ON SYMPLECTIC INVARIANTS ASSOCIATED TO ZOLL MANIFOLDS

by

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PUSH START BUTTON

To 1069—the *cooler* first-year office.

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ABSTRACT

On Symplectic Invariants Associated to Zoll Manifolds

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In this thesis, we provide a partial classification for M. Audin's polarized symplectic manifolds, which are smooth symplectic manifolds endowed with a Morse-Bott function having only two critical values—a minimum, which is attained on a Lagrangian submanifold, and a maximum, which is attained on a symplectic submanifold of codimension 2. We provide examples via Lerman's symplectic cut construction in which the Lagrangian minima are the Zoll manifolds, i.e. Riemannian manifolds all of whose geodesics are simply closed and of the same period. Given a polarized symplectic manifold with some additional assumptions on the Morse-Bott function, we prove that the Lagrangian minimum must be Zoll and obtain a local equivalence of such manifolds on a neighborhood of the Lagrangian. We then extend the equivalence out towards the symplectic maximum using gradient flows. The main tools we use are arguments in symplectic geometry and Morse(-Bott) theory. In low dimensions, we make use of a result due to McDuff and Lalonde to classify the 4-dimensional polarized symplectic manifolds up to symplectomorphism.

CHAPTER I

Introduction

1.1 A Summary of What To Expect

Symplectic geometry and topology came into being in the 18th century as the language of classical mechanics. Since then, the field has picked up considerably thanks to the work of Gromov, Arnold, Maslov, Moser, Floer, McDuff, and many other great mathematicians. The area has seen remarkable progress over the past few decades with the development of methods in J-holomorphic curves, Symplectic Floer Theory, etc.

This thesis aims to further our understanding of the polarized symplectic manifolds introduced by Audin in [2] and [3], which are smooth manifolds together with a type of Morse-Bott function having only two critical values, imposing topological constraints on the possibilities for the manifold and its critical submanifolds. These manifolds appear as special cases of the Lagrangian barriers investigated by Biran in [7]. In the aforementioned papers, Audin provides examples (via Lerman's symplectic cut construction) in which the Lagrangian skeletons appearing are none other than Riemannian manifolds all of whose geodesics are simply closed and of the same period—that is, Zoll manifolds. (Victor Guillemin was reportedly glad to hear that Zoll manifolds were "still alive and well.")

We begin in Chapter 2 with a review of relevant definitions and theorems from symplectic geometry and provide an brief introduction to Zoll manifolds. Readers with previous exposure to these areas should be able to skip this chapter. Those unfamiliar with symplectic geometry are encouraged to consult Ana Cannas da Silva's lecture notes [11] for an accessible, more comprehensive overview of this exciting area of mathematics. For more information on Zoll manifolds, the book by Besse [6] serves as a standard reference. A brief review of Morse-Bott theory is also included at the end of this chapter in the hopes that this thesis will be mostly self-contained.

We also take the time to define and review some geometric properties of the compact rank one symmetric spaces (CROSSes), in particular the projective spaces $\mathbb{KP}^n$, where $\mathbb{K}$ is one of the normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$, or $\mathbb{O}$ (the real numbers, the complex numbers, the quaternions, and the octonions). Although the octonions are a non-associative algebra, they are 2-associative (any two elements generate an associative subalgebra), which allows for just enough associativity to define the octonionic projective line and plane. The expository article by Baez [4] is an excellent introduction to the octonions and the exciting role they play in algebra, geometry, topology, and physics.

The main objects of study, polarized symplectic manifolds, are defined in Chapter 3, and some known examples are given, including those resulting as symplectic cuts via what we call the "Zoll cut construction," introduced by Audin in [3]. The CROSSes make a reappearance in this chapter as Lagrangian critical submanifolds in these examples. This immediately raises the question "Do all polarized symplectic manifolds contain a CROSS as the minimal submanifold?" There are partial results in this direction, some of which are due to Audin, others of which to the present author in the subsequent chapters.

The heart of this thesis is contained in Chapter 4, which provides a partial classification for Audin's polarized symplectic manifolds, at least under certain (restrictive) assumptions of periodicity on the flow associated to the square root of the Morse-Bott function and derivatives of the Morse-Bott function itself. To this end, we consider an oriented blow-up of the original manifold along the Lagrangian submanifold, thus introducing degeneracies to the inherited "symplectic" form. Calculations are performed on the blown-up space, where we identify two S^1-actions (one induced from the given Morse-Bott function, the other from its Hessian) via "conjugation" and deduce a sort of equivariant Moser-type result. In this way, we are able to show that associated to each Zoll manifold is one of these polarized symplectic manifolds resulting from the symplectic cut of the cotangent bundle using the geodesic flow, unique up to equivariant symplectomorphism, so we can think of the "Zoll cut" as a symplectic invariant. We end the chapter by introducing some new terminology for polarized symplectic manifolds with a geometrically natural discrete symmetry.

To complete the picture of these polarized symplectic manifolds, Chapter 5 begins by discussing the local picture nearby the symplectic submanifold V and results that are already known about it. We also include a brief discussion on the manifold of geodesics of Zoll spheres in low dimensions ($n = 3$ and 4). One can imagine that a similar local classification to the one provided in Chapter 4 near L can be obtained by analyzing the symplectic structure near the symplectic maximum V, where by the symplectic neighborhood theorem one has that a neighborhood of V is determined by the isomorphism class of its normal bundle N_V in W. In this case, one wonders what the analogue of the symplectic cut construction would be, since to recover W from the normal bundle of V, one would have to "compactify" it with an n-dimensional Lagrangian.

One can also ask how the two pictures fit together—a neighborhood of L looks like a neighborhood of the zero section of $T^\star L$, a neighborhood of V is also standard and looks like a neighborhood of the zero section of N_V in W, and the total space of the sphere bundles agree. Heuristically, we would like to find some way to glue the two neighborhoods together symplectically along the sphere bundles to recover the original manifold. Those familiar with Milnor's construction of exotic 7-spheres [24] might ask if such a gluing construction could result in some unexpected differentiable structure on W. In Chapter 5, we show that this is not the case, at least under the additional assumptions we have imposed on the Morse-Bott function. We achieve this by performing an "inverse cut" of W along V, resulting in a manifold with boundary that is diffeomorphic to a closed disk subbundle of the cotangent bundle $T^\star L$. We then extend the local equivalence from Chapter 4 out to the rest of the manifold and show that the resulting symplectic cuts are S^1-equivariantly equivalent.

At least in lower dimensions, there is still some room for some simple calculations involving the intersection of classes of certain symplectically embedded 2-spheres, thanks to a result of McDuff-Lalonde [17]. Chapter 6 recalls the results of their paper and includes a new (to the author's knowledge) result improving on Audin's classification of polarized symplectic manifolds in the case where the Lagrangian is a surface. Other lower dimensional cases are briefly explored in this chapter as well.

1.2 Future Directions

Despite the length of this thesis, many questions remain unanswered. The conditions we imposed on the Morse-Bott function are implied quite naturally by the Zoll cut construction, and it would be nice to understand why there is some insistence on evenness as noted in Section 4.5. One also wonders what sorts of classification results can be obtained from dropping the periodicity assumption. For example, there is still the question of whether any polarized symplectic manifold is obtained from a Zoll cut and if one can obtain a Finsler metric on L from the Morse-Bott function (or even a Riemannian metric, under the assumption that the polarized symplectic manifold is even).

We hope to explore these new directions ourselves in the near future or that other interested parties will take over where this thesis has left off. For now, the author is content to share her current findings in the present text.

CHAPTER II

Background Definitions and Results

Unless noted explicitly otherwise, all manifolds are assumed to be smooth and without boundary (but not necessarily compact).

2.1 Basic Results from Symplectic Geometry

The purpose of this chapter is to remind the reader of some of the relevant definitions and theorems from symplectic geometry. It is by no means a substitute for a course in the subject. Proofs are often omitted, but most can be found in introductory texts. The author recommends [11] for a quick introduction.

Definition II.1. A *symplectic manifold* (W, ω) is a smooth manifold endowed with a closed, non-degenerate 2-form ω.

Remark II.2. Non-degeneracy of ω is an algebraic condition forcing the n-th exterior power ω^n of the symplectic form to give rise to a *volume form* on W. Thus, (W, ω) comes with an orientation. Without loss of generality, we assume that $\int_W \omega^n > 0$. The quantity $\int_W \frac{\omega^n}{n!}$ will be referred to as the *symplectic volume* of (W, ω).

Remark II.3. Closedness gives a differential equation $d\omega$ that forces symplectic manifolds of equal dimension to be locally indistinguishable (see: Theorem II.26 below). This can be thought of as an analytical condition.

Definition II.4. A vector field X is *symplectic* if $\mathcal{L}_X\omega = 0$, which is equivalent to $i_X\omega$ being closed. X is said to be a *Hamiltonian* vector field if, morever, $i_X\omega = dH$ is exact, and H is called a Hamiltonian function for X. (Some authors prefer to use the sign convention $i_X\omega = -dH$.)

One can also think of a symplectic form as a way to assign vector fields X_H on a manifold M to smooth functions $H : M \to \mathbb{R}$. Non-degeneracy of ω guarantees that one can solve the equation $i_{X_H}\omega = dH$ for a well-defined X_H.

Example II.5. The most basic example of a symplectic manifold is the vector space $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and symplectic form given by

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j = -d\alpha,$$

where $\alpha = \sum_{j=1}^{n} y_j dx_j$ is referred to as the *Liouville 1-form*. (Some authors may choose to define ω_0 with the opposite sign.) Note in particular that in this example, the symplectic form is not only closed but also exact.

In a sense, every symplectic manifold looks like $(\mathbb{R}^{2n}, \omega_0)$ locally (see: Darboux Theorem below).

Example II.6. Endow $\mathbb{C}^n$ with the form $\omega_0 = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j$. Letting $z_j = x_j + iy_j$, one easily verifies that this example is equivalent to the previous one.

Example II.7. Consider the complex projective space $\mathbb{CP}^n$ with homogeneous coordinates $\{[z_0 : z_1 : \ldots : z_n] \,|\, z_j \text{ not all zero}\}$. Set $U_j = \{z_j \neq 0\} \subset \mathbb{CP}^n$. The maps $\phi_j : U_j \to \mathbb{C}^n$ given by $\phi_j([z_0 : z_1 : \ldots : z_n]) = (\frac{z_0}{z_j}, \frac{z_1}{z_j}, \ldots, \overline{1}, \ldots, \frac{z_n}{z_j})$ (the j-th coordinate is omitted) gives coordinates on the patch U_j. The *Fubini-Study form* ω_{FS} is given by the pullbacks $\phi_j^\star\omega_0$ of the standard form on $\mathbb{C}^n$. One checks that ω_{FS} is well-defined on overlaps $U_j \cap U_k$

The next basic example of a symplectic manifold, which happens to be non-compact, is the cotangent bundle of a differentiable manifold with a canonically defined symplectic form.

Example II.8. *(Cotangent bundles)* Consider a manifold L and the total space of its cotangent bundle $T^\star L$, where the projection is denoted $\pi : T^\star L \to L$. There is a canonical 1-form, commonly denoted α and called the *Liouville 1-form*, given at the point $p = (x, \xi) \in T^\star L$ by

$$\alpha_p = (d\pi_p)^\star \, \xi \in T_p^\star T^\star L,$$

so that $\alpha_p(v) = \xi((d\pi_p)v)$ for $v \in T_p T^\star L$. Then taking $\omega = -d\alpha$ gives a symplectic form.

One can also obtain a description of α and ω in terms of coordinates. Consider a coordinate chart $(U, x_1, \ldots, x_n)$ on the manifold L, yielding associated coordinates on the cotangent bundle $(T^\star U, x_1, \ldots, x_n, y_1, \ldots, y_n)$. Then

$$\alpha = \sum_{j=1}^{n} y_j dx_j$$

and

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j = -d\alpha.$$

Note that the previous example of $(\mathbb{R}^{2n}, \omega_0)$ is really just the case where $L = \mathbb{R}^n$ and $T^\star \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$.

Example II.9. It may be instructive to consider non-examples of symplectic manifolds. As noted in Remark II.2, the n-th exterior power ω^n gives rise to a volume form on a $2n$-dimensional symplectic manifold M, so that if M is compact, the cohomology class $[\omega] \in H^2(M; \mathbb{R})$ (and its powers $[\omega]^k \in H^{2k}(M; \mathbb{R})$, for $0 \leq k \leq n$) must be non-zero. Thus, compact manifolds with trivial even cohomology cannot be symplectic.

In particular, the even-dimensional spheres S^{2n} for $n > 1$ cannot be symplectic. (Nor can the odd-dimensional ones be, for obvious reasons.)

Example II.10. *(Coadjoint orbits)* Many examples of symplectic manifolds can be constructed as orbits of the coadjoint action of a Lie group G on its Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. Recalling that G acts on $\mathfrak{g}$ by the adjoint action, the fundamental vector field associated to $X \in \mathfrak{g}$ is given by

$$^{\mathfrak{g}}\underline{X}_Y = [X, Y].$$

Transposing the adjoint G-action on $\mathfrak{g}$ gives the coadjoint action on $\mathfrak{g}^\star$. For $\xi \in \mathfrak{g}^\star$ and $X \in \mathfrak{g}$,

$$\langle \mathrm{Ad}^\star \xi, X \rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} X \rangle.$$

Then the fundamental vector field on $\mathfrak{g}^\star$ associated with $X \in \mathfrak{g}$ is then given by

$$\langle {}^{\mathfrak{g}^\star}\underline{X}_\xi, Y \rangle = \langle \xi, [Y, X] \rangle.$$

We can define, for any $\xi \in \mathfrak{g}^\star$ a skew-symmetric bilinear form by

$$\omega_\xi(X, Y) = \langle \xi, [X, Y] \rangle,$$

the kernel of which is g_ξ, the stabilizer of ξ for the coadjoint representation. Then ω_ξ is non-degenerate on $\mathfrak{g}/\mathfrak{g}_\xi$, a vector space that can be identified with $T_\xi(G \cdot \xi) \subseteq \mathfrak{g}^\star$. This yields a canonical symplectic structure on each coadjoint orbit. (For details, see the introductory text by Audin [1], for example.)

In this way, the complex projective spaces can also be realized as symplectic manifolds as orbits of the coadjoint action of the group $U(n)$ on the real Lie algebra $\mathfrak{u}(n)^\star$, identified with the space $\mathcal{H} = i\mathfrak{u}(n)$ Hermitian matrices.

It turns out that we can always endow (M,ω) with an *almost complex structure* J, i.e. a smooth field of structures on the tangent space $J_p : T_pM \to T_pM$ $(p \in M)$ such that $J_p^2 = -\mathrm{Id}$.

Definition II.11. Let (M,ω) be a symplectic manifold and let J be an almost complex structure on M. Then J is called *ω-compatible* if for all $p \in M$, $\omega_p(u,v) := \omega_p(u, J_p v)$ defines a Riemannian metric on M.

If the almost complex structure J is integrable (meaning that it is induced by a complex structure on M) and ω-compatible, then M is a *Kähler manifold* and ω is called a *Kähler form*.

Often one is interested in special submanifolds of a symplectic manifold (M^{2n},ω).

Definition II.12.

1. A submanifold $P \subset M$ is *isotropic* if the symplectic form vanishes identically on it, i.e. $i^\star\omega = 0$, where $i : P \hookrightarrow M$ is the inclusion map.
2. A submanifold $L \subset M$ is *Lagrangian* if it is isotropic of maximal dimension, i.e. $i^\star\omega = 0$ and $\dim(L) = n$.
3. A submanifold $Q \subset M$ is *coisotropic* if it contains its symplectic complement $Q^\omega = \{p \in M \,|\, \omega_p(v,w) = 0 \,\forall\, v \in T_pQ, w \in T_pM\}$.
4. A submanifold $N \subset M$ is *symplectic* if ω restricts to a symplectic form on N, i.e. $i^\star\omega$ is non-degenerate for $i : N \hookrightarrow M$ the inclusion map.

Example II.13. *(Examples of special submanifolds of a symplectic manifold)* Consider $(M,\omega) = (\mathbb{R}^{2n},\omega)$ with the standard basis given by $(e_1,\dots,e_n,f_1,\dots,f_n)$ and symplectic form given by $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$, $\omega_0(e_i,e_j) = 0 = \omega_0(f_i,f_j), \omega_0(e_i,f_j) = \delta_{ij}$.

1. $\mathrm{Span}\{(e_1,0,\dots,0)\}$ an isotropic subspace (submanifold).
2. $\mathrm{Span}\{(e_1,\dots,e_n,0,\dots,0)\}$ a Lagrangian subspace.

3. $\mathrm{Span}\{(e_1, \ldots, e_n, f_1, 0, \ldots, 0)\}$ is coisotropic.

4. $\mathrm{Span}\{(e_1, 0, \ldots, 0, f_1, 0, \ldots, 0)\}$ is symplectic.

We can consider several notions of equivalence in the symplectic category.

Definition II.14. A diffeomorphism $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ between two symplectic manifolds is a *symplectomorphism* if $\phi^\star \omega_2 = \omega_1$. We denote the group of symplectomorphisms from (M, ω) to itself by $\mathrm{Symp}(M, \omega) \subset \mathrm{Diff}(M)$.

In general, requiring that $[\omega_0] = [\omega_1]$ does not imply that the two given symplectic structures are symplectomorphic. To investigate this question further, there are other notions of equivalence.

Definition II.15. (Notions of equivalence of symplectic forms)

1. Two symplectic forms ω_0, ω_1 on a manifold M are *deformation equivalent* if they can be joined by a smooth family $\omega_t, 0 \leq t \leq 1$, of symplectic forms.

2. ω_0 and ω_1 are *isotopic* if there is a 1-parameter family ω_t joining ω_0 to ω_1 such that the de Rham cohomology class $[\omega_t] \in H^2(M; \mathbb{R})$ is constant.

3. ω_0 and ω_1 are *strongly isotopic* if there is a smooth homotopy (isotopy) $\phi_t : M \to M, 0 \leq t \leq 1$, beginning at the identity, such that $\phi_1^\star \omega_1 = \omega_0$.

Remark II.16. There exist examples starting in dimension 6 of cohomologous symplectic forms that are non-deformation equivalent, as well as examples of deformation equivalent forms which are non-isotopic. [21]

One sees that by definition, two isotopic forms are automatically deformation equivalent, while being strongly isotopic implies being symplectomorphic, as well as isotopic by the homotopy invariance of de Rham cohomology (take $\omega_t = \phi_t^\star \omega_0$).

It turns out that on *compact* manifolds, being isotopic implies being strongly isotopic. The proof is due to Moser [25] and uses what is called *Moser's trick* or *Moser's method*, a technique that appears frequently in symplectic geometry.

Theorem II.17. ***(Moser)*** *Let M be a compact manifold, and suppose ω_0 and ω_1 are isotopic through a family of forms ω_t, where $[\omega_t] = [\omega_0]$ is independent of t. Then ω_0 and ω_1 are strongly isotopic.*

Proof. By assumption, $[\frac{d\omega_t}{dt}] = [\dot{\omega}_t] = 0$. Then $\dot{\omega}_t = -d\alpha_t$, for a smooth family of 1-forms α_t that can be chosen by Poincaré's lemma. By non-degeneracy, there exists an associated time-dependent vector field X_t such that $i_{X_t}\omega_t = \alpha_t$. By compactness of M, there is a well-defined flow ϕ_t of X_t, where

$$X_t = \frac{d\phi_t}{dt} \circ \phi_t^{-1}.$$

By Cartan's formula for the Lie derivative, one has

$$\frac{d}{dt}\left(\phi_t^\star \omega_t\right) = \phi_t^\star \left(\mathcal{L}_{X_t}\omega_t\right) + \phi_t^\star \dot{\omega}_t = \phi_t^\star \left(d\imath_{X_t}\omega_t + \dot{\omega}_t\right) = \phi_t^\star \left(d\alpha_t - d\alpha_t\right) = 0$$

Since ϕ_0 is the identity and ϕ_t gives the desired isotopy satisfying $\phi_1^\star \omega_1 = \omega_0$. □

2.1.1 Symplectic Vector Bundles

Definition II.18. A *symplectic vector space* is a (finite-dimensional, real) vector space V together with a bilinear form ω satisfying the following two properties:

- (skew-symmetry) for any vectors $u, v \in V$, $\omega(u, v) = -\omega(v, u)$, and
- (non-degeneracy) for any $u \in V$, $\omega(u, v) = 0 \, \forall \, v \in V \Rightarrow u = 0$.

Remark II.19. As before, non-degeneracy of ω forces V to be even-dimensional. If $\dim(V) = 2n$, then non-degeneracy of the skew-symmetric bilinear form ω is equivalent to the condition $\omega^n = \omega \wedge \ldots \wedge \omega \neq 0$.

Example II.20. Euclidean space $V = \mathbb{R}^{2n}$ together with $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$ from before gives a symplectic vector space. Explicitly, if $u = (x_1, \ldots, x_n, y_1, \ldots, y_n)^T$ and $v = (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n)^T$, then

$$\omega_0(u, v) = \sum_{i=1}^{n} (x_i y'_i - x'_i y_i) = -u^T J_0 v,$$

where $J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ and I_n denotes the $n \times n$ identity matrix.

Definition II.21. A *linear symplectomorphism* $\psi : V \to V$ of a symplectic vector space (V, ω) is a vector space isomorphism preserving the symplectic form, i.e.

$$\omega(\psi u, \psi v) = \omega(u, v), \quad \forall u, v \in V.$$

The group of linear symplectomorphisms of (V, ω) is denoted by $\mathrm{Sp}(V, \omega)$. When $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$, this group is simply denoted by $\mathrm{Sp}(2n)$ and can be identified with the group of $2n \times 2n$ symplectic matrices Ψ satisfying $\Psi^T J_0 \Psi = J_0$.

It turns out that up to linear symplectomorphism, the standard Euclidean case is the only example of a symplectic vector space. The proof is not difficult and is covered in any basic course in symplectic geometry, so we omit it here.

Theorem II.22. *Let (V, ω) be a symplectic vector space of dimension $2n$. Then there exists a vector space isomorphism $\psi : \mathbb{R}^{2n} \to V$ satisfying*

$$\omega_0(u, v) = \omega(\psi u, \psi v), \qquad \forall u, v \in \mathbb{R}^{2n}.$$

Thus, $Sp(V, \omega) \simeq Sp(2n)$.

Definition II.23. A *symplectic vector bundle* $\pi : E \to M$ is a real vector bundle over a smooth manifold M together with a smooth section ω of $E^\star \wedge E^\star$ such that for each point $p \in M$, (E_p, ω_p) is a symplectic vector space. ω is called a *symplectic bilinear form* on E.

Two symplectic vector bundles (E_1, ω_1) and (E_2, ω_2) are *isomorphic* if there is a vector bundle isomorphism $\phi : E_1 \to E_2$ and such that $\phi^\star \omega_2 = \omega_1$.

The usual constructions (pullback, restriction to submanifolds of M, direct sum, etc.) used in bundle theory work equally well for symplectic vector bundles. A symplectic subbundle is a subbundle F of E such that for $p \in M$, $(F_p, \omega_p|_{F_p})$ is a symplectic vector space. Its symplectic complement F^ω is a real vector bundle isomorphic to the quotient bundle E/F and is given by

$$F^\omega = \cup_{p \in M} \{u \in E_p \,|\, \omega_p(u, v) = 0, \qquad \forall v \in F_p\}.$$

Example II.24. (*The tangent bundle of a symplectic manifold*) For any symplectic manifold (M, ω), ω is a smooth section of $T^\star M \wedge T^\star M$. Being non-degenerate, it gives the tangent bundle (TM, ω) the structure of a symplectic vector bundle.

Example II.25. (*The normal bundle of a symplectic submanifold*) For a symplectic submanifold $N \subset (M, \omega)$, TN is a symplectic subbundle of $(TM|_N, \omega|_N)$. The normal bundle $\nu_N \equiv TM|_N / TN$ of N in M is also a symplectic subbundle of $(TM|_N, \omega|_N)$ and we have

$$TM|_N = TN \oplus \nu_N.$$

It turns out that since $\mathrm{Sp}(2n)$ and $GL(n, \mathbb{C})$ both retract onto the maximal compact subgroup $U(n)$, a symplectic vector bundle can always be given the structure of a complex vector bundle (and vice versa). Thus, it makes sense to talk about the Chern classes of a symplectic vector bundle.

2.1.2 Local Theorems

In this section, we remind the reader of various neighborhood theorems that are useful in symplectic geometry.

The Darboux theorem gives a local picture for symplectic manifolds nearby a point. In particular, the only local invariant is dimension, in stark contrast to the case of Riemannian geometry (where there are local invariants like curvature).

Theorem II.26. ***(Darboux Theorem)*** *Let (W, ω) be a symplectic manifold of real dimension $2n$. Then for every point $p \in W$, there exist a neighborhood $\mathcal{U}$ and a coordinate chart $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ centered at p such that ω can be written as*

$$\omega = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

The theorem below gives that a submanifold which is Lagrangian with respect to two different symplectic forms has neighborhoods on which the forms are symplectomorphic.

Theorem II.27. ***(Weinstein Lagrangian Neighborhood Theorem)*** *Suppose W is a $2n$-dimensional manifold and X is a compact n-dimensional submanifold. Let $i : L \hookrightarrow W$ denote the inclusion map, and consider two symplectic forms ω_0, ω_1 on W with $i^*\omega_0 = i^*\omega_1 = 0$. Then there exist neighborhoods $\mathcal{U}_0, \mathcal{U}_1$ of L in W and a symplectomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that φ restricts to the identity on L and $\varphi^*\omega_1 = \omega_0$.*

The next theorem is a standard result and provides us with a picture for how a symplectic manifold looks locally nearby a Lagrangian submanifold. It makes apparent the importance of the example of the cotangent bundle with its canonical symplectic form.

Theorem II.28. ***(Weinstein Tubular Neighborhood Theorem)*** *Let L be a compact Lagrangian submanifold of a symplectic manifold (W, ω). Let i denote the inclusion $i : L \hookrightarrow W$ and $i_0 : L \hookrightarrow T^*L$ the Lagrangian embedding of L as the zero section. Then there exist neighborhoods $\mathcal{U}_0$ of L in T^*L and $\mathcal{U}_1$ of L in W and a symplectomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that $\varphi^*\omega = \omega_0$ and $\varphi \circ i_0 = i$, where ω_0 is the Liouville form.*

One also has a standard local picture around *symplectic* submanifolds in terms of normal bundles. In particular, the neighborhood of a symplectic submanifold is determined by the isomorphism class of its symplectic normal bundle.

Theorem II.29. ***(Symplectic Neighborhood Theorem)*** *Let $(W_j, \omega_j), j = 1, 2,$ be symplectic manifolds with a compact symplectic manifold Q_j. Suppose further that there exists an isomorphism $\Phi : (\nu_{Q_1}, \omega_1) \to (\nu_{Q_2}, \omega_2)$ of normal bundles covering a symplectomorphism $\phi : (Q_1, \omega_1) \to (Q_2, \omega_2)$. Then there exist neighborhoods $N_1 \supseteq Q_1$ and $N_2 \supseteq Q_2$ and a symplectomorphism $\Psi : (N_1, \omega_1) \to (N_2, \omega_2)$ extending ϕ such that $d\Psi = \Phi$ on ν_{Q_1}.*

Similar results about neighborhoods of isotropic and coisotropic submanifolds exist, but we do not use them in the following chapters and so omit their statements. Equivariant versions of the preceding neighborhood theorems also exist.

2.2 Symplectic and Hamiltonian Actions and Moment Maps

Let the Lie group G act on a symplectic manifold (M, ω) so that we have a map

$$\psi : G \to \mathrm{Diff}(M)$$

$$g \mapsto \psi_g.$$

If in fact $\psi(G) \subset \mathrm{Symp}(M, \omega)$, then the action is said to be *symplectic*. For fixed g, let X be the derivative of ψ_g at the identity element e. Denote the *fundamental vector field* associated to X by $X^\#$, which is the vector field on M generated by the one-parameter subgroup $\{\exp(tX) \,|\, t \in \mathbb{R}\} \subset M$. The action of g on M being symplectic is equivalent to $\mathcal{L}_{X^\#}\omega = di_{X^\#}\omega = 0$. Here we used Cartan's formula for the Lie derivative of differential forms

$$\mathcal{L}_Y\omega = di_Y\omega + i_Y d\omega$$

along with the closedness of ω.

The action is said to be *Hamiltonian* if, furthermore, $i_{X^\#}\omega = dH$ for some function H, unique up to an additive constant.

Let $\mathfrak{g} = T_eG$ be the Lie algebra $\mathrm{Lie}(G)$ and let $\mathfrak{g}^\star$ its dual vector space. Suppose there is a map $\mu : M \to \mathfrak{g}^\star$ satisfying the following conditions:

1. For every $X \in \mathfrak{g}$, let μ^X be given by $\mu^X(p) = \langle \mu(p), X\rangle$. Then $d\mu^X = i_{X^\#}\omega$. That is, μ^X is a Hamiltonian function for the fundamental vector field $X^\#$.

2. Equivariance intertwining the given action and coadjoint action on $g^\star$. That is, for every $g \in G$, $\mu \circ \psi_g = Ad_g^\star \circ \mu$.

Then we call (M, ω, G, μ) a *Hamiltonian G-space* with moment map μ. We will mostly be concerned with the case where $G = S^1$, in which equivariance will just mean invariance. (As G is abelian in this case, the coadjoint action is trivial.)

Example II.30. The Lie group $U(n)$ acts on $(\mathbb{C}^n, \omega_0)$ in the usual way. Identify $\mathfrak{u}(n)$ with its dual $\mathfrak{u}(n)^\star$ via the inner product $\langle A, B\rangle = Tr(A^\star B)$. Then the moment map $\mu : \mathbb{C}^n \to \mathfrak{u}(n) \simeq \mathfrak{u}(n)^\star$ is given by

$$\mu(z) = \frac{i}{2}zz^\star.$$

2.3 Some Definitions from Morse(-Bott) Theory

We include here a quick review of definitions from Morse-Bott theory. Knowing that a function on a manifold M is Morse-Bott provides information about the topology of M. For more information about Morse theory, the author recommends the exposition by Milnor [23] and the survey by Bott [9].

Definition II.31. A smooth function $f : M \to \mathbb{R}$ is a *Morse function* if all of its critical points are non-degenerate. That is, if $df_a = 0$ at a point $a \in M$, then the matrix of second partial derivatives $\text{Hess}(f)$ is non-singular.

Lemma II.32. ***(Morse Lemma)*** Let $a \in \text{Crit}(f) \subset M$ be a critical point of a Morse function f. Then there exists a coordinate chart $(x_1, \ldots, x_n)$ valid on a neighborhood U of a such that $x(a) = 0$ and f takes the form

$$f(x_1, \ldots, x_n) = f(a) - x_1^2 - \ldots - x_\alpha^2 + x_{\alpha+1}^2 + \ldots + x_n^2.$$

Definition II.33. The number α is called the *index* of the critical point a.

From the Morse lemma, one easily concludes the following:

Corollary II.34. *Non-degenerate critical points are isolated.*

Given a Morse function f on a manifold, one can reconstruct the manifold, at least topologically, knowing the critical points of f and their indices. This also turns out to be the case for a more general class of functions that are allowed to have non-degenerate submanifolds of critical points.

Definition II.35. A function f on a manifold M is called a *Morse-Bott* function if its critical set is a submanifold of M and the second derivative is non-degenerate in the transverse directions. This means that the kernel of the Hessian at a critical point is the tangent space to the critical submanifold.

There is a version of the Morse lemma for Morse-Bott functions.

Proposition II.36. ***(Morse Lemma With Parameter)*** Let $f : M \to \mathbb{R}$ be a Morse-Bott function and let N be a critical submanifold of f. Then for each point $p \in N$, there exist local coordinates (x, y) on a neighborhood of p sending p to $(0, 0)$ such that N can locally be described by $N = \{y = 0\}$ and f takes the form

$$f(x, y) = f(p) + Q_x(y),$$

where Q_x is a non-degenerate quadratric form in the y-variables, the transverse directions to N.

The *index* $\alpha(N)$ of a connected component of a critical submanifold is the index (number of negative eigenvalues) of the second derivative. It is equal to the rank of the *negative normal bundle* of N in the ambient manifold M, which can then be used to reconstruct the topological type of M from the Morse-Bott function f. We do not use this construction explicitly, so we refer the reader to the literature for details.

2.4 The CROSSES from a Geometric Point of View

The compact rank one symmetric spaces (CROSSes) are classical examples of symmetric spaces and are in a sense “very homogeneous.” They are precisely the Euclidean spheres, the projective spaces $\mathbb{KP}^n$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (where $\mathbb{H}$ denotes the quaternions), and the exceptional Cayley plane $\mathbb{OP}^2$ (also denoted by $\mathrm{Ca}\mathbb{P}^2$). We remind the reader of some of the properties of the CROSSes. More information can be found in Chapter 3 of Besse [6] or the comprehensive book by Helgason [15].

2.4.1 Defining The Projective Spaces

Recall that $\mathbb{K}^{n+1}$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ has a right vector space structure given by

$$x \cdot \lambda = (x_1 \cdot \lambda, \ldots, x_{n+1} \cdot \lambda).$$

One also obtains a Hermitian inner product given by

$$\langle x, y\rangle = \sum_{j=1}^{n+1} \overline{x}_j y_j = \overline{x}^T y$$

yielding a real inner product

$$\langle x, y\rangle_{\mathbb{R}} = \mathrm{Re}\langle x, y\rangle$$

from which the Riemannian structure of $\mathbb{KP}^n$ comes (Section 2.4.4 below). One can interpret the projective spaces $\mathbb{KP}^n$ as the space of $\mathbb{K}$-lines in $\mathbb{K}^{n+1}$ through the origin.

Definition II.37. The *projective space* $\mathbb{KP}^n$ is the orbit space for the right action of the group $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$ on $\mathbb{K}^{n+1} \setminus \{0\}$. That is,

$$\mathbb{KP}^n = \left(\mathbb{K}^{n+1} \setminus \{0\}\right) / \sim,$$

where $x \sim y \Leftrightarrow x = y \cdot \lambda$ for some $\lambda \in \mathbb{K} \setminus \{0\}$. We denote the quotient map by π, so that points in $\mathbb{KP}^n$ are of the form $\pi(x)$ for $x \in \mathbb{K}^{n+1} \setminus \{0\}$.

Remark II.38. This definition cannot be used to define the Cayley plane $\mathbb{OP}^2$ due to the non-associative nature of $\mathbb{O}$. However, we can modify the definition slightly (see Section 2.4.5 below) to take advantage of its 2-associativity.

2.4.2 Riemannian Symmetric Spaces

Definition II.39. A connected Riemannian manifold M is *symmetric* (in the sense of E. Cartan) if given any $x \in M$, there exists an involution $s_x : M \to M$, which is a global isometry such that

- $s_x(x) = x$ and
- $(ds_x)_\star = -Id : T_xW \to T_xW$.

Remark II.40. If M is symmetric, then M is homogeneous (the isometry group acts transitively) and is complete.

Example II.41. The round sphere $M = S^n = \{x_1^2 + \ldots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$ is a Riemannian symmetric space. Consider the points $x = (0, \ldots, 0, 1)$ (the "North Pole") and let s_x be given by the $(n+1) \times (n+1)$ matrix

$$\left(\begin{array}{c|c} -I_n & 0 \\ \hline 0 & 1 \end{array}\right).$$

We fix some notation for the remainder of this section. Let M be connected such that the identity component of its isometry group $G = \mathrm{Isom}_0(M)$ is compact. Fix some point $o \in M$, and let K denote the isotropy group at o, which is a compact subgroup of G by a standard result in Lie theory. Then one can write $M \cong G/K$ under the correspondence $g \cdot o \mapsto g \cdot K$. (For more details, see [15].)

The structure of a symmetric space is encoded in its Lie algebra. Since s_o is an involution about the point o, $s_o \in K$, and one gets an action on G by conjugation given by $g \mapsto s_o g s_o^{-1}$. Differentiating at o, one obtains an involution $\theta : \mathfrak{g} \to \mathfrak{g}$. Denote by $\mathfrak{k}$ the (+1)-eigenspace of θ (its fixed set) and by $\mathfrak{p}$ the (-1)-eigenspace. One has the diagram below.

$$\begin{array}{ccc} G & \mathrm{Lie}(G) = \mathfrak{g} & \theta^2 = Id \\ & & \mathfrak{k} = V_{+1}(\theta) \\ \cup & \cup & \\ & & \mathfrak{p} = V_{-1}(\theta) \\ K & \mathrm{Lie}(K) = \mathfrak{k} & \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \end{array}$$

Definition II.42. A *maximal flat* in M is the image under the exponential map of a maximal abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$. (Recall that $\mathfrak{a}$ is abelian if $[\mathfrak{a}, \mathfrak{a}] = 0$.)

Definition II.43. The *rank* of a symmetric space M is defined to be the maximal dimension of a totally geodesic submanifold in M, i.e.

$$\operatorname{rank}(M) = \dim(\mathfrak{a}).$$

Example II.44. Returning to the example $M = S^n$, we choose o to be the North Pole. Then $G = \mathrm{Isom}_0 S^n = SO(n+1)$, and the isotropy subgroup K that preserves o consists of matrices of the form

$$\left(\begin{array}{c|c} SO(n) & 0 \\ \hline 0 & 1 \end{array}\right)$$

One deduces that $M = S^n \cong SO(n+1)/SO(n) \times 1$ and

$$\mathfrak{k} = \left(\begin{array}{c|c} \mathfrak{so}(n) & 0 \\ \hline 0 & 0 \end{array}\right).$$

One checks that a maximal abelian subalgebra $\mathfrak{a}$ can be given by

$$\mathfrak{a} = \left\{ \left(\begin{array}{c|cc} 0 & & \\ \hline & 0 & \lambda \\ & -\lambda & 0 \end{array}\right) \,\middle|\, \lambda \in \mathbb{R} \right\},$$

so that $\operatorname{rank}(M) = \operatorname{rank}(S^n) = 1$, and we have that $\exp_o \mathfrak{a} = T^1 = S^1$.

There is also a conjugacy theorem of Cartan:

Theorem II.45. ***(Cartan)*** *If $\mathfrak{a}, \mathfrak{a}' \subset \mathfrak{p}$ are maximal abelian subalgebras, then there is some $k \in K$ such that $k \cdot \mathfrak{a} = \mathfrak{a}'$.*

In particular, if the rank of M is 1, any line in $\mathfrak{p}$ is conjugate by K to any other line in $\mathfrak{p}$. To see why this is relevant to our discussion, we note that $T_o M = T_o(G/K) = \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p}$. Since K acts transitively on the unit sphere bundle $S_o(M)$, all geodesics are isometric and are all orbits of conjugate 1-parameter subgroups. This implies that all geodesics are closed and of equal length (as in the case of the round sphere), so that the manifold M is what is known as a *Zoll manifold*. (See Section 2.5 below.)

2.4.3 The Projective Spaces as Symmetric Spaces

Let $M = \mathbb{KP}^n \neq \mathbb{OP}^2$. Denote by $G = \mathrm{U}(n+1, \mathbb{K})$ the subgroup of $\mathrm{GL}(n+1, \mathbb{K})$ leaving the Hermitian inner product $\langle\, ,\, \rangle$ fixed.

Remark II.46. Using this notation, we have

$$\begin{aligned} \mathrm{U}(n+1,\mathbb{R}) &= \mathrm{O}(n+1), \\ \mathrm{U}(n+1,\mathbb{C}) &= \mathrm{U}(n+1), \\ \mathrm{U}(n+1,\mathbb{H}) &= \mathrm{Sp}(n+1). \end{aligned}$$

We also note that the group $\mathrm{U}(n+1, \mathbb{K})$ acts transitively on the unit sphere $S\mathbb{K}^{n+1}$, so that this action descends to a transitive action on $\mathbb{KP}^{n+1}$.

Now let $e_1, \ldots, e_{n+1}$ denote the canonical basis of $\mathbb{K}^{n+1}$, and set $K = \mathrm{Isot}(\pi(e_{n+1}))$. Consider the map $p : U(n+1, \mathbb{K}) \to \mathbb{KP}^n$ defined by $p(A) = \pi A\,(e_{n+1})$. Standard Lie theory gives that there is a diffeomorphism (in fact, an isometry)

$$\Phi : \mathrm{U}(n+1, \mathbb{K})/K \to \mathbb{KP}^n,$$

where if $A \in K$, one has that

$$\begin{aligned} & A(e_{n+1}) = e_{n+1} \cdot \lambda, \lambda \in \mathrm{U}(1, \mathbb{K}) = S\mathbb{K} = S^{a-1} \\ \Rightarrow\ & A = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & \lambda \end{array}\right), B \in \mathrm{U}(n, \mathbb{K}) \\ \Rightarrow\ & K \cong \mathrm{U}(n, \mathbb{K}) \times \mathrm{U}(1, \mathbb{K}). \end{aligned}$$

Thus as symmetric spaces, we have the relations

$$\begin{aligned} \mathbb{RP}^n &\cong \mathrm{O}(n+1)/\mathrm{O}(n) \times \{\pm 1\}, \\ \mathbb{CP}^n &\cong \mathrm{U}(n+1)/\mathrm{U}(n) \times \mathrm{U}(1), \\ \mathbb{HP}^n &\cong \mathrm{Sp}(n+1)/\mathrm{Sp}(n) \times \mathrm{Sp}(1), \end{aligned}$$

and all of these spaces have a symmetric structure with the involution on $\mathrm{U}(n+1,\mathbb{K})$ given by

$$\theta(A) = SAS^{-1}, \qquad S = \left(\begin{array}{c|c} -I_n & 0 \\ \hline 0 & 1 \end{array}\right)$$

such that $\theta(A) = A$ if and only if $A \in K$. A decomposition of the Lie algebra $\mathfrak{u}(n+1,\mathbb{K}) \cong \mathfrak{k} \oplus \mathfrak{p}$ is given by

$$\begin{aligned} \mathfrak{k} &= \mathfrak{u}(n,\mathbb{K}) \oplus \mathfrak{u}(1,\mathbb{K}) \\ \mathfrak{p} &= \left\{ \left(\begin{array}{c|c} 0 & \xi \\ \hline -\xi^T & 0 \end{array}\right) \Big| \, \xi \in \mathbb{K}^n \right\} \\ &\cong \mathbb{K}^n, \end{aligned}$$

from which we deduce that $\dim_{\mathbb{R}}(\mathfrak{a}) = 1$, so that the $\mathbb{KP}^n$ are of rank one.

2.4.4 Obtaining the Standard Homogeneous Metric on Projective Space

Recalling from before that $T_{\pi(e_{n+1})}\mathbb{KP}^n \cong T_o(G/K) \cong \mathfrak{p} \cong \mathbb{K}^n$, we have an adjoint action of K on $\mathfrak{p}$ given by

$$\mathrm{ad}\left(\begin{array}{c|c} B & 0 \\ \hline 0 & \lambda \end{array}\right)(\xi) = B(\xi) \cdot \overline{\lambda}.$$

$\langle \xi, \eta \rangle_{\mathbb{R}}$ gives an $\mathrm{ad}(K)$-invariant inner product for $\xi, \eta \in \mathfrak{p} \cong \mathbb{K}^n$, since

$$\langle B\xi \cdot \overline{\lambda}, B\eta \cdot \overline{\lambda} \rangle_{\mathbb{R}} = \langle B\xi, B\eta \rangle_{\mathbb{R}} = \langle \xi, \eta \rangle_{\mathbb{R}},$$

where the first equality in the previous line is due to the $S\mathbb{K}$-invariance of $\langle\, , \rangle_{\mathbb{R}}$. Then $\langle\, , \rangle_{\mathbb{R}}$ pushes forward to a metric on $\mathrm{U}(n+1,\mathbb{K})/K \cong \mathbb{KP}^n$.

2.4.5 The Octonionic Projective Spaces

The case of the last (and strangest) of the normed division algebras requires some care due to its non-associativity. One would still like to think of the projective spaces

$\mathbb{O}\mathbb{P}^n$ as the space of $\mathbb{O}$-lines through the origin in $\mathbb{O}^{n+1}$, but now one must be careful when defining what a "line" is. We start by considering lines in $\mathbb{O}^2$. The line through the origin containing a point $(x, y) \in \mathbb{K}^2$ is in general not given by

$$L = \{(\alpha x, \alpha y) \,|\, \alpha \in \mathbb{K}\}$$

unless $\mathbb{K}$ is associative. It is true, however, when either $x = 1$ or $y = 1$. This motivates us to define an equivalence relation on $\mathbb{K}^2 \setminus \{(0, 0)\}$ by setting $(x, y) \sim (1, x^{-1}y)$ for $x \neq 0$ and $(x, y) \sim (y^{-1}x, 1)$ for $y \neq 0$. Its structure as a smooth manifold is given by the two coordinate charts

- U_x consisting of points of the form $[(1, y)]$ and
- U_y consisting of points of the form $[(x, 1)]$,

together with the transition function $x \mapsto x^{-1}$ for points such that $[(x, 1)] = [(1, y)]$.

We define $\mathbb{O}\mathbb{P}^2$ similarly. For any nonzero element $(x, y, z) \in \mathbb{O}^3$ with $(xy)z = x(yz)$, we normalize it so that

$$||x||^2 + ||y||^2 + ||z||^2 = 1$$

to obtain a point $[(x, y, z)] \in \mathbb{O}\mathbb{P}^2$. We can cover the octonionic projective plane by three coordinate charts:

- U_x consisting of points of the form $[(1, y, z)]$,
- U_y consisting of points of the form $[(x, 1, z)]$, and
- U_z consisting of points of the form $[(x, y, 1)]$,

thus making it into a smooth manifold. Lines in $\mathbb{O}\mathbb{P}^2$ are just copies of $\mathbb{O}\mathbb{P}^1$ and so are 8-spheres.

Equivalently, in a similar way to how we defined the other projective spaces, we can consider the symmetric and reflexive relation $\sim$ on $\mathbb{O}^3$ given by

$$(x, y, z) \sim (u, v, w) \Longleftrightarrow \exists \lambda \in \mathbb{O}^\times \text{ such that } x = u\lambda, y = v\lambda, z = w\lambda. \tag{2.1}$$

One can check that $\sim$ is transitive restricting to the union of the subsets

$$\{1\} \times \mathbb{O} \times \mathbb{O}, \mathbb{O} \times \{1\} \times \mathbb{O}, \text{ and } \mathbb{O} \times \mathbb{O} \times \{1\}$$

in $\mathbb{O}^3$, despite the non-associativity of $\mathbb{O}$. In terms of the octonion-valued coordinate functions on the charts $U_j/\sim$ (for $j = x, y, z$), one can define a quadratic form which gives rise to a Riemannian metric on $\mathbb{OP}^2$. For example, on $U_x = \{[1, y, z]\}$, we have octonion-valued 1-forms dy and dz and define

$$ds^2 = \frac{|dy|^2(1 + |z|^2) + |dz|^2(1 + |y|^2) - 2Re[(y\bar{z})(dy\, d\bar{z})]}{(1 + |y|^2 + |z|^2)^2}. \tag{2.2}$$

We can define a quadratic form on U_y and U_z similarly and check the compatibility of the definitions with respect to the transition maps.

The octonionic projective spaces can also be defined in terms of projections in the formally real Jordan algebras $\mathfrak{h}_2(\mathbb{O})$ and $\mathfrak{h}_3(\mathbb{O})$ of Hermitian $n \times n$ matrices with entries in $\mathbb{O}$. We refer the interested reader to Baez's article [4] and the original works of P. Jordan and H. Freudenthal, the results of which allow us to identify $\mathbb{OP}^2$ with the rank one Riemannian symmetric space $F_4/\mathrm{Spin}(9)$. More details about the Riemannian structure of the octonionic plane can be found in [14].

2.5 Zoll Manifolds

For a more complete (but admittedly outdated) treatment of Zoll manifolds and related topics, we refer the reader to Besse's book. [6]

Definition II.47. A *Zoll manifold* is a Riemannian manifold endowed with a metric such that all geodesics are closed with the same minimal period.

Remark II.48. As a consequence of the Hopf-Rinow theorem, Zoll manifolds are automatically compact, being geodesically complete.

Example II.49. Besides the obvious example of the sphere with its metric of constant curvature (the round metric), with respect to which all geodesics are great circles obtained by intersecting S^n with 2-planes through the origin in $\mathbb{R}^{n+1}$, the other CROSSes with their standard metrics are also Zoll.

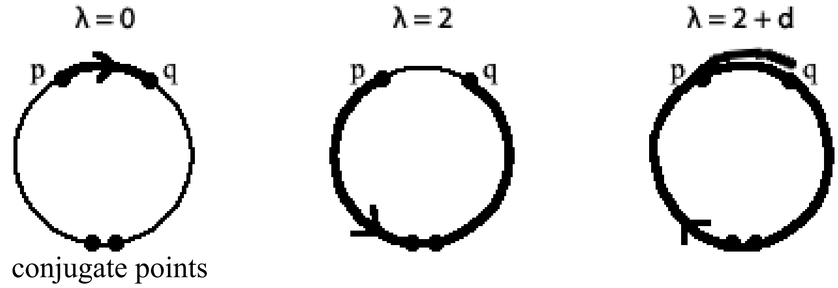
The first non-trivial examples were discovered by Otto Zoll in his 1901 thesis. [33]

Example II.50. A *Zoll surface* is a Riemannian surface homeomorphic to S^2 but not necessarily isometric to the round sphere. Zoll exhibited an infinite-dimensional family of deformations of the round metric giving rise to Zoll metrics on S^2.

Later on, Weinstein showed the existence of non-trivial Zoll metrics on spheres of every dimension. One would like to know if these are all the Zoll manifolds. The projective spaces $\mathbb{KP}^n$ appear to behave differently from the case of the spheres, as Tsukamoto [31] proved that there are no other Zoll metrics "near" the standard metric on the projective spaces. Whether the standard metric on the projective spaces is the only metric such that all geodesics are simply closed and of the same period is still unknown. However, there are some results that point in this direction, at least on a homological level (Theorem II.53 below).

Raoul Bott [8] and Hans Samelson [28] considered (C^3-)Riemannian manifolds whose geodesics issuing from *a single point* p were all simple, closed, and of the same minimal period. By considering the space of paths Ω from the point p to a nearby point q, Bott used Morse theoretic methods to compute the homology of the space of paths using a Leray spectral sequence. By looking at the Leray spectral sequence associated to the Serre projection, he was able to recover the $\mathbb{Z}$-cohomology of the

Figure 2.1: Indices of various geodesic segments σ



original manifold.

Before stating the theorem, we introduce some notation from Berger [5]. First observe that for all points q chosen close enough to p, the geodesics going from p to q are all coverings of one unique periodic geodesic.

Definition II.51. The *index* $\lambda(\gamma)$ of a geodesic segment γ is the weighted sum of points conjugate to the distinguished point p along γ.

Notation II.52. Let σ denote a geodesic segment connecting p to a point q nearby such that p and q are not conjugate along σ, and let d be the dimension of the manifold M.

The Bott-Samelson theorem considers the integer $\lambda = \lambda(\gamma - \sigma)$, the index of the complement of σ in γ and shows that this integer determines the (co)homology of M. (See Figure 2.5.)

Theorem II.53. ***(Bott [8], Samelson [28])*** *Let M be a Riemannian manifold of dimension $d = dim\, M$ such that all geodesics through a point $p \in M$ are periodic and of the same minimal length. Let λ be the index of the complement $\gamma - \sigma$ of the geodesic segment σ as above. The integer λ does not depend on the choice of the point q, and if $\lambda > 0$, then M is simply connected, and its integral cohomology has exactly one generator. Moreover, there are only the following possibilities:*

Index	***Dimension***	***Topology***
$\lambda = 1$	$d = 2n$	*M has the homotopy type of* $\mathbb{CP}^n$
$\lambda = 3$	$d = 4n$	*M has the homology ring of* $\mathbb{HP}^n$
$\lambda = 7$	$d = 16$	*M has the integral cohomology ring of* $\mathbb{OP}^2$
$\lambda = d - 1$	*d arbitrary*	*M has the homotopy type of* S^d *and is thus homeomorphic to* S^d
$\lambda = 0$	*d arbitrary*	*M is diffeomorphic to* $\mathbb{RP}^d$

The above theorem gives that, at least homologically, Zoll manifolds look like CROSSes. In the case of the sphere S^d, it is not known if M must be diffeomorphic to the standard sphere in general. For example, when $d = 7$, Milnor exhibited examples of exotic differentiable spheres as S^3-bundles over S^4. [24] However, one has in dimensions $d = 1, 2, 3, 5,$ and 6 that S^d must be diffeomorphic to the standard sphere, due to the Generalized Poincaré Conjecture in the differentiable setting.

CHAPTER III

Polarized Symplectic Manifolds in the Sense of Audin

3.1 Polarized Symplectic Manifolds

Polarized symplectic manifolds appear as examples exhibiting Lagrangian barriers in certain Kähler manifolds with integral Kähler form in a paper of Biran's. [7] We use the definition given by Audin in [2] and [3].

Definition III.1. A *polarized symplectic manifold* (W, ω) is a (connected) symplectic manifold endowed with a Morse-Bott function

$$H : W \to \mathbb{R}$$

with exactly two critical submanifolds. We require that the minimum of H be obtained along a Lagrangian submanifold L and the maximum on a symplectic submanifold of V. (In the literature, Audin often uses the notation f for the Morse-Bott function.)

The label *polarized* is more restrictive in this setting than in [7] because the complement $W \setminus V$ is required to retract (for example, under the gradient flow of H associated to some Riemannian metric on W) onto the Lagrangian submanifold L, rather than an isotropic skeleton.

3.2 Known Examples

In the following examples of polarized symplectic manifolds, the Lagrangian submanifolds L are precisely the CROSSes mentioned above. (The ambient symplectic manifolds W are in fact polarized *Kähler* manifolds in the sense of Biran. [7]) The Morse-Bott function is often given by the square of the norm of the moment map for a Hamiltonian action.

If one endows the CROSSes with their standard "homogeneous" metrics (Section 2.4.4), with respect to which they are Zoll manifolds, one can perform the symplectic cut construction on the cotangent bundle to obtain these examples using the co-geodesic flow. This construction and its relevance to the problem at hand are discussed in Section 3.4.

Example III.2. $L = S^n, V = Q^{n-1}, W = Q^n$

Consider $Q^n \subset \mathbb{CP}^{n+1}$ sitting inside complex projective space and take homogeneous coordinates $\{[z_0 : z_1 : \ldots : z_{n+1}] \,|\, z_j \text{ not all zero}\}$. As usual, we let $z_j = x_j + iy_j$. Then $W = Q^n$ is given as the zero locus of the homogenous polynomial $z_0^2 + \ldots + z_{n+1}^2$, and the symplectic form is given by restriction of the Fubini-Study form. The symplectic submanifold $V = Q^{n-1}$ is the hyperplane at "infinity"

$$V = W \cap \{z_0 = 0\} = \{z_1^2 + \ldots z_{n+1}^2 = 0\},$$

while the Lagrangian sphere is given by $\{[i : x_1 : \ldots : x_{n+1}] \,|\, x_j \in \mathbb{R}\}$. The function

$$H([z_0 : \ldots : z_{n+1}]) = 1 - \frac{|z_0|^2}{\sum_{j=0}^{n+1} |z_j|^2} = 1 - \frac{x_0^2 + y_0^2}{\sum_{j=0}^{n+1} (x_j^2 + y_j^2)}$$

is Morse-Bott. It reaches its minimum along L and its maximum along V.

The space of unparametrized oriented geodesics of S^n with the round metric is known to be diffeomorphic to a complex quadric (which can be identified with the

Grassmanian of oriented 2-planes in $\mathbb{R}^{n+1}$). In particular,

$$Q^{n-1} = \tilde{G}_2(\mathbb{R}^{n+1}) = \mathrm{SO}(n+1)/\mathrm{SO}(n-1) \times \mathrm{SO}(2).$$

Example III.3. $L = \mathbb{RP}^n, V = Q^{n-1}, W = \mathbb{CP}^n$

With the metric on $\mathbb{RP}^n$ induced from that of the round sphere S^n, one sees that the manifold of unparametrized, oriented geodesics is again Q^{n-1}, since geodesics in S^n are sent to geodesics in $\mathbb{RP}^n$ under the antipodal action on S^n.

Choosing homogeneous coordinates $[z_0 : z_1 : \ldots : z_n]$ on $W = \mathbb{CP}^n$ and setting $z_j := x_j + iy_j$, let

$$V = Q^{n-1} = \{z_0^2 + z_1^2 + \ldots + z_n^2 = 0\} = \left\{ \sum_{j=0}^{n} (x_j^2 - y_j^2) + 2i \sum_{j=0}^{n} x_j y_j = 0 \right\}.$$

Here, $\mathbb{CP}^n$ is given the Fubini-Study form $\omega_{FS} = \{\omega_k = i\partial\overline{\partial} \log \frac{\sum |z_i|^2}{|z_k|^2} \text{ on } U_k\}$. If we choose $x, y \in \mathbb{R}^{n+1}$ to be orthogonal such that $||x|| = 1$, then the function

$$H([x+iy]) = \frac{||y||^2}{(1+||y||^2)^2}$$

attains its minimum along $L = \mathbb{RP}^n = \{y = 0\}$ and its maximum along Q^{n-1}, where $x \cdot y = 0$ and $||y|| = ||x|| = 1$.

Example III.4. $L = \mathbb{CP}^n, V = \mathbb{F}(1, n, n+1), W = \mathbb{CP}^n \times \mathbb{CP}^n$

Here, V is identified with an incidence variety (a flag manifold). It is a bidegree (1,1)-hypersurface of W. (Audin [3] uses the notation M^{4n-2}.) It is a submanifold of $\mathbb{CP}^n \times \mathbb{CP}^n$ of (real) codimension 2 defined by the equation

$$\sum_{i=0}^{n} z_i \bar{w}_i = 0. \tag{3.1}$$

$\mathbb{CP}^n \times \mathbb{CP}^n$ is endowed with the symplectic form $\omega_{FS} \oplus -\omega_{FS}$, where ω_{FS} is the Fubini-Study form (or some scalar multiple of it).

Example III.5. $L = \mathbb{HP}^n, V = \mathrm{Gr}_{\mathrm{isot}}(2, 2n+2), W = \mathrm{Gr}_{\mathbb{C}}(2, 2n+2)$

Here, W is the Grassmannian of all complex 2-planes in $\mathbb{C}^{2n+2}$, and the symplectic submanifold V can be described as follows. Let i, j, k denote the usual quaternionic units acting on $\mathbb{C}^{2n+2} \cong \mathbb{H}^n$. Then right multiplication by j sends a left complex subspace (with respect to i) to another such complex subspace. This induces an antiholomorphic map of order 2 (conjugation) such that the j-stable 2-planes (i.e. the quaternionic lines in $\mathbb{HP}^n$) are the real points fixed under the conjugation.

Example III.6. $L = \mathbb{OP}^2 = F_4/\mathrm{Spin}(9), V = F_4/\mathrm{SO}(2) \times \mathrm{Spin}(7), W = E_6^{\mathbb{C}}$

The case of the octonionic (Cayley) plane is an anomaly among the projective spaces and so must be treated with care. As discussed in Section 2.4.5, it is a bit painful to construct. Its geodesics are the great circles in its projective lines. (For a more detailed discussion, see [20].) The exceptional compact Lie group F_4 is the automorphism group of both $\mathbb{OP}^2$ and the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$, and Spin(9) is realized as a subgroup, where the group Spin(n) is the usual double cover of $SO(n)$ given by

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 1.$$

The ambient manifold $W = E_6^{\mathbb{C}} = E_6/\mathrm{SO}(2) \times \mathrm{SO}(10)$ is one of the Hermitian symmetric spaces of compact type appearing in Cartan's list of irreducible Riemannian globally symmetric spaces. Here, E_6 refers to the exceptional Lie group. Geometrically, $E_6^{\mathbb{C}}$ can be interpreted as the complexification $(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2$ of $\mathbb{OP}^2$ and so has real dimension 32.

3.3 The Symplectic Submanifold V of W

In the above examples, it turns out that W is a projective algebraic manifold that can be endowed with a conjugation σ such that V is a σ-invariant divisor and L is the set of fixed points under this conjugation. It turns out that V can be identified with the space of unparametrized oriented closed geodesics in the Zoll manifold L. We will see in the next section a way to construct examples of polarized symplectic manifolds in a way that makes apparent V's role as a moduli space of geodesics in L.

3.4 The Symplectic Cut Construction

We remind the reader of the symplectic cut construction, first introduced by Eugene Lerman in [19].

Recall that if a symplectic manifold (M, ω) is endowed with a free Hamiltonian circle action with moment map $\mu : M \to \mathbb{R}$, one forms the cut (at a regular level) by considering the product symplectic manifold $(M \times \mathbb{C}, \omega \oplus \frac{i}{2} dw \wedge d\bar{w})$. The circle action extends to a free action of S^1 on the product manifold by $e^{i\theta} \cdot (z, w) = (e^{i\theta} z, e^{i\theta} w)$ with moment map $\Phi(z, w) = \mu(z) + \frac{1}{2}|w|^2$. (Here we use the sign convention $i_{X_H}\omega = -dH$.)

One fixes a value ϵ and considers the level set $\{\Phi(z, w) = \epsilon\}$, which can be identified with the disjoint union

$$\begin{aligned} \Phi^{-1}(\epsilon) &= \{(z, 0) \in M \times \mathbb{C} \,|\, \mu(z) = \epsilon\} \\ &\sqcup \{(z, w) \in M \times \mathbb{C} \,|\, \mu(z) < \epsilon \,\&\, w = e^{i\theta} \cdot \sqrt{2\,(\epsilon - \mu(z))}\}. \end{aligned}$$

The first of these sets is simply $\{\mu(z) = \epsilon\}$, while the latter is equivariantly diffeomorphic to the product $M_{\mu<\epsilon} \times S^1$, where $M_{\mu<\epsilon}$ denotes the open submanifold $\{z \in M \,|\, \mu(z) < \epsilon\}$. The symplectic cut (at level ϵ) is (topologically) the quotient manifold

$$\overline{M}_{\mu \leq \epsilon} = \{(z, w) \in M \times \mathbb{C} : \Phi(z, w) = \epsilon\}/S^1,$$

which, by the above discussion, can be identified with the disjoint union

$$\Phi^{-1}(\epsilon)/S^1 \quad = \quad \overline{M}_{\mu=\epsilon} \sqcup M_{\mu<\epsilon}$$

of the reduced space $\overline{M}_{\mu=\epsilon}$, a codimension 2 symplectic submanifold with an inherited symplectic form $\overline{\omega_\epsilon}$ characterized by its pullback to $\mu^{-1}(\epsilon)$ being equal to the restriction of ω to $\mu^{-1}(\epsilon)$. The symplectic form on the open dense submanifold $M_{\mu<\epsilon}$ is just given by the restriction of the original symplectic form ω.

To obtain examples of polarized symplectic manifolds, one applies the cut construction to the cotangent bundle $T^\star L$ of a Zoll manifold L, the periodic geodesic flow of which induces a free S^1-action away from the zero section (denoted by 0_L). We call this process the *Zoll cut construction.* The Hamiltonian is given by

$$\begin{aligned} H : T^\star L \backslash 0_L &\to \mathbb{R} \\ (x, y) &\mapsto ||y|| \end{aligned}$$

and one performs the cut at a regular level, say $H = 1$. The Morse-Bott function could then be given by $f = H^2 = ||y||^2$.

The symplectic reduction V embeds symplectically into the cut space as a codimension 2 submanifold and is exactly the maximum locus of f. Taking the quotient of the sphere bundle $S(T^\star L)$ by the S^1-action induced by the (co-)geodesic flow identifies points in $T^\star L$ if and only if they define the same oriented geodesic in L. Thus, $V = S(T^\star L)/S^1$ can be thought of as the space of unparametrized, oriented geodesics in L. Its complement in the symplectic cut is then easily seen to be the open unit disk bundle of $T^\star L$.

Remark III.7. Since the cut construction affects only a codimension 1 submanifold (the cut level), the S^1-action induced by the geodesic flow descends to the rest of the cut manifold.

More generally, suppose we have two Hamiltonian actions of Lie groups G and H on a symplectic manifold (M, ω) with corresponding moment maps μ and ψ. Suppose further that the actions commute, in the sense that μ is H-invariant and ψ is G-invariant. Assuming further that G acts freely on the reduction level and denoting the reduced space by $(M_{\mu=c}, \omega_{\mu=c})$, the action of H descends to the reduced space. This is because H preserves the level set $\{\mu = c\}$ and ψ commutes with the action of G (so that the restriction of ψ to the reduction level descends to a moment map on the reduced manifold $\psi_{\mu=c} : M_{\mu=c} \to \mathfrak{h}^\star$).

In our case explicitly, the S^1-action on the cut space $\overline{M}_{\mu\leq\epsilon} := \Phi^{-1}(\epsilon)/S^1$ is induced by the product action of the original S^1 on M with the trivial action on the $\mathbb{C}$ factor. This product action (H in the previous paragraph) commutes with free S^1-action (the above G) that was used to perform the cut and so descends to the cut space.

Some of the examples of polarized symplectic manifolds which arise from this construction have already been worked out in [3]. In particular, Audin proved:

Proposition III.8. ***(Audin [3])*** Let (W, ω) be the symplectic manifold resulting as the symplectic cut of the cotangent bundle $T^\star L$ by the (co-)geodesic flow of a Zoll manifold L. Then W is symplectomorphic to one of the following:

- $W = \mathbb{CP}^n$ if $L = \mathbb{RP}^n$; the reduced space V is the quadric Q^{n-1};
- $W = Q^n$ if $L = S^n$, the round sphere; the reduced space V is the quadric Q^{n-1};
- $W = \mathbb{CP}^n \times \mathbb{CP}^n$ (with the symplectic form $\omega_{FS} \oplus -\omega_{FS}$) if $L = \mathbb{CP}^n$; the reduced space V is the submanifold defined by equation (3.1).

CHAPTER IV

The Lagrangian Critical Submanifold L

We aim to at least partially address a question posed by Audin in [2,3]. Given that (W, ω) is a polarized symplectic manifold with Morse-Bott function H and critical submanifolds L and V as before, can we say (up to some notion of equivalence) what L, V and W "are"?

The Morse-Bott function H on W should in principal encode the necessary information about L and V. Being Morse-Bott, the Hessian Hess(H) is non-degenerate in the transverse directions to the critical submanifolds, in particular, to L. In local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, where $y_i(p) = 0$ for $p \in L$, this means the matrix of mixed second partial derivatives of H has $\big(H_{y_i,y_j}(x,0)\big) > 0$. This yields a metric on (the vertical tangent bundle of) L. This also gives a metric on the tangent bundle TL (for example, the Sasaki metric), and thus a metric on T^*L.

Remark IV.1. We also have (after choosing a compatible almost complex structure J) a metric induced by the original symplectic form ω on W. A natural question to ask is: In what way are these two metrics related?

Having a metric on T^*L, we can speak of its unit disk and unit sphere bundles, $D(T^*L)$ and $S(T^*L)$, respectively. Using the fact that L is Lagrangian in W, Theorem II.28 gives that a tubular neighborhood of L is symplectomorphic to a disk subbundle of the cotangent bundle $T^\star L$ with the canonical symplectic form.

In the rest of this chapter, we work mostly on this neighborhood of the zero section in $T^\star L$ and impose a further constraint on the Morse-Bott function H to determine what possible manifolds the Lagrangian L can be. In particular, we require that the flow of the vector field $X_{\sqrt{H}}$ generate a free S^1-action away from the critical sets L and V in W and show that this gives rise to a Zoll metric on L.

4.1 Blowing Up to Resolve Singularities

We begin with a polarized symplectic manifold (W, ω) in the sense of Audin [3] under the assumption that the flow of the symplectic vector field $X_{\sqrt{H}}$ corresponding to the square root of the Morse-Bott function H is freely periodic outside of its critical set, of period, say, 2π. (The actual period does not factor into the following calculations.) Recall that H has exactly two critical submanifolds, where the minimum value is attained along a smooth Lagrangian submanifold L and the maximum along a smooth symplectic submanifold V.

The flow is thus singular at the minimum critical set L (where $dH = 0$) and also at the maximum set V. In this discussion, we focus on the minimum. To examine the nature of the singularity there, we perform a real oriented blow-up along L in $T^\star L$. (As this is a local construction, it is akin to blowing up along L in the original manifold W.) The resulting blown-up manifold, denoted by $\widehat{T^\star L}$ (or $\widehat{W}$), is then a manifold with boundary $\partial\widehat{T^\star L}$ (sometimes denoted $\widehat{L}$), and one can perform calculations as in Melrose's b-calculus on it.

We set the following sign conventions and notation:

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$$

$$i_{X_H}\omega = -dH,$$

so that X_H with respect to the standard symplectic form ω_0 is just

$$X_H = -H_y\partial_x + H_x\partial_y$$

and the corresponding Hamiltonian equations are

$$\dot{x} = -\frac{\partial H}{\partial y}$$
$$\dot{y} = \frac{\partial H}{\partial x}.$$

We aim to show that the Lagrangian L must be a Zoll manifold (a Riemannian manifold all of whose geodesics are closed of the same minimal period). Since H is Morse-Bott, its Hessian is non-degenerate in the directions transverse to the critical submanifolds. As L is Lagrangian, by an equivariant version of Theorem II.28, an open neighborhood of it can be identified equivariantly symplectically with an open neighborhood of the zero section of the cotangent bundle $T^\star L$ with the standard symplectic form ω_0.

Working in the cotangent bundle $T^\star L$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ (corresponding to the canonical form $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$), where L corresponds to the zero section $(x_1, \ldots, x_n, 0, \ldots, 0)$, by the non-degeneracy of the Morse-Bott condition, we can write the Hessian of H along the zero-section L as

$$\mathrm{Hess}(H) = \begin{pmatrix} 0 & 0 \\ 0 & \left(\frac{\partial^2 H}{\partial y_i \partial y_j}\right) \end{pmatrix}_{(x_1,\ldots,x_n,0,\ldots,0)}.$$

The submatrix $\left(\frac{\partial^2 H}{\partial y_i \partial y_j}\right)$ is positive definite and symmetric, and therefore defines a Riemannian metric $(g_{ij}(x))$ on the Lagrangian L. With this metric, we consider a new Hamiltonian $H_0 = \frac{1}{2}\sum_{i,j} g_{ij}(x) y_i y_j$ (which by construction differs from the original function H by terms of order 3 or higher in the y-variables), its square root $\sqrt{H_0}$, and the flows of the associated vector fields X_{H_0} and $X_{\sqrt{H_0}}$ on the cotangent bundle $T^\star L$ where they are defined. Some difficulty lies in the singularity (non-smoothness) of $\sqrt{H_0}$ and $\sqrt{H}$ at the zero section so that the Hamiltonian flows are not defined there. However, we check that each orbit of the flow for both vector fields has a limit approaching the zero section, even if the vector fields are not smooth there.

To do this, we perform an oriented blow-up (denoted by $\widehat{T^\star L}$) of $T^\star L$ along the zero section, replacing each point of L with the space of directions at that point. (The construction is analogous to the blow-up construction familiar to complex geometers.) We work in "polar coordinates" defined on patches $U^{(k)}$ where $y_k \neq 0$.

Without loss of generality, assume $k = 1$ so that $y_1 \neq 0$. The calculations for other values of k are similar (and equally tedious). Recall that in a coordinate chart, the blowing up map along the zero section is given by

$$B : (x_1, \ldots, x_n, y_1, \ldots, y_n)^T \mapsto \left(x_1, \ldots, x_n, r, \frac{y_2}{y_1}, \ldots, \frac{y_n}{y_1}\right)^T,$$

where $r = \sqrt{y_1^2 + \ldots + y_n^2}$. Let $\eta_j = \eta_j^{(1)} = \frac{y_j}{y_1}$. The corresponding blowing down map (which is a diffeomorphism away from the boundary) is given by

$$\beta : (x_1, \ldots, x_n, r, \eta_2, \ldots, \eta_n)^T \mapsto \left(x_1, \ldots, x_n, \frac{r}{||\eta||}, \frac{r}{||\eta||}\eta_2, \ldots, \frac{r}{||\eta||}\eta_n\right)^T.$$

Then in the coordinate patch $U^{(1)}$, we compute the lift of the Hamiltonian vector field X_h corresponding to a Hamiltonian h in terms of the coordinates $(x_1, \ldots, x_n, r, \eta_2, \ldots, \eta_n)$, push the lift $\widehat{X_h}$ back down to the original space, and compute the limit of the resulting vector field as $r \to 0$. We will see that the limit of the vector field, and hence of the corresponding flow, is well behaved approaching the zero section.

In the above coordinates, the matrix corresponding to the derivative map dB is given by

$$\begin{pmatrix} I_n & 0 & 0 & 0 & \dots & 0 \\ & \frac{y_1}{r} & \frac{y_2}{r} & \frac{y_3}{r} & \dots & \frac{y_n}{r} \\ & -\frac{y_2}{y_1^2} & \frac{1}{y_1} & 0 & 0 & 0 \\ 0_n & \vdots & 0 & \frac{1}{y_1} & 0 & 0 \\ & \vdots & \vdots & & \ddots & \\ & -\frac{y_n}{y_1^2} & 0 & \dots & 0 & \frac{1}{y_1} \end{pmatrix}.$$

Then the lift of X_h corresponding to a Hamiltonian function h can be calculated to be

$$dB \cdot X_h = \begin{pmatrix} I_n & 0 & 0 & 0 & \dots & 0 \\ & \frac{y_1}{r} & \frac{y_2}{r} & \frac{y_3}{r} & \dots & \frac{y_n}{r} \\ & -\frac{y_2}{y_1^2} & \frac{1}{y_1} & 0 & 0 & 0 \\ 0_n & \vdots & 0 & \frac{1}{y_1} & 0 & 0 \\ & \vdots & \vdots & & \ddots & \\ & -\frac{y_n}{y_1^2} & 0 & \dots & 0 & \frac{1}{y_1} \end{pmatrix} \begin{pmatrix} -h_{y_1} \\ \vdots \\ -h_{y_n} \\ h_{x_1} \\ \vdots \\ h_{x_n} \end{pmatrix}$$

$$
(4.1) \qquad = \begin{pmatrix} -h_{y_1} \\ \vdots \\ -h_{y_n} \\ \frac{y_1}{r}h_{x_1} + \ldots + \frac{y_n}{r}h_{x_n} \\ -\frac{y_2}{y_1^2}h_{x_1} + \frac{1}{y_1}h_{x_2} \\ -\frac{y_3}{y_1^2}h_{x_1} + \frac{1}{y_1}h_{x_3} \\ \vdots \\ -\frac{y_n}{y_1^2}h_{x_1} + \frac{1}{y_1}h_{x_n} \end{pmatrix},
$$

We first consider $h = \sqrt{H_0} = \sqrt{\frac{1}{2}\sum_{i,j} g_{ij}(x)y_i y_j}$ and handle the situation for $h = \sqrt{H}$ later on in this section. In the present case, the partial derivatives that appear in (4.1) are

$$
\begin{aligned}
h_{x_k} &= \frac{1}{4h}\sum_{i,j}\frac{\partial g_{ij}}{\partial x_k}y_i y_j \quad \text{and} \\
h_{y_k} &= \frac{1}{2h}\sum_{i=1}^{n} g_{ik}(x)y_i.
\end{aligned}
$$

In terms of the blow-up coordinates (x, r, η) and letting $||\eta|| = \sqrt{1 + \eta_2^2 + \ldots + \eta_n^2}$ and $\eta_1 = 1$, h_{x_k} and h_{y_k} become

$$
\begin{aligned}
h_{x_k} &= \frac{r}{||\eta||}\frac{1}{4\sqrt{\frac{1}{2}\sum_{i,j} g_{ij}\eta_i\eta_j}}\left(\sum_{i,j=1}^{n}\frac{\partial g_{ij}}{\partial x_k}\eta_i\eta_j\right) \quad \text{and} \\
h_{y_k} &= \frac{1}{2\sqrt{\frac{1}{2}\sum_{i,j} g_{ij}\eta_i\eta_j}}\left(\sum_{i=1}^{n} g_{ki}(x)\eta_i\right).
\end{aligned}
$$

For aesthetic purposes, let us denote the quantity $\sqrt{\frac{1}{2}\sum_{i,j} g_{ij}\eta_i\eta_j}$ by G. Then in the blow-up coordinates, $\widehat{X_h} = dB \cdot X_h$ becomes

$$\widehat{X_h} = \begin{pmatrix} -\frac{1}{2G}\left(\sum_{i=1}^{n} g_{1i}(x)\eta_i\right) \\ \vdots \\ -\frac{1}{2G}\left(\sum_{i=1}^{n} g_{ni}(x)\eta_i\right) \\ \frac{r}{||\eta||^2}\frac{1}{4G}\sum_k\sum_{i,j}\frac{\partial g_{ij}}{\partial x_k}\eta_i\eta_j\eta_k \\ \frac{1}{4G}\sum_{i,j}\left(-\eta_2\frac{\partial g_{ij}}{\partial x_1}+\frac{\partial g_{ij}}{\partial x_2}\right)\eta_i\eta_j \\ \vdots \\ \frac{1}{4G}\sum_{i,j}\left(-\eta_n\frac{\partial g_{ij}}{\partial x_1}+\frac{\partial g_{ij}}{\partial x_n}\right)\eta_i\eta_j \end{pmatrix}.$$

We now push the vector field back down to $T^\star L$ under the blowing down map

$$\beta^{(1)} = \beta : (x_1,\dots,x_n,r,\eta_2,\dots,\eta_n)^T \mapsto \left(x_1,\dots,x_n,\frac{r}{||\eta||},\frac{r}{||\eta||}\eta_2,\dots,\frac{r}{||\eta||}\eta_n\right)^T,$$

the corresponding differential $d\beta$ of which has matrix representation

$$d\beta = \begin{pmatrix} I_n & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{||\eta||} & -\frac{r\eta_2}{||\eta||^3} & -\frac{r\eta_3}{||\eta||^3} & \cdots & -\frac{r\eta_n}{||\eta||^3} \\ 0 & \frac{\eta_2}{||\eta||} & \frac{r}{||\eta||^3}(||\eta||^2-\eta_2^2) & -\frac{r\eta_2\eta_3}{||\eta||^3} & \cdots & -\frac{r\eta_2\eta_n}{||\eta||^3} \\ 0 & \frac{\eta_3}{||\eta||} & -\frac{r\eta_3\eta_2}{||\eta||^3} & \frac{r}{||\eta||^3}(||\eta||^2-\eta_3^2) & \cdots & -\frac{r\eta_2\eta_n}{||\eta||^3} \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \frac{\eta_n}{||\eta||} & -\frac{r\eta_n\eta_2}{||\eta||^3} & -\frac{r\eta_n\eta_3}{||\eta||^3} & \cdots & \frac{r}{||\eta||^3}(||\eta||^2-\eta_n^2) \end{pmatrix}.$$

The calculation for $d\beta\cdot\widehat{X_h}$ is tedious, but for completeness, we include it here. We first note that the coefficients for the ∂_{x_k} entries are unaffected, since $d\beta$ is the identity on the x-coordinates. In the coordinate patch $U^{(1)}$, we calculate the coefficients for ∂_{y_1} and ∂_{y_2} (in terms of (x,r,η)) and take the limit as $r\to 0$ to rule out any messy behavior. The calculations for the other ∂_{y_k} terms are completely analogous to the $k = 2$ case.

First, the ∂_{y_1} term works out to be

$$\begin{aligned}
\text{coefficient for } \partial_{y_1} &= \frac{r}{||\eta||^3} \cdot \frac{1}{4G} \sum_k \sum_{i,j} \frac{\partial g_{ij}}{\partial x_k} \eta_i \eta_j \eta_k \\
&\quad - \frac{r}{||\eta||^3} \sum_k \eta_k \sum_{i,j} \left(-\eta_k \frac{\partial g_{ij}}{\partial x_1} + \frac{\partial g_{ij}}{\partial x_k} \right) \eta_i \eta_j \cdot \frac{1}{4G} \\
&= \frac{r}{||\eta||^3} \cdot \frac{1}{4G} \sum_k \sum_{i,j} \left(\frac{\partial g_{ij}}{\partial x_k} \eta_i \eta_j \eta_k + \frac{\partial g_{ij}}{\partial x_1} \eta_k^2 \eta_i \eta_j - \frac{\partial g_{ij}}{\partial x_k} \eta_i \eta_j \eta_k \right) \\
&= \frac{r}{||\eta||^3} \frac{1}{4G} \left(\sum_k \eta_k^2 \right) \left(\sum_{i,j} \frac{\partial g_{ij}}{\partial x_1} \eta_i \eta_j \right) \\
&= \frac{r}{||\eta||} \frac{1}{4G} \sum_{i,j} \frac{\partial g_{ij}}{\partial x_1} \eta_i \eta_j \to 0 \text{ as } r \to 0.
\end{aligned}$$

We note that the above expression, when expressed in terms of the original coordinates (x, y), is just $\frac{1}{4h} \sum_{i,j} \frac{\partial g_{ij}}{\partial x_1} y_i y_j = h_{x_1}$ as expected. The calculation for ∂_{y_2} is as follows (recalling that $\eta_1 = 1$):

$$
\begin{aligned}
\text{coefficient for } \partial_{y_2} &= \frac{r\eta_2}{||\eta||^3}\frac{1}{4G}\sum_k\sum_{i,j}\frac{\partial g_{ij}}{\partial x_k}\eta_i\eta_j\eta_k \\
&\quad + \frac{r}{||\eta||^3}\left(||\eta||^2 - \eta_2^2\right)\frac{1}{4G}\sum_{i,j}\left(-\eta_2\frac{\partial g_{ij}}{\partial x_1} + \frac{\partial g_{ij}}{\partial x_2}\right)\eta_i\eta_j \\
&\quad - \frac{r}{||\eta||^3}\eta_2\eta_3\frac{1}{4G}\sum_{i,j}\left(-\eta_3\frac{\partial g_{ij}}{\partial x_1} + \frac{\partial g_{ij}}{\partial x_3}\right)\eta_i\eta_j - \dots \\
&\quad - \frac{r}{||\eta||^3}\eta_2\eta_n\frac{1}{4G}\sum_{i,j}\left(-\eta_n\frac{\partial g_{ij}}{\partial x_1} + \frac{\partial g_{ij}}{\partial x_n}\right)\eta_i\eta_j \\
&= \frac{r}{||\eta||^3}\frac{1}{4G}\left\{\begin{array}{l}
\eta_2\sum_k\eta_k\sum_{i,j}\frac{\partial g_{ij}}{\partial x_k}\eta_i\eta_j \\
+||\eta||^2\sum_{i,j}\left(-\eta_2\frac{\partial g_{ij}}{\partial x_1} + \frac{\partial g_{ij}}{\partial x_2}\right) \\
-\eta_2^2\sum_{i,j}\left(-\eta_2\frac{\partial g_{ij}}{\partial x_1} + \frac{\partial g_{ij}}{\partial x_2}\right)\eta_i\eta_j \\
-\eta_2\eta_3\sum_{i,j}\left(-\eta_3\frac{\partial g_{ij}}{\partial x_1} + \frac{\partial g_{ij}}{\partial x_3}\right)\eta_i\eta_j \\
\vdots \\
-\eta_2\eta_n\sum_{i,j}\left(-\eta_n\frac{\partial g_{ij}}{\partial x_1} + \frac{\partial g_{ij}}{\partial x_n}\right)\eta_i\eta_j
\end{array}\right\} \\
&= \frac{r}{||\eta||^3}\frac{1}{4G}\left\{\begin{array}{l}
\eta_2\sum_{k=1}^n\eta_k\sum_{i,j}\frac{\partial g_{ij}}{\partial x_k}\eta_i\eta_j \\
+||\eta||^2\sum_{i,j}\left(-\eta_2\frac{\partial g_{ij}}{\partial x_1} + \frac{\partial g_{ij}}{\partial x_2}\right)\eta_i\eta_j \\
-\eta_2\sum_{k=2}^n\eta_k\sum_{i,j}\left(-\eta_k\frac{\partial g_{ij}}{\partial x_1} + \frac{\partial g_{ij}}{\partial x_k}\right)\eta_i\eta_j
\end{array}\right\} \\
&= \frac{r}{||\eta||^3}\frac{1}{4G}\left\{\begin{array}{l}
\eta_2\sum_{k=1}^n\eta_k\sum_{i,j}\frac{\partial g_{ij}}{\partial x_k}\eta_i\eta_j \\
-\eta_2||\eta||^2\sum_{i,j}\frac{\partial g_{ij}}{\partial x_1}\eta_i\eta_j \\
+||\eta||^2\sum_{i,j}\frac{\partial g_{ij}}{\partial x_2}\eta_i\eta_j \\
+\eta_2\sum_{k=2}^n\eta_k^2\sum_{i,j}\frac{\partial g_{ij}}{\partial x_1}\eta_i\eta_j \\
-\eta_2\sum_{k=2}^n\eta_k\sum_{i,j}\frac{\partial g_{ij}}{\partial x_k}\eta_i\eta_j
\end{array}\right\}
\end{aligned}
$$

$$= \frac{r}{||\eta||^3}\frac{1}{4G}\left\{\begin{array}{l} \eta_2 \sum_{k=1}^n \eta_k^2 \sum_{i,j} \dfrac{\partial g_{ij}}{\partial x_1}\eta_i\eta_j \\ -\eta_2||\eta||^2 \sum_{i,j} \dfrac{\partial g_{ij}}{\partial x_1}\eta_i\eta_j \\ +||\eta||^2 \sum_{i,j} \dfrac{\partial g_{ij}}{\partial x_2}\eta_i\eta_j \end{array}\right\}$$

$$= \frac{r}{||\eta||^3}\frac{1}{4G}||\eta||^2 \sum_{i,j} \frac{\partial g_{ij}}{\partial x_2}\eta_i\eta_j$$

$$= \frac{r}{||\eta||}\frac{1}{4G} \sum_{i,j} \frac{\partial g_{ij}}{\partial x_2}\eta_i\eta_j \to 0 \text{ as } r \to 0.$$

We also note that after expressing the above in terms of the coordinates (x, y), we recover h_{x_2}. Thus the Hamiltonian vector field $X_{\sqrt{H_0}}$ has a limit approaching the zero section from above and its lift $\widehat{X}_{\sqrt{H_0}}$ is well behaved in the blown-up space $\widehat{T^\star L}$. Furthermore, since H and H_0 differ only by higher order terms in r, we see that their flows agree in the limit (as $r \to 0$) approaching the boundary $\partial\widehat{T^\star L}$.

We can see this more explicitly by calculating the lift of $X_{\sqrt{H}}$ using $h = \sqrt{H}$ in (4.1), the entries of which are in terms of the partial derivatives h_{x_k} and h_{y_k}. We will soon see that matrix entries agree with those of $\widehat{X}_{\sqrt{H_0}}$ at $r = 0$.

By construction, $H = H_0 + O_y(3)$. Converting to blow-up coordinates, we have that $H = H_0 + O_r(3)$, since each factor of y_k contributes a factor of r in the new coordinates under the change $y_k \mapsto \eta_k \cdot y_1 = \eta_k \cdot \frac{r}{||\eta||}$. Setting $h = \sqrt{H} = \sqrt{H_0 + O_r(3)}$, we compute the x_k- and y_k-partial derivatives in terms of the blow-up coordinates. (There are also some more detailed calculations in Section 4.3 below.)

$$\begin{aligned}
h_{x_k} &= \frac{\partial}{\partial x_k}\left(\sqrt{H}\right) \\
&= \frac{1}{2\sqrt{H_0 + O_y(3)}}\left[\frac{\partial}{\partial x_k}\left(H_0 + O_y(3)\right)\right] \\
&= \frac{1}{2\sqrt{H_0 + O_y(3)}}\left[\frac{1}{2}\sum_{i,j}\frac{\partial g_{ij}}{\partial x_k}y_i y_j + \frac{\partial}{\partial x_k}O_y(3)\right] \\
&= \frac{1}{2\sqrt{H_0 + O_y(3)}}\left[\frac{1}{2}\sum_{i,j}\frac{\partial g_{ij}}{\partial x_k}y_i y_j + O_y(3)\right] \\
&= \frac{1}{2\sqrt{\frac{r^2}{||\eta||^2}\frac{1}{2}\sum_{i,j} g_{ij}\eta_i\eta_j + O_r(3)}}\left[\frac{r^2}{||\eta||^2}\frac{1}{2}\sum_{i,j}\frac{\partial g_{ij}}{\partial x_k}\eta_i\eta_j + O_r(3)\right] \\
&= \frac{1}{2\sqrt{\frac{1}{2}\sum_{i,j} g_{ij}\eta_i\eta_j + O_r(1)}}\left[\frac{r}{||\eta||}\frac{1}{2}\sum_{i,j}\frac{\partial g_{ij}}{\partial x_k}\eta_i\eta_j + O_r(2)\right]
\end{aligned}$$

$$\begin{aligned}
h_{y_k} &= \frac{\partial}{\partial y_k}\left(\sqrt{H}\right) \\
&= \frac{1}{2\sqrt{H_0 + O_y(3)}}\left[\frac{\partial}{\partial y_k}\left(H_0 + O_y(3)\right)\right] \\
&= \frac{1}{2\sqrt{H_0 + O_y(3)}}\left[\sum_{i=1}^{n} g_{ik}(x)y_i + \frac{\partial}{\partial y_k}O_y(3)\right] \\
&= \frac{1}{2\sqrt{H_0 + O_y(3)}}\left[\sum_{i=1}^{n} g_{ik}(x)y_i + O_y(2)\right] \\
&= \frac{1}{2\sqrt{\frac{r^2}{||\eta||^2}\frac{1}{2}\sum_{i,j} g_{ij}\eta_i\eta_j + O_r(3)}}\left[\frac{r}{||\eta||}\sum_{i=1}^{n} g_{ik}(x)\eta_i + O_r(2)\right] \\
&= \frac{1}{2\sqrt{\frac{1}{2}\sum_{i,j} g_{ij}\eta_i\eta_j + O_r(1)}}\left[\sum_{i=1}^{n} g_{ik}(x)\eta_i + O_r(1)\right]
\end{aligned}$$

It should be clear from the above expressions that the partial derivatives of $\sqrt{H_0}$ and $\sqrt{H}$ agree as $r \to 0$, so that the their flows agree in the blow-up at $\{r = 0\}$.

4.2 Polarized Symplectic Manifolds Whose Lagrangians Are Zoll

Using the above framework, if we impose one more (restrictive) condition on the Morse-Bott function H, we can show that L must be a Zoll manifold. The calculations towards the end of Section 4.1 show that $\widehat{X}_{\sqrt{H}}$ and $\widehat{X}_{\sqrt{H_0}}$ (the latter corresponding to the co-geodesic flow of the metric induced by the Hessian of H) are equal along $\{r = 0\}$. The orbits of $\widehat{X}_{\sqrt{H_0}}$ correspond to geodesics on L so that if all orbits are periodic of the same period, then the corresponding geodesics on L are periodic and of equal length.

Assuming further that the flow of $X_{\sqrt{H}}$ is freely periodic away from L and V in W, we get that the flow of $X_{\sqrt{H_0}}$ is also freely periodic away from L and V. We aim to show that no short orbits are introduced in the blow-up construction. Then interpreting the flow of $X_{\sqrt{H_0}}$ as the co-geodesic flow on $T^\star L$ yields periodic geodesics of the same length on L under the projection $\pi : T^\star L \to L$.

Theorem IV.2. *Let (W, ω) be a polarized symplectic manifold (in the sense of Audin), and suppose the Morse-Bott function H is such that the Hamiltonian flow of $X_{\sqrt{H}}$ generates a free S^1-action off of L and V. Then L is a Zoll manifold.*

Proof. As before, we consider the function $h = \sqrt{H}$ on the oriented blow-up $\widehat{T^\star L}$ along L in $T^\star L$. By assumption, on $\widehat{T^\star L}$, all orbits of the flow of $\widehat{X}_{\sqrt{H}}$ are closed with some common period l, but perhaps not a common *least* period.

The key point to note here is that if there exists a short orbit, it must lie on the boundary level set $\partial\widehat{T^\star L}$. Suppose it has period l/k, for k some integer larger than 1. We examine the Poincaré map for this offending geodesic and consider its flow map on a transverse Poincaré section. The picture is a half-space of sorts.

Without loss of generality, fix an S^1-invariant metric on the total space $\widehat{T^\star L}$ (i.e.

such that S^1 acts by isometries). Here, we are implicitly identifying a neighborhood of the zero section of the normal bundle of the boundary with a neighborhood of the boundary in $\widehat{T^\star L}$ via the exponential map. We want the S^1-action induced by the flow to be free on the boundary. For v normal to the boundary and invariant under the S^1-action, we have

$$g \cdot \exp_\xi(tv) = \exp_{g\cdot\xi}(tv), \qquad t \in [0,\epsilon), \xi \in \partial\widehat{T^\star L}, g \in S^1$$

since $g \in S^1$ is an isometry (i.e. $d(\xi, \exp_\xi(tv)) = d(g\cdot\xi, g\cdot\exp_\xi(tv))$, and the exponential map is a radial isometry. Suppose by way of contradiction that there is some element $g \in S^1$ which is not the identity, such that $g \cdot \xi = \xi$ for some ξ in the boundary (so that the orbit of ξ represents a short orbit).

$$\begin{aligned} g \cdot \exp_\xi(tv) &= \exp_{g\cdot\xi}(tv) \quad \text{by the above observation,} \\ &= \exp_\xi(tv) \quad \text{since } g\cdot\xi = \xi. \end{aligned}$$

In particular, for $t > 0$, $g \neq Id$ fixes $\exp_\xi(tv) \notin \partial\widehat{T^\star L}$, contradicting the freeness of the action away from the boundary. Thus, there can be no short orbits of $\widehat{X}_{\sqrt{H}}$ on the boundary. By the calculations in Section 4.1, since $\widehat{X}_{\sqrt{H}}$ and $\widehat{X}_{\sqrt{H_0}}$ agree on the boundary, there are consequently no short orbits of $\widehat{X}_{\sqrt{H_0}}$, so that the metric induced by the Hessian is indeed Zoll. $\square$

The above theorem gives that the Lagrangian L in this case cannot be a lens space (except for the case of $\mathbb{RP}^n$). It is still possible that L could be a lens space under weaker assumptions on the Morse-Bott function, however (see [3], Proposition 2.7, which considers the possibility of branched covers by the case where the Lagrangian is a sphere).

4.3 The Geodesic Flow of the Metric vs. the Hamiltonian Flow of H

Up to this point, we have three polarized symplectic manifolds: the given (W, ω), the symplectic cut performed at a regular level using the S^1-action associated to the periodic flow of $X_{\sqrt{H}}$, and the Zoll cut performed again at a regular leveel using the S^1-action from the flow of $X_{\sqrt{H_0}}$. They are, of course, diffeomorphic, being compactifications of diffeomorphic disk bundles, but we would like to say more.

In this section, we aim to show that the two symplectic cuts mentioned above are equivariantly symplectomorphic under the additional assumption that H and H_0 differ only by terms of order 4 and higher in the y-variables. We do not quite achieve this symplectomorphism due to the non-smoothness of $\sqrt{H}$ and $\sqrt{H_0}$ along the critical sets, however we do obtain an equivariant symplectomorphism in the blown-up space. The proof closely follows the arguments of Weinstein [32] with modifications to deal with the S^1-action and the non-smoothness of $\sqrt{H}$.

We first identify the two S^1-actions (one coming from $\sqrt{H}$ and the other from $\sqrt{H_0}$) by conjugation via exponential maps in order to send the level sets of one Hamiltonian to the other. To do this, we require metrics g_0, g_1 on the normal bundle to $\partial\widehat{T^\star L} = \widehat{L}$ that are invariant with respect to each circle action. This yields exponential functions that allow us to identify a neighborhood of the zero section of the normal bundle to a neighborhood of $\widehat{L}$ in the blown-up space. We take the following steps.

1. We show that the two vector fields $\widehat{\xi_1} = \widehat{X}_{\sqrt{H}}$ and $\widehat{\xi_0} = \widehat{X}_{\sqrt{H_0}}$ differ only up to higher order terms, i.e. $\widehat{\xi_1} - \widehat{\xi_0} = O_r(2)$, in the sense that the Taylor series of the coefficients differ only by second and higher order terms in r.

2. We show that $\mathcal{L}_{\widehat{\xi_1} - \widehat{\xi_0}} g_0 = O_r(1)$.

3. We can take $g_1 = \frac{1}{2\pi}\int_0^{2\pi}\left(\Phi_t^{\xi_1}\right)^\star g_0\,dt$ to obtain a metric invariant under the flow of $\widehat{\xi_1}$, which will agree with g_0 along $\{r=0\}$ and give the same unit normal vector by the above calculation.

4. Conjugating the two S^1-actions given by the flows induces $df_\star = Id$ on $T\widehat{T^\star L}$ along $\widehat{L}$, where $f = \exp_{g_0}\circ(Id_{\widehat{L}}\times Id_I)\circ\exp_{g_1}^{-1}$ takes $\widehat{\xi_1}$ to $\widehat{\xi_0}$ (by which we mean $f_\star\widehat{\xi_1} = \widehat{\xi_0}$).

5. Then $\widehat{\omega_0} - (f^{-1})^\star\widehat{\omega_0} = \widehat{\omega_0} - \widehat{\omega_1} = O_r(1)$, where we use the (hopefully) obvious notation $\widehat{\omega_0}$ to denote the lift of the symplectic form ω_0 to the blown-up space. We give an equivariant Moser argument in the blow-up, given that $[\widehat{\omega_0}] = [(f^{-1})^\star\widehat{\omega_0}]$ so the forms differ by an exact form as in the Poincare Lemma.

Now assume that $H = H_0 + O_y(4)$, where $H_0 = \frac{1}{2}\sum_{i,j=1}^n g_{ij}(x)y_iy_j$ and $g_{ij}(x)$ is given by the Hessian of H after setting $y=0$, as before. We know that $H = H_0 + O_r(4)$ (since each factor of y_i contributes a factor of r when converting to blow-up coordinates). We may write

$$H = r^2F + r^4G, \qquad H_0 = r^2F,$$

for some smooth functions $F \neq 0$ and G. Then we have $\sqrt{H_0} = \sqrt{r^2F} = rF^{1/2}$ and

$$\begin{aligned}
\sqrt{H} &= \sqrt{r^2F + r^4G} \\
&= r\sqrt{F + r^2G} \\
&= r(F + r^2G)^{1/2} \\
&= r\left(F\left(1 + r^2\frac{G}{F}\right)\right)^{1/2} \\
&= rF^{1/2}\left(1 + r^2\frac{G}{F}\right)^{1/2}.
\end{aligned}$$

Using the binomial series expansion around $\frac{r^2G}{F}=0$, we obtain

$$\begin{aligned}
&= rF^{1/2}\left(1+\frac{1}{2}r^2\frac{G}{F}+\frac{1}{2}\cdot-\frac{1}{2}r^4\frac{G^2}{F^2}+\frac{1}{2}\cdot-\frac{1}{2}\cdot-\frac{3}{2}r^6\frac{G^3}{F^3}+\ldots\right)\\
&= rF^{1/2}\left(1+\frac{1}{2}\frac{G}{F}r^2-\frac{1}{4}\frac{G^2}{F^2}r^4+\frac{3}{8}\frac{G^3}{F^3}r^6-\ldots\right)\\
&= rF^{1/2}+\frac{1}{2}\frac{G}{F^{1/2}}r^3-\frac{1}{4}\frac{G^2}{F^{3/2}}r^5+\frac{3}{8}\frac{G^4}{F^{5/2}}r^7-\ldots.
\end{aligned}$$

Together, we have

$$\begin{aligned}
\sqrt{H}-\sqrt{H_0} &= \frac{1}{2}\frac{G}{F^{1/2}}r^3-\frac{1}{4}\frac{G^2}{F^{3/2}}r^5+\frac{3}{8}\frac{G^4}{F^{5/2}}r^7-\ldots\\
&= O_r(3).
\end{aligned}$$

This gives us that

$$\frac{\partial}{\partial x_k}\left(\sqrt{H}-\sqrt{H_0}\right)=O_r(3),$$

and using equation (4.1), since the coefficient for the component of $\widehat{\xi_1}-\widehat{\xi_0}$ in the $\frac{\partial}{\partial r}$ direction is given in terms of the x_k partial derivatives, we get that

$$\left(\widehat{\xi_1}-\widehat{\xi_0}\right)^r=O_r(3). \tag{4.2}$$

(The superscript just denotes the directional component.)

Writing the coefficient $\frac{y_1}{r}h_{x_1}+\ldots+\frac{y_n}{r}h_{x_n}$ in terms of the blow-up coordinates $(x_1,\ldots,x_n,r,\eta_2,\ldots,\eta_n)$ does not introduce or reduce any powers of r coming from the x_k partial derivatives (because each $\frac{y_k}{r}$ becomes $\eta_k\frac{r}{||\eta||}\cdot\frac{1}{r}$, where $\eta_1=1$).

Equation (4.2) also gives that the vector fields differ by $O_r(3)$ terms in the $\frac{\partial}{\partial y_k}$ directions, since these coefficients depend only on the x_k partial derivatives. Referring to the calculation for $dB\cdot X_h$, we see that lifting to the blow-up gives the following coefficient for the $\frac{\partial}{\partial\eta_k}$ direction:

$$-\frac{y_k}{y_1^2}h_{x_1}+\frac{1}{y_1}h_{x_k}.$$

Converting to blow-up coordinates introduces a factor of $\frac{1}{r}$ so that the coefficients of the difference $\widehat{\xi_1} - \widehat{\xi_0}$ differ by $O_r(2)$-terms in the η_k-directions.

The blow-up map is simply the identity in the x_k-directions, so the $\frac{\partial}{\partial x_k}$ component of $\widehat{\xi_1} - \widehat{\xi_0}$ is given by

$$\frac{\partial}{\partial y_k}\left(\sqrt{H} - \sqrt{H_0}\right) = \frac{\partial}{\partial r}\left(\sqrt{H} - \sqrt{H_0}\right) \cdot \frac{\partial r}{\partial y_k},$$

by the Chain Rule. Since $r = \sqrt{y_1^2 + \ldots + y_n^2}$, a quick calculation gives $\frac{\partial r}{\partial y_k} = \frac{\eta_k}{||\eta||}$. Thus, differentiating with respect to y_k decreases the order of r in the difference $(\sqrt{H} - \sqrt{H_0}) = O_r(3)$ by 1, so that the $\frac{\partial}{\partial x_k}$ components of $\widehat{\xi_1} - \widehat{\xi_0}$ are $O_r(2)$.

We now consider the metric g_0, which is S^1-invariant with respect to the S^1-action induced by the flow of $\widehat{\xi_0} = \widehat{X}_{\sqrt{H_0}}$. We relabel the coordinates $(x_1, \ldots, x_n, \eta_2, \ldots, \eta_n, r)$ as $(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n-1}, x_{2n} = r)$ and write $g_0 = g_{ij} dx^i \otimes dx^j$, where summation is implied when an index appears both raised and lowered. Since g_0 is invariant under the flow of $\widehat{\xi_0}$, we have $\mathcal{L}_{\widehat{\xi_0}} g_0 = 0$.

Claim IV.3. $\mathcal{L}_{\widehat{\xi_1} - \widehat{\xi_0}}(g_0) = O_r(1)$.

Proof. Recall that the Lie derivative along a vector field $X = X^i \frac{\partial}{\partial x^i}$ can be calculated in terms of the Lie bracket as follows:

$$\begin{aligned} \mathcal{L}_{X^i \frac{\partial}{\partial x^i}}\left(\tfrac{\partial}{\partial x^k}\right) &= \left[X^i \tfrac{\partial}{\partial x^i}, \tfrac{\partial}{\partial x^k}\right] \\ &= -\frac{\partial X^i}{\partial x^k} \frac{\partial}{\partial x^i}. \end{aligned}$$

Then we can calculate $\mathcal{L}_{X^i \frac{\partial}{\partial x^i}}(dx^l)$:

$$\begin{aligned} 0 &= \mathcal{L}_{X^i \frac{\partial}{\partial x^i}} \delta_l^k = \mathcal{L}_{X^i \frac{\partial}{\partial x^i}} \langle \tfrac{\partial}{\partial x^k}, dx^l \rangle \\ &= \langle \mathcal{L}_{X^i \frac{\partial}{\partial x^i}} \tfrac{\partial}{\partial x^k}, dx^l \rangle + \langle \tfrac{\partial}{\partial x^k}, \mathcal{L}_{X^i \frac{\partial}{\partial x^i}} dx^l \rangle \\ &= \langle -\tfrac{\partial X^i}{\partial x^k} \tfrac{\partial}{\partial x^i}, dx^l \rangle + \langle \tfrac{\partial}{\partial x^k}, \mathcal{L}_{X^i \frac{\partial}{\partial x^i}} dx^l \rangle \\ \langle \tfrac{\partial}{\partial x^k}, \mathcal{L}_{X^i \frac{\partial}{\partial x^i}} dx^l \rangle &= \langle \tfrac{\partial X^i}{\partial x^k} \tfrac{\partial}{\partial x^i}, dx^l \rangle. \end{aligned}$$

The left-hand side of the last equation gives the coefficient a_k of the k-th component of the Lie derivative, while the pairing on the right is only non-zero for $i = l$. This gives $a_k = \frac{\partial X^l}{\partial x^k}$ so that

$$\mathcal{L}_{X^i \frac{\partial}{\partial x^i}}(dx^l) = \frac{\partial X^l}{\partial x^k} dx^k.$$

In our case, $X = \widehat{\xi}_1 - \widehat{\xi}_0 = \sum_{i=1}^{2n-1} r^2\sigma^i \frac{\partial}{\partial x^i} + r^3\tau\frac{\partial}{\partial r}$, and we would like to take the Lie derivative of the $(0,2)$-tensor $g_0 = g_{ij}dx^i \otimes dx^j$ (recalling that $x^{2n} = r$).

$$\begin{aligned}
\mathcal{L}_X\left(g_{jk}dx^j \otimes dx^k\right) &= \mathcal{L}_X\left(g_{jk}dx^j\right) \otimes dx^k + g_{jk}dx^j \otimes \mathcal{L}_X dx^k \\
&= \left[X(g_{jk})dx^j + g_{jk}\mathcal{L}_X(dx^j)\right] \otimes dx^k + g_{jk}dx^j \otimes \mathcal{L}_X(dx^k) \\
&= \left[\underbrace{X(g_{jk})}_{O_r(2)} dx^j + g_{jk}(\underbrace{\tfrac{\partial X^j}{\partial x^l}}_{O_r(1)} dx^l)\right] \otimes dx^k + g_{jk}dx^j \otimes (\underbrace{\tfrac{\partial X^k}{\partial x^l}}_{O_r(1)} dx^l) \qquad (4.3) \\
&= O_r(1),
\end{aligned}$$

where by the last equality, we mean each coefficient of $dx^j \otimes dx^k$ is divisible by r. $\square$

What this calculation shows in particular is that when $r = 0$ (that is, along $\partial\widehat{T^\star L}$), $\mathcal{L}_{\widehat{\xi}_1} g_0 = 0$. To obtain a new metric g_1 that is invariant with respect to the S^1-action induced by ξ_1, we average the metric g_0 over this S^1. That is,

$$g_1 = \int_0^{2\pi} (\phi_t^{\widehat{\xi}_1})^\star g_0 \, dt,$$

where the flow of $\widehat{\xi}_1$ is given by $\phi_t^{\widehat{\xi}_1}$. Since $\mathcal{L}_{\widehat{\xi}_1} g_0 = 0$ along $\{r = 0\}$, averaging over the flow of $\widehat{\xi}_1$ also preserves the unit normal to the boundary.

The exponential maps associated to each of the two metrics g_0 and g_1 (call them $\exp_{g_0}$ and $\exp_{g_1}$, respectively) allow us to identify a neighborhood of the boundary $\partial\widehat{T^\star L}$ with a tubular neighborhood of its normal bundle in the total space, which will be of the form $\widehat{L} \times I$ for some interval I.

In order to identify the two S^1-actions as one, we define the conjugating map

$$f = \exp_{g_0} \circ \left(Id_{\widehat{L}} \times Id_I \right) \circ \exp_{g_1}^{-1},$$

where $Id_{\widehat{L}}$ just sends the point (x, c) in the normal bundle to the corresponding point $x \in \widehat{L}$ and Id_I sends (x, c) to c in the interval I. The induced map $df_\star$ is the identity on $T\widehat{T^\star L}$ along $\widehat{L}$ and subsequently $[(f^{-1})^\star \widehat{\omega_0}] = [\widehat{\omega_0}]$.

After applying f, we have an S^1-action on a neighborhood of the zero section in $(\widehat{T^\star L}, \widehat{\omega_0})$, where $\widehat{\omega_0}$ is S^1-invariant, and the same S^1 acting on a neighborhood of the zero section in $(\widehat{T^\star L}, \widehat{(f^{-1})^\star \omega_0})$, where $(f^{-1})^\star \widehat{\omega_0}$ is S^1-invariant. We would like to say that the two symplectic forms $\widehat{\omega_0}$ and $(f^{-1})^\star \widehat{\omega_0}$ are equivariantly symplectomorphic. To achieve this, we use a slightly modified equivariant Moser's trick in the blow-up.

4.4 Moser's Trick and the Poincaré Lemma

Inspired by the proof of the Poincaré Lemma (see Bott-Tu [10]) and Moser's trick (Section II.17), we find a time-independent 1-form $\mu_t = \mu$ that allows us to show that our two symplectic forms on (a neighborhood of) $\partial \widehat{T^\star L}$ are locally equivariantly symplectically equivalent.

Given the two symplectic forms $\widehat{\omega}_{\text{canonical}} = \widehat{\omega_0}$ and $(f^{-1})^\star \widehat{\omega_0} = \widehat{\omega_1}$, we apply an equivariant version of Moser's trick on the blow-up $\widehat{T^\star L}$ to show that they are locally equivalent up to equivariant symplectomorphism. Since $df_\star$ acts as the identity near $\widehat{L}$, including in the normal direction, we have that $[\widehat{\omega_0}] = [\widehat{\omega_1}]$, and consider the 1-parameter family of symplectic forms

$$\begin{aligned} \widehat{\omega_t} &= (1-t)\widehat{\omega_0} + t\widehat{\omega_1} \\ &= \widehat{\omega_0} + t(\widehat{\omega_1} - \widehat{\omega_0}), \qquad 0 \leq t \leq 1. \end{aligned}$$

Then $[\widehat{\omega_t}] = [\widehat{\omega_0}]$ for all t. We want

$$\begin{aligned} \phi_t^\star \widehat{\omega_t} &= \widehat{\omega_0}, \qquad \forall t \\ \mathcal{L}_{v_t}\widehat{\omega_t} &= \underbrace{i_{v_t} d\widehat{\omega_t}}_{0} + di_{v_t}\widehat{\omega_t} + \dot{\widehat{\omega_t}} = 0 \\ di_{v_t}\widehat{\omega_t} &= -\dot{\widehat{\omega_t}} = \widehat{\omega_0} - \widehat{\omega_1} \\ &= -d\mu. \end{aligned}$$

To find μ, we apply methods from the proof of the Poincaré Lemma to the situation at hand in order to use an equivariant Moser's trick in the blow-up. We first identify a neighborhood of $\partial\widehat{T^\star L}$ in $\widehat{T^\star L}$ with a collar neighborhood $U_0 = \widehat{L} \times I$ for some $I = [0, \epsilon]$ corresponding to the (normal) r direction.

Since $[\widehat{\omega_1}] = [\widehat{\omega_0}]$, we know that $\widehat{\omega_1}$ and $\widehat{\omega_0}$ differ by an exact form. That is,

$$\widehat{\omega_1} - \widehat{\omega_0} = d\mu,$$

for some 1-form $\mu \in \Omega^1(U_0)$, which is 0 along $\widehat{L} \times \{0\}$. We would like to apply Moser's method to the family of cohomologous forms

$$\widehat{\omega_t} = \widehat{\omega_0} + t(\widehat{\omega_1} - \widehat{\omega_0}) = \widehat{\omega_0} + td\mu,$$

which we can assume to be symplectic (perhaps after shrinking neighborhoods) since non-degeneracy is an open property.

We can use the usual homotopy formula as in the case of the proof of the Poincaré Lemma to construct the exact form $d\mu$ as follows. Let $\iota_0 : \widehat{L} \hookrightarrow U_0$ be the inclusion of $\widehat{L}$ as the zero section $\widehat{L} \times \{0\}$ and let $\pi_0(x, r) = x$ be the projection onto the first factor. We define the retraction ρ_t for $t \in [0, 1]$ by

$$\begin{aligned} \rho_t : U_0 &\to U_0 \\ (x, r) &\mapsto (x, tr). \end{aligned}$$

Then ρ_t fixes $\widehat{L} \times \{0\}$ for all t, $\rho_0 = \iota_0 \circ \pi_0$ has image $\widehat{L} \times \{0\}$, and ρ_1 is the identity map.

Recall that a (de Rham) *homotopy operator* between $\rho_0 = \iota_0 \circ \pi_0$ and $\rho_1 = Id$ is a linear map

$$K : \Omega^k(U_0) \to \Omega^{k-1}(U_0)$$

that satisfies the homotopy formula

$$Id^\star - (\iota_0 \circ \pi_0)^\star = dK + Kd.$$

Since $dK + Kd$ maps closed forms to exact forms, it induces zero in cohomology. Then $(\iota_0 \circ \pi_0)^\star$ is chain homotopic to the identity. Setting $\Omega = \widehat{\omega_1} - \widehat{\omega_0}$, we have $d\Omega = 0$ and $\iota_0^\star \Omega = 0$, so that the above formula reduces to $\Omega = dK\Omega$.

We explicitly set

$$K\Omega = \int_0^1 \rho_t^\star \left(i_{X_t} \Omega \right) dt, \tag{4.4}$$

where $X_t(q)$ is the vector tangent to $\rho_s(p)$ at $s = t$ for $q = \rho_t(p)$. In this particular case, the vector field is just $X_t = r\frac{\partial}{\partial r}$. Since $X_t = 0$ for all points in $\widehat{L} \times \{0\} = \{r = 0\}$, $\mu_x = (K\Omega)_x = 0$ for all $x \in \widehat{L}$. One readily verifies that this definition for K satisfies the homotopy formula with the help of Cartan's formula. Let τ be a k-form.

$$\begin{aligned} Kd\tau + dK\tau &= \int_0^1 \rho_t^\star \left(i_{X_t} d\tau \right) dt + d \int_0^1 \rho_t^\star \left(i_{X_t} \tau \right) dt \\ &= \int_0^1 \rho_t^\star \left(i_{X_t} d\tau + d i_{X_t} \tau \right) dt \\ &= \int_0^1 \rho_t^\star \mathcal{L}_{X_t} \tau dt \\ &= \rho_1^\star \tau - \rho_0^\star \tau. \end{aligned}$$

The last equality follows from the Fundamental Theorem of Calculus and

$$\frac{d}{dt} \rho_t^\star \tau = \rho_t^\star \mathcal{L}_{X_t} \tau.$$

Without loss of generality, we can assume that μ and $d\mu$ are S^1-invariant (for example, by averaging $\mu = K\Omega$ over S^1).

Away from $\{r = 0\}$, solving Moser's equation

$$i_{v_t}\widehat{\omega_t} = -\mu$$

works as in the standard case. However, we would like to know that the equation can be solved so that $v_t = 0$ at $r = 0$ smoothly. To verify that we can do this, as before, we work in a coordinate neighborhood of the blown-up space and verify that the coefficients of v_t in these coordinates have limit 0 as r tends to 0.

Working as usual in the coordinate patch given by $y_1 \neq 0$, consider the matrix expression for $\widehat{\omega_0}$ in the coordinates $(x_1, \ldots, x_n, r, \eta_2, \ldots, \eta_n)$. Then 2-forms will have basis elements $dx_i \wedge dx_j$, $dx_i \wedge dr$, $dx_i \wedge d\eta_j$, $dr \wedge d\eta_i$, and $d\eta_i \wedge d\eta_j$. In terms of this basis and setting $||\eta|| = \sqrt{1 + \eta_2^2 + \ldots + \eta_n^2}$, the skew-symmetric matrix A_0 representing $\widehat{\omega_0}$ is given by:

$$A_0 = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix},$$

where A is the $n \times n$ matrix

$$\begin{pmatrix}
\frac{1}{||\eta||} & -\frac{r}{||\eta||^3}\eta_2 & -\frac{r}{||\eta||^3}\eta_3 & \cdots & -\frac{r}{||\eta||^3}\eta_n \\
\frac{\eta_2}{||\eta||} & \frac{r}{||\eta||}(1 - \frac{\eta_2}{||\eta||^2}) & -\frac{r}{||\eta||}\eta_2\eta_3 & \cdots & -\frac{r}{||\eta||}\eta_2\eta_n \\
\frac{\eta_3}{||\eta||} & -\frac{r}{||\eta||}\eta_3\eta_2 & \frac{r}{||\eta||}(1 - \frac{\eta_3}{||\eta||^2}) & \cdots & -\frac{r}{||\eta||}\eta_3\eta_n \\
\vdots & \vdots & & \ddots & \vdots \\
\frac{\eta_n}{||\eta||} & -\frac{r}{||\eta||}\eta_n\eta_2 & \cdots & -\frac{r}{||\eta||}\eta_n\eta_{n-1} & \frac{r}{||\eta||}(1 - \frac{\eta_n}{||\eta||^2})
\end{pmatrix}.$$

Recall that $\widehat{\omega_t} = \widehat{\omega_0} + td\mu$. In order to invert Moser's equation, which in matrix form looks like

$$A_t V_t = -N, \tag{4.5}$$

we need to know that the matrix A_t representing $\widehat{\omega_t}$ is "smooth enough" in the blow-up such that, when applied to the matrix $-N$ representing $-\mu$, we get a vector field V_t that is 0 along $\widehat{L}$. Since the definition of A_t ($\widehat{\omega_t}$) depends on N (μ), it is enough to consider the powers of r that appear in the Taylor expansions at $r = 0$ for the entries in N corresponding to the coefficients of μ.

Claim IV.4. r^2 divides (the Taylor series for the coefficients of) μ.

Proof. In the homotopy formula (4.4), contraction with $X_t = r\frac{\partial}{\partial r}$ introduces one factor of r. Since $\Omega = \widehat{\omega_1} - \widehat{\omega_0} = 0$ along $\{r = 0\}$, the Taylor series for the coefficients of Ω in the integral are also divisible by r, so that r^2 divides μ. □

In particular, r divides $d\mu$. Thus, we can express $\widehat{\omega_t}$ in matrix form as

$$\begin{aligned} A_t &= A_0 + t \cdot M = A_0 + t \cdot rE \\ &= A_0(Id + t \cdot rA_0^{-1}E) \end{aligned} \tag{4.6}$$

for some matrix E with smooth entries. In the above, A_0^{-1} is given by

$$A_0^{-1} = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}, \tag{4.7}$$

where B is the $n \times n$ matrix given by

$$\begin{pmatrix} -\frac{1}{||\eta||} & \frac{\eta_2||\eta||}{r} & \frac{\eta_3||\eta||}{r} & \dots & \frac{\eta_n||\eta||}{r} \\ -\frac{\eta_2}{||\eta||} & -\frac{||\eta||}{r} & 0 & \dots & 0 \\ -\frac{\eta_3}{||\eta||} & 0 & -\frac{||\eta||}{r} & \dots & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 \\ -\frac{\eta_n}{||\eta||} & 0 & \dots & 0 & -\frac{||\eta||}{r} \end{pmatrix}, \tag{4.8}$$

from which it is apparent that A_0^{-1} has entries with at worst a pole of order 1 in r. (Off-diagonal entries in the lower right $(n-1) \times (n-1)$ submatrix are all zero.) By considering only small values of r, we expand (4.6) using the the Neumann series, which generalizes the geometric series for operators of small enough norm. We consider the matrices below as linear operators on the tangent space $T_x(\widehat{T^\star L})$ for each $x \in \widehat{L} \times [0, \epsilon]$.

$$
\begin{aligned}
A_t^{-1} &= (Id + t \cdot rA_0^{-1}E)^{-1}A_0^{-1} \\
&= (Id - t \cdot \underbrace{rA_0^{-1}}_{\text{smooth}} E + (t \cdot rA_0^{-1}E)^2 - \ldots)A_0^{-1},
\end{aligned}
\tag{4.9}
$$

which is smooth since E and rA_0^{-1} are smooth. For equality to hold, we need to know that the series converges. To prove this, we use a few notions from functional analysis.

Definition IV.5. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ denote the (real or complex) eigenvalues of a matrix $A \in M_{N\times N}(\mathbb{C})$. Then the *spectral radius* $\rho(A)$ of A is defined to be the maximum modulus of its eigenvalues,

$$\rho(A) \overset{\text{def}}{=} \max_{1\leq j\leq N} (|\lambda_j|) \,.$$

Definition IV.6. A linear transformation between normed vector spaces $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ is said to be *bounded* if there exists a constant $M > 0$ such that for all $v \in X$,

$$||Lv||_Y \leq M||v||_X.$$

Taking the smallest value M gives the *operator norm* $||L||_{op}$ of L.

In the present situation, the matrix $B = rA_0^{-1}E$ at $\widehat{L} \times [0, \epsilon]$ takes the tangent space at x to itself, so that both norms are the same. Then M can be taken to be the spectral radius of B at x, so that B is bounded.

Claim IV.7. For the convergence of the Neumann series (4.9), it suffices to show that $\rho(B^2) < 1$.

Proof. The result is a consequence of Gelfand's Formula, which states that for a bounded linear operator A and the operator norm $||\cdot|| = ||\cdot||_{op}$,

$$\rho(A) = \lim_{n\to\infty} ||A^n||^{1/n}.$$

Then $\rho(A) < 1$ implies that there is some k such that $||A^k|| < 1$. One can then rewrite the series

$$\sum_{j=0}^{\infty} A^j = (1 + A + \ldots + A^{k-1}) \sum_{j=0}^{\infty} A^{kj}.$$

Since $||A^k|| < 1$, the sum $\sum_{j=0}^{\infty} A^{kj}$ converges so that the sum on the left converges.

In particular, we rewrite (4.9) as

$$\begin{aligned} A_t^{-1} &= (Id - tB + (tB)^2 - (tB)^3 + \ldots)A_0^{-1} \\ &= (1 - tB) \sum_{j=0}^{\infty} (t^2B^2)^j. \end{aligned}$$

Since $||t^2B^2|| \le ||B^2||$ for $0 \le t \le 1$, the above sum will converge if $\rho(B^2) < 1$. $\square$

To show that $\rho(B^2) < 1$ for small enough r, we show that r divides each of the entries in B^2. Then taking ϵ close enough to 0, the eigenvalues of B will be small in modulus.

Using (4.7) and (4.8), we see that rA_0^{-1} takes the form

$$\left(\begin{array}{c|c}
0_{n\times n} & \begin{array}{c|cccc} r & \star & \star & \dots & \star \\ r & \star & 0 & \dots & 0 \\ \vdots & 0 & \star & 0 & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ r & 0 & \dots & 0 & \star \end{array} \\ \hline
\begin{array}{ccccc} -r & -r & \dots & \dots & -r \\ \hline \star & \star & 0 & \dots & 0 \\ \star & 0 & \star & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \star & 0 & \dots & 0 & \star \end{array} & 0_{n\times n}
\end{array}\right), \tag{4.10}$$

where an r indicates that the entry is divisible by r and an asterisk indicates a term of order zero in r. From the definition of μ in (4.4) and Claim IV.4, $d\mu$ has the form

$$\begin{aligned} d\mu &= d(r^2\alpha), \qquad \text{for some 1-form } \alpha \\ &= 2rdr \wedge \alpha + r^2 d\alpha \\ &= r(2dr \wedge \alpha + rd\alpha), \end{aligned}$$

so that the matrix E represents the 2-form $2dr \wedge \alpha + rd\alpha$ and thus it takes the form

$$
\left(\begin{array}{c|c|c}
 & \star & \\
 & \vdots & \\
r_{n\times n} & & r_{n\times(n-1)} \\
 & \vdots & \\
 & \star & \\ \hline
\star \;\; \dots \;\; \dots \;\; \star & 0 & \star \;\; \dots \;\; \star \\ \hline
 & \star & \\
r_{(n-1)\times n} & \vdots & r_{(n-1)\times(n-1)} \\
 & \star &
\end{array}\right), \tag{4.11}
$$

where $\star$ indicates coefficients that are of order 0 in r and $r_{m,n}$ indicates that each entry in an $m \times n$ submatrix is divisible by r (and possibly equal to 0, as in the case of diagonal entries).

From (4.10) and (4.11), we find that $rA_0^{-1}E$ has the form

$$
B = \left(\begin{array}{c|c|c}
 & \star & \\
 & \vdots & \\
r_{n\times n} & & r_{n\times(n-1)} \\
 & \vdots & \\
 & \star & \\ \hline
 & r & \\ \cline{2-2}
 & \star & \\
r_{n\times n} & \vdots & r_{n\times(n-1)} \\
 & \star &
\end{array}\right), \tag{4.12}
$$

where the distinguished column is the $(n+1)$-th column. From this it follows that B^2 has the form

$$
B^2 = \left(\begin{array}{c|c|c}
 & r & \\
 & \vdots & \\
r_{n\times n} & & r_{n\times(n-1)} \\
 & \vdots & \\
 & r & \\
\hline
 & r & \\
 & \vdots & \\
r_{n\times n} & & r_{n\times(n-1)} \\
 & \vdots & \\
 & r &
\end{array}\right), \tag{4.13}
$$

so that all entries are divisible by r. By taking $\epsilon > 0$ small enough (that is, by working on a small enough neighborhood of $\widehat{L}$), one can guarantee that the eigenvalues of B^2 are small enough in size to guarantee that $\rho(B^2) < 1$. Then by Claim IV.7, A_t is invertible for $0 \leq t \leq 1$ and is given by (4.9).

Applying A_t^{-1} to both sides of (4.5) gives

$$
\begin{aligned}
A_t^{-1} A_t V_t &= A_t^{-1}(-N) \\
V_t &= -A_t^{-1} N.
\end{aligned}
$$

It is clear from (4.9) that A_t^{-1} has at worst a pole of order 1, and we have shown previously that r^2 divides N, so that r divides V_t. Thus, the vector field v_t in Moser's equation vanishes along $\widehat{L}$. Since μ was chosen to be S^1-invariant and $\Omega = \widehat{\omega_1} - \widehat{\omega_0}$ is S^1-invariant, v_t is an S^1-invariant vector field and is smooth up to $\widehat{L}$ and vanishes along $\widehat{L}$.

By integrating v_t, one can locally solve for the flow $\phi_t, 0 \leq t \leq 1$ (shrinking the neighborhood U_0, if necessary), which is the identity on $\widehat{L}$ from the above discussion. Since v_t is S^1-invariant, the flow ϕ_t is automatically S^1-equivariant.

Since $\phi_1^\star \widehat{\omega_1} = \widehat{\omega_0}$, we get that the two forms are locally equivariantly symplectomorphic, and the two S^1-actions can be identified after applying the conjugating map $f = \exp_{g_0} \circ (Id_{\widehat{L}} \times Id_I) \circ \exp_{g_1}^{-1}$. Since $f_\star \widehat{\xi_1} = \widehat{\xi_0}$, that is, $f_\star \widehat{X}_{\sqrt{H}} = \widehat{X}_{\sqrt{H_0}}$, the two functions are related by $\sqrt{H} = \sqrt{H_0} + C$ (for a constant C) locally since they give rise to the same vector field $\widehat{\xi_0}$ after applying the conjugator. This gives that, after normalizing to account for the constant C, performing a symplectic cut in the blow-up for small values of $\sqrt{H}$ and $\sqrt{H_0}$ (with respect to the structures $(f^{-1})^\star \widehat{\omega_0}$ and $\widehat{\omega_0}$, respectively) will yield equivariantly symplectomorphic cut manifolds. (Notice that a different choice of metric g_0, from which g_1 is defined, will just yield another equivariantly symplectic form.)

We would like to say that $\widehat{\omega_1} = (f^{-1})^\star \widehat{\omega_0}$ descends to a smooth symplectic form under the blow-down map. Its smoothness is not entirely clear, but by construction, df is the identity along the boundary, the only place where the blowing down operation is not well-behaved. The descended form will at least be C^0 or C^1 and identically zero along L.

We summarize this result as follows:

Theorem IV.8. *Let (W, ω) be a polarized symplectic manifold and suppose its Morse-Bott function H is such that*

- *$X_{\sqrt{H}}$ generates a free S^1-action away from L and V, and*
- *the Taylor expansions along L for H and H_0 defined by the Hessian as above differ only by terms of order 4 and higher in transverse coordinates y_k.*

Then there is a neighborhood of L in W such that (after normalizing $\sqrt{H}$ by a constant) for small $\epsilon > 0$ the symplectic cut under the S^1-action induced by $X_{\sqrt{H}}$ is equivariantly equivalent (symplectomorphic in the blow-up) to the Zoll cut of a neighborhood of the zero section in $(T^\star L, \omega)$ with respect to the geodesic flow given *by $X_{\sqrt{H_0}}$ at levels ϵ.*

We end this chapter by introducing a subfamily of polarized symplectic manifolds with a specific type of symmetry.

4.5 Even Polarized Symplectic Manifolds

Inspired by the Zoll cut examples of polarized symplectic manifolds, we introduce some new terminology to describe a special class of polarized symplectic manifolds defined in terms of a very natural geometric constraint.

Recall that in the Zoll cut construction associated to a Zoll manifold L, the Morse-Bott function is given in local coordinates on the cotangent bundle $(T^\star L, \omega_0)$ with its canonical symplectic form by $H_0(x, y) = ||y||^2$ so that $H_0(x, -y) = H_0(x, y)$. There is an obvious $\mathbb{Z}_2$-action on the space given by $\sigma_0(x, y) = (x, -y)$ that preserves H_0 and is antisymplectic, i.e. $\sigma_0^\star \omega_0 = -\omega_0$. Since this $\mathbb{Z}_2$-action on $T^\star L$ commutes with the geodesic flow and preserves levels of $\sqrt{H_0} = ||y|| = c$, it descends to the symplectic cut at level $||y|| = \epsilon$ and preserves the reduced manifold V, which as noted before is just the space of unparametrized, oriented geodesics on L. Moreover, the induced action of σ_0 interchanges points on V corresponding to the same unparametrized geodesics with opposite orientations.

The above discussion motivates us to define the following:

Definition IV.9. An *even polarized symplectic manifold* is a polarized symplectic manifold (W, ω) (in the sense of Audin) together with an involution $\sigma : W \to W$ such that

- $\sigma^\star \omega = -\omega$ (σ is antisymplectic),
- σ fixes L and preserves V set-wise ($\sigma|_L = Id_L$ and $\sigma(V) = V$), and
- $\sigma^\star H = H$ (the Morse-Bott function H is invariant under σ).

When a function H on a symplectic manifold (W, ω) is invariant under an antisymplectic involution σ (i.e. $\sigma^\star H = H \circ \sigma = H$), H is said to be a *reversible Hamiltonian.*

It is well known that the fixed point set of an antisymplectic involution of a symplectic manifold, when non-empty, must be a Lagrangian submanifold. (See, for example, Meyer [22].) Forgetting about the Morse-Bott function H for a brief moment, it turns out that a neighborhood of the Lagrangian L is equivariantly symplectomorphic to the standard example of a neighborhood of the zero section of L in $T^\star L$ with the action of σ_0 sending a covector to its negative.

Theorem IV.10. ***(Meyer [22])*** *Let (W, ω) be a symplectic manifold with an antisymplectic involution $\sigma : W \to W$ with non-empty fixed point set L. Then there exists a neighborhood U_0 of the zero section in $(T^\star L, \omega_0)$ with the canonical symplectic structure, a neighborhood U of L in W, and an equivariant symplectomorphism $\varphi : U_0 \to U$ such that $\varphi^\star \omega = \omega_0$ and $\varphi^\star \sigma = \sigma \circ \varphi = \sigma_0$.*

The proof of the theorem uses a modified version of Weinstein's argument in [32], much like the proof of Theorem IV.8, in which there is an S^1-action off of the critical set instead of a $\mathbb{Z}_2$-action. For details, the reader is referred to Meyer's paper [22]. Thus, in the present case, we also have a standard local picture.

We can now say more about polarized symplectic manifolds with $X_{\sqrt{H}}$ generating a free S^1-action off of L and V given the additional constraint that H is reversible. In the coordinates on (a neighborhood of the zero section in) $T^\star L$ supplied by Theorem IV.10, σ is just given by $(x, y) \mapsto (x, -y)$, so that in these coordinates, σ-invariance of H means that $H(x, -y) = H(x, y)$. That is, H is an even function in the fiber variables y so that its Taylor expansion about $y = 0$ contains only even powers of y.

Thus, we find ourselves in the setting of Theorem IV.8 since $H = H_0 + O_y(4)$

as a consequence of Meyer's theorem. By Theorem IV.2, we know that L is a Zoll manifold. Theorem IV.8 then gives that a neighborhood of L is S^1-equivariantly symplectomorphic to a neighborhood of the zero section of the standard Zoll model with the S^1-action coming from the geodesic flow such that the symplectic cuts at corresponding levels are the same.

Since the symplectic cut of the cotangent bundle $(T^\star L, \omega_0)$ with respect to the geodesic flow of a Zoll manifold L comes with the standard antisymplectic involution σ_0, we have exhibited these particular even polarized symplectic manifolds as a sort of (local) symplectic invariant to Zoll manifolds.

CHAPTER V

The Symplectic Critical Submanifold V

Up to this point, we have focused on a neighborhood of the Langrangian L in the polarized symplectic manifold (W, ω) while largely ignoring the symplectic submanifold V corresponding to the maximum of H. There are a few nice results about what manifolds V must be in very specific cases for L, and we mention them here. We also describe a normal form for the symplectic normal bundle of V in W, as presented in Biran's paper [7]. We then work on extending the local equivalence in Theorem IV.8 in Section 5.3.

5.1 Known Results About V

For basic topological reasons, the symplectic submanifold V is a codimension 2 symplectic submanifold of W. Note that this condition was not a priori insisted upon in the definition of polarized symplectic manifolds. The following result is due to Audin:

Proposition V.1. ***(Audin [2])*** *Let W be a polarized symplectic manifold, and let $H : W \to \mathbb{R}$ denote its Morse-Bott function, whose minimum is reached along a Lagrangian submanifold L and whose maximum is reached along a symplectic submanifold V. Then V has (real) codimension 2.*

Proof. The proof is by soft topological techniques. Suppose $\text{codim}V = 2k \geq 4$. Consider the preimage of a regular value c of f, and let $\mathcal{C} = f^{-1}(c)$. Then $p : \mathcal{C} \to V$ is an oriented S^{2k-1}-bundle, and the Gysin exact sequence gives that the map

$$p^* : H^2(V, \mathbb{Z}) \to H^2(\mathcal{C}, \mathbb{Z})$$

is injective. However, the Mayer-Vietoris sequence

$$\begin{array}{ccccc} H^2(W, \mathbb{R}) & \longrightarrow & H^2(L, \mathbb{R}) \oplus H^2(V, \mathbb{R}) & \longrightarrow & H^2(\mathcal{C}, \mathbb{R}) \\ [\omega] & \longmapsto & (0, j^*[\omega]) & \overset{p^*}{\longmapsto} & 0 \end{array}$$

gives a nontrivial element in the kernel of p^*, a contradiction. Thus codim $V = 2$. $\square$

There are some results classifying V up to some notion of isomorphism in very specific cases. For example, when the Lagrangian L is a Zoll 3-sphere, Ono [27] proved that V, interpreted as a moduli of geodesics in L, must be *symplectomorphic* to the product $S^2 \times S^2$ of 2-spheres of equal area. To achieve this, he used a result of Ohta-Ono [26] on the classification of monotone symplectic 4-manifolds. (Recall that a symplectic manifold (V, ω) is monotone if $c_1(X) = \lambda[\omega], \lambda > 0$.)

In a similar vein, Sato [29] was also able to show that the space of geodesics of a Zoll 4-sphere is *diffeomorphic* to the complex quadric Q^3.

One can ask whether similar results hold in higher dimensions and whether the classification is up to homeomorphism, diffeomorphism, symplectomorphism, etc. If the Zoll metric on $L = S^n$ is obtained as a deformation of the standard metric, then Moser's stability theorem (modulo scaling) gives that the resulting manifolds of geodesics V are all symplectomorphic. But no one yet knows what the space of Zoll metrics on the general n-sphere looks like.

5.2 The Local Picture Near V

As a result of the Symplectic Neighborhood Theorem (Theorem II.29), a neighborhood of V in W is symplectomorphic to a disk subbundle (E, ω_0) of the symplectic normal bundle N_V of V in W. There is a well-known construction giving a normal form for ω_0, which we briefly recall here.

Choosing a compatible almost complex structure J of W in a neighborhood of V, one gets that a tubular neighborhood of V is diffeomorphic to the disk bundle $\pi : E \to V$, of the normal bundle, which is a complex line bundle since V is of real codimension 2 and also a symplectic bundle by earlier remarks. The form ω_0 is characterized by requiring that its restriction to the zero section be equal to $\omega|_V$, that all of the fibers be symplectic with the same area $\frac{1}{k}$, and that it remain invariant under the S^1-action by rotation along the fibers.

The bundle $\pi : (E \backslash 0) \to V$ can be endowed with a connection 1-form α satisfying:

- a normalization condition $i_X \alpha \equiv 1$ for the fundamental vector field X of the S^1-action and
- the curvature condition $d\alpha = -\pi^\star \omega|_V$.

Then ω_0 is expressed by the following formula:

$$\omega_0 = \pi^\star \omega|_V + \frac{1}{k} d(r^2 \alpha),$$

where r is the radial distance (defined using the metric coming from J) from the zero section in any given fiber. This construction gives what Biran calls a *standard symplectic disk (disc) bundle* in [7].

Biran proves some nice results in the case where (W, ω) is *Kähler*, in which case the almost complex structure J can be chosen to be integrable (that is, a legitimate complex structure) compatible with ω. In this case, V is a divisor and so defines a holomorphic line bundle $\mathcal{L} = \mathcal{O}_W(V)$. Then one has a holomorphic section $s : W \to E$ with zero set equal to V. The section s is defined up to a complex, non-zero factor, and one obtains a smooth Morse-Bott function $||s(x)||^2$ near V. The relevant result that Biran proved is:

Theorem V.2. ***(Biran [7])*** *Let (W, ω) be a closed Kähler manifold with integral $[\omega]$ and a complex hypersurface V which represents the Poincaré dual to $k[\omega]$ for some $k \in \mathbb{N}$. Suppose further that the function $||s||^2$ is Morse-Bott on $W \setminus V$. Then the union of all the unstable manifolds corresponding to critical points of $||s||^2$ in $W \setminus V$ is an isotropic cellular subspace, which can be replaced by a homotopically equivalent, isotropic CW-complex.*

In the context of polarized symplectic manifolds, we would like to think of the function $||s||^2$ as $1 - H$ (where H is assumed to have been normalized to take values in $[0, 1]$). The union of all the unstable manifolds corresponding to critical points of $||s||^2$ would correspond to a Lagrangian L if it has dimension n (being an isotropic subspace of maximal dimension).

The proofs in Biran's paper do not immediately transfer over to the symplectic (non-Kähler) case, as we lose the association between the symplectic "divisor" V and (almost) holomorphic line bundles with the correct Chern class. We do, however, still have the infinitesimal S^1-action by rotation in the fibers, which we would like to show is Hamiltonian and gives rise to a Morse-Bott function like $1 - H$. We investigate this property in more detail in the next few sections.

There remains the question of how to "compactify" the normal bundle N_V by a Lagrangian L in a manner similar to the symplectic cut construction on the cotangent bundle of a Zoll L to obtain W. This problem is reminiscent of Milnor's gluing construction [24]. In our setting, we would like to take the two ends of the polarized symplectic manifold (a neighborhood $\mathcal{U}$ that retracts onto L and a neighborhood $\mathcal{V}$ that retracts onto V) and glue the two pieces together along their boundary (symplectically) to recover the original manifold W.

We avoid this problem in the context of Theorem IV.8 by extending the local equivalence Φ to a larger neighborhood of L and closing up the disk bundle via the symplectic cut construction to recover V. What this tells us is that the situation is somehow much more rigid (diffeomorphically) than the case with the exotic 7-spheres because of the added structure coming from the symplectic form ω.

5.3 Extending the Local Symplectomorphism

For the remainder of this chapter, we aim to extend the local equivalence Φ obtained in Theorem IV.8.

5.3.1 W is a Symplectic Cut

Pick up from where we were in Chapter 4, so that we have two polarized symplectic manifolds: (W, ω) with Morse-Bott function H such that $X_{\sqrt{H}}$ generates a free S^1-action away from L and V and (W_0, ω_0), the symplectic cut of $T^\star L$ at level $\sqrt{H_0} = 1$ by the S^1-action coming from the geodesic flow on L. Assume further that (W, ω) is *even* and that $H = 0$ on L and $H = 1$ on V so that H differs from H_0 by only even-powered terms of order 4 and higher in the variables transverse to L. In this setting, the conditions of Theorem IV.8 are satisfied.

We show that W is itself a symplectic cut via an *inverse symplectic cut construction*, which amounts to taking the oriented blow-up of the manifold W along V in the fibers of the normal bundle N_V. The blown-up manifold will inherit a degenerate symplectic form, which we will work around via a coordinate change to obtain a new "twisted" symplectic manifold with boundary with an inherited Hamiltonian

S^1-action, which we can then use to cut the manifold and recover the original W.

5.3.2 The Model Case

We begin by discussing the inverse cut construction in the model case, where V is 0-dimensional, i.e. a point. In this case, the symplectic normal bundle $N_W V$ is simply $\mathbb{R}^2$, V is the origin, and one can find coordinates (x, y) such that $\omega = dx \wedge dy$. One has the obvious Hamiltonian S^1-action by rotation about the origin with Hamiltonian given by $H = \frac{1}{2}(x^2 + y^2)$. Taking the oriented blow-up is equivalent to working in polar coordinates $(x, y) \mapsto (r, \theta)$. The form $\omega = dx \wedge dy$ becomes $\widehat{\omega} = rdr \wedge d\theta$ in the blow-up, the Hamiltonian becomes $\widehat{H} = \frac{1}{2}r^2$, and $\widehat{\mathbb{R}^2}$ can be identified with the cylinder $C = [0, \infty) \times S^1$ for $r \in [0, \infty)$ and $0 \leq \theta < 2\pi$.

The form $\widehat{\omega} = rdr \wedge d\theta$ is clearly degenerate along the boundary $\{r = 0\}$, so we introduce the coordinate change $\tau(r, \theta) = (\frac{1}{2}r^2, \theta) = (\rho, \theta)$. The resulting space can still be identified with a cylinder ${}^{\tau}C = [0, \infty) \times S^1$ and the Hamiltonian simplifies to $H = \rho$, but the symplectic form ${}^{\tau}\widehat{\omega} = d\rho \wedge d\theta$ is now easily seen to be non-degenerate along the boundary $\{\rho = 0\}$ (and elsewhere).

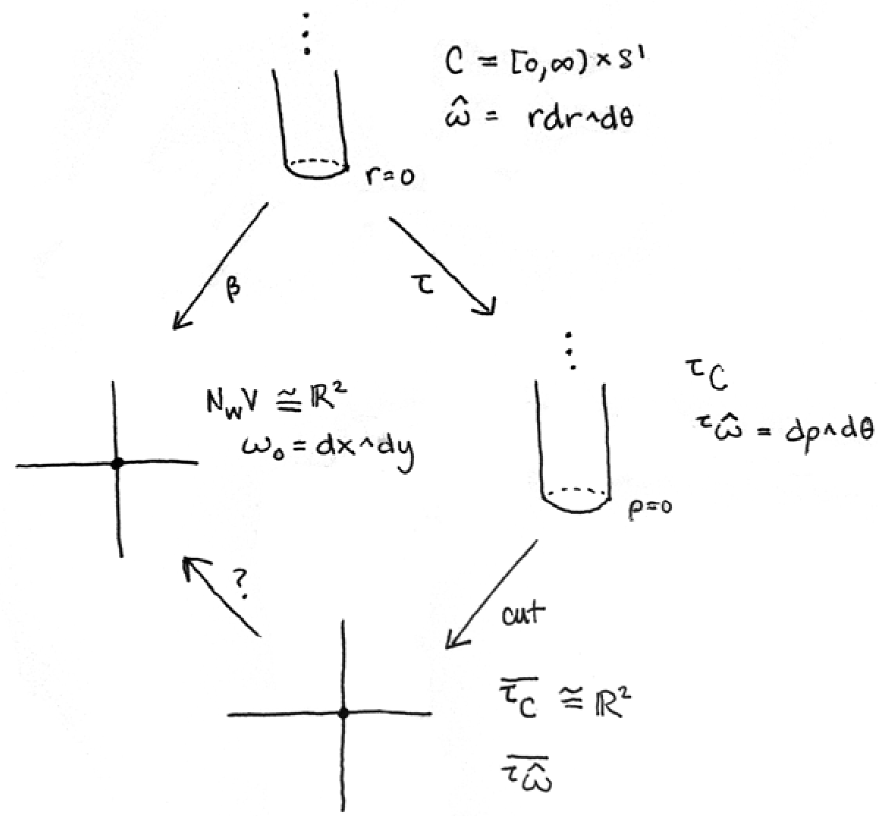
The Hamiltonian vector field defining the S^1-action off of V defined by

$$i_{X_H}\omega = -dH$$

is given by $X_H = -\frac{\partial H}{\partial y}\frac{\partial}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial}{\partial y} = -y\partial_x + x\partial_y$. In the blow-up C, this lifts to $\widehat{X}_H = \frac{\partial}{\partial \theta}$, and after twisting by the map τ, the action is generated on ${}^{\tau}C$ by ${}^{\tau}\widehat{X}_H = \frac{\partial}{\partial \theta}$.

We can now perform the symplectic cut at $\rho = 0$ on ${}^{\tau}C$ and verify that we recover the original manifold, recalling Lerman's construction. We consider the larger space ${}^{\tau}C \times \mathbb{C}$ with the symplectic form ${}^{\tau}\widehat{\omega} \oplus \frac{i}{2}dz \wedge d\bar{z}$, on which we have the extended Hamiltonian $\tilde{H} = \rho - \frac{1}{2}|z|^2$, where S^1 acts on the $\mathbb{C}$ factor by $e^{-iT}z$. S^1 acts freely

Figure 5.1: Oriented blow-up along V followed by taking the symplectic cut



on the level set

$$\begin{aligned}\tilde{H}^{-1}(0) &= \{\rho = 0 \ \&\ z = 0\} \sqcup \{((\rho,\theta), z) \in^\tau C \times \mathbb{C} \,|\, \rho > 0 \ \&\ z = e^{-iT}\sqrt{2\rho}\} \\ &\cong S^1 \sqcup Int({}^\tau C) \times S^1,\end{aligned}$$

so we may take the quotient $\tilde{H}^{-1}(0)/S^1 \cong \mathbb{R}^2$.

As a symplectic reduced space, $\tilde{H}^{-1}(0)/S^1$ inherits the symplectic form ${}^\tau\widehat{\omega}_{red}$ satisfying $\pi^\star({}^\tau\widehat{\omega}_{red}) = i^\star({}^\tau\widehat{\omega} \oplus \frac{i}{2} dz \wedge d\bar{z})$. But on $\{\rho = 0 \ \&\ z = 0\}$, the product form restricts to 0, while on $\{((\rho,\theta), z) \in^\tau C \times \mathbb{C} \,|\, \rho > 0 \ \&\ z = e^{-iT}\sqrt{2\rho}\}$, the $\frac{i}{2} dz \wedge d\bar{z}$ factor of the product form is 0 because it is a 2-form restricted to a 1-dimensional space, so that ${}^\tau\widehat{\omega}_{red}$ is equal to the restriction of the original ${}^\tau\widehat{\omega}$ on $Int({}^\tau C) = \{\rho > 0\}$.

Converting the coordinates back to $(\rho,\theta) \mapsto (\sqrt{2\rho}, \theta)$, one easily verifies that

$rdr \wedge d\theta$ pulls back to $d\rho \wedge d\theta$ and that converting back to (x, y) coordinates recovers the original symplectic form $dx \wedge dy$.

The above discussion gives us a local picture of the inverse cut at each point along the symplectic submanifold V. We observe that we need to replace the variable r with $\rho = \frac{1}{2}r^2$ to obtain a non-degenerate symplectic form on the resulting manifold with boundary. In the more general case, we use the S^1-invariance of the symplectic form ω on W and the function $\sqrt{H}$ generating the S^1-action.

First, we will need some preparations.

5.3.3 Extending the Equivalence Via Gradient Flows

From the results in Chapter 4, we have that a neighborhood near L in a polarized symplectic manifold (W, ω) with $X_{\sqrt{H}}$ generating a free S^1-action off of the critical submanifolds can be (equivariantly symplectomorphically in the blow-up) identified with a neighborhood of the zero section L in the standard $(T^\star L, \omega_0)$. Ideally, we would be able to extend this symplectomorphism by pushing it out away from L to the V side of W so that we can identify W with the Zoll cut at level 1, which we denote by W_0. We do this by working in the blown-up space ${}^\tau\widehat{W}$ from the previous section, so that we are working with a symplectic manifold with boundary. We identify the complement $W_0 \setminus V_0$ with the open unit disk bundle $(T^\star_{<1}L, \omega_0)$, where V_0 denotes the reduced symplectic submanifold at infinity arising from the Zoll cut construction, so that W_0 is the Zoll cut of the closed unit disk bundle $(T^\star_{\leq 1}L, \omega)$.

The classical way of moving from one critical submanifold to another is to follow the gradient flow of a function with respect to some metric. Recall that for a smooth function f and a Riemannian metric g, one can define the gradient vector field ∇f

of f with respect to the metric g by the equation

$$g(\nabla f, \cdot) = df.$$

On a region of the manifold where $\nabla f \neq 0$, the flowline $\gamma(t)$ of the vector field $\xi_f := \frac{\nabla f}{|\nabla f|^2}$ takes on a simple form. Namely, since $\xi_f(f) \equiv 1$ and

$$\dot{\gamma}(t) = \xi_f(\gamma(t)),$$

the Fundamental Theorem of Calculus gives that $f(\gamma(t)) = f(x_0) + t$ for $\gamma(0) = x_0$. Explicitly,

$$\begin{aligned} \frac{d}{dt} f(\gamma(t)) &= \dot{f}(\gamma(t))\dot{\gamma}(t) = \dot{f}(\gamma(t))\xi_f(\gamma(t)) \\ &= df(\xi_f)(\gamma(t)) = \xi_f(f(\gamma(t))) \equiv 1. \end{aligned}$$

Since we have the additional structure of a symplectic S^1-action on (W, ω), we would like to find an S^1-invariant metric that is also compatible with the symplectic form and such that the gradient flow of $\sqrt{H}$ will commute with the S^1-action.

To achieve this, we will utilize a bit of Gromov's J-holomorphic technology to obtain an almost complex structure J on W compatible with ω that is S^1-invariant away from L. This will give rise to a pseudo-Kähler metric $\tilde{g}$. We will use the metric to define the gradient flow of $\sqrt{H}$, which will commute with the flow associated to the S^1-action (again, away from L), and we will follow the (normalized) gradient flow from a neighborhood of L towards a neighborhood of V to extend the equivariant equivalence obtained near L.

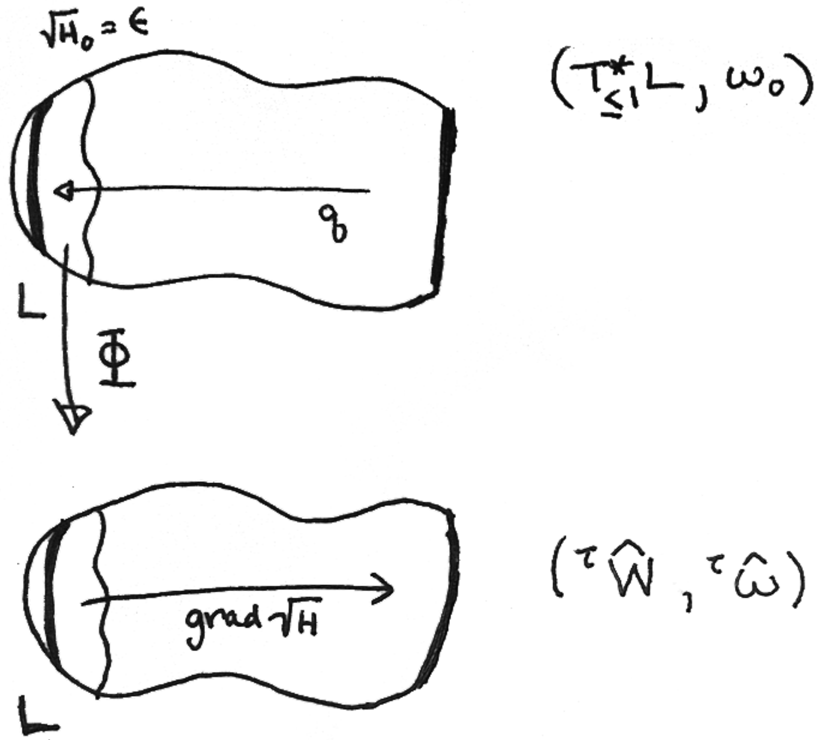
On the unit disk bundle $(T^\star_{\leq 1}L, \omega_0)$, there is an obvious way to scale down the symplectic form along the fiber directions. If Φ is the equivariant equivalence given by Theorem IV.8 valid on a neighborhood of the zero section L (up through the level

$\sqrt{H_0} = \epsilon \leq 1$, for example), then we can use the retraction map $q_\epsilon : T^\star_{\leq 1}L \to T^\star_{\leq \epsilon}$ given by $q_\epsilon(x,y) = (x, \epsilon y)$. This map gives us an identification

$$(T^\star_{\leq \epsilon}L, \omega_\epsilon) \simeq (T^\star_{\leq 1}L, \epsilon\omega_0).$$

Applying Φ from Theorem IV.8 sends the neighborhood of the zero section to a neighborhood of L in W. We can then push this neighborhood out towards V by following the (normalized) gradient flow of $\sqrt{H}$ for the appropriate length of time. (See Figure 5.3.3 below.)

Figure 5.2: Extending the local symplectomorphism via gradient flows



More explicitly, assume we have an S^1-invariant pseudo-Kähler metric on (W, ω). (We give more details on how to obtain such a metric in Section 5.3.4 below.) Consider the regular level set $W_c := \{\sqrt{H} = c\}$ for $0 < c < 1$. (Recall that we assumed that H attains its maximum value of 1 along V.) If $\phi_t^{X_{\sqrt{H}}}$ denotes the flow of $X_{\sqrt{H}}$

on W (where defined), then we can define a map $\psi(x,t) = \phi_{t-c}^{X_{\sqrt{H}}}(x)$, which is a diffeomorphism of $W_c \times [c,1)$ onto the set $\{c \leq \sqrt{H} < 1\} \subset W$.

Under this identification, the t-coordinate on $W_c \times [c,1)$ is sent to $\sqrt{H}$. That is, $\psi^\star \sqrt{H} = t$. We can then show that the pullback $\psi^\star \omega$ is a smooth, non-degenerate symplectic form on the manifold with boundary $W_c \times [c,1]$, and the symplectic cut by S^1 of this manifold will just be $\{\sqrt{H} \geq c\} \subset W$. The manifold with boundary is thus globally well-defined, so now all that is left to show for the inverse cut construction is that $\psi^\star\omega$ is smooth and non-degenerate on $W_c \times [c,1]$. It suffices to show this in a local model.

By an equivariant version of the symplectic neighborhood theorem, we can trivialize the S^1-action in a neighborhood of the symplectic submanifold V. (The construction is akin to the Riemannian version of geodesic normal coordinates.) Let (x_1, x_2) denote the normal coordinates to V spanning a symplectic plane, so that V is locally cut out by the equations $V = \{x_1 = x_2 = 0\}$. Take symplectic coordinates $x' = (x_3, \ldots, x_{2n})$ on a patch on V such that the circle action can be written linearly as the standard action in the first two variables, i.e.

$$e^{i\theta} \cdot (x_1, x_2, x_3, \ldots, x_{2n}) = (\cos\theta\, x_1 - \sin\theta\, x_2, \sin\theta\, x_1 + \cos\theta\, x_2, x_3, \ldots, x_{2n}).$$

We will need the following well-known lemma, the more general statement of which can be found in Schwarz's paper [30].

Lemma V.3. ***(Schwarz [30])*** *Let S^1 act on $\mathbb{R}^{2n}$ locally as above and let $r^2 = x_1^2 + x_2^2$. Let $f = f(x_1, \ldots, x_{2n})$ be a smooth S^1-invariant function defined near 0. Then there exists a smooth function $F = F(t, x_3, \ldots, x_{2n})$ of $2n-1$ variables such that $f = F(r^2, x_3, \ldots, x_{2n})$.*

As a result of the above lemma, there exist local functions $\zeta = \zeta(u, x')$ and

$\eta = \eta(v, x')$ of $2n-1$ variables such that $r^2 = \zeta(1-\sqrt{H}, x')$ and $1-\sqrt{H} = \eta(r^2, x')$, where $\zeta(0, x') = \eta(0, x') = 0$, $\frac{\partial\zeta}{\partial u}(0, x') > 0$ and $\frac{\partial\eta}{\partial v}(0, x') > 0$. Lemma V.3 thus allows us to treat the blow-up in a local model so that all we have to do is prove that ω can be written in terms of $d\theta, d\rho, dx_3, \ldots, dx_n$ and that the coefficient functions are smooth in the variables $\rho, x_3, \ldots, x_{2n}$ up to and including the boundary $\rho = 0$.

We first consider a basis of S^1-invariant 1-forms on $\mathbb{R}^{2n} = \mathbb{R}^2 \times \mathbb{R}^{2n-2}$ off of $\{0\} \times \mathbb{R}^{2n-2}$ given by

$$d\rho = \frac{1}{2}d(r^2) = x_1dx_1 + x_2dx_2,\ d\theta = \frac{-x_2dx_1 + x_1dx_2}{r^2}, dx_3, \ldots, dx_{2n}. \tag{5.1}$$

Then ω can be written

$$\omega = Ad\rho \wedge d\theta + \sum_{j=3}^{2n} B_j \wedge dx_j + \sum_{i,j=3}^{2n} C_{i,j}dx_i \wedge dx_j,$$

where A is an S^1-invariant function that is smooth on $\{\mathbb{R}^2 \setminus \{0\}\} \times \mathbb{R}^{2n-2}$, the B_j are S^1-invariant 1-forms of the form $ad\rho + bd\theta$, and the $C_{i,j}$ are S^1-invariant functions satisfying $C_{i,j} = -C_{j,i}$. Since (5.1) form a basis, the aforementioned functions and forms are unique and S^1-invariant. Lemma V.3 gives that we can write $C_{i,j} = C_{i,j}(\rho, x')$ for all i, j. These terms are thus in the form we want to show, so we may subtract them off and focus on the remaining terms $\Omega = Ad\rho \wedge d\theta + \sum_{j=3}^{j=2n} B_j \wedge dx_j$.

Using (5.1), the 2-form $Ad\rho \wedge d\theta$ can be seen to have a smooth extension of the form $cdx_1 \wedge dx_2$ for $c = c(x_1, x_2, x')$ an S^1-invariant function with $c(0, 0, x') \neq 0$. Thus, Lemma V.3 gives that $c = \tilde{c}(r^2, x')$ for some function $\tilde{c}(t, x')$ that is smooth in the $2n-1$ variables (t, x'). Thus, $Ad\rho \wedge d\theta = 2\tilde{c}d\rho \wedge d\theta$ is of the form we want.

Since the vector fields $\frac{\partial}{\partial x_j}$ were chosen to be S^1-invariant, each $B_j = i_{\frac{\partial}{\partial x_j}}\Omega$ is an invariant 1-form. Each B_j is a linear combination of $d\rho$ and $d\theta$ and has a smooth extension to $\mathbb{R}^{2n}$, i.e. across V.

Lemma V.4. *Let $Ad\rho + Bd\theta$ be an S^1-invariant 1-form that has a smooth extension across V locally. Then there exist smooth functions $a = a(u, x')$ and $b = b(u, x')$ such that $A = a(r^2, x')$ and $B = b(r^2, x')$.*

Proof.

$$Ad\rho + Bd\theta = fdx_1 + gdx_2, \tag{5.2}$$

where $f = f(x_1, x_2, x')$ and $g = g(x_1, x_2, x')$ are smooth functions defined locally on $\mathbb{R}^{2n}$. Using (5.1), we rewrite (5.2) in terms of dx_1 and dx_2.

$$A(x_1dx_1 + x_2dx_2) + B\frac{-x_2dx_1 + x_1dx_2}{x_1^2 + x_2^2} = fdx_1 + gdx_2$$

Combining coefficients, we obtain

$$\begin{aligned} x_1A - x_2\frac{B}{r^2} &= f, \\ x_2A + x_1\frac{B}{r^2} &= g. \end{aligned}$$

Solving for A and B, we get

$$\begin{aligned} A &= \frac{x_1f + x_2g}{r^2} \\ \frac{B}{r^2} &= \frac{-x_2f + x_1g}{r^2}, \end{aligned}$$

from which we can see that $B = -x_2f + x_1g$ is both smooth and S^1-invariant, so that we may write $B = b(r^2, x')$ as given by Lemma (V.3). Similarly, $r^2A = x_1f + x_2g$ is smooth and S^1-invariant, so that $r^2A = \tilde{a}(r^2, x')$ for some smooth $\tilde{a}(u, x')$. Since $\tilde{a}(0, x') \equiv 0$, we set $a(u, x') = \frac{\tilde{a}(u,x')}{u}$, which is smooth, and so $A = a(r^2, x')$. $\square$

Thus the blow-up ${}^{\tau}\widehat{W}$ is a well-defined symplectic manifold with boundary, and performing the symplectic cut at the level 1 recovers the original manifold W by the same observations as in the model case.

5.3.4 Obtaining the Almost Complex Structure

We use linear algebra to choose an almost complex structure compatible with a given symplectic form. The discussion below is fairly standard and can be found in most introductory texts on symplectic geometry. We will use the notation for the polarized symplectic manifold (W, ω) with Morse-Bott function H.

Fix some S^1-invariant Riemannian metric on W. By non-degeneracy, one can find an invertible matrix A satisfying

$$\omega(\cdot, \cdot) = g(A\cdot, \cdot),$$

where A is S^1-invariant and skew-symmetric with respect to g, i.e. $A^\star = -A$. This is because

$$\omega(x, y) = g(Ax, y) = g(x, A^\star y) = g(A^\star y, x) = \omega(y, x) = -\omega(x, y) = -g(Ax, y).$$

Then $AA^\star = -A^2$ is symmetric and positive definite and therefore diagonalizable with strictly positive, real eigenvalues λ_i. Then it makes sense to take the square root $\sqrt{AA^\star} = \mathrm{diag}(\sqrt{\lambda_i})$ and define $J = (\sqrt{AA^\star})^{-1}A$. This gives a polar decomposition

$$A = \sqrt{AA^\star}J.$$

The matrix A commutes with $\sqrt{AA^\star}$. To see why, denote the λ_i-eigenspace of $AA^\star$ (and hence, the $\sqrt{\lambda_i}$-eigenspace of $\sqrt{AA^\star}$) by V_i. Then for all $v \in V_i$, Av is also in V_i because

$$(AA^\star)Av = -A^3 v = A(AA^\star)v = \lambda_i Av.$$

It follows that J also commutes with A and $\sqrt{AA^\star}$ and is skew-symmetric because

$$J^\star = A^\star(\sqrt{AA^\star})^{-1} = -A(\sqrt{AA^\star})^{-1} = -(\sqrt{AA^\star})^{-1}A = -J.$$

J is also orthogonal because

$$J^\star J = A^\star(\sqrt{AA^\star})^{-1}(\sqrt{AA^\star})^{-1}A = Id,$$

and so $J^2 = -J^\star J = -Id$, so that J is an almost complex structure. It is S^1-invariant because A is and compatible with ω because

$$\omega(Ju, Jv) = g(AJu, Jv) = g(JAu, Jv) = g(Au, Av) = \omega(u, v)$$

and

$$\begin{aligned}
\omega(u, Ju) &= g(Au, Ju) = g(-JAu, u) = g(-(\sqrt{AA^\star})^{-1}AAu, u) \\
&= g(\sqrt{AA^\star})^{-1}(AA^\star)u, u) = g(\sqrt{AA^\star}u, u) > 0.
\end{aligned}$$

We define a pseudo Kähler metric $\tilde{g}$ by

$$\tilde{g}(\cdot, \cdot) = g(\sqrt[4]{AA^\star}\cdot, \sqrt[4]{AA^\star}\cdot).$$

One checks that $\tilde{g}(\cdot, \cdot) = \omega(\cdot, J\cdot)$ and $\tilde{g}(J\cdot, J\cdot) = \tilde{g}(\cdot, \cdot)$.

$$\begin{aligned}
\tilde{g}(\cdot, \cdot) &= g(\sqrt[4]{AA^\star}\cdot, \sqrt[4]{AA^\star}\cdot) = g(\sqrt{AA^\star}\cdot, \cdot) \\
&= g(\sqrt{AA^\star}J\cdot, J\cdot) = g(A\cdot, J\cdot) = \omega(\cdot, J\cdot) \\
\tilde{g}(J\cdot, J\cdot) &= g(\sqrt[4]{AA^\star}J\cdot, \sqrt[4]{AA^\star}J\cdot) \\
&= g(\sqrt{AA^\star}J\cdot, J\cdot) = g(A\cdot, J\cdot) = \omega(\cdot, J\cdot) = \tilde{g}(\cdot, \cdot)
\end{aligned}$$

Thus, $(\omega, J, \tilde{g})$ is a compatible triple (without requiring that J be integrable), so we have a pseudo-Kähler structure on W. Denote the gradient of the function $\sqrt{H}$ with respect to the metric $\tilde{g}$ by $\nabla\sqrt{H}$.

Claim V.5. $\nabla\sqrt{H} = -JX_{\sqrt{H}}$

Proof. By definition,

$$\tilde{g}(\cdot,\cdot) = \omega(\cdot, J\,\cdot) = \omega(-J\,\cdot,\cdot). \tag{5.3}$$

We also have

$$\omega(X_{\sqrt{H}},\cdot) = -d\sqrt{H} = -\tilde{g}(\nabla\sqrt{H},\cdot)$$

Substituting formula (5.3) in the above, one obtains

$$\omega(X_{\sqrt{H}},\cdot) = -\omega(-J\nabla\sqrt{H},\cdot) = \omega(J\nabla\sqrt{H},\cdot),$$

so that $X_{\sqrt{H}} = J\nabla\sqrt{H}$ by non-degeneracy of ω and the claim follows. □

Since J is S^1-invariant, $[-JX_{\sqrt{H}}, X_{\sqrt{H}}] = 0 = [\nabla\sqrt{H}, X_{\sqrt{H}}]$, so that the gradient flow and the flow associated to $X_{\sqrt{H}}$ commute. Flowing by the gradient will then send levels of $\sqrt{H}$ to levels of $\sqrt{H}$ while preserving the S^1-action.

Taking the Zoll cut of $T^\star_{\leq 1}L$ and the symplectic cut of ${}^\tau\widehat{W}$ (which we identify with W), we obtain an equivariant diffeomorphism, which we also denote by $\Phi : W_0 \to W$ such that $\Phi^\star H = H_0$. The cut of $T^\star_{\leq 1}L$ inherits a form, which we continue to denote by ω_0, while W inherits ω. Using the identification under Φ, we can now work in a setting where we assume there is only one manifold, call it W_0, one S^1-action, and one Hamiltonian $\sqrt{H_0}$. However, we now have two (cohomologous) symplectic forms ω_0 and $\omega_1 = \Phi^\star\omega$ on W_0 that are identical near L.

5.3.5 Duistermaat-Heckman

We now aim to show that ω_0 and ω_1 on W_0 as defined above are equivariantly symplectomorphic via a standard Moser argument. To apply equivariant Moser, we first need to obtain a smooth family of cohomologous symplectic forms ω_t connecting ω_0 to ω_1. We will need a theorem of Duistermaat and Heckman relating the cohomology class of symplectic forms of reduced spaces. (The theorem is more general than what is stated below, but we will only need it for the case of a Hamiltonian S^1-action.)

Theorem V.6. ***(Duistermaat-Heckman [12])*** *Fix a regular value ϵ_0 of the moment map for a Hamiltonian S^1-action $\mu : M \to \mathbb{R}$ and identify the diffeomorphism type of the reduced spaces $\mu^{-1}(\epsilon)/S^1 := M_\epsilon \simeq M_{\epsilon_0}$. Then*

$$[\omega_\epsilon] = [\omega_{\epsilon_0}] + (\epsilon - \epsilon_0)\beta,$$

where $\beta \in H^2(W, \mathbb{Z})$ denotes the (common) first Chern class of the S^1-fibration $p_\epsilon : \mu^{-1}(\epsilon) \to M_\epsilon$.

Recalling that the symplectic cut at a level $\sqrt{H_0} = t$ is a special case of the process of symplectic reduction in a larger space, we can apply the Duistermaat-Heckman theorem to our current setting. Taking symplectic cuts of $T^\star_{\leq 1}L$ at different levels $\sqrt{H_0} = t$, and using the fact that the cuts are all diffeomorphic and so can be identified with W_0, we obtain two 1-parameter families of symplectic forms $\omega_{0,t}$ and $\omega_{1,t}$ that are in the same cohomology class for each level.

We will need the following result of Audin's.

Proposition V.7. ***(Audin, Proposition 2.3.5 [2])*** *Suppose (W, ω) is a connected polarized symplectic manifold with $H, L,$ and V as before. Assume further that V is simply connected. Then W is simply connected, V is dual to a multiple of the symplectic form, and $\pi_1(L)$ is at most (finite) cyclic.*

In our case, the class β is given by $c_1(E) \in H^2(W)$, where $E \to W_0$ is a complex line bundle that trivially extends the normal bundle $N_V \to V$ of V in W_0 and $c_1(E) = k[\omega_1]$ is a non-zero multiple of the class of the symplectic form (as in the proof of the above proposition in [2]). Fix ϵ small enough so that the symplectic cuts with respect to ω_0 and ω_1 agree. Then the Duistermaat-Heckman theorem gives that for the symplectic cut at level $\sqrt{H_0} = t$ with $T^\star_{\leq 1}L$ endowed with the symplectic form ω_0,

$$\begin{aligned}
[\omega_{0,t}] &= [\omega_{0,\epsilon}] + (t-\epsilon)\cdot k[\omega_0] \\
&= c_\epsilon[\omega_0] + (t-\epsilon)\cdot k[\omega_0], \quad \text{for some constant } c_\epsilon \\
&= C[\omega_0], \quad \text{for } C = C(t) \text{ a constant depending on } t.
\end{aligned}$$

Then the family of forms $\frac{1}{C(t)}\omega_{0,t}$ for $\epsilon \leq t \leq 1$ are all symplectic and $[\frac{1}{C(t)}\omega_{0,t}] = [\omega_0] = [\omega_1]$ by construction. Since the symplectic cuts agree at level ϵ by choice of ϵ, we have $\frac{1}{C(\epsilon)}\omega_{0,\epsilon} = \frac{1}{C(\epsilon)}\omega_{1,\epsilon}$. By a second application of Duistermaat-Heckman and Audin's proposition, we obtain another family of symplectic forms $\frac{1}{C(t)}\omega_{1,t}$, also in the cohomology class $[\omega_1] = [\omega_0]$.

We set

$$\omega_t = \begin{cases} \frac{1}{C(1-2(\epsilon-1)t)}\omega_{0,1-2(\epsilon-1)t} & \text{if } 0 \leq t \leq \frac{1}{2} \\ \frac{1}{C(\epsilon+2(1-\epsilon)(t-\frac{1}{2}))}\omega_{1,\epsilon+2(1-\epsilon)(t-\frac{1}{2})} & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

We now have a family ω_t of cohomologous symplectic forms on the compact manifold W_0 interpolating between ω_0 and ω_1. By the standard arguments involved in equivariant Moser's method, which we will not repeat here, we obtain an equivariant symplectomorphism $\Psi : W_0 \to W_0$ such that $\Psi^\star\omega_1 = \omega$. Equivariance and the constancy of the S^1-action give that $\Psi^\star H_0 = H_0$. Then $\Psi^\star\omega_1 = \Psi^\star\Phi^\star\omega = \omega_0$, where Ψ and Φ are both equivariant, so that W is equivariantly equivalent to the Zoll cut W_0. To summarize, what we have shown is:

Theorem V.8. *Let (W,ω) be an even polarized symplectic manifold and suppose its Morse-Bott function H is such that $X_{\sqrt{H}}$ generates a free S^1-action away from L and V. Assuming further that $H|_L = 0$ and $H|_V = 1$, then there exists an S^1-equivariant* *map* $\Pi : W_0 \to W$ *between the Zoll cut of* $(T^\star L, \omega_0)$ *at* $H_0 = 1$ *and* W *such that* $\Pi^\star \omega = \omega_0$ *and* $\Pi^\star H = H_0$. *(However, the diffeomorphism* Π *will only be* C^1 *along* L.*)*

CHAPTER VI

Low Dimensions

We end this thesis with a quick summary of what is known in low dimensions about polarized symplectic manifolds (W^{2n}, ω). The case $n = 1$ is fairly uninteresting, so we do not spend much time on it. We then improve on a result of Audin's classifying the $n = 2$ case using results from Gromov, McDuff, and Lalonde and a little intersection theory. Lastly, we mention a result of Audin's in the case $n = 3$.

6.1 Case $n = 1$

In this case, dim $L = 1$ and W is a surface endowed with an area form (which determines the symplectic form). Since the symplectic submanifold V has codimension 2, it is just a finite collection of points. There is really only one example, and in this case, the area of W completely determines its symplectic structure up to isotopy (according to Moser's theorem).

6.2 Case $n = 2$

When the Lagrangian L is a surface (orientable or not), the polarized symplectic manifold W has (real) dimension 4. The symplectic submanifold V is also a surface but, being symplectic, must be oriented.

Audin has already classified the possible polarized symplectic manifolds in this case up to homeomorphism. [3]

Proposition VI.1. ***(Audin [3])*** *Suppose* (W, ω) *is a connected, compact polarized symplectic manifold with Morse-Bott function* H *reaching its minimum along a Lagrangian* L *and maximum along a symplectic submanifold* V. *Then* V *is a 2-sphere and there are only two possibilities for* L *(up to diffeomorphism):*

- *Either* L *is diffeomorphic to the 2-sphere and* W *is homeomorphic to* $S^2 \times S^2$,
- *or* L *is diffeomorphic to the real projective plane* $\mathbb{RP}^2$ *and* W *is homeomorphic to* $\mathbb{CP}^2$.

In fact, one can show more. For symplectic manifolds of dimension 4, McDuff and Lalonde exhibited criteria for determining when a symplectic 4-manifold is a blow-up of a rational ruled surface or of $\mathbb{CP}^2$. We are interested in the case where the 4-manifold is minimal in the sense that it contains no symplectically embedded 2-spheres of self-intersection -1. [18] We first briefly recall some definitions.

Definition VI.2. A 4-manifold (W, ω) is *ruled* if it is the total space of an S^2-fibration $\pi : W \to \Sigma$, where Σ is a Riemann surface. The symplectic form ω on W is said to be *compatible* with the ruling if it is non-degenerate on the fibers. Such a ruling π is then said to be *symplectic*.

We are now ready to state the relevant theorem.

Theorem VI.3. ***(Gromov-McDuff-Lalonde [13] [18])*** *Suppose* (W, ω) *is a closed, connected symplectic 4-manifold, and let* C *be a symplectically embedded 2-sphere such that* $C.C \geq 0$. *If* $W - C$ *is minimal, then* (W, ω) *is symplectomorphic to* $\mathbb{CP}^2$ *with its standard Kahler structure or is a ruled symplectic manifold with* ω *isotopic to a standard Kahler form.*

The proof modifies a deformation of symplectic forms in the same cohomology class by a homotopy relative endpoints into an isotopy using a method called "inflation." The process involves taking a topologically trivial symplectic sum of the original manifold W along some symplectic divisor D with a ruled surface. (Algebraic geometers would recognize this as a deformation to the normal cone of D.) The interested reader is encouraged to consult [18] for details of the proof.

In the case where $L = S^2$, we need only to exhibit a symplectically embedded sphere of non-negative self-intersection, and McDuff's result will give us that W is isotopic to $S^2 \times S^2$ with a standard product form. An obvious candidate would be the symplectic submanifold V. Using Audin's result, since W is homeomorphic to $S^2 \times S^2$, we know the structure of its cohomology ring and hence, its intersection form.

Since L and V are both compact and orientable, they represent 2-cycles in homology $[L]$ and $[V]$. As $L = S^2$ is Lagrangian, it has self-intersection -2. (All Lagrangian spheres of a given dimension are symplectically equivalent. [16]) We also have that $[L].[V] = 0$, since L and V are disjoint. Choosing a basis for homology $f_1 = [S^2 \times \{\text{pt}\}], f_2 = [\{\text{pt}\} \times S^2]$ such that $f_i.f_j = \delta_{ij}$, we may express $[L], [V]$ and the intersection form Q_W in terms of this basis.

$$Q_W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$[V] = af_1 + bf_2$$

$$[L] = cf_1 + bf_2$$

A simple calculation using $[L].[L] = -2$ shows that $c = -d = \pm 1$, and $[L]$ spans the (-1)-eigenspace of the matrix Q_W. Since $[L].[V] = 0$, $[V]$ must span the

$(+1)-$eigenspace and so $[V].[V] > 0$. By McDuff's result, W must be isotopic to $S^2 \times S^2$ with a standard product form. Since W is compact, Moser's theorem gives that they are in fact strongly isotopic and, hence, symplectomorphic.

When $L = \mathbb{RP}^2$ and W is homeomorphic to $\mathbb{CP}^2$, the intersection form reduces to a scalar, so the above argument does not work. We instead look at a regular level surface $\mathcal{C} = H^{-1}(c)$ of a regular value c for the given Morse-Bott function. Since a neighborhood of the Lagrangian $\mathbb{RP}^2$ is diffeomorphic to a neighborhood of the zero section of $T^\star\mathbb{RP}^2$ with the canonical symplectic form, $\mathcal{C}$ must be diffeomorphic to the unit sphere bundle $S(T^\star\mathbb{RP}^2) \approx L(4,1)$. (One can think of the lens space $L(4,1)$ as the unit sphere $S^3 \subset \mathbb{C}^2$ quotiented out by the action of the fourth roots of unity.)

In this case, we use the fact that $p : \mathcal{C} \to V$ can be viewed as a principal S^1-bundle over $V \approx S^2$. We use the Euler class of this bundle (which coincides with the Euler class of the normal bundle of V in W) to compute the self-intersection of V.

By Van Kampen's theorem, since $\pi_1(V) = \{1\}$ we may write

$$\pi_1(\mathcal{C}) = \langle c \,|\, c^m = 1 \rangle,$$

where c is the image of a fiber and m is plus or minus the Euler class of the fibration $p : \mathcal{C} \mapsto V$. But $\pi_1(\mathcal{C}) = \pi_1(\mathrm{L}(4,1)) = \mathbb{Z}_4$, so $m = \pm 4$. To see why m must be positive, we make use of a result of Audin's that we already encountered in the previous chapter:

Proposition VI.4. ***(Audin, Proposition 2.3.5 [2])*** *Suppose (W, ω) is a connected polarized symplectic manifold with H, L, and V as before. Assume further that V is simply connected. Then W is simply connected, V is dual to a multiple of the symplectic form, and $\pi_1(L)$ is at most (finite) cyclic.*

Writing $\mathrm{PD}[V] = k[\omega]$ for $k \neq 0$, where $\mathrm{PD}[V]$ denotes the Poincare dual of the class of $[V]$, and using the duality of the intersection form with the cup product in cohomology, we have:

$$[V].[V] = k[\omega] \smile k[\omega] = k^2[\omega] \smile [\omega] \neq 0.$$

By assumption on the orientation of (W, ω), $\omega^2 > 0$ gives a positive volume form on W. Thus $[V].[V] > 0$, and by the result of Gromov, W is symplectomorphic to $\mathbb{CP}^2$ with the standard Kahler form.

Theorem VI.5. *Let (W, ω) be a polarized symplectic manifold (in the sense of Audin). Then there are only two possibilities, up to symplectomorphism:*

- *The Lagrangian submanifold L is diffeomorphic to S^2, and W is symplectomorphic to $S^2 \times S^2$ with a standard product form; or*
- *The Lagrangian L is diffeomorphic to $\mathbb{RP}^2$, and W is symplectomorphic to $\mathbb{CP}^2$ with a standard Kahler form.*

In particular, the Zoll cut construction applied the cotangent bundle of a Zoll sphere L will be symplectomorphic to the cut with the respect to round metric.

6.3 Case $n = 3$

In this case, the symplectic submanifold V is of dimension 4. Audin has some classification results under the restrictive assumption that there is a Hamiltonian action of $SU(2)$ or $SO(3)$ on a compact connected W of dimension 6, where the moment map for the action μ is such that $H = ||\mu||^2$ has only two critical values, one of which is along a Lagrangian submanifold.

Proposition VI.6. ***(Audin, Proposition 2.4.1 [2])*** *Under the above assumptions, the minimum of H is zero and corresponds to the Lagrangian L, which is either*

S^3 or $\mathbb{RP}^3$, while the maximum corresponds to a symplectic $S^2 \times S^2$. The manifold W is then either $\tilde{G}_2(\mathbb{R}^5) = Q^3$ or $\mathbb{CP}^3$.

It is noteworthy that none of the above classification results provides any information on the uniqueness of the Morse-Bott function H in the definition of polarized symplectic manifolds.

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