty \int_0^\infty e^{-x_2(t+u)} e^{ix_1(t-u)} t^{\alpha-1} u^{\beta-1}\,\mathrm{dt}\,\mathrm{du}.$$

A change of variables $t \mapsto \frac{1}{2}(u+t), u \mapsto \frac{1}{2}(u-t)$ yields

$$\frac{1}{(x_2 - ix_1)^\alpha (x_2+ix_1)^\beta} = \frac{2^{1-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_{-\infty}^\infty \int_{u>|t|} e^{-x_2 u}(u+t)^{\alpha-1}(u-t)^{\beta-1}\,\mathrm{du}\ e^{ix_1 t}\,\mathrm{dt}.$$

Thus the function

$$x_1 \mapsto \frac{1}{(x_2 - ix_1)^\alpha (x_2 + ix_1)^\beta}$$

is the Fourier transform of the function

$$t \mapsto \frac{2^{1-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_{u>|t|} e^{-x_2 u}(u+t)^{\alpha-1}(u-t)^{\beta-1}\,\mathrm{du}.$$

On applying Fourier inversion formula

$$g(\xi) = \int_{-\infty}^{\infty} f(x) e^{ix\xi}\mathrm{dx} \implies f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\xi) e^{-ix\xi}\mathrm{d}\xi,$$

we find that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\mathbf{e}(-\xi x_1)}{(x_2 - ix_1)^\alpha (x_2 + ix_1)^\beta}\,\mathrm{dx}_1 \\
&= \frac{2\pi \cdot 2^{1-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_{u>|2\pi\xi|} e^{-x_2 u}(u+2\pi\xi)^{\alpha-1}(u-2\pi\xi)^{\beta-1}\,\mathrm{du} \\
&= \frac{(2\pi)^{\alpha+\beta}\, 2^{1-\alpha-\beta}}{\Gamma(\alpha)\Gamma(\beta)} \int_{u>|\xi|} e^{-2\pi x_2 u}(u+\xi)^{\alpha-1}(u-\xi)^{\beta-1}\,\mathrm{du}.
\end{aligned}$$

Consider the latter integral. If $\xi = 0$, the integral is

$$\int_0^\infty e^{-2\pi x_2 u} u^{\alpha+\beta-2}\,\mathrm{du} = (2\pi x_2)^{1-\alpha-\beta}\Gamma(\alpha+\beta-1).$$

If $\xi > 0$, the integral is

$$\begin{aligned}
&\int_\xi^\infty e^{-2\pi x_2 u}(u+\xi)^{\alpha-1}(u-\xi)^{\beta-1}\,\mathrm{du} \\
&= e^{-2\pi\xi x_2}\int_0^\infty e^{-2\pi x_2 u}(u+2\xi)^{\alpha-1}u^{\beta-1}\,\mathrm{du} \\
&= e^{-2\pi\xi x_2}(2\pi x_2)^{1-\alpha-\beta}\int_0^\infty e^{-u}(u+4\pi\xi x_2)^{\alpha-1}u^{\beta-1}\,\mathrm{du} \\
&= e^{-2\pi\xi x_2}(2\pi x_2)^{1-\alpha-\beta}\ \upsilon(4\pi\xi x_2;\alpha,\beta).
\end{aligned}$$

If $\xi < 0$, the integral is

$$\int_{|\xi|}^\infty e^{-2\pi x_2 u}(u-|\xi|)^{\alpha-1}(u+|\xi|)^{\beta-1}\,\mathrm{du} = e^{-2\pi|\xi|x_2}(2\pi x_2)^{1-\alpha-\beta}\ \upsilon(4\pi|\xi|x_2;\beta,\alpha).$$

This concludes (5.10) for $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, and $\mathrm{Re}(\alpha+\beta) > 1$. But both sides of

(5.10) are analytic functions of α, β in the domain $\{(\alpha, \beta) \in \mathbb{C}^2 : \mathrm{Re}(\alpha+\beta) > 1\}$. Thus (5.10) holds for $\mathrm{Re}(\alpha + \beta) > 1$ by analytic continuation. The proof is complete. $\square$

5.7 Fourier expansion and analytic continuation of $Z(x, y; \alpha, \beta)$

We are in a position to deduce the Fourier development of the genaralized Lerch zeta function $Z(x, y; \alpha, \beta)$, thereby proving its analytic continuation.

Recall that $\mathbb{C}' = \mathbb{C} - \mathbb{R}_{\leq 0}$.

Lemma 5.7.1. *(Poisson summation formula) Suppose that $y \in \mathbb{R}$, that $f \in L^1(\mathbb{R})$ and that f is continuous on $\mathbb{R}$. Suppose further that*

$$\sum_{n=-\infty}^{\infty} f(n+x)\,\mathbf{e}((n+x)y)$$

converges absolutely and uniformly for $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, and that

$$\sum_{\xi=-\infty}^{\infty} |\hat{f}(\xi - y)| < \infty.$$

Then

$$\sum_{n=-\infty}^{\infty} f(n+x)\,\mathbf{e}(ny) = \sum_{\xi=-\infty}^{\infty} \hat{f}(\xi - y)\,\mathbf{e}((\xi - y)x).$$

Proof. Put $g(x) = \sum_{n=-\infty}^{\infty} f(n+x)\,\mathbf{e}((n+x)y)$, so g is a continuous function on $\mathbb{T}$. Its Fourier coefficient $\hat{g}(\xi)$, where $\xi \in \mathbb{Z}$, is

$$\begin{aligned}
\hat{g}(\xi) &= \int_0^1 g(x)\,\mathbf{e}(-\xi x)\,\mathrm{dx} \\
&= \int_0^1 \sum_{n=-\infty}^{\infty} f(n+x)\,\mathbf{e}((n+x)y)\,\mathbf{e}(-\xi x)\,\mathrm{dx} \\
&= \sum_{n=-\infty}^{\infty} \int_0^1 f(n+x)\,\mathbf{e}((n+x)y)\,\mathbf{e}(-\xi x)\,\mathrm{dx} \\
&= \int_{-\infty}^{\infty} f(x)\,\mathbf{e}((y-\xi)x)\,\mathrm{dx} \\
&= \hat{f}(\xi - y).
\end{aligned}$$

By hypothesis, the Fourier coefficients of g are absolutely summable. Hence the

Fourier series of g converges uniformly to g, namely

$$g(x) = \sum_{\xi=-\infty}^{\infty} \hat{g}(\xi)\,\mathbf{e}(\xi x).$$

Thus

$$\sum_{n=-\infty}^{\infty} f(n+x)\,\mathbf{e}(ny) = \sum_{\xi=-\infty}^{\infty} \hat{f}(\xi - y)\,\mathbf{e}((\xi - y)x).$$

The proof is complete. $\square$

Theorem 5.7.2. *Suppose that $x = x' + ix'' \in \mathbb{H}$ and that $y \in \mathbb{R}$. Suppose further that $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha + \beta) > 1$. Then*

$$Z(x, y; \alpha, \beta) = \mathbf{e}(-x'y) \sum_{n=-\infty}^{\infty} c_n\ \mathbf{e}(nx') \tag{5.11}$$

where the Fourier coefficients $c_n = c_n(x'', y; \alpha, \beta)$ are given by

$$c_n = 2\pi\, e^{\frac{1}{2}\pi i(\beta-\alpha)}\, (2x'')^{1-\alpha-\beta} \begin{cases} \frac{e^{-2\pi(n-y)x''}}{\Gamma(\alpha)\,\Gamma(\beta)}\ \upsilon(4\pi(n-y)x''; \alpha, \beta) & \text{if } n > y \\ \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\,\Gamma(\beta)} & \text{if } n = y \\ \frac{e^{-2\pi(y-n)x''}}{\Gamma(\alpha)\,\Gamma(\beta)}\ \upsilon(4\pi(y-n)x''; \beta, \alpha) & \text{if } n < y. \end{cases} \tag{5.12}$$

Proof. We begin the proof by writing

$$\begin{aligned} Z(x, y; \alpha, \beta) &= \sum_{n=-\infty}^{\infty} \frac{1}{(n+x)^{\alpha}(n+\overline{x})^{\beta}}\ \mathbf{e}(ny) \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{(n+x'+ix'')^{\alpha}(n+x'-ix'')^{\beta}}\ \mathbf{e}(ny) \\ &= e^{\frac{1}{2}\pi i(\beta-\alpha)} \sum_{n=-\infty}^{\infty} \frac{1}{(x''-i(n+x'))^{\alpha}(x''+i(n+x'))^{\beta}}\ \mathbf{e}(ny) \\ &= e^{\frac{1}{2}\pi i(\beta-\alpha)} \sum_{n=-\infty}^{\infty} g(x'+n)\ \mathbf{e}(ny) \end{aligned}$$

where $g(x') := \frac{1}{(x''-ix')^{\alpha}(x''+ix')^{\beta}}$; note that the third equality follows from Lemma

5.2.1. It follows from Proposition 5.6.1 that

$$\hat{g}(\xi) = 2\pi\ (2x'')^{1-\alpha-\beta} \begin{cases} \frac{e^{-2\pi\xi x''}}{\Gamma(\alpha)\Gamma(\beta)}\, \upsilon(4\pi\xi x''; \alpha, \beta) & \text{if } \xi > 0 \\ \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)} & \text{if } \xi = 0 \\ \frac{e^{-2\pi|\xi|x''}}{\Gamma(\alpha)\Gamma(\beta)}\, \upsilon(4\pi|\xi|x''; \beta, \alpha) & \text{if } \xi < 0. \end{cases}$$

We are in a position to apply Poisson summation formula as stated in Lemma 5.7.1. Lemma 5.2.3 shows that

$$\sum_{n=-\infty}^{\infty} g(n+x')\,\mathbf{e}((n+x')y)$$

converges absolutely and uniformly for $x' \in \mathbb{R}$. Proposition 5.4.3 (iv) shows that

$$\sum_{\xi=-\infty}^{\infty} |\hat{g}(\xi - y)| < \infty.$$

It now follows from Lemma 5.7.1 that

$$\sum_{n=-\infty}^{\infty} g(n+x')\,\mathbf{e}(ny) = \sum_{\xi=-\infty}^{\infty} \hat{g}(\xi - y)\mathbf{e}((\xi - y)x').$$

Thus

$$Z(x, y; \alpha, \beta) = e^{\frac{1}{2}\pi i(\beta-\alpha)} \sum_{\xi=-\infty}^{\infty} \hat{g}(\xi - y)\ \mathbf{e}(x'(\xi - y)).$$

This proves the theorem. □

Theorem 5.7.3. (i) *For fixed $x \in \mathbb{H}$ and fixed $y \in \mathbb{R} - \mathbb{Z}$, the function $Z(x, y; \alpha, \beta)$ extends to an analytic function for $\alpha \in \mathbb{C}, \beta \in \mathbb{C}$.*

(ii) *For fixed $x \in \mathbb{H}$ and fixed $y \in \mathbb{Z}$, the function*

$$\tilde{Z}(x, y; \alpha, \beta) = Z(x, y; \alpha, \beta) - 2\pi\, e^{\frac{1}{2}\pi i(\beta-\alpha)}\,(2x'')^{1-\alpha-\beta}\ \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}$$

extends to an analytic function for $\alpha \in \mathbb{C}, \beta \in \mathbb{C}$.

Proof. By Theorem 5.7.2, $Z(x, y; \alpha, \beta)$ admits a Fourier expansion given by (5.11).

Therefore

$$Z(x,y;\alpha,\beta) - \delta_{y\in\mathbb{Z}}\ 2\pi\, e^{\frac{1}{2}\pi i(\beta-\alpha)}\,(2x'')^{1-\alpha-\beta}\ \frac{\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)} \tag{5.13}$$

$$= \mathbf{e}(-x'y) \sum_{\substack{n=-\infty \\ n\neq y}}^{\infty} c_n\ \mathbf{e}(nx') \tag{5.14}$$

where $c_n = c_n(x'', y; \alpha, \beta)$ is given by (5.12). If $n \neq y$, then

$$c_n = 2\pi\, e^{\frac{1}{2}\pi i(\beta-\alpha)}\,(2x'')^{1-\alpha-\beta} \begin{cases} \frac{e^{-2\pi(n-y)x''}}{\Gamma(\alpha)\,\Gamma(\beta)}\ \upsilon(4\pi(n-y)x''; \alpha, \beta) & \text{if } n > y \\ \frac{e^{-2\pi(y-n)x''}}{\Gamma(\alpha)\,\Gamma(\beta)}\ \upsilon(4\pi(y-n)x''; \beta, \alpha) & \text{if } n < y. \end{cases}$$

In view of Lemma 5.4.1 (iv), the second member of (5.13) is an analytic function for $\alpha \in \mathbb{C}, \beta \in \mathbb{C}$. Moreover, Proposition 5.4.3 (iii) implies that the Fourier coefficients $c_n = c_n(x'', y, \alpha, \beta)$, which are given by (5.12), are of exponential decay as $|n| \to \infty$. This proves the theorem. □

5.8 Relation of $Z(x, y; \alpha, \beta)$ to the Lerch zeta function

The Lerch zeta function $\zeta(s, y, x)$ is given for $\mathrm{Re}(s) > 1$, $0 < x < 1$, $y \in \mathbb{R}$ as the Dirichlet series

$$\zeta(s,y,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}\, \mathbf{e}(ny).$$

The Lerch zeta function $\zeta(s, y, x)$ has two closely related variants given by the following

Definition 5.8.1. The (real-analytic) Lerch zeta function is defined by the series

$$\zeta_{\mathbb{R}}(s,y,x) = \sum_{n=0}^{\infty} \frac{1}{|n+x|^s}\, \mathbf{e}(ny). \tag{5.15}$$

Here $\frac{1}{|n+x|^s} = \exp(-s \log |n + x|)$ for the principal branch logarithm.

The series (5.15) converges on the domain

$$\mathcal{D} = \{x \in \mathbb{C}^+, y \in \mathbb{H}\}$$

to an entire function of $s \in \mathbb{C}$. For each fixed $s \in \mathbb{C}$, the function $\zeta_{\mathbb{R}}(s, y, x)$ is complex analytic in y and real analytic in x on the domain $\mathcal{D}$.

If $y \in \mathbb{R}$ and $x \in \mathbb{C}^+$, then the series (5.15) converges to an analytic function for $\mathrm{Re}(s) > 1$. For fixed s with $\mathrm{Re}(s) > 1$ and fixed $y \in \mathbb{R}$, the function $\zeta_{\mathbb{R}}(s, y, x)$ is real analytic in $x \in \mathbb{C}^+$.

Definition 5.8.2. The (complex-analytic) Lerch zeta function is defined by the series

$$\zeta_{\mathbb{C}}(s, y, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \mathbf{e}(ny). \tag{5.16}$$

Here $\log(n+x) = \log|n+x| + i\arg(n+x)$ with $-\pi \le \arg(n+x) < \pi$, and $\frac{1}{(n+x)^s} = \exp(-s\log(n+x))$.

The series (5.16) converges on the domain $\mathcal{D}$ to an entire function of $s \in \mathbb{C}$. For each fixed $s \in \mathbb{C}$, the function $\zeta_{\mathbb{C}}(s, y, x)$ is complex analytic in both x and y on the domain $\mathcal{D}$.

If $y \in \mathbb{R}$ and $x \in \mathbb{C}^+$, then the series (5.16) converges to an analytic function for $\mathrm{Re}(s) > 1$. For fixed s with $\mathrm{Re}(s) > 1$ and fixed $y \in \mathbb{R}$, the function $\zeta_{\mathbb{C}}(s, y, x)$ is complex analytic in $x \in \mathbb{C}^+$.

Remark 5.8.3. If $x, y \in \mathbb{R}$ with $0 < x < 1$, then

$$\zeta_{\mathbb{R}}(s, y, x) = \zeta_{\mathbb{C}}(s, y, x) = \zeta(s, y, x).$$

Recall

$$Z(x, y; \alpha, \beta) = \sum_{n=-\infty}^{\infty}{}' \frac{1}{(n+x)^{\alpha}(n+\overline{x})^{\beta}} \mathbf{e}(ny)$$

which, by Lemma (5.2.3), converges for $\mathrm{Re}(\alpha+\beta) > 1$ provided that $y \in \mathbb{R}$ and that $x \in \mathbb{C}^+$. In Theorem (5.7.3) we analytically continue it to $(\alpha, \beta) \in \mathbb{C}^2$. We now study how $Z(x, y; \alpha, \beta)$ relates to $\zeta(s, y, x)$, $\zeta_{\mathbb{R}}(s, y, x)$ and $\zeta_{\mathbb{C}}(s, y, x)$.

For $\mathrm{Re}(s) > 1, x \in \mathbb{H}, y \in \mathbb{R}$, we have

$$Z\left(x, y; \frac{s}{2}, \frac{s}{2}\right) = \sum_{n=-\infty}^{\infty} \frac{1}{|n+x|^s} \mathbf{e}(ny).$$

For $\mathrm{Re}(s) > 1, x \in \mathbb{H}, y \in \mathbb{R}$, we have

$$Z(x, y; s, 0) = \sum_{n=-\infty}^{\infty} \frac{1}{(n+x)^s} \mathbf{e}(ny)$$

Lemma 5.8.4. (i) *Suppose that* $\mathrm{Re}(s) > 1$*, that* $x \in \mathbb{H}^{\pm}$ *satisfies* $0 < \mathrm{Re}(x) < 1$*, and that* $y \in \mathbb{R}$*. Write* $x = x' + ix''$ *with* $x', x'' \in \mathbb{R}$*, so* $0 < x' < 1$*. Then*

$$Z(x, y; \frac{s}{2}, \frac{s}{2}) = \zeta_{\mathbb{R}}(s, y, x) + \mathbf{e}(-y)\,\zeta_{\mathbb{R}}(s, -y, 1 - x).$$

(ii) *Suppose that* $\mathrm{Re}(s) > 1$*, that* $x \in \mathbb{H}$ *satisfies* $0 < \mathrm{Re}(x) < 1$*, and that* $y \in \mathbb{R}$*. Write* $x = x' + ix''$ *with* $x', x'' \in \mathbb{R}$*, so* $0 < x' < 1$*. Then*

$$Z(x, y; s, 0) = \zeta_{\mathbb{C}}(s, y, x) + \mathbf{e}(-y)\, e^{-\pi i s}\,\zeta_{\mathbb{C}}(s, -y, 1 - x).$$

(iii) *Suppose that* $\mathrm{Re}(s) > 1$*, that* $x \in \mathbb{H}^{-}$ *satisfies* $0 < \mathrm{Re}(x) < 1$*, and that* $y \in \mathbb{R}$*. Write* $x = x' + ix''$ *with* $x', x'' \in \mathbb{R}$*, so* $0 < x' < 1$*. Then*

$$Z(x, y; s, 0) = \zeta_{\mathbb{C}}(s, y, x) + \mathbf{e}(-y)\, e^{\pi i s}\,\zeta_{\mathbb{C}}(s, -y, 1 - x).$$

Proof. We first show (i). If the hypotheses are satisfied, then $x \in \mathbb{H}^{\pm} \cap \mathbb{C}^{+}$ and so

$$\begin{aligned} Z\left(x, y; \frac{s}{2}, \frac{s}{2}\right) &= \sum_{n=-\infty}^{\infty} \frac{1}{|n + x|^s}\,\mathbf{e}(ny) \\ &= \sum_{n=0}^{\infty} \frac{1}{|n + x|^s}\,\mathbf{e}(ny) + \sum_{n=-\infty}^{-1} \frac{1}{|n + x|^s}\,\mathbf{e}(ny) \\ &= \zeta_{\mathbb{R}}(s, y, x) + \mathbf{e}(-y)\,\zeta_{\mathbb{R}}(s, -y, 1 - x). \end{aligned}$$

This proves (i).

We next show (ii). If the hypotheses are satisfied, then $x \in \mathbb{H} \cap \mathbb{C}^{+}$ and so

$$\begin{aligned} Z\left(x, y; s, 0\right) &= \sum_{n=-\infty}^{\infty} \frac{1}{(n + x)^s}\,\mathbf{e}(ny) \\ &= \sum_{n=0}^{\infty} \frac{1}{(n + x)^s}\,\mathbf{e}(ny) + \sum_{n=-\infty}^{-1} \frac{1}{(n + x)^s}\,\mathbf{e}(ny) \\ &= \zeta_{\mathbb{C}}(s, y, x) + \mathbf{e}(-y)\, e^{-\pi i s}\,\zeta_{\mathbb{C}}(s, -y, 1 - x). \end{aligned}$$

The last equality follows from Lemma 5.2.1 since

$$\begin{aligned}\sum_{n=-\infty}^{-1} \frac{1}{(n+x)^s}\, \mathbf{e}(ny) &= \mathbf{e}(-y) \sum_{m=0}^{\infty} \frac{1}{(-m-1+x)^s}\, \mathbf{e}(-my) \\ &= \mathbf{e}(-y)\, e^{-\pi i s} \sum_{m=0}^{\infty} \frac{1}{(m+1-x)^s}\, \mathbf{e}(-my).\end{aligned}$$

This proves (ii).

Finally we show (iii). If the hypotheses are satisfied, then $x \in \mathbb{H} \cap \mathbb{C}^{+}$ and so

$$\begin{aligned} Z\,(x, y; s, 0) &= \sum_{n=-\infty}^{\infty} \frac{1}{(n+x)^s}\, \mathbf{e}(ny) \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}\, \mathbf{e}(ny) + \sum_{n=-\infty}^{-1} \frac{1}{(n+x)^s}\, \mathbf{e}(ny) \\ &= \zeta_{\mathbb{C}}(s, y, x) + \mathbf{e}(-y)\, e^{\pi i s}\, \zeta_{\mathbb{C}}(s, -y, 1-x).\end{aligned}$$

The last equality follows from Lemma 5.2.1 since

$$\begin{aligned}\sum_{n=-\infty}^{-1} \frac{1}{(n+x)^s}\, \mathbf{e}(ny) &= \mathbf{e}(-y) \sum_{m=0}^{\infty} \frac{1}{(-m-1+x)^s}\, \mathbf{e}(-my) \\ &= \mathbf{e}(-y)\, e^{\pi i s} \sum_{m=0}^{\infty} \frac{1}{(m+1-x)^s}\, \mathbf{e}(-my).\end{aligned}$$

This proves (iii). ☐

We derive a relation between the limit behaviour of $Z\left(x, y; \frac{s}{2}, \frac{s}{2}\right)$ (resp. $Z(x, y; s, 0)$) to the Lerch zeta function $\zeta(s, y, x)$ where one fixes $y \in \mathbb{R}$ and lets x approach the segment $0 < x < 1$ from the upper or lower half-plane.

Proposition 5.8.5. (i) *Suppose that* $\mathrm{Re}(s) > 1$, *that* $x \in \mathbb{H}^{\pm}$ *satisfies* $0 < \mathrm{Re}(x) < 1$, *and that* $y \in \mathbb{R}$. *Write* $x = x' + ix''$ *with* $x', x'' \in \mathbb{R}$, *so* $0 < x' < 1$. *Then*

$$\lim_{x'' \to 0} Z(x' + ix'', y; \frac{s}{2}, \frac{s}{2}) = \zeta(s, y, x') + \mathbf{e}(-y)\, \zeta(s, -y, 1-x').$$

(ii) *Suppose that* $\mathrm{Re}(s) > 1$, *that* $x \in \mathbb{H}$ *satisfies* $0 < \mathrm{Re}(x) < 1$, *and that* $y \in \mathbb{R}$. *Write* $x = x' + ix''$ *with* $x', x'' \in \mathbb{R}$, *so* $0 < x' < 1$. *Then*

$$\lim_{x'' \to 0} Z(x' + ix'', y; s, 0) = \zeta(s, y, x') + \mathbf{e}(-y)\, e^{-\pi i s}\, \zeta(s, -y, 1-x').$$

(iii) *Suppose that* $\mathrm{Re}(s) > 1$*, that* $x \in \mathbb{H}^-$ *satisfies* $0 < \mathrm{Re}(x) < 1$*, and that* $y \in \mathbb{R}$*. Write* $x = x' + ix''$ *with* $x', x'' \in \mathbb{R}$*, so* $0 < x' < 1$*. Then*

$$\lim_{x'' \to 0} Z(x' + ix'', y; s, 0) = \zeta(s, y, x') + \mathbf{e}(-y)\, e^{\pi i s}\, \zeta(s, -y, 1 - x').$$

Proof. We first show (i). If the hypotheses are satisfied, then by Lemma 5.8.4

$$Z\left(x, y; \frac{s}{2}, \frac{s}{2}\right) = \zeta_{\mathbb{R}}(s, y, x) + \mathbf{e}(-y)\, \zeta_{\mathbb{R}}(s, -y, 1 - x)$$

For fixed s with $\mathrm{Re}(s) > 1$ and fixed $y \in \mathbb{R}$, the function $\zeta_{\mathbb{R}}(s, y, x)$ is real analytic in $x \in \mathbb{H} \cap \mathbb{C}^+$. On taking the limit as $x'' \to 0$

$$\lim_{x'' \to 0} Z(x' + ix'', y; \frac{s}{2}, \frac{s}{2}) = \zeta_{\mathbb{R}}(s, y, x') + \mathbf{e}(-y)\, \zeta_{\mathbb{R}}(s, -y, 1 - x').$$

This, in view of Remark 5.8.3, shows (i).

We next show (ii). If the hypotheses are satisfied, then by Lemma 5.8.4

$$Z\,(x, y; s, 0) = \zeta_{\mathbb{C}}(s, y, x) + \mathbf{e}(-y)\, e^{-\pi i s}\, \zeta_{\mathbb{C}}(s, -y, 1 - x)$$

For fixed s with $\mathrm{Re}(s) > 1$ and fixed $y \in \mathbb{R}$, the function $\zeta_{\mathbb{C}}(s, y, x)$ is complex analytic in $x \in \mathbb{H} \cap \mathbb{C}^+$. On taking the limit as $x'' \to 0$

$$\lim_{x'' \to 0} Z(x' + ix'', y; s, 0) = \zeta_{\mathbb{C}}(s, y, x') + \mathbf{e}(-y)\, e^{-\pi i s}\, \zeta_{\mathbb{C}}(s, -y, 1 - x').$$

This, in view of Remark 5.8.3, shows (ii).

Finally we show (iii). If the hypotheses are satisfied, then by Lemma 5.8.4

$$Z\,(x, y; s, 0) = \zeta_{\mathbb{C}}(s, y, x) + \mathbf{e}(-y)\, e^{\pi i s}\, \zeta_{\mathbb{C}}(s, -y, 1 - x)$$

For fixed s with $\mathrm{Re}(s) > 1$ and fixed $y \in \mathbb{R}$, the function $\zeta_{\mathbb{C}}(s, y, x)$ is complex analytic in $x \in \mathbb{H} \cap \mathbb{C}^+$. On taking the limit as $x'' \to 0$

$$\lim_{x'' \to 0} Z(x' + ix'', y; s, 0) = \zeta_{\mathbb{C}}(s, y, x') + \mathbf{e}(-y)\, e^{\pi i s}\, \zeta_{\mathbb{C}}(s, -y, 1 - x').$$

This, in view of Remark 5.8.3, shows (iii). $\square$

References

[1] T. M. Apostol, *On the Lerch zeta function*, Pacific J. Math. **1** (1951), 161–167.

[2] F. Bruhat, *Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes $\wp$-adiques*, Bull. Soc. Math. France **89** (1961), 43–75.

[3] D. Bump, *Automorphic Forms and Representations*, Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1998.

[4] P. Cartier, *Quantum Mechanical Commutation Relations and Theta Functions*, in: Algebraic Groups and Discontinuous Subgroups, Proc. Symp. Pure Math. Vol. 9, Amer. Math. Soc.: Providence, RI 1966.

[5] J. W. S. Cassels and A. Fröhlich, *Algebraic number theory*, Proceedings of the instructional conference held at the University of Sussex, Brighton, September 1–17, 1965. Academic Press Inc., London, 1986.

[6] S. Kudla, *Tate's thesis.* In *An introduction to the Langlands program (Jerusalem, 2001)*, Birkhäuser, Boston, 2003.

[7] J. C. Lagarias and W.-C. Winnie Li, *The Lerch zeta function* I. *Zeta integrals*, Forum Math. **24** (2012), no. 1, 1–48.

[8] J. C. Lagarias and W.-C. Winnie Li, *The Lerch zeta function* II. *Analytic continuation*, Forum Math. **24** (2012), no. 1, 49–84.

[9] J. C. Lagarias and W.-C. Winnie Li, *The Lerch zeta function* III. *Polylogarithms and special values*, preprint.

[10] S. Lang, *Algebraic number theory*, GTM, 110. Springer-Verlag, New York, 1994.

[11] A. Laurinčikas and R. Garunkštis, *The Lerch zeta function.* Kluwer Academic Publishers: Dordrecht 2002.

[12] M. Lerch, *Note sur la function* $\mathfrak{R}(w,x,s) = \sum_{k=0}^{\infty} \frac{e^{2k\pi ix}}{(w+k)^s}$, Acta Math. **11** (1887), no. 1–4, 19–24.

[13] R. Lipschitz, *Untersuchung einer aus vier Elementen gebildeten Reihe*, J. Reine Angew. **54** (1857), 313–328.

[14] R. Lipschitz, *Untersuchung der Eigenshaften einer Gattung von undenlichen Reihen*, J. Reine Angew. **105** (1889), 127–156.

[15] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **121** (1949), 141–183.

[16] H. Maass, *Automorphe Funktionen von mehreren Veränderlichen und Dirichletsche Reihen*, Abh. Math. Sem. U. Hamburg **16** (1949), 72–100.

[17] H. Maass, *On Siegel's modular functions*, Tata Lectures on Mathematics, 3. Tata Institute of Fundamental Research, Bombay, 1955.

[18] H. Maass, *Lectures on modular functions of one complex variable*, Tata Lectures on Mathematics, 29. Tata Institute of Fundamental Research, Bombay, 1964.

[19] H. Maass, *Siegel's modular forms and Dirichlet series*, Lecture Notes in Mathematics, 216. Springer-Verlag, Berlin, New York, 1971.

[20] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory.* I. *Classical theory*, Cambridge Studies in Advanced Mathematics, 97. Cambridge University Press, Cambridge, 2007.

[21] E. Oberhettinger, *Note on the Lerch zeta function*, Pacific J. Math. **6** (1956), 117–120.

[22] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, editors *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010.

[23] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark, editors *NIST Digital Library of Mathematical Functions*. Online companion to [22]: http://dlmf.nist.gov/

[24] D. Ramakrishnan and R. J. Valenza, *Fourier Analysis on Number Fields*, GTM, 186. Springer-Verlag, New York, 1999.

[25] J. P. Serre, *Algebraic groups and class fields*, GTM, 117. Springer-Verlag, New York, 1988.

[26] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Iwanami Shoten and Princeton University Press, 1971.

[27] G. Shimura, *Confluent hypergeometric functions on tube domains*, Math. Ann. **260** (1982), no. 3, 269–302.

[28] G. Shimura, *On Eisenstein series*, Duke J. Math. **50** (1983), 417–476.

[29] G. Shimura, *On the Eisenstein series of Hilbert modular groups*, Rev. Mat. Iberoamericana **1** (1985), no. 3, 1–42.

[30] G. Shimura, *Elementary Dirichlet series and modular forms*, Springer Monographs in Mathematics, Springer, New York, 2007.

[31] G. Shimura, *The critical values of certain Dirichlet series*, Doc. Math. **13** (2008), 775–794.

[32] G. Shimura, *The critical values of generalizations of the Hurwitz zeta function*, Doc. Math. **15** (2010), 489–506.

[33] J. T. Tate, *Fourier analysis in number fields and Hecke's zeta functions*, Ph.D. Thesis (1950), 60pp.

[34] J. T. Tate, *Residues of differentials on curves*, Ann. Sci. École Norm. Sup. **4** (1968), 149–159.

[35] J. T. Tate, *Local constants*, in: Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), 89–131. Academic Press, London, 1977.

[36] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Second edition. Cambridge University Press, Cambridge, 1966.

[37] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Fourth edition, 1927. Cambridge University Press, Cambridge, Reprint 1965.

[38] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 143–211.

[39] A. Weil, *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1–87.

[40] A. Weil, *Basic number theory*, Die Grundlehren der Mathematischen Wissenschaften, Band 144. Springer-Verlag, New York, 1974.

[41] A. Weil, *Fonction zêta et distributions*, Séminaire Bourbaki **9** (1995), no. 312, 523–531.

[42] A. Weil, *Elliptic functions according to Eisenstein and Kronecker*, Classics in Mathematics. Springer-Verlag, Berlin, 1999.