Complex Geometric Invariants Associated to Zoll Manifolds

by

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ABSTRACT

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In this thesis, we explore several invariants associated to Zoll manifolds such as the manifold of geodesics and the ways to embed them into complex spaces. We study the construction by Lebrun and Mason on Zoll surfaces and the Grauert tube constructions in the CROSSes. We prove that the only Zoll spheres with an infinite tube are the round ones. Using a similar argument, we prove a special case of Burns' algebraicization conjecture. In the study of the Lebrun-Mason construction, we obtain some new ideas about the embedded holomorphic disks on the quadric $\mathcal{Q}^2 \subset \mathbb{CP}^3$.

CHAPTER I

Introduction

It is well-known that the round sphere S^n has many symmetries. One of the "symmetries" that the sphere possesses is that it is a Zoll manifold, i.e., all of its geodesics are simply closed and of equal length. In fact, it is well-known that all compact rank-one symmetric spaces (CROSSes) are Zoll. It seems that the Zoll condition is very strong, which may suggest that Zoll manifolds are not abundant at all. But to the contrary even we restrict to the surface in the case of $n = 2$ with an S^1-action of isometries, i.e., a surface of revolution, there is already an infinite dimensional family of Zoll metrics on S^2 [Bes78]. In this thesis, our main focus will be on the $n = 2$ Zoll case, although some of the theorems here apply to the case of arbitrary dimension.

It is also well-known that the fundamental group of a Zoll manifold must be finite [Bes78]. Indeed one can show that any Zoll manifold is compact and the universal cover of a Zoll manifold will have the property that all the geodesics are closed. This in turn shows that the universal cover is also compact, thus the original Zoll manifold will have a finite fundamental group. This means all the Zoll *surfaces* must be diffeomorphic to the real projective space or the sphere. The original Blaschke conjecture, proved by Leon Green [Gre63], asserts that, up to isometries and rescaling, the only Zoll metric on $\mathbb{RP}^2$ is the standard one. On the other hand, not much is known about the Zoll metrics on S^2. It is known that, by a theorem of Guillemin [Gui76], originally suggested by Funk [Fun13], modulo isometries and rescalings, a general Zoll perturbation of the round metric on S^2 depends on an odd function on S^2.

Given a Zoll manifold (M, g) of real dimension n, one can define the moduli space of oriented geodesics N_g^+ as a symplectic manifold [Bes78]. We would naturally ask whether we could put (M, g) and N_g^+ together and study their correspondence.

One would start looking at the tangent bundle TM of M. It is well-known that TM is equipped with a canonical symplectic form (which is the pull-back of the Liouville form on the cotangent bundle by the metric). With respect to this symplectic form, the Hamiltonian flow of the energy function $E := \frac{1}{2}g_x(v, v)$ is well-known to be the geodesic flow of (M, g). Since (M, g) is Zoll, the geodesic flow is periodic in the slit tangent bundle $TM - 0_M$. Thus we can perform the symplectic reduction at the level $E = \frac{1}{2}$. The resulting manifold is exactly N_g^+, the moduli space of oriented geodesics. One could also perform the symplectic cut at $E = \frac{1}{2}$ to get a $(2n)$-dimensional manifold X. Notice that M is identified as the zero section in TM; and N^+ is identified as the symplectic reduction and both are submanifolds of X. One interesting observation here is that M is Lagrangian and N is symplectic. [Aud07]

On the other hand, Lebrun and Mason [LM02, LM10] introduced a different approach to Zoll surfaces. Given a Zoll surface M, we look at the projectivized tangent bundle $\mathbb{P}TM$ being embedded into the projectivized complexified tangent bundle $\mathbb{P}T_{\mathbb{C}}M$. The canonical foliation map $\mathbb{P}TM \to N$, where N denotes the moduli space of unoriented geodesics, gives an embedding of N into $\mathbb{CP}^2$, which is obtained by blowing down the closure of a connected component of $\mathbb{P}T_{\mathbb{C}}M - \mathbb{P}TM$ along the projection $\mathbb{P}TM \to N$. Then the locus $\{g_{\mathbb{C}}(v,v) = 0\} \subset \mathbb{P}T_{\mathbb{C}}M$, which descends to $\mathbb{CP}^2$, traces out a complex submanifold, which can be identified with M, with the conformal structure determined by the Zoll metric. This shows that there is an opposite way to embed M and N in $\mathbb{CP}^2$, with M being complex and N being totally real and Lagrangian (with respect to a sign-ambiguous symplectic form on $\mathbb{CP}^2 - M$).

Lebrun and Mason showed part of the converse: if N is a totally real surface diffeomorphic to $\mathbb{RP}^2$ satisfying some transversality and Lagrangian conditions (they say N is *docile*), then one can reconstruct a Zoll manifold (M, g) such that $N \subset \mathbb{CP}^2$ is given by the above construction. Later Rochon [Roc11] showed that the Zoll manifold constructed is the unique one leading to N by the Lebrun-Mason construction. Since docility is an open condition in the (infinite dimensional) space of all totally real embeddings of $\mathbb{RP}^2$ into $\mathbb{CP}^2$, we at least have some idea about a large open set of the space of all the Zoll metrics on S^2.

One would start to think if there is a construction that put the world of complex geometry and symplectic geometry together: Lempert and Szőke [LS91, Sző91], together with Guillemin and Stenzel [GS91, GS92], investigated how to put a complex structure in a neighborhood of the zero section of the tangent bundle, such that for each geodesic γ, its tangent bundle $T\gamma$ is a complex submanifold of TM where the complex structure is defined. They together showed that the energy $E := \frac{1}{2}g_x(v,v)$ is strictly plurisubharmonic (s.p.s.h.) and its square root is plurisubharmonic (p.s.h.) and satisfies the homogeneous complex Monge-Ampère (HCMA) equation off the zero-section. In this complex structure, the zero section M is embedded in an open set in TM as a totally real submanifold. The complex structure is called the *adapted* *complex structure* and the open neighborhood of M in TM is called the *Grauert tube* of M. It is well-known (for example [LS91]) that for the CROSSes, the adapted complex structure can be defined on the whole tangent bundle. By compactifying the tangent bundle, M can be embedded in a compact complex manifold X. The complement $X - TM$ corresponds exactly to the manifold of oriented geodesics N^+. In this way, M is a totally real submanifold and N^+ is a codimension one complex submanifold of X. For example, if $M = S^n$, $TM = \mathcal{Q}^n_{aff}$ and $X = \mathcal{Q}^n$, where $\mathcal{Q}^n$ is the (non-singular) hyperquadric in $\mathbb{CP}^{n+1}$ and $\mathcal{Q}^n_{aff}$ is the affine part. Here $N^+ = \mathcal{Q}^n - \mathcal{Q}^n_{aff} = \mathcal{Q}^{n-1}$. One would say that this is the "cut at infinity" as the resulting compactified manifold is Kähler.

In these cases, there are always extra structures corresponding to the embedding, namely foliations. In the Grauert tube picture, there are Riemannian foliations on $X - M$ such that all the leaves are closed and could be extended across M. Each leaf corresponds to a compactified complexified geodesic - $T\gamma$ compactified with 2 extra points. (The same applies to the symplectic cut picture for a general M.) Following the Lebrun-Mason construction, there are foliations of $\mathbb{CP}^2 - N$ by holomorphic disks centered in M with boundaries in N. For a totally real $M \subset X$, let $\mathcal{M}$ be the moduli space of all embedded holomorphic disks in $X - M$ with boundaries on M. In the Lebrun-Mason construction, $X = \mathbb{CP}^2$, M is a docile surface and $\mathcal{M} \approx S^2$.

In this thesis, we investigate these pictures and present some interesting results. In Chapter II, we review some basic facts about Zoll manifolds. In Chapter III, we review the Grauert tube picture described in [LS91]. It is clear that there exists a family of surface of revolutions that has infinite Grauert tubes[Sző91]. But what about Zoll surfaces? The round n-spheres are Zoll and they have infinite Grauert tubes. Are there any others? The answer is no:

Theorem III.20. *Let M be diffeomorphic to S^n be a Zoll manifold with period 2π, such that the adapted complex structure is defined on the whole tangent bundle $(R = \infty)$. Then M is isometric to the round n-sphere.*

In Chapter 3.5, we generalize this theorem by localizing the Zoll properties of u, the solution of the homogeneous Monge-Ampère equation, near infinity. In particular, we prove a special case of Burns' algebraicization conjecture. In Chapter IV, we review the Lebrun-Mason construction and discuss our related work in Chapter V about holomorphic disks. Lastly, we outline some possible further research directions in Chapter VI.

CHAPTER II

Zoll Manifolds and their Properties

2.1 Definitions and manifold of geodesics

We would start by stating the basic definitions of Zoll manifolds and the objects associated to them.

Definition II.1. Let (M^n, g) be a Riemannian manifold of dimension n. M is a *Zoll* manifold if all of whose geodesics associated to g are simply closed curves of equal length.

Remark II.2. In [Bes78], Zoll manifolds with all of whose geodesics are of length l are called SC_l-manifolds.

Let TM be the tangent bundle of a Riemannian manifold (M, g). We have the Liouville form α_g given by the pullback of the canonical Liouville form α on T^*M by g. Let (x^i) be a local coordinates on M and (p^i) be the coordinates on TM cooresponding to the basis $\dfrac{\partial}{\partial x^i}$. In this local coordinate system, we have

$$g = \sum_{i,j} g_{ij} dx^i \otimes dx^j. \tag{2.1}$$

Then

$$\alpha_g = \sum_{i,j} g_{ij} p^i dx^j. \tag{2.2}$$

Let $x \in M$ and $v \in T_xM$. Let $E(x, v) = \frac{1}{2}g_x(v, v)$ be the energy function on TM. Denote $\mathbb{S}TM$ be the unit tangent bundle $E^{-1}(\frac{1}{2})$. From the viewpoint of symplectic geometry, the geodesic flow on TM is given by the hamiltonian flow of E with respect to the symplectic form $\omega_g := d\alpha_g$. Restricting the flow to the slit tangent bundle $TM - 0_M$, by the Zoll property of M, the orbits of the hamiltonian flow are all periodic and of equal length. This means that by symplectic reduction, we can define

Definition II.3. N^+ as the moduli space of the orbits of the flow. In particular,

$$N^+ = \mathbb{S}TM/S^1, \tag{2.3}$$

is a (smooth) manifold, where S^1 is action of the geodesic flow, i.e. Hamiltonian flow.

Remark II.4. Note that symplectic reduction reduces the dimension by 2, which means $\dim N^+ = 2n - 2$.

In this case, each orbit corresponds to an oriented geodesic. This means that N^+ cooresponds to the manifold of oriented geodesics.

One could also consider the map a on N^+ to itself by reversing the direction of the geodesic. The map a is an involution on N^+ and thus we can

Definition II.5. Define

$$N := N^+/a \tag{2.4}$$

be the manifold of unoriented geodesics.

One can also think of a as the map induced by N_{-1} on TM, in which for $s \in \mathbb{R}$, let N_s denote the map that multiplies the vectors in TM by s. In other words, for

$x \in M$ and $v \in T_x M$,

$$N_s(x, v) = (x, sv). \tag{2.5}$$

The map N_{-1} preserves the level sets of E and thus it defines an involution on $\mathbb{S}TM$. Using this map, we can consider the free action of $\mathbb{Z}_2 \times S^1$ on $\mathbb{S}TM$ and this gives N. It is clear that N^+ is a two-sheeted covering of N.

As one can see that the tangent space of N^+ captures the geodesic variations. Thus one intuitively would think that the tangent space of N^+ is related to the space of Jacobi fields. This is indeed the case:

Proposition II.6. *([Bes78] Theorem 2.13) The tangent space at a point γ in N^+ is naturally isomorphic to the space of normal Jacobi fields along the geodesic γ in M.*

Remark II.7. We exhibit a local chart for N^+: Let $\gamma \in N^+$. Then γ is a unit speed geodesic on M. Let $m = \gamma(0)$. Let $N_m\gamma$ be the vector subspace of T_mM normal to $\dot\gamma(0)$. Define a map φ from $\mathbb{R}^{2n-2}$ to $E^{-1}(\frac{1}{2})$ as follows: for $(u, v) \in N_m\gamma \times N_m\gamma$,

$$\varphi(u, v) = \frac{d}{ds} \exp_{\exp_m u}\left(sP_u \left(\frac{\dot\gamma(0) + v}{\sqrt{1 + g_m(v, v)}} \right)\right)\Bigg|_{s=0}, \tag{2.6}$$

where P_u is the parallel transport along the geodesic $t \mapsto \exp_m tu$. Via the projection map $q : E^{-1}(\frac{1}{2}) \to N^+$, the map $q \circ \varphi$ is a differentiable map from $\mathbb{R}^{2n-2}$ into N^+ with maximal rank [Bes78], thus it is a local chart of N^+ around γ. See Fig. 2.15 of [Bes78].

The manifold of oriented geodesics N^+, constructed by symplectic reduction, is clearly a symplectic manifold with the induced symplectic form ω. For a point $\gamma \in N^+$ and $J_1, J_2 \in T_\gamma N^+$, where we identifies tangent vectors to normal Jacobi fields J_1 and J_2, we have

$$\omega(J_1, J_2) = g(J_1, J_2') - g(J_2, J_1'), \tag{2.7}$$

where J' denotes the covariant derivative of J along γ.

2.2 Compact Rank One Symmetric Spaces

The standard examples of Zoll manifolds are the spaces that are called the Compact Rank One Symmetric Spaces (CROSSes). We start off by introducing basic definitions. Let M be a Riemannian manifold and $p \in M$. Define $s : T_pM \to T_pM$ by $s(X) = -X$ for $X \in T_pM$ (Note that $s = N_{-1}$ from the previous section).

Definition II.8. A connected Riemannian manifold M is a **symmetric space** if for each $p \in M$ there is an involutive isometry $s_p : M \to M$ such that

$$s_p \circ \exp_p = \exp_p \circ s. \tag{2.8}$$

Clearly if M is symmetric then M is complete.

Remark II.9. If M is a symmetric space, then M can be written as G/K, where G is the Lie group of isometries of M and K is the isotropy group of the action of G on M fixing a point $p \in M$.

Definition II.10. The **rank** of a symmetric space M is the maximal dimension of a flat, totally geodesic submanifold of M.

We restrict our focus on the symmetric spaces of compact type.

Proposition II.11. *(Classification of CROSSes) Let M be a compact rank one sym-*

metric space. Then M must be one of the followings:

M	G	K
$\mathbb{RP}^n$ $(n \geq 1)$	$O(n+1)$	$O(n) \times O(1)$
S^n $(n \geq 1)$	$O(n+1)$	$O(n)$
$\mathbb{CP}^n$ $(n \geq 1)$	$U(n+1)$	$U(n) \times U(1)$
$\mathbb{HP}^n$ $(n \geq 1)$	$Sp(n+1)$	$Sp(n) \times Sp(1)$
$\mathbb{OP}^2$	F_4	$Spin(9)$

Proposition II.12. *The CROSSes are Zoll.*

2.3 Examples of non-trivial Zoll manifolds

It seems very difficult for a manifold to be Zoll. The CROSSes have a lot of symmetries and so one would think that they are naturally Zoll. But in fact there are Zoll manifolds with much smaller (or even trivial) isometry groups. We are going to describe a family of Zoll surfaces. With an S^1 action by isometries, one would study the surfaces of revolution on S^2.

Let M be diffeomorphic to S^2 having S^1 as an effective isometry group, i.e. the metrics of revolution. Since $\chi(S^2) = 2$ and by Hopf's index theorem, this equals to the sum of the indices at the zeroes of any vector field with isolated singularities. The vector field generated by the S^1 isometry action will have isolated zeroes and the indices of those would be 1. This means that there are two fixed points of the S^1 action. We call them N and S, which denote the north and south pole.

A local chart of $M - \{N, S\}$ can be given by

$$(u, \theta) \in (0, \pi) \times (0, 2\pi) \to M - \{N, S\}. \tag{2.9}$$

The S^1 action is given by the (constant) translation in the θ variable.

We have the following classification of Zoll surfaces of revolution.

Proposition II.13. *In the above coordinate system, g is a Zoll metric of revolution with all the period of length 2π if and only if*

$$g = [1 + h(\cos r)]^2 \, dr^2 + \sin^2 r \, d\theta^2, \tag{2.10}$$

where $h : [-1, 1] \to (-1, 1)$ is an odd function such that $h(1) = h(-1) = 0$. The metric is C^k if and only if h is C^k on $[-1, 1]$.

This shows that there is an infinite dimensional family of Zoll surfaces even we restrict our objects of interest to the surfaces of revolution.

The theorem can be extended to showing that there are nontrivial Zoll metrics on S^n:

Proposition II.14. *Let $h : [-1, 1] \to (-1, 1)$ be a C^∞ odd function with $h(-1) = h(1) = 0$. Then the metric*

$$g = (1 + h(\cos\theta))^2 \, d\theta + \sin^2 \theta \; can_{d-1} \tag{2.11}$$

extends to a Zoll metric on S^d. Here can_{d-1} is the canonical metric on S^{d-1}. For details, see [Bes78].

For general Zoll metric on S^2, we let $g_t = \exp(\rho_t) g_0$ be a smooth family of Zoll metrics on S^2, where g_0 is the round metric on S^2. Then Weinstein [Wei74] showed the following:

Proposition II.15. *Let $g_t = \exp(\rho_t) g_0$ be a smooth family of Zoll metrics with geodesics of length 2π on a compact manifold M, with $\rho_0 = 0$. Denote $\dot\rho = \left.\dfrac{d\rho_t}{dt}\right|_{t=0}$. Then for any closed geodesic γ of the metric g_0,*

$$\int_0^{2\pi} \dot\rho(\gamma(s)) ds = 0. \tag{2.12}$$

For any Zoll perturbation of the round metric of S^2, Funk[Fun13] showed (2.12) holds if and only if $\dot{\rho}$ is an odd function of S^2, by means of Radon transform. One would naturally ask that given any odd function f on S^2, does there exist a family $g_t = \exp(\rho_t)g_0$ such that $\dot{\rho} = f$? Funk gave a method for this by expanding $\exp(\rho_t)$ analytically but his proof was only formal. He did not succeed in proving the convergence of the series for t small. Later, Guillemin[Gui76] proved the following results by looking into the Nash-Moser implicit function theorem:

Theorem II.16. *For every odd function f on S^2, there exists a smooth one-parameter family of C^∞ functions ρ_t such that $\rho_0 = 0$, $\dot{\rho} = f$ and $\exp(\rho_t)g_0$ is a Zoll metric for small t.*

After some more work, one can show [Bes78] that there exists an open dense subset in the space of odd functions on S^2 such that for any f odd on S^2, $\exp(\rho_t)g_0$ admits no non-trivial isometry for t small.

CHAPTER III

Grauert Tubes

3.1 Monge-Ampère model and adapted complex structures

Following [LS91], we discuss the Monge-Ampère model.

Definition III.1. Let X be a complex manifold of complex dimension n, M a real analytic maximally totally real submanifold of X such that $M = \{u = 0\}$ for a non-negative plurisubharmonic exhaustion function u of X. Moreover, u^2 is smooth on all of X and strictly plurisubharmonic and that u satisfies the Homogeneous complex Monge-Ampère equation

$$(\partial\bar{\partial}u)^n = 0 \tag{3.1}$$

on $X - M$. In this case, we say that (X, M, u) is a **Monge-Ampère model**. Here we call M the **center**.

We have

$$\frac{\partial^2(u^2)}{\partial z^i \partial \bar{z}^j} = 2\left(u\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} + \frac{\partial u}{\partial z^i}\frac{\partial u}{\partial \bar{z}^j}\right).$$

Let $s = (s^i)$ such that

$$\sum_{i,j} \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} s^i \bar{s}^j = 0.$$

Then by the strict plurisubharmonicity of u^2, we have $\sum_i s^i \frac{\partial}{\partial z^i} \neq 0$. But there is at most 1 dimension of s satisfying this condition. Thus we have that the kernel of

$\partial\bar{u}$ is of one dimension. This gives a foliation of $X - M$ with leaves of one complex dimension and $\partial\bar{\partial}u(v, -) = 0$ for v tangent to each leaf. We call this the Monge-Ampère foliation.

Since u^2 is strictly plurisubharmonic, there is a Kähler metric on X whose Kähler form is $-i\partial\bar{\partial}u^2$. We call this metric the u^2-metric or the metric associated to u. To study the Monge-Ampère model, one would define the *Riemmanian foliation* of the slit tangent bundle of some manifolds.

Example III.2. Let (M, g) a compact real-analytic Riemannian manifold with g real analytic, and TM be its tangent bundle. Let $\tau \in \mathbb{R}$ and recall that from (2.5), $N_\tau : TM \to TM$ be the smooth mapping defined by multiplication by τ in the fibers. If $\gamma : \mathbb{R} \to M$ is a geodesic, we define an immersion $\psi_\gamma : \mathbb{C} \to TM$ by

$$\psi_\gamma(\sigma + i\tau) = N_\tau \dot{\gamma}(\sigma). \tag{3.2}$$

The images of $\mathbb{C} - \mathbb{R}$ under the mapping ψ_γ defines a smooth foliation of $TM - 0_M$ by (real) surfaces. Each leaf of the foliation extends across M but the leaves will intersect each other on M. We called this foliation the *Riemannian foliation*. We call each leaf a *complexified geodesic* and we denote it by C_γ for the leaf containing a geodesic γ. (or just C if the context is clear). Let

$$T^R M = \{v \in TM : g(v, v) < R^2\}. \tag{3.3}$$

and recall from Chapter II that $E : TM \to \mathbb{R}$ denotes half the riemannian length:

$$E(x, v) = \frac{1}{2} g_x(v, v), \tag{3.4}$$

Then we have:

Proposition III.3. *([LS91] Theorem 3.1) Given a Monge-Ampère model (X, M, u), such that $R = \sup u \leq +\infty$, let g be the restriction of the u^2-metric on M. Then* *there is a diffeomorphism $\phi : T^R M \to X$ such that $2E = (u \circ \phi)^2$ and ϕ maps the leaves of the Riemannian foliation associated to g biholomorphically onto leaves of the Monge-Ampère foliation.*

Notice that, a priori, $T^R M$ does not carry a complex structure. This means ϕ is only a diffeomorphism, not a biholomorphism. Pulling back the complex structure on X to $T^R M$ via ϕ gives a complex structure on $T^R M$, in which the leaves of the Riemannian foliations are complex submanifolds:

Definition III.4. An **adapted complex structure** on $T^R M$ is a smooth complex structure on $T^R M$ such that the leaves of the Riemannian foliation are complex submanifolds in this structure.

The adapted complex structure is studied by [LS91, Sző91] and at the same time by [GS91, GS92].

Remark III.5. P. Dombrowski [Dom62, Agu96] introduced another way to define an almost complex structure J on TM, by the following:

Let $X, Z \in TM$, and $X^h_Z, X^v_Z \in T_Z TM$ be the horizontal and vertical lifts of X at Z. Then we define

$$JX^h_Z = X^v_Z; \quad JX^v_Z = -X^h_Z. \tag{3.5}$$

Dombrowski[Dom62] shows that J is integrable if and only if (M, g) is flat.

Here we want to show the uniqueness of the adapted complex structure.

Definition III.6. A **parallel vector field** ξ on a complexified geodesic C is a vector field on $T^R M$ along C such that ξ is invariant under N_τ and ϕ_s, where ϕ_s is the geodesic flow.

Let $z \in T^R M - M$ and $\tilde{\xi} \in T_z T^R M$, then there exists a parallel vector field ξ such that $\xi(i) = \tilde{\xi}$.

A parallel vector field can also be visualized as geodesic variations. Let $z \in T^RM - M$ and $\tilde{\xi} \in T_zT^RM$. Then $\tilde{\xi}$ is a tangent vector at z of a smooth curve $z(t) \in TM$ with $z(0) = z$. Denote $\gamma_t : \mathbb{R} \to M$ as a geodesic variation such that for each t, $\dot{\gamma}_t(0) = z(t)$. For every t, the mapping ψ_{γ_t} defined by

$$\psi_{\gamma_t}(\sigma + i\tau) = N_\tau \dot{\gamma}_t(\sigma) \tag{3.6}$$

are holomorphic from open subsets of $\mathbb{C}$ to TM and $\psi_{\gamma_t}(i) = z(t)$. Then we have

$$\xi = \frac{d}{dt}\psi_{\gamma_t}\bigg|_{t=0}. \tag{3.7}$$

Then we also have $\xi(i) = \tilde{\xi}$. Let $\pi : TM \to M$ is the projection map and $K : T(TM) \to TM$ is the connection map.

Lemma III.7. *$\xi|_{\mathbb{R}}$ is a Jacobi field Y along γ_0, with $Y(\sigma) = \pi_*(\xi(\sigma + i))$ and $Y'(\sigma) = K(\xi(\sigma + i))$.*

Proof. Using the above notation we have

$$\xi(\sigma) = \frac{d}{dt}N_0\dot{\gamma}_t(\sigma)\bigg|_{t=0} = \frac{d}{dt}\gamma_t(\sigma)\bigg|_{t=0}. \tag{3.8}$$

Thus $\xi|_{\mathbb{R}}$ is a Jacobi field Y with

$$Y(\sigma) = \xi(\sigma) = (N_0)_*(\xi(\sigma + i)) = \pi_*(\xi(\sigma + i)) \tag{3.9}$$

and

$$Y'(\sigma) = K\frac{d}{d\sigma}\xi(\sigma) = K\frac{d}{d\sigma}\frac{d\gamma_t(\sigma)}{dt}\bigg|_{t=0} = K\,\frac{d}{dt}\dot{\gamma}_t(\gamma)\bigg|_{t=0} = K\xi(\sigma + i) \tag{3.10}$$

since $N_0 = \pi$. □

Conversely,

Lemma III.8. *Let Y be a Jacobi field along γ with $Y(0) = u$ and $Y'(0) = v$, there exists a unique parallel vector field ξ along C_γ such that $\xi|_{\mathbb{R}}(\sigma) = Y(\sigma)$*

Proof. At the point $z = \dot{\gamma}(0) \in C_\gamma$, T_zTM is the direct sum of H_zTM and V_zTM, the horizontal subspace and vertical subspace of T_zTM. Therefore there is a unique $\xi(i) \in T_zTM$ such that $K\xi(i) = Y'(0)$ and $\pi_*\xi(i) = Y(0)$. Extend this to a parallel vector field ξ so that it restricts to $\mathbb{R}$ is a Jacobi field $\tilde{Y}$ with $\tilde{Y}(0) = Y(0)$ and $\tilde{Y}'(0) = Y'(0)$. By the uniqueness of Y, $Y = \tilde{Y}$. $\square$

Notice that if ξ is tangent to the leaf at a point, then ξ is tangent to the leaf at all points.

Recall that α_g is the canonical one form on TM (*c.f.* (2.2)). Using Lempert-Szőke's notation, we rename α_g to Θ:

$$\Theta(v) = g(z, \pi_* v) \quad v \in T_z(TM). \tag{3.11}$$

Then $\Omega := d\Theta$ is the canonical symplectic form on TM. For $z \in TM$, we define

$$V_z = \ker(\Theta)_z \cap \ker(dE)_z \subset T_z(TM). \tag{3.12}$$

By Theorem 5.3 in [LS91], we know that V_z is a J-invariant subspace. Let $z = \dot{\gamma} \in TM$. We also notice that ξ is a parallel vector field such that $\xi(i) \in V_z$ if and only if $\xi|_{\mathbb{R}}$ is a normal Jacobi field along γ.

Let γ be a arc-length parametrized geodesic in M and let $z = \dot{\gamma}(0) \in T_{\gamma(0)}M$. Choose $w_j \in T_{\gamma(0)}M$, $1 \leq j \leq n$ be a set of orthonormal basis such that $w_n = z$. Define parallel vector fields ξ and η such that

$$\pi_*\xi_j(i) = w_j \quad K\xi_j(i) = 0; \tag{3.13}$$

and

$$\pi_*\xi_j(i) = 0 \quad K\xi_j(i) = w_j. \tag{3.14}$$

Let $Y_j = \xi_j|_{\mathbb{R}}$ and $Z_j = \eta_j|_{\mathbb{R}}$ be Jacobi fields along γ. Notice that Y_j's are pointwise linearly independent (except perhaps on a discrete subset S of $\mathbb{R}$) Jacobi fields along

γ. The Z_j's are also smooth vector fields and hence there exist smooth functions $\tilde{a}_{jk}$ such that

$$Z_k = \sum \tilde{a}_{jk} Y_j \tag{3.15}$$

on $\mathbb{R} - S$. The presence of an adapted complex structure ensure that [LS91] $\tilde{a}_{jk}$ has a (unique) meromorphic extension a_{jk} over the domain

$$D = \{\sigma + i\tau \in \mathbb{C}, |\tau| < R/\sqrt{2E(z)} = R\}$$

such that the poles of a_{jk} lie on $\mathbb{R}$ and the matrix $\Im m\,(a_{jk})$ is symmetric and positive definite (hence invertible) in $D - \mathbb{R}$. Let $(e_{jk}) = (\Im m\,(a_{jk}))^{-1}$ and from [LS91], for any point $p = \psi_\gamma(\sigma + i\tau)$, $0 < \tau < R$, we have

(3.16)

$$J_p \xi_h(\sigma + i\tau) = \sum e_{kh}(\sigma + i\tau) \left[\eta_k(\sigma + i\tau) - \left(\sum (\Re e\, a_{jk}(\sigma + i\tau)) \xi_j(\sigma + i\tau) \right) \right].$$

This equation shows that

Proposition III.9. *Given a complete Riemannian manifold (M, g) and R such that $0 < R \leq \infty$, there is at most one adapted complex structure on $T^R M$.*

We also notice that $\xi^{1,0} = \frac{1}{2}(\xi - iJ\xi) \in T^{1,0}X$ is a holomorphic section of $T^{1,0}(TM)$ along the geodesic. Regarding the adapted complex structures, we have the following relations.

Lemma III.10. *$\bar{\partial} E - \partial E = i\Theta$ and $\partial\bar{\partial} E = \frac{i}{2}\Omega$.*

Lemma III.11. *E is strictly plurisubharmonic and $\sqrt{E}$ is plurisubharmonic and satisfies the Monge-Ampère equation $(\partial\bar{\partial}\sqrt{E})^n = 0$.*

Lemma III.12. *N_{-1} is an antiholomorphic involution of $T^R M$.*

Remark III.13. Since N_{-1} is antiholomorphic, it follows that the zero section M is real analytic. This shows that analyticity is a necessary condition for the existence of the adapted complex structure.

By (III.9) and (III.3), every Monge-Ampère model (X, M, u) is biholomorphic to $(T^R M, M, \sqrt{2E})$. This means that a Monge-Ampère model is characterized by the center and u:

Proposition III.14. *Suppose* (X, M, u) *and* (X', M', u') *are two Monge-Ampère models such that* M *is isometric to* M' *in their metric associated to* u *and* u' *respectively. Assume theat* $\sup u = \sup u'$*. Then there is a biholomorphic map* $F : X \to X'$ *such that* $u = u' \circ F$*.*

Looking at the existence of an adapted complex structure, we have

Theorem III.15. *(Theorem 2.2 [Sző91]) Let* M *be a compact real analytic manifold equipped with a real analytic metric* g*. Then there exists* $R > 0$ *such that* $T^R M$ *carries a unique adapted complex structure. We call* $T^R M$ *the* ***Grauert tube*** *of* M*.*

Remark III.16. If there is an adapted complex structure on all of TM, then we say M has an *infinite tube*. In this case, TM admits a strictly plurisubharmonic exhaustion function, which implies that TM is Stein.

Remark III.17. This shows that the analyticity of M is a necessary and sufficient condition on the existence of a Grauert tube. One may wonder if the analyticity of g is a necessary condition. Lempert showed that it is indeed necessary [Lem93].

3.2 Complexifications of the CROSSes

It is well-known [PW91] that if M is a CROSS, then M has an infinite tube. Thus TM is Stein. Moreover, the symmetry of M allows TM to be openly embedded into a smooth projective variety X. The following is the list of all the CROSSes and their tube.

(3.17)

M	X	$M \subset X$	$TM \subset X$
$\mathbb{RP}^n (n \geq 2)$	$\mathbb{CP}^n$	$\mathbb{RP}^n \subset \mathbb{CP}^n$	$\mathbb{CP}^n - \mathcal{Q}^{n-1}$
$S^n (n \geq 2)$	$\mathcal{Q}^n$	$\mathcal{Q}^n \cap \mathbb{RP}^n$	$\mathcal{Q}^n - H^n$
$\mathbb{CP}^n (n \geq 1)$	$\mathbb{CP}^n \times \mathbb{CP}^n$	$\{([z], [\bar{z}])\}$	$\{([z], [w]) \mid z \cdot w \neq 0\}$
$\mathbb{HP}^n (n \geq 1)$	$Gr_{\mathbb{C}}(2, 2n)$	J-inv. elements in $Gr_{\mathbb{C}}(2, 2n)$	$Gr_{\mathbb{C}}(2, 2n) - H^{N-1}$

Remark III.18. There are several remarks regarding the above table.

1. Here $\mathbb{RP}^n \subset \mathbb{CP}^n$ is the fixed point of a standard conjugation in $\mathbb{CP}^n$
2. $\mathcal{Q}^n$ denotes the complex hyperquadric in $\mathbb{CP}^{n+1}$.
3. H^n is a hyperplane in the $\mathbb{CP}^{n+1}$.
4. J is a $\mathbb{C}$-antilinear map from $\mathbb{C}^{2n}$ to $\mathbb{C}^{2n}$ given by

$$((z_1, w_1), \ldots, (z_n, w_n)) = ((-\bar{w}_1, \bar{z}_1), \ldots, (-\bar{w}_n, \bar{z}_n)). \tag{3.18}$$

5. Via the Plücker embedding, $Gr_{\mathbb{C}}(2, 2n)$ can be embedded into $\mathbb{CP}^N$, where $N = n(2n-1) - 1$.
6. The case of $\mathbb{OP}^2$ is notationally very complicated and is omitted here.

3.3 Spheres of revolution with infinite Grauert tubes

One would wonder whether the existence of infinite Grauert tubes is abundant among various metrics on the 2-sphere. Szöke showed that [Sző91] among the surface of revolutions (there is an S^1-action on M by isometries), there are two parameters that characterize all the metrics with infinite tubes among the case of surfaces of revolution.

Theorem III.19. *There exists a two parameter family of different, real analytic metrics $g_{r,d}$ $(r > 0, d \geq 0)$ on the two sphere S^2 such that all of them give rise to an adapted complex structure on the whole TS^2. The metric $g_{r,0}$ gives the standard round sphere with constant curvature $1/r^2$ and $g_{r,d}$ is homogeneous in r, i.e., $g_{r.d} = rg_{1,d}$. Following the homogeneity, we have*

$$g_{1,d} = ds^2 + \frac{\sin s}{d\sin^2 s + 1} d\theta^2, \tag{3.19}$$

where $(s, \theta) \in (0, \pi) \times (0, 2\pi)$ is a local chart on $M - \{N, S\}$ and the free S^1-action on $M - \{N, S\}$ is given by the translation of θ.

This shows that modulo dilations, there is one-parameter family of spheres with infinite tubes.

3.4 Zoll spheres with infinite Grauert tubes

We already know that there exist "exotic" spheres which possess infinite Grauert tubes. In the follow sections, we show that among the Zoll spheres in arbitrary dimension, the only one with an infinite tube is the round one:

Theorem III.20. *Let M be diffeomorphic to S^n be a Zoll manifold with period 2π, such that the adapted complex structure is defined on the whole tangent bundle $(R = \infty)$. Then M is isometric to the round n-sphere.*

To prove this, we have to look at the properties of Zoll manifolds.

Fix an arc-length parametrized geodesic γ in M. The moduli space of geodesic N^+ is a manifold and the tangent space of N^+ corresponds to the space of normal Jacobi fields (c.f. Chapter II). Recall from (2.6) that a neighborhood of $\gamma \in N^+$ can be given as below.

For $m \in \gamma$ and $u, v \in N_\gamma M$ at m, where $N_\gamma M$ denotes the normal subspace with respect to g, we have

$$(3.20)\qquad (u,v) \mapsto \frac{d}{ds} \exp_{\exp_m u} s \left(P_u \frac{\dot\gamma(0) + v}{\sqrt{1+g(v,v)}} \right)\Bigg|_{s=0} \in TM.$$

then project to N^+. This map is of maximal rank and for any fixed (u, v),

$$(3.21)\qquad \exp_{\exp_m u} s \left(P_u \frac{\dot\gamma(0) + v}{\sqrt{1+g(v,v)}} \right)$$

is a geodesic. So this is a $(2n-2)$-parameter family of geodesics around γ.

Relative to a local frame $\{e_i\}$ of $N_\gamma M$, let u_i (resp. v_i) be the standard coordinates corresponding to u (resp. v) relatives to e_i, then $\frac{\partial}{\partial u_i}$ at $(u, v) = (0, 0)$ corresponds to the Jacobi field Y such that $Y(0) = e_i$ and $Y'(0) = 0$; $\frac{\partial}{\partial v_i}$ at $(0, 0)$ corresponds to the Jacobi field Z such that $Z(0) = 0$ and $Z'(0) = e_i$ [Bes78].

Now let $m = \gamma(0)$ and $z = \dot\gamma(0)$. Let $\{w_i\}$, $i = 1...n-1$ be an orthonormal basis of $N_\gamma M$ at m. Then define $\tilde\xi_i$ at $T_z TM$ to be the horizontal lift of w_i; and $\tilde\eta_i$ at $T_z TM$ to be the vertical vector corresponding to w_i. Define parallel vector fields ξ and η as in (3.13) and (3.14). Then we have the meromorphic functions a_{ij} on C_γ.

As C_γ is the complexified geodesic of γ, and γ is closed and of equal length of 2π, the Riemannian foliation is actually a map ψ_γ from $\mathbb{C}/2\pi\mathbb{Z}$ to TM, and thus it is isomorphic to $\mathbb{C}^*$ to TM. This shows that each C_γ is a copy of $\mathbb{C}^*$.

Now consider that following map $\psi : \mathbb{C}/2\pi\mathbb{Z} \times N_m M \times N_m M \to TM$.

$$(3.22)\qquad (\sigma + i\tau, u, v) \mapsto \frac{d}{ds} \exp_{\exp_m u} s\tau \left(P_u \frac{\dot\gamma(0) + v}{\sqrt{1+g(v,v)}} \right)\Bigg|_{s=\sigma}.$$

Here U is a small open neighborhood of $N_m M$ around 0. Notice that ψ restricted to $\{\tau > 0\} \times U \times U$ is a diffeomorphism onto its image when U is small enough. Let $u = \sum u_j w_j$ and $v = \sum v_j w_j$. Using this diffeomorphism, we can see that [Bes78]

$$(3.23)\qquad \xi_j(i) = \frac{\partial}{\partial u_j} \text{ at } (i, 0, 0); \quad \text{and}$$

$$\eta_j(i) = \frac{\partial}{\partial v_j} \text{ at } (i, 0, 0). \tag{3.24}$$

Since N_τ and ϕ_σ commutes with $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$, we have the above two at any points in the leaf C_γ. This can also be interpreted in the way that $\frac{\partial}{\partial u_j}$ and $\frac{\partial}{\partial v_j}$ are given by Jacobi fields. As geodesic flows and reparametrizations preserve Jacobi field (after reparametrizations), we have $\xi_j = \frac{\partial}{\partial u_j}$ and $\eta_j = \frac{\partial}{\partial v_j}$ everywhere.

Now we want to compactify TM by adding some points as below. Using the fact that ψ restricted to $\tau > 0$ is a diffeomorphism onto its image, we compose ψ with a conformal map in $\mathbb{C}$ then add the points in the center of disk. Explicitly, locally around a geodesic γ, define the map $\tilde{\psi}$ as follows:

$$\tilde{\psi} : D^* \times U \times U \xrightarrow{-i \log z \times id \times id} \mathbb{C}/2\pi\mathbb{Z} \times U \times U \xrightarrow{\psi} TM,$$

where $D \subset \mathbb{C}$ is the open unit disk. This map is a diffeomorphism. Then locally around γ, let $X_\gamma := (D \times U \times U) \cup_{\tilde{\psi}} TM$. It is clear that X_γ is locally a manifold. Let X be pasting all the X_γ together. This makes X a real compact manifold with $N^+ := X - TM$ corresponds to the set of oriented geodesics. Notice that along each $C_\gamma \cong \mathbb{C}^*$, we add 0 and ∞ in N^+ so that C_γ is completed to a "compactified complexified geodesic". We will abuse some notation and denote each compactified complexified geodesic by C_γ or just C. Notice that along each C, $C \cong \mathbb{P}^1$.

Lemma III.21. *Let ξ_j and η_j be parallel vector fields along $C_\gamma - \{0, \infty\}$ defined above. Then they can be extended to a smooth vector field along C_γ.*

Proof. This is clear since using the map $\tilde{\psi}$, ξ_j and η_j corresponds to $\frac{\partial}{\partial u_j}$ and $\frac{\partial}{\partial v_j}$ along $D^* \times \{0\} \times \{0\}$. Then it is clear that $\frac{\partial}{\partial u_j}$ and $\frac{\partial}{\partial v_j}$ can be extended to $D \times \{0\} \times \{0\}$. $\square$

Lemma III.22. *Let J be the adapted complex structure on TM. Then J can be extended to a complex structure on X.*

Proof. If J can be extended to X smoothly (or C^1), then J is integrable on X by the C^1 condition. So we just want to extend J smoothly to X.

Along C, the complex structure J can be extended using the structure in $\mathbb{P}^1$. It suffices to extend the structure to the normal direction of C. We can do this by looking at the action of J to $\frac{\partial}{\partial u_j}$ and $\frac{\partial}{\partial v_j}$ locally in the $D \times U \times U$. We already know the action of J on $D^* \times U \times U$ as in (3.16). To extend the J smoothly, we just have to extend a_{ij} smoothly to the origin such that $\Im m\,(a_{ij})$ is invertible at the origin.

We know that $\Im m\,(a_{ij})$ is symmetric and positive definite in D^*. Any diagonal entry in a positive definite matrix must be positive, which means $\Im m\, a_{ii} > 0$ in D^*. But a_{ii} is holomorphic in D^* which $\Im m\, a_{ii} > 0$. This shows that a_{ii} can be extended to a holomorphic function in D by Little Picard theorem and the fact that the upper half plane does not contain a neighborhood of ∞. Any principal minor of a symmetric positive definite matrix is positive. Let $i < j$, a 2×2 principal minor

$$\begin{pmatrix} \Im m\, a_{ii} & \Im m\, a_{ij} \\ \Im m\, a_{ij} & \Im m\, a_{jj} \end{pmatrix} \tag{3.25}$$

is positive definite with positive diagonal entries. This means

$$(\Im m\, a_{ij})^2 < (\Im m\, a_{ii})(\Im m\, a_{jj}). \tag{3.26}$$

We already know a_{ii} are holomorphic, hence bounded in a neighborhood of 0. This means in a neighborhood of 0, $\Im m\, a_{ij}$ is bounded. Using the same argument (Little Picard and ∞ neighborhood), we can see that a_{ij} can be extended to a holomorphic function on D.

It suffices to show that $\Im m\, a_{ij}$ is invertible at 0. Fix $v \in \mathbb{C}^n - \{0\}$ and $A = \Im m\,(a_{ij})$ be the matrix. Then $v^t A v$ is a harmonic function on D. Since A is positive definite in D^* we have $v^t A v > 0$ in D^*. By the maximum principle of harmonic functions, we have $v^tAv > 0$ in D, which shows that A is positive definite in D, in particular A is invertible. This shows that J can be extended along each leaf of the compactified Riemannian foliation.

To show that J is continuous, we use the Cauchy integral formula. Since these a_{ij}^{γ} is continuous in $D^* \times N_mM \times N_mM$ which can be extended to $D \times N_mM \times N_mM$ continuously along each D, by the Cauchy integral formula, the a_{ij}^{γ} is continuous (smooth). Hence J is smooth and this completes the proof. □

By Lemma III.12, we know that N_{-1} is antiholomorphic in TM. We can extend this to an antiholomorphic map on X by mapping 0 to ∞ along each leaf. Again this is clearly antiholomorphic.

Lemma III.23. *N^+ is a complex submanifold of X.*

Proof. By construction, $\frac{\partial}{\partial u_j}$ and $\frac{\partial}{\partial v_j}$ are tangent vectors to N^+ in X. By (3.16), we have J preserves TN^+. As N^+ is a real submanifold with J preserving TN^+, N^+ is a complex submanifold of X. □

To show that the pair (X, N^+) is the same as $(\mathcal{Q}^n, \mathcal{Q}^{n-1})$ in $\mathbb{P}^{n+1}$ (c.f. Table 3.17), we want to show that X is Kähler. $\mathcal{Q}^n$ is Kähler with its Fubini-Study metric with potential $\log(1 + \|z\|^2)$. Here in an affine chart of $\mathcal{Q}^n \subset \mathbb{CP}^{n+1}$, $\|z\|^2 = \sum_i z^i \bar{z}^i$. On the other hand, in [LS91] and [PW91] we have $\cosh^{-1} \|z\|^2 = 2\sqrt{2E}$, we should look at the potential function $\rho := \log(1 + \cosh 2\sqrt{2E})$ in X. In [LS91], we know that $E = (\sqrt{E})^2$ is strictly plurisubharmonic and thus ρ is strictly plurisubharmonic.

Proposition III.24. *The potential function $\rho := \log(1 + \cosh 2\sqrt{2E})$ defines a Kähler form on TM, which extends to a Kähler form in X.*

Proof. To prove this, first we want to show that the Kähler form extends to X continuously. To do this, look at each leaf in the Riemannian foliation and let

$z = \sigma + i\tau \in \mathbb{C}/2\pi\mathbb{Z}$. Here σ is the unit speed geodesic parameter and τ is the length. Hence $\tau = \sqrt{2E}$. Locally near a point in $X - TM$ in each leaf can be given by $\zeta = e^{iz}$. Thus $\zeta = 0$ corresponds to the point p in $X - TM$ in each leaf and let U be a neighborhood of p. Then we know $e^{-\tau}e^{-i\sigma} = \zeta$ and thus $\tau = -\log|\zeta|$. In this coordinate, in $U - \{p\}$,

$$\begin{aligned}
\rho &= \log(1+\cosh 2\tau) \\
&= \log(1+\cosh(-\log|\zeta|^2)) \\
&= \log\left(1+\frac{|\zeta|^2+|\zeta|^{-2}}{2}\right) \\
&= \log\frac{(1+|\zeta|^2)^2}{2|\zeta|^2}.
\end{aligned}$$

Notice that ζ is not holomorphic in X, but ζ restrict to each leaf is holomorphic. Let w be a holomorphic coordinate in U such that $w = 0$ corresponds to $X - TM$ and $dw \neq 0$ at $w = 0$. Then restricted to the leaf, both ζ and w are holomorphic and vanish at degree 1 at p. Thus we can write $\zeta = fw$ for some smooth function along the leaf and $f(p) \neq 0$, in which f is holomorphic along each leaf. It is easy to see that f locally a smooth function near p in X by Cauchy integral formula.

Returning to our function ρ, we have

$$\rho = \log\frac{(1+|\zeta|^2)^2}{2|\zeta|^2} = \log\frac{(1+|fw|^2)^2}{2|fw|^2} = \log\frac{(1+|fw|^2)^2}{2|f|^2} - \log|w|^2.$$

Since w is a holomorphic coordinate in X and by simple calculation that $\partial\bar{\partial}\log|w|^2 = 0$, we can see that the function $\log\frac{(1+|fw|^2)^2}{2|f|^2}$ defines the same Kähler form as ρ does in $X - TM$. But this function is smooth near $p \in TM$ and thus it defines a form in X near p. Repeat this process for different $p \in X - TM$ will give a form on X that restrict to TM is a Kähler form.

Now we have to show that this form is positive definite at $p \in X - TM$. To show this, we use the method of brute force. Notice that the form is given by

$$\begin{aligned}\omega: \ &= -i\partial\bar{\partial}\log(1+\cosh 2\sqrt{2E})\\ &= -i\partial\left(\frac{2\sqrt{2}\sinh 2\sqrt{2E}}{1+\cosh 2\sqrt{2E}}\bar{\partial}\sqrt{E}\right)\\ &= -i\left(\frac{2\sqrt{2}\sinh 2\sqrt{2E}}{1+\cosh 2\sqrt{2E}}\partial\bar{\partial}\sqrt{E}+\frac{8}{1+\cosh 2\sqrt{2E}}\partial\sqrt{E}\wedge\bar{\partial}\sqrt{E}\right).\end{aligned}$$

We have

$$\partial\bar{\partial}\sqrt{E} = \partial\left(\frac{1}{2\sqrt{E}}\bar{\partial}E\right) = \frac{1}{2\sqrt{E}}\partial\bar{\partial}E - \frac{1}{4E\sqrt{E}}\partial E\wedge\bar{\partial}E$$

and

$$\partial\sqrt{E}\wedge\bar{\partial}\sqrt{E} = \frac{1}{4E}\partial E\wedge\bar{\partial}E.$$

Using Lemma III.10 and Corollary 5.5 in [LS91], we have

$$\partial E\wedge\bar{\partial}E = \frac{1}{2}(\partial E+\bar{\partial}E)\wedge(\bar{\partial}E-\partial E) = \frac{1}{2}dE\wedge i\Theta = \frac{i}{2}dE\wedge\Theta$$

and

$$\partial\bar{\partial}E = \frac{i}{2}\Omega.$$

Thus we have

$$\begin{aligned}\partial\bar{\partial}\sqrt{E} \ &= \frac{1}{2\sqrt{E}}\partial\bar{\partial}E - \frac{1}{4E\sqrt{E}}\partial E\wedge\bar{\partial}E\\ &= \frac{i}{4\sqrt{E}}\Omega - \frac{i}{8E\sqrt{E}}dE\wedge\Theta\\ &= \frac{i}{8E\sqrt{E}}(2E\Omega - dE\wedge\Theta).\end{aligned}$$

and

$$\partial\sqrt{E}\wedge\bar{\partial}\sqrt{E} = \frac{1}{4E}\partial E\wedge\bar{\partial}E = \frac{i}{8E}dE\wedge\Theta.$$

So we have

$$\begin{aligned}\omega \ &= \ -i\partial\bar{\partial}\log(1+\cosh 2\sqrt{2E})\\ &= \ -i\left(\frac{2\sqrt{2}\sinh 2\sqrt{2E}}{1+\cosh 2\sqrt{2E}}\partial\bar{\partial}\sqrt{E}+\frac{8}{1+\cosh 2\sqrt{2E}}\partial\sqrt{E}\wedge\bar{\partial}\sqrt{E}\right)\\ &= \ \frac{\sqrt{2}\sinh 2\sqrt{2E}}{4E\sqrt{E}(1+\cosh 2\sqrt{2E})}(2E\Omega - dE\wedge\Theta)\\ &\quad +\frac{1}{E(1+\cosh 2\sqrt{2E})}dE\wedge\Theta.\end{aligned}$$

Look at the parallel vector fields ξ_j and η_j generated by w_j again $(1 \le j \le n-1)$, we know that ξ_j and η_j restricted to $\mathbb{R}$ are normal Jacobi field and thus they lie in $\ker\Omega \cap \ker dE$. We also know that ξ_j and η_j extends to X smoothly to independent vectors at $X - TM$. We also have

$$\Omega(\xi_j, \eta_j) = g(\pi_*\xi_j, K\eta_j) - g(\pi_*\eta_j, K\xi_j) = g(w_j, w_j) = 1$$

at $z = i$. Since $N_s^*\Omega = s\Omega$ for $s \in \mathbb{R}$, we have $\Omega(\xi_j, \eta_j) = \sqrt{2E}$ at any point in the leaf of $TM - M$. Thus we have

$$\omega(\xi_j, \eta_j) = \frac{\sqrt{2}\sinh 2\sqrt{2E}}{2\sqrt{E}(1+\cosh 2\sqrt{2E})}\Omega(\xi_j, \eta_j) = \frac{\sinh 2\sqrt{2E}}{(1+\cosh 2\sqrt{2E})}.$$

When $E \to \infty$ we have $\omega(\xi_j, \eta_j) \to 1 \neq 0$. Since $\Omega(\xi_j, \xi_k) = \Omega(\eta_j, \eta_k) = 0$ and $\Omega(\xi_j, \eta_k) = 0$ for $j \neq k$, by continuity $\omega(\xi_j, \xi_k) = \omega(\eta_j, \eta_k) = 0$ and $\omega(\xi_j, \eta_k) = 0$ for $j \neq k$. This applies to ξ_n and η_n in the sense that $\omega(\eta_n, \xi_j) = \omega(\eta_n, \eta_j) = 0$ and similarly for ξ_n $(1 \le j \le n-1)$. We also have $\omega(\eta_n, \xi_n) \neq 0$ because this is the standard Fubini-Study metric on the C. Thus ω is non-degenerate n all directions at $p \in X - TM$. □

Using the details of this proof, we can conclude that

Lemma III.25. *$\mathcal{O}(N^+)$ is a positive line bundle.*

Proof. To prove that $\mathcal{O}(N^+)$ is a positive line bundle, we want to construct a hermitian metric h on $\mathcal{O}(N^+)$ such that $-i\partial\bar{\partial}\log h$ is the Kähler form above. As in the proof above, let $p \in N^+$ and U_α be a neighborhood of p in X. Using the notation above we set

$$h_\alpha(x) = \frac{(1 + |f_\alpha(x)w_\alpha(x)|^2)^2}{2|f_\alpha(x)|^2}$$

for $x \in U_\alpha$. Let g_α be a (holomorphic) defining function of N^+. Notice that we can pick $g_\alpha = w_\alpha$ since w_α is also a defining function on U_α. This means, for another neighborhood U_β of p, and $x \in U_\alpha \cap U_\beta - N^+$, we have

$$\begin{aligned} h_\alpha &= \frac{(1+|f_\alpha w_\alpha|^2)^2}{2|f_\alpha|^2} \\ &= |w_\alpha|^2 \frac{(1+|f_\alpha w_\alpha|^2)^2}{2|f_\alpha w_\alpha|^2} \\ &= |w_\alpha|^2 \frac{(1+|\zeta|^2)^2}{2|\zeta|^2} \\ &= \frac{|w_\alpha|^2}{|w_\beta|^2} \frac{(1+|f_\beta w_\beta|^2)^2}{2|f_\beta|^2} \\ &= \frac{|w_\alpha|^2}{|w_\beta|^2} h_\beta. \end{aligned}$$

Since all of h_α, h_β and $\frac{|w_\alpha|^2}{|w_\beta|^2}$ are continuous in $U_\alpha \cap U_\beta$, we have

$$h_\alpha = \frac{|w_\alpha|^2}{|w_\beta|^2} h_\beta$$

for all $x \in U_\alpha \cap U_\beta$. Since $\frac{w_\alpha}{w_\beta}$ is the transition function of $\mathcal{O}(N^+)$, we successfully define a hermitian matrix such that $\mathcal{O}(N^+)$ is positive in a neighborhood of $\mathcal{O}(N^+)$.

To complete the proof, we have to define h away from N^+. This is easy, as we can just set

$$h = \frac{(1+|\zeta|^2)^2}{2|\zeta|^2} = \frac{1}{|w_\alpha|^2} h_\alpha$$

Since 1 is a defining function of N^+ away from N^+, we have a hermitian metric such that $-i\partial\bar{\partial}\log h = \omega$ is a Kähler form, hence positive. This completes the proof. □

Remark III.26. We have

$$c_1(\mathcal{O}(N^+)) = -\frac{i}{2\pi}\partial\bar{\partial}\log h = \frac{1}{2\pi}\omega \in H^2(X, \mathbb{Z}).$$

Since X is Kähler and has a positive bundle $\mathcal{O}(N^+)$, by Kodaira embedding theorem, X is projective and thus algebraic. In the following discussion we rename N^+ as V to fit in the algebraic geometry world.

Next we have to figure out what the cohomologies of X are.

Proposition III.27. *If n be even, then*

$$H^i(X,\mathbb{Z}) = \begin{cases} 0 & \textit{if} \quad i \textit{ is odd} \\ \mathbb{Z}\oplus\mathbb{Z} & \textit{if} \quad i = n \\ \mathbb{Z} & \textit{if} \quad \textit{otherwise.} \end{cases}$$

If n is odd, then

$$H^i(X,\mathbb{Z}) = \begin{cases} 0 & \textit{if} \quad i \textit{ is odd} \\ \mathbb{Z} & \textit{if} \quad i \textit{ is even} \end{cases}$$

Proof. c.f. [Aud07]. By the construction, the symplectic cut W at $E = \frac{1}{2}$ of TM in the liouville symplectic form on TM is diffeomorphic to X. Then $M \subset W$ is Lagrangian and the symplectic reduction at $E = 2$ is a symplectic codimension-2 submanifold, with E a Morse-Bott function with only M and V as critical sets. This means W is a polarized symplectic manifold. By [Aud07], it has the above cohomology groups. Since X is diffeomorphic to W, X has the above cohomology group. □

Remark: In [Aud07], the above cohomology groups are generated by the Poincaré dual of $[N^+]^i$ over $\mathbb{Q}(\frac{1}{2}\mathbb{Z}$ to be clear) if they are isomoprhic to $\mathbb{Z}$ (with $[V]^n = 2$). $H^n(X,\mathbb{Z})$ is generated by the Poincaré dual of $[V]^{n/2}$ and $[M]$ if n is even, with $[V]^n = 2$, $[V].[M] = 0$ and $[M]^2 = 0$ if n is odd and $[M]^2 = -2$ if n is even. (c.f. [Aud07])

Lemma III.28. *Let C be a compactified complexified geodesic and we know $C \cong \mathbb{CP}^1$. Let $\mathcal{N}$ be the (holomorphic) normal bundle of C in X. Then $\det\mathcal{N} = \mathcal{O}_{\mathbb{CP}^1}(2n-2)$, where n is the complex dimension of X.*

Proof. Using previous notation, we have ξ_j, $1 \le j \le n-1$ be parallel vector fields along C. Then $\xi_j^{1,0}$ is a holomorphic section of $T^{1,0}(TM)$ ([LS91] Prop 5.1). By continuity in our extension, it is a holomorphic section of $T^{1,0}X$. Since these parallel fields are not in TC, $\xi_j^{1,0}$ define holomorphic sections of N. Now $\xi_j^{1,0}$ are linearly independent over $\mathbb{C}$ except a discrete set of points in M, in which the $\xi_j^{1,0}$ vanishes. This shows that $\bigwedge \xi_j^{1,0}$ defines a holomorphic section of $\det \mathcal{N} = \bigwedge^{n-1} \mathcal{N}$. $\bigwedge \xi_j^{1,0}$ has the vanishing order the sum of the vanishing order of $\xi_j^{1,0}$ because they are linearly independent over $\mathbb{C}$ except where one of them vanishes. But ξ_j are given by Jacobi fields along M and each Jacobi field vanishes at exactly 2 points. This shows that $\bigwedge \xi_j^{1,0}$ has vanishing order of $2n-2$ and therefore $\det \mathcal{N} = \mathcal{O}_{\mathbb{CP}^1}(2n-2)$. □

Lemma III.29. $K_X = \mathcal{O}(-nV)$

Proof. Case $n > 2$: By adjunction formula we have $K_C = K_X|_C + \det \mathcal{N}$. Since $C = \mathbb{P}^1$, we have $K_C = \mathcal{O}(-1-1) = \mathcal{O}(-2)$ and $\det \mathcal{N} = \mathcal{O}(2n-2)$. This shows that $K_X|_C = \mathcal{O}(-2n)$. Since we know that $c_1(K_X)$ defines a class in $H^2(X)$ and $h^2(X) = 1$ and it is generated by $[V]$, we must have $K_X = rV$ for some $r \in \mathbb{Z}$. Since V intersects C transversely at 2 points, we have $V|_C = \mathcal{O}(2)$. This shows that $\mathcal{O}(-2n) = K_X|_C = rV|_C = \mathcal{O}(2r)$ and so $r = -n$.

Case $n = 2$: We also have

$$\begin{aligned} c_1(X)|_V &= c_1(TV \oplus N_{V|X}) \\ &= c_1(V) + c_1(N_{V|X}) \\ &= c_1(\mathbb{P}^1) + [V] \cdot [V] \\ &= 2 + 2 = 4 \in H^2(V, \mathbb{Z}). \end{aligned}$$

This shows that $c_1(X) \cdot [V] = 4$. Since $[V]$ and $[M]$ are generators of $H^2(X, \mathbb{Z})$, we can write $c_1(X) = \alpha[V] + \beta[M]$. From $c_1(X) \cdot [V] = 4$, we see that $\alpha = 2$. By Proposition III.27, the Euler characteristics of X is 4. Thus by Thom-Hirzebruch Signature Theorem, we have

$$c_1^2(X) = 2\chi + 3\tau = 2(4) + 3(0) = 8.$$

Thus we have $\beta = 0$. This shows that $c_1(X) = 2[V]$ and equivalently $K_X = \mathcal{O}(-2V)$.

$\square$

Proposition III.30. $h^0(X, \mathcal{O}_X(V)) = n + 2$.

Proof. Case $n > 2$: Since $\mathcal{O}(V)$ is positive and by Kodaira vanishing theorem, we have

$$H^i(X, \mathcal{O}(kV) \otimes K_X) = H^i(X, \mathcal{O}((k-n)V)) = 0 \text{ for } i, k > 0.$$

This is equivalent to

$$H^i(X, \mathcal{O}(rV)) = 0 \text{ for all } r > -n \text{ and } i > 0.$$

We also have $H^0(X, \mathcal{O}(rV)) = 0$ for any $r < 0$ and $H^0(X, \mathbb{C}) = \mathbb{C}$. Let

$$\chi(r) = \chi(X, \mathcal{O}(rV)) = \sum (-1)^i h^i(X, \mathcal{O}(rV)).$$

Then we have $\chi(0) = 1$ and $\chi(r) = 0$ for $-n < r < 0$. We also have $\chi(1) = h^0(X, \mathcal{O}(V))$.

By Serre duality and Kodaira vanishing theorem, we have

$$h^i(X, \mathcal{O}(-nV)) = h^{n-i}(X, \mathcal{O}(nV) \otimes K_X) = 0 \text{ for } n - i > 0.$$

On the other hand, we have

$$h^n(X, \mathcal{O}(-nV)) = h^0(X, \mathcal{O}(nV) \otimes K_X) = h^0(X, \mathcal{O}) = 1.$$

This shows that $\chi(-n) = (-1)^n$.

The following table summarizes all the information about $h^i(X, \mathcal{O}(rV))$.

	$i = 0$	$0 < i < n$	$i = n$	χ
$r = -n$	0	0	1	$(-1)^n$
$-n < r < 0$	0	0	0	0
0	1	0	0	1
1	?	0	?	$h^0(X, \mathcal{O}(V))$

Let $c_i(X) = \lambda_i[V]^i$ for $\lambda_i \in \mathbb{Q}$, for $i \neq n/2$. For n even, we have

$$c_{n/2}(X) = \lambda_{n/2}[V]^{n/2} + \lambda[M].$$

Then we have

$$\mathrm{td}(X) = \sum \mu_i[V]^i + \mu[M] + \nu[M]^2$$

and

$$\mathrm{ch}(\mathcal{O}(kV)) = \sum \frac{c_1^i(\mathcal{O}(kV))}{i!} = \sum \frac{k^i[V]^i}{i!}.$$

(Notice that $\mu = \nu = 0$ if n is odd.) The Riemann-Roch Theorem states that

$$\chi(k) = \mathrm{ch}(\mathcal{O}(kV))\mathrm{td}(X) = \nu[M]^2 \sum_{i=0}^{n} \frac{k^i \mu_{n-i}[V]^n}{i!} = \nu[M]^2 + \sum_{i=0}^{n} k^i \frac{2\mu_{n-i}}{i!} = \sum_{i=0}^{n} a_i k^i.$$

Since we know $\chi(k)$ for $-n \leq k \leq 0$, we can uniquely solve for a_i.

By simple calculations, when $X = \mathcal{Q}^n$ and $V = \mathcal{Q}^n \cap H$, we have $[V]^n = 2$, $\chi(1) = n+2$, $\mathcal{O}(V)$ positive and $K_X = \mathcal{O}(-nV)$.

This shows that the values of $\chi(k)$ $(-n \leq k \leq 0)$ are the same as the case when $X = \mathcal{Q}^n$. Since the $\chi(k)$'s uniquely determine the values of a_i, in which they uniquely determine $\chi(1)$. This shows that $\chi(1) = n+2$ as in the case of $X = \mathcal{Q}^n$ and $V = \mathcal{Q}^n \cap H$. But $\chi(1) = h^0(X, \mathcal{O}(V))$, we have $h^0(X, \mathcal{O}(V)) = n+2$.

Case $n = 2$: The exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(V) \to \mathcal{O}_V(V) \to 0$$

induces that exact sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(V)) \to H^0(V, \mathcal{O}_X(V)) \to H^1(X, \mathcal{O}_X).$$

Since X is Kähler, we have $H^1(X, \mathcal{O}_X) = H^{0,1}(X, \mathbb{C}) \subset H^1(X, \mathbb{C}) = 0$. Then

$$h^0(X, \mathcal{O}_X(V)) = h^0(X, \mathcal{O}_X) + h^0(V, \mathcal{O}_X(V)) = 1 + h^0(\mathbb{CP}^1, \mathcal{O}(V.V)) = 1 + 3 = 4,$$

by $[V]^2 = 2$ and $V \cong \mathbb{P}^1$ since V can be identified with the manifold of oriented geodesics of M. $\square$

Using Kachi and Kollár's argument [KK00], X is biholomorphic to the nonsingular quadric $\mathcal{Q}^n$ by looking at the map $X \to \mathbb{P}H^0(X, \mathcal{O}_X(V))^*$.

Lemma III.31. *For $s \in H^0(X, \mathcal{O}_X(V))$, define $Ns := \overline{N_{-1}^* s}$. Then N is a conjugate linear involution from $H^0(X, \mathcal{O}_X(V))$ to itself.*

Proof. Let $\overline{X}$ and $\overline{V}$ be X and V with opposite complex structure. Then N_{-1} is a holomorphic map from X to $\overline{X}$ and from V to $\overline{V}$. Since $\mathcal{O}_{\overline{X}}(\overline{V}) = \overline{\mathcal{O}_X(V)}$, we have N_{-1}^* mapping $H^0(X, \mathcal{O}(V))$ to $H^0(\overline{X}, \overline{\mathcal{O}(V)})$. This means $N_{-1}^* s$ is an antiholomorphic map from X to $\overline{\mathcal{O}(V)}$ such that composing with the bundle map is the identity (the bundle map is antiholomorphic). Conjugating the line bundle, we have $Ns = \overline{N_{-1}^* s}$ to be a holomorphic map from X to $\mathcal{O}(V)$ such that the composition with the bundle map is identity. This mean $Ns \in H^0(X, \mathcal{O}(V))$. N is clearly conjugate-linear from the conjugation of the line bundle. $\square$

Lemma III.32. *The set of fixed points of N is a maximal totally real subspace $H^0(X, \mathcal{O}_X(V))_{\mathbb{R}}$ of $H^0(X, \mathcal{O}_X(V))$.*

Proof. Since all complex vector spaces are isomorphic to $\mathbb{C}^n$, we just assume if N is a conjugate linear map from $\mathbb{C}^n$ to $\mathbb{C}^n$, then its fix point is totally real.

N is an involution of $\mathbb{R}^{2n}$ such that $NJ = -JN$ for the complex structure J. Since N is an involution, the possible eigenvalues are ± 1. Their correpsonding eigenspace are denoted by Z_+ and Z_-. Notice that J sends Z_+ to Z_- and the other way around. Since J is of full rank with $J^2 = -I$, we have $\dim Z_+ = \dim Z_-$. Any element v in $\mathbb{R}^{2n}$ can be written as $\frac{1}{2}(v + Nv) + \frac{1}{2}(v - Nv) \in Z_+ + Z_-$, and $Z_+ \cap Z_- = 0$. This shows that $\dim Z_+ = n$. Since J sends Z_+ to Z_-, Z_+ is maximally totally real in $\mathbb{C}^n$. □

By the above, we can choose $\tilde{s}_0, \dots \tilde{s}_{n+1} \in H^0(X, \mathcal{O}_X(V))_{\mathbb{R}}$ and it span $H^0(X, \mathcal{O}_X(V))$ over $\mathbb{C}$. We can let s_0 be the defining section of V (since V is N-invariant). Thus we have $f : x \mapsto [\tilde{s}_0(x) : \dots : \tilde{s}_{n+1}(x)]$ and f commutes with the standard conjugation in $\mathbb{P}^{n+1}$. This shows that N_{-1} extends to the standard conjugation in $\mathbb{CP}^{n+1}$.

Proof of Theorem III.20. Without loss of generality, we can set $\mathcal{Q} = \{\sum_{i=0}^{n+1} z_i^2 = 0\}$. Since $\mathcal{Q} \subset \mathbb{P}^{n+1}$ is an embedding, we can say that $X \subset \mathbb{CP}^{n+1}$. By above we know that $N_{-1} : X \to X$ extends to a conjugation τ in $\mathbb{CP}^{n+1}$. Let σ be the standard conjugation in the z_i coordinates: $[z_i] \mapsto [\bar{z}_i]$. Then $\tau \circ \sigma$ is a bihomolorphic map from $\mathbb{CP}^{n+1}$ to itself fixing $\mathcal{Q}$. This shows that $\tau \circ \sigma$ is a projective linear transformation fixing $\mathcal{Q}$. This shows that there exists a linear biholomorphism $G : \mathcal{Q} \to \mathcal{Q}$ such that it maps M, the fixed point of N_{-1}, to the fixed point of σ, which is the standard case of round sphere.

Since G is linear, it maps complex lines to complex lines. But the complex lines are complexified compactified geodesics and their restriction to M are geodesics. This means that the geodesics of M are the same as those of the round sphere and thus M must be isometric to the round sphere, modulo dilations. □

Remark III.33. Most of the above procedures above can be applied to all the Zoll manifolds with an infinite tube. Indeed, one can use the same argument to show that any Zoll manifold with an infinite tube can be embedded into a projective complex manifold, with the space of oriented geodesics being a positive divisor.

3.5 Generalizations

One could ask whether the extended complex normal Jacobi fields are crucial in the proof of III.22 that the complex structure extends to the points at infinity. Consider the following scenario, originally suggested by Stoll [Sto77, Sto80]:

Let X be a complex Stein manifold of complex dimension n with a strictly plurisubharmonic exhaustion function $\tau > 0$. Then we can define a Kähler metric Ω, or a "τ-metric", with Kähler form equal to:

$$\frac{i}{2}\partial\bar{\partial}\tau = dd^c\tau > 0.$$

In local coordinates,

$$\Omega = \sum_{i,j} \tau_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

where $\tau_i = \frac{\partial \tau}{\partial z^i}$ and $\tau_{\bar{i}} = \frac{\partial \tau}{\partial \bar{z}^i}$.

Define $\tau^{i\bar{j}}$ by the relation

$$\sum_j \tau^{i\bar{j}} \tau_{k\bar{j}} = \delta^i_k.$$

We consider the vector field of type $(1,0)$:

$$Y := \sum_{i,j} \tau^{i\bar{j}} \tau_{\bar{j}} \frac{\partial}{\partial z^i}.$$

Let $\xi = \Re e\, Y$ and $\eta = \Im m\, Y$, with flow ψ and ϕ respectively. Let u be a real-valued function on X such that $e^u = \tau$. Assume that there exists a $\tau_0 < +\infty$ such that whenever $\tau \geq \tau_0$,

1. u is plurisubharmonic and satisfies the homogeneous complex Monge-Ampère equation

$$(\partial\bar{\partial}u)^n = 0.$$

2. η has a periodic flow ϕ and there is a free action of S^1 on $X_0 := X \cap \{\tau \geq \tau_0\}$ corresponding to this flow.

By the uniqueness of the solution to ODE, the flow ϕ is simply periodic with the same period. Now since $\tau = e^u$, similar to the previous section, we have

$$\tau_{i\bar{j}} = e^u(u_{i\bar{j}} + u_i u_{\bar{j}}).$$

Let $z = (z^i)$ such that

$$\sum_{i,j} u_{i\bar{j}} z^i \bar{z}^j = 0. \tag{3.27}$$

Then by the strict plurisubharmonicity of τ, we must have $\sum_i u_i z^i \neq 0$. But there is at most 1 dimension of $z's$ satisfying condition 3.27 and thus we have that the kernel of $u_{i\bar{j}}$ is one dimension. This gives a foliation of X_0 with leaves of one complex dimension and $dd^c u(v, -) = 0$ for all v tangential to each leaf. We call this the Monge-Ampère foliation. We denote a generic leaf of the foliation by C.

Notice that by Stoll [Sto80], Y is tangent to the foliation in the sense that $Y \in T^{1,0}C$. This means that ξ and η are parallel to the leaves and they commute with each other. We also have that

$$Y\tau = \tau.$$

Thus, we have

$$\xi\tau = \tau, \text{ and } \eta\tau = 0.$$

This means that η preserves τ and thus u.

Lemma III.34. *Recall that $\Omega = dd^c\tau > 0$. We have $\psi_t^*\Omega = e^t\Omega$ and $\phi_t^*\Omega = \Omega$.*

Proof. For ϕ_t, we have

$$\phi_t^*\Omega = \phi_t^* dd^c\tau = dd^c\phi_t^*\tau = dd^c\tau = \Omega,$$

here we used ϕ_t preserves τ.

For ψ_t, pick a point $x \in X_0$ and consider $w(t) = \tau(\psi_t(x))$. Then we have

$$\begin{aligned} w(T) - w(0) &= \tau(\psi_T(x)) - \tau(x) = \int_0^T \frac{d}{dt}\tau(\psi_t(x))dt \\ &= \int_0^T d\tau_{\psi_t(x)}(\xi)dt \\ &= \int_0^T \xi_{\psi_t(x)}(\tau)dt \\ &= \int_0^T \tau(\psi_t(x))dt \\ &= \int_0^T w(t)dt. \end{aligned}$$

Differentiate $w(T)$ with respect to T gives a differential equation

$$w'(T) = w(T).$$

Thus $w(T) = w(0)e^T$, which means

$$\psi_t^*\tau = \tau \circ \psi_t = e^t\tau.$$

This shows that

$$\psi_t\Omega = \psi_t^* dd^c\tau = dd^c\psi_t^*\tau = e^t dd^c\tau = e^t\Omega.$$

□

This also shows that $\psi_t^* u = t + u$.

Following Stoll's work [Sto80], we have for the followings:

Lemma III.35. *Let C be a leaf of the foliation F. Then $C \cap \{\tau > \tau_0\} \in X_0$ is biholomorphic to a punctured holomorphic disk.*

Proof. Let $x \in X_0$ such that $\tau(x) = \tau_0$. Let C be the leaf of the foliation containing x. Then C is a one-dimensional complex manifold, and τ_0 is a regular value of τ since $\xi\tau = \tau \neq 0$. Thus the set $\tau^{-1}(\tau_0) \subset C$ is a one real dimensional manifold. Let $y \in \tau^{-1}(\tau_0) \cap C$ and recall ϕ is periodic and non-trivial. Thus the connected components of $\tau^{-1}(\tau_0) \cap C$ are closed and thus they are diffeomorphic to S^1. If there are more than one connected components, say it contains more than one copy of S^1, by Morse theory, as it has no critical point in C, C will be more than one copy of a cylinder. This is a contradiction because C as a (maximal) leaf of a foliation, must be connected. This also shows that C is a cylinder.

It is clear that the flow of ϕ is periodic of the same period for each punctured disk. This is because ϕ_s commutes with ψ_t: Assume ϕ has a period of s_0 at $x \in X_0$. Then

$$\psi_t(x) = \psi_t(\phi_{s_0}(x)) = \phi_{s_0}(\psi_t(x)),$$

which shows that at $\psi_t(x)$, ϕ has the same period by continuity of t. Thus there is a diffeomorphism

$$\begin{array}{ccccc} \mathbb{R}/s_0\mathbb{Z} & \times & \mathbb{R}_{>0} & \to & C \\ (s & , & t) & \mapsto & \phi_s\psi_t(x). \end{array}$$

Since ψ and ϕ are real and imaginary components of a holomorphic vector field on C, by giving the set $\mathbb{R}/s_0\mathbb{Z} \times \mathbb{R}_{>0}$ the standard complex structure, i.e. $J\frac{\partial}{\partial s} = \frac{\partial}{\partial t}$, the above map is a biholomorphism from $\mathbb{R}/s_0\mathbb{Z} \times \mathbb{R}_{>0}$ to C. Let $u_0 = u(x)$. Then by composing with the exponential map

$$\zeta = e^{-\frac{2\pi}{s_0}(t+u_0-is)},$$

we have a map from a holomorphic disk to C, where

$$|\zeta| = e^{-\frac{2\pi}{s_0}(t+u_0)} = e^{-\frac{2\pi u}{s_0}}.$$

□

Thus we can compactify each leaf:

Corollary III.36. *Each leaf can be compactified by filling the hole of each punctured disk. Thus X can be compactified to a real $(2n)$-dimensional manifold $\hat{X}$ by filling all the holes of the disks of the leaves in the foliation.*

Proof. We have that each leaf can be compactified by filling the hole of each punctured disk. Now in the neighboring leaf, ζ can be defined as a smooth coordinate around C which restricts to holomorphic coordinates in each leaf. This can be done because the period s_0 is constant. Thus a neighborhood of C is given by $D^* \times \mathbb{R}^{2n-2}$. Thus $\hat{X}$ is a smooth manifold. $\square$

Following Lempert and Szőke [LS91], we want to describe the complex structure J as we did in the previous section. But first, we define the notion of parallel vector fields.

Pick a point $x \in X_0$ such that $\tau(x) = \tau_0$ and C be the leaf containing x. Let $\tilde{v} \in T_x X_0$. Recall ψ denotes the flow of ξ and ϕ denotes the flow of η. Since ϕ and ψ commute and ϕ is periodic along the leaf C, we have a well-defined vector field v along C such that it is invariant under ϕ and ψ.

Let $V_x = \ker du \cap \ker d^c u$. We have

Lemma III.37. *V_x is a J-invariant subspace and $T_x X_0 = V_x \oplus T_x C$.*

Proof. Let $V_{x\mathbb{C}} = V_x \otimes \mathbb{C} = V_x^{1,0} \oplus V_x^{0,1}$. Then we have $V_{x\mathbb{C}} = \ker \partial u \cap \ker \bar{\partial} u$.

Let $v \in V_x$. Then $v \in V_{x\mathbb{C}}$ and we can write $v = v^{1,0} + v^{0,1}$. Then

$$\partial u(Jv) = \partial u(Jv^{1,0} + Jv^{0,1}) = \partial u(iv^{1,0}) = i\partial u(v^{1,0}) = 0$$

and similarly for $\bar{\partial}$. Thus $Jv \in \ker \partial u \cap \ker \bar{\partial} u = V_{x\mathbb{C}}$. But Jv is a real vector and thus $Jv \in V_x$. As $d^c u(\eta) \neq 0$ and $du(\xi) \neq 0$, we have $T_x X_0 = V_x \oplus T_x C$. $\square$

Let ξ be a parallel vector field along C such that $\xi_x \in V_x$ at a point x. Then by the relations $\psi_t^* u = t + u$ and $\phi_s^* u = u$, we have $\xi_y \in V_y$ for all $y \in C$.

Lemma III.38. *Any parallel vector fields can be extended to the compactification $\hat{X}_0 = \cap X_0 \cup \{\hat{X} - X\}$.*

Proof. Let $\gamma(r)$ be a curve in X_0 such that $\gamma(0) = x$ and $\gamma'(0) = \tilde{v}$. Extend $\tilde{v}$ to a parallel vector field v along the leaf. Then v at the corresponding point would be defined as $(\psi_t \phi_s \gamma)'(0)$. To extend the parallel vector field, one defines it to be $\lim_{t \to +\infty} (\psi_t \gamma)'(0)$. The limit exists in the compactification $\hat{X}_0$ and is well-defined because $\lim_{t\to+\infty} \psi_t = \lim_{t\to+\infty} \psi_t \psi_s$ for any s. $\square$

Following Lempert-Szőke, choose $\tilde{\xi}_i \in T_x X_0$, $1 \le i \le n$, such that the vectors $\{\tilde{\xi}_i^{1,0}\}, 1 \le i \le n$ form a basis of $T_x^{1,0} X_0$. We can assume $\tilde{\xi}_n = \xi$ at x. Extend $\tilde{\xi}_i$ to parallel vector fields ξ_i along C. Choose n more vectors $\tilde{\eta}_i \in T_x X_0$ such that together with $\tilde{\xi}_j$ they form a basis when $1 \le i, j \le n$. Extend them to parallel vector fields η_i along C. Again, assume $\tilde{\eta}_n = \eta$ at x. The holomorphic sections $\xi_i^{1,0}$ are independent at x over $\mathbb{C}$, hence they are pointwise independent except on a discrete subset S on C. Therefore there are meromorphic functions a_{jk} on $C - S$ such that

$$\eta_k^{1,0} = \sum_j a_{jk} \xi_j^{1,0} \qquad \text{on } C - S$$

Taking the real part, we have

$$\eta_k = \sum_j (\Re e\, a_{jk}) \xi_j + \sum_j (\Im m\, a_{jk}) J \xi_j .$$

Let $(e_{jk}) = (\Im m\, a_{jk})^{-1}$, then we have

$$J\xi_h = \sum_k e_{kh} \left(\eta_k - \sum_j (\Re e\, a_{jk}) \xi_j \right).$$

If we can show that the meromorphic functions a_{jk} can be extended to the compactifications near $\hat{X}_0 - X_0$ and their imaginary parts are invertible, we can extend J to $\hat{X}_0$ and hence $\hat{X}$ as an almost complex structure. (Here J is at least C^1 by the Cauchy integral formula, similar to the proof of Lemme III.22 in the above section.) Since $V := \hat{X} - X$ is of measure zero, by continuity, the Nijenhius tensor will also vanish at V and therefore J is integrable everywhere. Lastly, V is J invariant by the above formula, which in turn implies that V is a complex submanifold of X.

To show that a_{jk} can be extended, we have to choose the $\tilde{\xi}_i$'s and $\tilde{\eta}_i$'s wisely. We first establish a series of lemmas.

Lemma III.39. *Let ξ and η be parallel fields along C. If $\Omega(\xi, \eta) = 0$ at a point $y \in C$, then $\Omega(\xi, \eta) = 0$ at all points in C.*

Proof. This is clear by the above theorem and by the fact that any parallel vector fields are ϕ_s and ψ_t invariant. □

Lemma III.40. *$\Omega(v, w) = 0$ for $v \in V_x$ and $w \in T_xC$. Hence $\Omega = dd^c\tau$ restricts to a symplectic form in V_x.*

Proof. Since $\tau = e^u$, we have

$$\bar{\partial}\partial\tau = e^u(\partial\bar{\partial}u + \bar{\partial}u \wedge \partial u).$$

Since $\partial\bar{\partial}u = 0$ along C and $V_x = \ker du \cap \ker d^c u$, we have $\Omega(v, w) = 0$. □

Since V_x is J-invariant, symplectic with a symplectic form Ω, Ω is a $(1,1)$-form defined by a strictly plurisubharmonic function, we have Ω restricted to V_x tamed and compatible with J. Thus there exists an orthonormal basis with respect to the metric $\Omega(-, J-)$

$$\{\tilde{\xi}_1, J\tilde{\xi}_1, \ldots, \tilde{\xi}_{n-1}, J\tilde{\xi}_{n-1}\}$$

such that $\Omega(\tilde{\xi}_i, \tilde{\xi}_j) = 0$ and $\Omega(\tilde{\xi}_i, J\tilde{\xi}_j) = \delta_{ij}$. Then $\{\tilde{\xi}_i^{1,0}\}_{1 \le i \le n-1}$ span $T^{1,0}V_x$. Together with $\tilde{\xi}_n^{1,0} = \xi^{1,0}$, it will span $T_x^{1,0}X_0$. Let $\tilde{\eta}_i = J\tilde{\xi}_i$ and extend to parallel vector fields ξ and η along C. Then $a_{jk} = i\delta_{jk}$ at x.

Lemma III.41. *There vector fields ξ_j and $J\xi_k$ are (pointwise) linearly independent over $\mathbb{R}$ at all of C. (Here $J\xi_k$ are not parallel)*

Proof. Assume that $\xi_0 = \sum b_i \xi_i = \sum c_j J\xi_j$ at some point in C, for some nonzero b_i and c_j. Then

$$\Omega(\xi_0, J\xi_0) = \sum b_i c_j \Omega(J\xi_j, J\xi_i) = 0.$$

But Ω is a Kähler form and thus it is a contradiction. □

Lemma III.42. *Using this choice of ξ_i and η_j, we have (a_{ij}) holomorphic and invertible on all of C.*

Proof. Since ξ_j and $J\xi_k$ is linearly independent over $\mathbb{R}$ on all of C, we have $\{\xi_j^{1,0}\}$ linearly independent over $\mathbb{C}$ on C. Similarly $\{\eta_j^{1,0}\}$ is linearly independent over $\mathbb{C}$ on C. Thus (a_{ij}) carries linearly independent vectors to linearly independent vectors are therefore it is invertible. (This suggests that $S = \emptyset$.) And at each point a_{ij} is well-defined because of the linear independence of ξ_j's. □

Lemma III.43. *(e_{ij}) is a symmetric matrix on all of C. Thus (a_{ij}) is symmetric on all of C.*

Proof. We have

$$\Omega(\xi_i, J\xi_h) = \Omega(\xi_i, e_{kh}(\eta_k - (\Re e\, a_{jk})\xi_j)) = \Omega(\xi_i, e_{kh}\eta_k) = e_{kh}\Omega(\xi_i, \eta_k).$$

At $x' \in X_0$ such that $x' = \psi_t \phi_s(x)$, $\Omega_{x'} = e^t \Omega_x$. Thus at x', we have

$$\Omega(\xi_i, J\xi_h) = e_{kh}\Omega_{x'}(\xi_i, \eta_k) = e_{kh}e^t\Omega_x(\xi_i, \eta_k) = e_{kh}e^t\delta_{ik} = e_{ih}e^t.$$

But Ω is Kähler and therefore

$$\Omega(\xi_i, J\xi_h) = \Omega(\xi_h, J\xi_i),$$

Thus we have $e_{ih} = e_{hi}$ and therefore $a_{jk} = a_{kj}$. □

Corollary III.44. *$\Im m\,(a_{jk})$ is symmetric and positive definite on all of C.*

Proof. Since $\Im m\,(a_{jk}) = Id$ at x, and the matrix is invertible on all of C, it is positive definite on all of C. $\square$

Corollary III.45. *J can be extended to a complex structure on $\hat{X}_0$.*

Proof. The proof is completely the same as Lemma III.22. $\square$

Now we claim that

Proposition III.46. *The potential function $\rho := \log(1 + \cosh \frac{4\pi u}{s_0})$ defines a Kähler form on X_0, which extends to a Kähler form on $\hat{X}$.*

Proof. In a neighborhood of V, we have τ is strictly plurisubharmonic, and $u = \log \tau$ is plurisubharmonic and it satisfies the Homogeneous Complex Monge-Ampère equation. Using the strictly plurisubharmonicity of τ, we have

$$\begin{aligned}
(\partial\bar{\partial}\tau)^n &= (\partial\bar{\partial}e^u)^n \\
&= e^{nu}(\partial\bar{\partial}u + \partial u \wedge \bar{\partial}u)^n \\
&= e^{nu}((\partial\bar{\partial}u)^n + n\partial u \wedge \bar{\partial}u \wedge (\partial\bar{\partial}u)^{n-1}) \\
&= ne^{nu}\partial u \wedge \bar{\partial}u \wedge (\partial\bar{\partial}u)^{n-1} \\
&\neq 0.
\end{aligned}$$

This means

$$ne^{nu}\partial u \wedge \bar{\partial}u \wedge (\partial\bar{\partial}u)^{n-1} \neq 0.$$

Then we have

$$\begin{aligned}
(\partial\bar{\partial}u^2)^n &= 2^n(u\partial\bar{\partial}u + \partial u \wedge \bar{\partial}u)^n \\
&= 2^n(u^n(\partial\bar{\partial}u)^n + nu^{n-1}\partial u \wedge \bar{\partial}u \wedge (\partial\bar{\partial}u)^{n-1}) \\
&= 2^n nu^{n-1}\partial u \wedge \bar{\partial}u \wedge (\partial\bar{\partial}u)^{n-1} \\
&\neq 0.
\end{aligned}$$

Thus near V, u^2 is strictly plurisubharmonic and so is $\frac{16\pi^2}{s_0^2}u^2$. Thus by Lemma 3.1 from Patrizio-Wong [PW91], we have that ρ is strictly plurisubharmonic near V. Since X is a Stein manifold, we can extend ρ to a strictly plurisubharmonic function on all of X. To do this, first we pick a smooth function ρ_1 on $X-X_0$ such that it agrees with ρ on the boundary $\{\tau=\tau_0\}$. The set $X-X_0$ is holomorphically convex, thus there exists a strictly plurisubharmonic function ρ_2 on $X-X_0$ such that it vanishes at the boundary. Since $\overline{X-X_0}$ is compact, there exists C_1 such that $\rho_1+C_1\rho_2$ is strictly plurisubharmonic on $X-X_0$ such that it agrees on ρ on the boundary. Thus ρ can be extended to a function which is strictly plurisubharmonic on X except on $\{\tau=\tau_0\}$. Smoothing ρ near the boundary gives a strictly plurisubharmonic function on X such that it agrees with ρ when τ is large enough. Thus it defines a Kähler metric on X.

Notice that we have

$$\rho=\log(1+\cosh\frac{4\pi u}{s_0})=\log(1+\cosh(-\log|\zeta|^2)),$$

and from the proof of Proposition III.24,

$$\omega:=-i\partial\bar\partial\rho$$

can be extended to a form on $\hat{X}$.

To prove the form is positive definite, the method is exactly the same as in the proof of Proposition III.24:

$$\begin{aligned}\omega: &= -i\partial\bar\partial\log\left(1+\cosh\frac{4\pi u}{s_0}\right)\\ &= -i\left(\frac{8\pi i}{s_0e^u}\frac{\sinh\frac{4\pi u}{s_0}}{1+\cosh\frac{4\pi u}{s_0}}\Omega+\frac{4\pi}{s_0}\frac{\frac{4\pi}{s_0}-\sinh\frac{4\pi u}{s_0}}{1+\cosh\frac{4\pi u}{s_0}}\partial u\wedge\bar\partial u\right).\end{aligned}$$

For parallel vector fields ξ_j and η_j lying in V_y for all $y\in C$, we have

$$\Omega(\xi_j,\eta_j)=1$$

at x. Since $\psi_t^*\Omega = e^t\Omega$, and $u = u_0 + t$, we have

$$\begin{aligned}\omega(\xi_j, \eta_j) &= \frac{8\pi}{s_0 e^u}\frac{\sinh\frac{4\pi u}{s_0}}{1+\cosh\frac{4\pi u}{s_0}}\Omega(\xi_j,\eta_j)\\ &= \frac{8\pi}{s_0\tau_0}\frac{\sinh\frac{4\pi u}{s_0}}{1+\cosh\frac{4\pi u}{s_0}}\\ &\to \frac{8\pi}{s_0\tau_0} \neq 0\end{aligned}$$

as $u \to +\infty$. Similarly we have $\Omega(\xi_j, \xi_k) = \Omega(\eta_j, \eta_k) = 0$ and the same holds for ξ_n and η_n, as in the proof of Proposition III.24. Thus ω defines a Kähler form on $\hat{X}$. $\square$

Similar to Proposition III.25 we can prove that

Lemma III.47. *$\mathcal{O}(V)$ is a positive line bundle.*

Proof. Similar to the proof of Lemma III.25. $\square$

Thus by Kodaira's embedding theorem, $\hat{X}$ is projective algebraic.

Notice that we have

$$\frac{1}{2\pi}\omega = c_1(\mathcal{O}(V)) \in H^2(X, \mathbb{Z}).$$

This in the end shows that $\hat{X}$ is projective algebraic.

This is one direction of a special case of Burn's algebraicization conjecture:

Conjecture III.48. *Let X be a Stein manifold, τ is a smooth strictly plurisubharmonic exhaustion function. Then the followings are equivalent:*

(a) There exists a $\tau_0 > 0$ such that on $X_0 = \{\tau \geq \tau_0\}$, we have $u := \log\tau$ is plurisubharmonic and satisfies the Homogeneous Complex Monge-Ampère equation $(\partial\bar{\partial}u)^n = 0$.

(b) X is canonically an affine algebraic variety with coordinate ring

$$R = \{f \in \mathcal{O}(X), |f| \leq C_f(1+\tau)^N \text{ for some } N < +\infty\} \tag{3.28}$$

We proved that

Proposition III.49. *If (a) is satisfied with the conditions that the action given by the flow of ϕ is free and periodic, we have (b).*

Proof. The only thing we have to prove here is the coordinate ring is given by 3.28. Any algebraic function on X is given by polynomials. Since X is quasiprojective, it can be extended to a meromorphic function g on $\hat{X}$. Since $\tau = \frac{1}{|\zeta|^2} = \frac{1}{|fw|^2}$, where w is a holomorphic coordinate near a point on V and f does not vanish at V, c.f. the proof of III.24. Thus g is locally bounded by a polynomial of τ. But $\hat{X}$ is compact. Thus g is bounded by a polynomial of τ.

Conversely, if g is bounded by a polynomial of τ, g is bounded by a polynomial of w, a holomoprhic coordinate system near a point on V. Thus g is meromorphic and thus it is a rational function which is holomorphic defined on X. Thus g is algebraic on X. $\square$

In the converse direction, Burns [Bur84] proved the conjecture in the case that $X \subset \mathbb{C}^N \subset \mathbb{CP}^N$ such that $V = \hat{X} \cap H$, where H is a generic hyperplane in $\mathbb{CP}^N$ ($\hat{X}$ intersects H transversally).

CHAPTER IV

Lebrun-Mason Twistor Correspondence

In 2002, Lebrun and Mason [LM02] discovered a way to embed a Zoll surface and its manifold of unoriented geodesics into complex projective manifold in a different way than what we have looked at in the Grauert tube case. Chapter 4.1 will all be about Lebrun and Mason's work [LM02] on the struction and chapter 4.2 will be about their work on docility in 2010[LM10]. We also present two small results (Proposition IV.17 and Proposition IV.20) related to their construction. This construction will be crucial for the picture in Chapter V. We will be using their terminologies for subsequent uses.

4.1 The construction

Lebrun and Mason did extend the notion of Zoll metric into Zoll projective structures. We include their discussion here for the convenience of the readers.

Definition IV.1. Two torsion-free affine connections ∇ and $\hat{\nabla}$ on a manifold M are said to be *projectively equivalent* if they have the same geodesics, considered as unparametrized curves.

Definition IV.2. A C^k projective structure on a smooth manifold is the projective equivalence class $[\nabla]$ of some torsion-free C^k affine connection ∇.

Definition IV.3. Let ∇ be a C^1 torsion-free affine connection on a smooth surface M. We will say that the projective equivalence class $[\nabla]$ of ∇ is a *Zoll* projective structure if the image $\mathfrak{C}$ of any maximal geodesic of ∇ is an embedded circle $S^1 \subset M$.

Let Z be the projective tangent bundle

$$\mathbb{P}TM = (TM - 0_M)/\mathbb{R}^*,$$

where TM denotes the tangent bundle of M and 0_M denotes its zero section.

If $c : (a, b) \to M$ is any immersed curve, its derivative is a well-defined element on Z. We call the map $t \mapsto [dc/dt] \in Z$ the *canonical lift* of c. Given a C^k Zoll projective structure $[\nabla]$ on M, the canonical lifts of its geodesics give us a C^k foliation $\mathcal{F}$ of Z by circles. Let $N := Z/S^1$ denote the leaf space. Notice that N can be realized as the space of *unoriented* geodesics.

Theorem IV.4. *Let M be a (real) surface with a Zoll projective structure. Then M is diffeomorphic to $\mathbb{RP}^2$ or S^2. N is a 2-dimensional real manifold, diffeomorphic to $\mathbb{RP}^2$.*

Denote $\mu : Z \to M$ the canonical projection of the bundle and $\nu : Z \to N$ be the quotient map of the foliation to the leaf space.

Theorem IV.5. *If $M = S^2$, $|\pi_1(Z)| = 4$. If $M = \mathbb{RP}^2$, $|\pi_1(Z)| = 8$.*

If $[\nabla]$ is a Zoll projective structure on a compact surface M, we know that N is diffeomorphic to $\mathbb{RP}^2$. Let

$$\ell_x = \nu(\mu^{-1}(x))$$

be the set of geodesics passing through $x \in M$. Note that the $\ell_x \subset N$ are embedded circles.

4.1.1 Case $M \approx \mathbb{RP}^2$

Now assume that $M = \mathbb{RP}^2$. Let $\mathcal{Z}$ denote the projectivized complexified tangent bundle

$$\mathcal{Z} := \mathbb{P}T_{\mathbb{C}}M = (\mathbb{C} \otimes TM - 0_M)/\mathbb{C}^*.$$

Note that Z is a hypersuface in $\mathcal{Z}$. Our goal is to embed N into a complex surface $\mathcal{N}$ by "blowing down" Z to N via the map ν.

Theorem IV.6. *Let $[\nabla]$ be a Zoll projective structure which is represented by a C^3 connection ∇ on $M \approx \mathbb{RP}^2$. Then the manifold of unoriented geodesics N can be embbeded into a complex surface $\mathcal{N}$, via the blowing down map Φ as illustrated below,*

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "\nu"] & \mathcal{Z} \arrow[d, "\Phi"] \\ N \arrow[r, hook] & \mathcal{N} \end{tikzcd}$$

such that Φ maps $\mathcal{Z} - Z$ diffeomorphically to $\mathcal{N} - N$. The map Φ is C^1 and $\mathcal{N}$ is biholomorphic to $\mathbb{CP}^2$.

We denote

$$\Sigma_x = \Phi(\mathbb{P}T_{x\mathbb{C}}M).$$

Then Σ_x is an embedded genus 0 complex curve in $\mathcal{N}$. Note that $\Sigma_x \cap N = \ell_x$ and we can think of Σ_x be the "complexification" of the embedded circles ℓ_x in N.

Theorem IV.7. *The complex conjugation in the fibers of $\mathcal{Z} = \mathbb{P}T_{\mathbb{C}}M$ induces an antiholomorphic involution of $\mathcal{N}$ with fixed point set N. Thus N is maximally totally real surface in $\mathcal{N}$. Consequently N can be identified with the standard $\mathbb{RP}^2$ embedded into $\mathbb{CP}^2$.*

Theorem IV.8. *The curves $\Sigma_x \subset \mathcal{N}$ are projective lines.*

Using the above theorem, we have the Blaschke conjecture on Zoll manifold on $\mathbb{RP}^2$:

Theorem IV.9. *Let $(M, [\nabla])$ be a compact 2-manifold diffeomorphic to $\mathbb{RP}^2$ with a Zoll projective structure. If ∇ is of differentiability class $C^{k,\alpha}$, for some $k \geq 3$ and some $\alpha \in (0, 1)$, then there is a $C^{k+2,\alpha}$ diffeomorphism $\Phi : M \to \mathbb{RP}^2$ such that $[\nabla] = [\Phi^*\nabla_0]$, where ∇_0 is the Levi-Civita connection of the standard, constant curvature Riemannian metric h on $\mathbb{RP}^2$.*

If we start with a Zoll metric g on $M = \mathbb{RP}^2$ instead of just a Zoll projective structure, the complex surface $\mathcal{N}$ comes equipped with a complex curve $\mathcal{Q} \subset \mathcal{N}$. Consider

$$\mathcal{C} = \{[v] \in \mathbb{P}T_{\mathbb{C}}M | g_{\mathbb{C}}(v, v) = 0\},$$

where $g_{\mathbb{C}}$ denotes the extension of g from TM to $T_{\mathbb{C}}M$ as a complex bilinear form. Let

$$\mathcal{Q} = \Phi(\mathcal{C}).$$

Then we have $\mathcal{C}$ is diffeomorphic to S^2 and it intersects each Σ_x at 2 points. Using this curve, we have the following

Theorem IV.10. *Let (M, g) be a $C^{k,\alpha}$ Riemannian 2-manifold whose geodesics are all embedded circles of length π, where $k \geq 4$ and $\alpha \in (0, 1)$. If M is not simply connected, there is a $C^{k+1,\alpha}$ diffeomorphism $\Phi : M \to \mathbb{RP}^2$ such that $g = \Phi^*h$, where h is the standard curvature 1 Riemannian metric on $\mathbb{RP}^2$.*

This in turn re-confirms[Gre63] that the only Zoll metric on $\mathbb{RP}^2$ is the standard one.

4.1.2 Case $M \approx S^2$

The construction when $M \approx S^2$ is similar to the construction in $\mathbb{RP}^2$ except that $\mathcal{Z} - Z$ is disconnected.

Let M be a real surface diffeomorphic to S^2 with a Zoll projective structure. Then $\mathcal{Z} - Z$ consists of two connected components $\mathcal{U}_+$ and $\mathcal{U}_-$. Consider the compact 4-manifold-with-boundary

$$\mathcal{Z}_+ := \mathcal{U}_+ \cup Z,$$

with $\partial \mathcal{Z}_+ = Z$. Then we have

Theorem IV.11. *Let $[\nabla]$ be a Zoll projective structure which is represented by a C^3 connection ∇ on $M \approx S^2$. Then the manifold of unoriented geodesics N can be embbeded into a complex surface (with no boundary) $\mathcal{N}$, via the blowing down map Φ as illustrated below,*

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d, "\nu"] & \mathcal{Z}_+ \arrow[d, "\Phi"] \\ N \arrow[r, hook] & \mathcal{N} \end{tikzcd}$$

such that Φ maps $\mathcal{U}_+$ diffeomorphically to $\mathcal{N} - N$. The map Φ is C^1 and $\mathcal{N}$ is biholomorphic to $\mathbb{CP}^2$. Moreover if ∇ is of class C^{2k+6}, then Φ is C^k.

Definition IV.12. A differentiable embedding $j : \mathbb{RP}^2 \hookrightarrow \mathbb{CP}^2$ is weaakly unknotted if there exists a diffeomorphism $\varphi : \mathbb{CP}^2 \to \mathbb{CP}^2$ such that $j = \varphi \circ j_0$, where $j_0 : \mathbb{RP}^2 \hookrightarrow \mathbb{CP}^2$ is the standard embedding $[x : y : z] \mapsto [\bar{x} : \bar{y} : \bar{z}]$.

Theorem IV.13. *Let $[\nabla]$ be a C^3 Zoll projective structure on an oriented surface $M \approx S^2$. Then, up to a projective linear transformation, the projective structure $[\nabla]$ uniquely determines a differentiable, totally real weakly unknotted embedding of the space of geodesics $N \approx \mathbb{RP}^2$ into $\mathbb{CP}^2$. If $[\nabla]$ is C^∞, so is the embedding. Moreover,* *the image of each of the circles $\ell_x \subset N$, $x \in M$, bounds a holomorphic embedding of the disk $D^2 \hookrightarrow \mathbb{CP}^2$, and the interiors of these disks foliate the complement $\mathbb{CP}^2 - N$.*

Note that the disks are images of the fibers of $\mathcal{Z}_+ \to M$ under Φ.

An open question remains: How to classify all Zoll surfaces using this picture? One would suspect that given *any* weakly unknotted, totally real surfaces $N \approx \mathbb{RP}^2$ in $\mathbb{CP}^2$, it corresponds to the manifold of unoriented geodesics from a Zoll surfaces $M \approx S^2$. In order to let this happen, one must first construct a family of disks in $\mathbb{CP}^2$, such that their boundaries are lying on N such that the interiors of the disks foliate the complement $\mathbb{CP}^2 - N$. The boundaries of the disks were realized as ℓ_x for $x \in M$ if we started with a Zoll surface. This means that points in M correspond to unparametrized disks in the foliation. This motivates one to look at the "moduli space" of these disks and we can re-construct M as the moduli space of the disks. To trace each geodesic on M, recall that a point on N corresponds to a geodesic on M. Tracing out the set

$$\mathfrak{C}_y = \{D \in M, y \in D \cap N\}$$

gives a family of curves in M which can be realized as geodesics on M.

Indeed this is true locally around the standard Zoll sphere:

Theorem IV.14. *If $N \subset \mathbb{CP}^2$ is the image of any embedding $\mathbb{RP}^2 \hookrightarrow \mathbb{CP}^2$ which is sufficiently closed to the standard one in the C^{2k+5} topology, then*

1. *N contains a unique family of embedded oriented circles $\ell_x \subset N$, $x \in S^2$, each of which bounds an embedded holomorphic disk $D^2 \subset \mathbb{CP}^2$. These disks are all embedded and their interiors foliate $\mathbb{CP}^2 - N$.*
2. *There is a unique C^k Zoll projective structure $[\nabla]$ on $M \approx S^2$ for which for any $y \in N$, $\mathfrak{C}_y$ is a geodesic.*

If we start with a Riemannian metric g, then $N \subset \mathbb{CP}^2$ is "Lagrangian" with respect to the sign-ambiguous symplectic form $\Omega = \Im m\, \Upsilon$ on $\mathbb{CP}^2 - \mathcal{Q}$, where

$$\Upsilon = \pm \frac{z_0 dz_1 \wedge dz_2 + z_1 dz_2 \wedge dz_0 + z_2 dz_0 \wedge dz_1}{\sqrt{(z_0^2 + z_1^2 + z_2^2)^3}},$$

and

$$\mathcal{Q} = \{z_0^2 + z_1^2 + z_2^2 = 0\}$$

is the non-singualr conic in $\mathbb{CP}^2$. It turns out that it is also true in the reverse direction.

Theorem IV.15. *Let $N \subset \mathbb{CP}^2$ be a totally real embedding of $\mathbb{RP}^2$ which arises from a $C^{k,\alpha}$ projective structure $[\nabla]$ on $M \approx S^2$, $k \geq 3$, $\alpha \in (0,1)$. Then there is a $C^{k+1,\alpha}$ Riemannian metric g on M whose Levi-Civita connection ∇ belongs to the projective class $[\nabla]$ iff, after a $PSL(3,\mathbb{C})$ transformation of $\mathbb{CP}^2$, the surface N avoids the conic $\mathcal{Q}$, and is "Lagrangian" with respect to $\pm\Omega$ on $\mathbb{CP}^2 - \mathcal{Q}$. Such a Lagrangian embedding completely determines the metric g up to an overall multiplicative constant.*

The theorem was strengthened by F. Rochon [Roc11]:

Theorem IV.16. *Let $N \subset \mathbb{CP}^2$ be a totally real embedding. If it exists a family of holomorphic disks that give rise to a Zoll surface, the family of disks is unique.*

To summarize, a smooth Zoll metric g on S^2 gives a unique embedding $N \subset \mathbb{CP}^2$ up to $PSL(3,\mathbb{C})$ transformation of $\mathbb{CP}^2$ avoiding $\mathcal{Q}$, which given any $N \subset \mathbb{CP}^2$ close to the round one (after a $PSL(3,\mathbb{C})$ transformation), it uniquely determines g on S^2.

Using Lebrun and Mason model in [LM02], one obtains the following result.

Proposition IV.17. *Let (M, g) be a Riemannian manifold diffeomorphic to S^2 such that all of its geodesics are simple closed curve and of equal length, i.e. Zoll. Assume that for every point $x \in M$, there exists another point $x' \in M$ such that all* *the geodesics passes through x coincide with all the geodesics passes through x' as unparametrized curve. Then M is isometric to the round sphere, modulo dilation.*

Proof. To prove this theorem, we first show that the map $x \mapsto x'$ is well-defined and differentiable. Using the notation of the work by Lebrun and Mason [LM02], let N be the manifold of unoriented geodesics. Then for $x \in M$, let ℓ_x be all the geodesics passing through x. Then ℓ_x is an embedded circle in N. By their construction, by embedding N into $\mathbb{CP}^2$, each embedded oriented circle bound a unique holomorphic disk which the interior of these disks foliate $\mathbb{CP}^2 - N$. The condition states that for any $x \in M$, there exists an $x' \in M$ such that $\ell_x = \ell_{x'}$. Since there are only 2 orientations on a curve, this shows that if there is at most 2 points such that there are at most 2 holomorphic disks that bound the embedded circle. This means that the map $x \mapsto x'$ is well-defined and it is an involution.

Let f be this map $x \mapsto x'$. Using theorem 2.15 of [LM02], we have a smooth diffeomorphism $\psi : \mathbb{S}TN \to \mathbb{P}TM$. Observe that the map $\sigma : \mathbb{S}TN \to \mathbb{S}TN$ that flip the direction of the vector in $\mathbb{S}TN$ is smooth. Let π be the projection map from $\mathbb{P}TM$ to M. Then we have $f \circ \pi \circ \psi = \pi \circ \psi \circ \sigma$. Since $\mathbb{P}TM \to M$ is a smooth circle bundle and σ is a smooth map, we conclude that f is a smooth map.

Now by Chapter 2 of [LM02], let ∇ be the Levi-Civita connection associated with g. Then $f^*\nabla$ is also a connection such that all the geodesics are the same as ∇ as unparametrized curve. Thus they belong to the same projective class of connection. Then the connection $\nabla + f^*\nabla$ also belong to the same class and it is f-invariant. This mean the connection descends to a connection on $S^2/f \cong \mathbb{RP}^2$ such that all the geodesics are simple closed curve. By Chapter 3 of [LM02], the connection corresponds to the round metric on $\mathbb{RP}^2$ and thus S^2 is round. □

This is a very similar to the results of Blaschke conjecture, which asserts that the only Wiedersehen spheres are the round ones. By Wiedersehen manifolds, it means the cut locus of every point x consists of a single point x'.

4.2 Docility and consequences

Several years after [LM02], Lebrun and Mason published an explicit open condition on N that gives rise to the Zoll metric [LM10].

Definition IV.18. A compact connected smoothly embedded two-manifold $N \subset \mathbb{CP}^2$ will be called a *docile* surface if

- N is a totally real submanifold of $\mathbb{CP}^2$;
- N is disjoint from the conic $\mathcal{Q}$; and
- N is transverse to each tangent projective line of the conic $\mathcal{Q}$.

The standard $\mathbb{RP}^2$ is docile and any docile surface N is diffeomorphic to $\mathbb{RP}^2$. Moreover, any docile surface is connected in the sense that any docile surface is isotopic to the standard $\mathbb{RP}^2 \subset \mathbb{CP}^2$ through a family of other docile surfaces N_t such that $N_1 = N$ and N_0 is the standard $\mathbb{RP}^2$. Notice that the docility condition is not affected by any $PSL(3, \mathbb{C})$ transformation of $\mathbb{CP}^2$.

The docility and Lagrangian conditions on N are sufficient to gaurantee that it arises from a Zoll metric by the Lebrun-Mason construction.

Theorem IV.19. *Let $N \subset \mathbb{CP}^2$ be any docile surface and let M denote the moduli space of all holomorphic disks in $\mathbb{CP}^2$ representing the generator of $H_2(\mathbb{CP}^2, N) \cong \mathbb{Z}$, with boundaries in N, modulo reparametrization. Then M is diffeomorphic to S^2. The interiors of these disks foliate $\mathbb{CP}^2 - N$. If N is "Lagrangian" with respect to $\pm\Omega$ then there is a unique Zoll metric g whose closed geodesics all have length 2π.*

Notice that docility is not a necessary condition. In the appendix of [LM10], one can find an N which is not docile which arises from a Zoll metric.

One can ask whether we can define a Monge-Ampère solution such that the Monge-Ampère foliation coincides with the Lebrun-Mason foliation for any docile N. By [LM02], there is a unique holomorphic disk $D \to \mathbb{CP}^2$ passing through x such that the interior of these disks foliates $\mathbb{CP}^2 - N$. Let $u = -\log|\zeta|$, where ζ is a coordinate of the disk, such that $\zeta = 0$ corresponds to the point where the disk meets M. Here ζ is not well-defined, but $|\zeta|$ is. Thus u is well-defined. We want to investigate if u is the function of Grauert tube picture. But this could only happen in the round case because of the following:

Proposition IV.20. *Let N be diffeomorphic to $\mathbb{RP}^2$ such that $N \subset \mathbb{CP}^2$ is disjoint from a nonsingular quadric $\mathcal{Q}^1$ in $\mathbb{CP}^2$. Assume that there is a function $u : \mathbb{CP}^2 - M \to \mathbb{R}_{\geq 0}$ such that*

1. *$u^{-1}(0) = N$ and $u(z) \to +\infty$ as $z \to M$;*
2. *u is smooth and p.s.h. and satisfies the HCMA equation on $\mathbb{CP}^2 - \{M \cup N\}$;*
3. *u^2 is smooth and s.p.s.h. on $\mathbb{CP}^2 - M$; and*
4. *each leaf of the Monge-Ampère foliation on $\mathbb{CP}^2 - \{M \cup N\}$, given by the kernel of $dd^c u$, extends across N and intersects N in a simple closed curve.*

Then N must be the standard $\mathbb{RP}^2 \subset \mathbb{CP}^2$.

Proof. By [PW91] Theorem 5.1, The Monge-Ampère foliation is totally geodesic with the u^2-metric, can be extended across N and so that the intersection of the foliations are geodesics on N. Thus N has the property that all the geodesics are closed in the Kähler metric defined by $i\partial\bar{\partial}u^2$ (we call this metric the "u^2-metric"). By the Blaschke conjecture or [LM02], N is isometric to the round $\mathbb{RP}^2$ up to a scale constant. By

[PW91] Theorem 5.2, there exists biholomorphism $F : \mathbb{CP}^2 - M \to \mathbb{CP}^2 - M$ such that $u \circ F = u_0$ and F sends the standard $\mathbb{RP}^2$ to N. Here u_0 is the standard Monge-Ampère solution associated to the round $\mathbb{RP}^2 \subset \mathbb{CP}^2 - \mathcal{Q}^1$.

Since we have F a biholomorphism from $\mathbb{CP}^2 - \mathcal{Q}^1$ to itself, and u_0 is the pullback of u, the Monge-Ampère foliation of u_0 is the pullback of the Monge-Ampère foliation of u. Let D^* (resp. D_0^*) be a leaf of the Monge-Ampére foliation of u (resp. u_0). Here D^* and D_0^* are punctured disks. Thus we have $F(D_0^*) = D*$. Using the biholomorphism F, we have

$$u_0|_{D_0^*} = u|_{D^*}.$$

Therefore, F restricted to each punctured disk can be extended to a biholomorphism $\tilde{F}$ of the whole disk and $\tilde{F}|_{D_0^*}(0) = 0 \in \overline{D^*}$. Thus $\tilde{F}$ is a map from $\mathbb{CP}^2$ to itself. Using the Cauchy integral formula, we see that $\tilde{F}$ is differentiable near $\mathcal{Q}^1$ and thus holomorphic. Thus $\tilde{F}$ is holomorphic on all of $\mathbb{CP}^2$. Thus $\tilde{F}$ is a biholomorphism from $\mathbb{CP}^2$ to itself. Thus $\tilde{F}$ is projective linear and $\tilde{F}$ sends $\mathbb{RP}^2$ to N linearly. Thus N is the standard $\mathbb{RP}^2$ modulo linear transformation. $\square$

It is curious that both the lagrangian condition and the docility condition come from the double branched cover of $\mathbb{CP}^2$, branching at $\mathcal{Q}$:

- Υ was originally defined on the double cover of $\mathbb{CP}^2 - \mathcal{Q}$ (notice that singularity of Υ at $\mathcal{Q}$). The form Ω is sign-ambigious because Υ does not descend to $\mathbb{CP}^2 - \mathcal{Q}$.
- Although the docility of N can be explicitly defined on $\mathbb{CP}^2$, the condition is much simpler in the branched cover of $\mathbb{CP}^2$. Consider the Segre embedding

$$\begin{aligned} \hat{\Pi} : \mathbb{CP}^1 \times \mathbb{CP}^1 &\to \mathbb{CP}^3 \\ ([u_0 : u_1], [v_0 : v_1]) &\mapsto [i(u_0v_0 + u_1v_1) : u_0v_0 - u_1v_1 : u_0v_1 + u_1v_0 : i(u_0v_1 - u_1v_0)] \end{aligned}$$

Let ϱ be the projection map

$$\begin{aligned} \varrho : \mathbb{CP}^3 &\to \mathbb{CP}^2 \\ [z_0 : z_1 : z_2 : z_3] &\mapsto [z_0 : z_1 : z_2] \end{aligned}$$

and $\Pi = \varrho \circ \hat{\Pi}$. Then Π is a two-to-one branched cover, ramified over the conic $\mathcal{Q}$. Let $\tilde{N} = \Pi^{-1}(N)$. Then N is docile if and only if $\tilde{N}$ is given by the graph of a fixed-point-free orientation reversing diffeomorphism. Indeed, let $\mathfrak{a}$ be the antipodal map of $\mathbb{CP}^1$

$$\mathfrak{a} : \mathbb{CP}^1 \to \mathbb{CP}^1 \tag{4.1}$$

$$[z_1 : z_2] \mapsto [-\bar{z}_2 : \bar{z}_1]. \tag{4.2}$$

Then $\tilde{N}$ can be thought of being the graph of $\varphi = \psi \circ \mathfrak{a} \circ \psi^{-1}$ for some orientation-preserving diffeomorphism ψ of $\mathbb{CP}^1$. Conversely, given any orientation-preserving diffeomorphism ψ, the graph of φ projects via Π to a docile surface N. Hence the docility condition can be given by aa orientation-preserving diffeomorphism ψ of $\mathbb{CP}^1$ to itself on the branched cover. Note that if the diffeomorphism is the identity, then the resulting docile surface is the standard $\mathbb{RP}^2$.

The discussion that both conditions arise from the branched cover, which is the quadric $\mathcal{Q}^2$ motivates us to work on the picture on $\mathcal{Q}^2$. We will discuss the work on $\mathcal{Q}^2$ in the next Chapter.

CHAPTER V

Holomorphic Disks

In this section we are going to simulate the Lebrun-Mason construction on $\mathbb{CP}^1 \times \mathbb{CP}^1$, the branched cover of $\mathbb{CP}^2$. Although it may not have any connections to Zoll surfaces, it is interesting in its own right in terms of global results on the existence of holomorphic disks. Via the Segre embedding, $\mathbb{CP}^1 \times \mathbb{CP}^1$ can be realized as $\mathcal{Q}^2$. Using the map Π as we have seen in the previous chapter, $\Pi^{-1}(\mathcal{Q})$ can be realized as $\mathcal{Q}^1 = \mathcal{Q}^2 \cap H$, where H is a hyperplane in $\mathbb{CP}^3$. If N is the standard $\mathbb{RP}^2$ embedded in $\mathbb{CP}^2$, then $\Pi^{-1}(N)$ is the standard S^2 in $\mathcal{Q}^2$ (in the picture of the Grauert tube of the CROSS, c.f. Section 3.2, Table 3.17). Moreover, the foliation by holomorphic disks in $\mathbb{CP}^2 - \mathbb{RP}^2$ will lift to $\mathcal{Q}^2 - S^2$. In the case when $N \approx \mathbb{RP}^2$, the holomorphic disks extend across $\mathbb{RP}^2$ and join with the holomorphic disks corresponding to the antipodal point of S^2.

We proved in Chapter III that any Zoll surface diffeomorphic to S^2 with an infinite tube is the round one. This means that the foliation by rational curves in $\mathcal{Q}^2$ is rigid in some sense, i.e., if we perturb the S^2 by a bit, we should not expect that there is a foliation of $\mathcal{Q}^2 - S^2$ by holomorphic disks with the circles of intersection corresponding to the geodesics of a Zoll metric on S^2. But looking at the Lebrun-Mason picture, one would suspect that we can find holomorphic disks, instead of rational curves, whose boundaries lie on S^2 such that the disks foliate $\mathcal{Q}^2 - S^2$.

For convenience, we will regard $\mathcal{Q}^2$ as $\mathbb{CP}^1 \times \mathbb{CP}^1$. The $\mathcal{Q}$ is represented by the diagonal $\Delta := \{u_1 v_0 = v_1 u_0\} \subset \mathbb{CP}^1 \times \mathbb{CP}^1$.

Following Lebrun-Mason definition, we have in the $\mathbb{CP}^1 \times \mathbb{CP}^1$ picture,

Definition V.1. A compact, connected, smoothly embedded two manifold $N \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is *docile* if N is given by the graph of $\psi \circ \mathfrak{a} \circ \psi^{-1} : \mathbb{CP}^1 \to \mathbb{CP}^1$ for some orientation-preserving ψ diffeomorphism of $\mathbb{CP}^1$ onto itself. Here $\mathfrak{a}$ is the antipodal map 4.1 of $\mathbb{CP}^1$ described in the previous chapter.

Clearly if N is docile, then N is diffeomorphic to S^2.

5.1 Existence of holomorphic disks

Consider the round S^2 picture:

$$S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{R}^3.$$

We can complexifiy $\mathbb{R}^3$ in $\mathbb{C}^3$ and give an embedding of

$$S^2 \subset \{z_1^2 + z_2^2 + z_3^2 = 1\} = \mathcal{Q}^2_{aff} \subset \{Z_1^2 + Z_2^2 + Z_3^2 = Z_0^2\} = \mathcal{Q}^2 \subset \mathbb{CP}^3,$$

where $z_i = Z_i/Z_0$. The geodesics on S^2 are given by the intersections of real planes in $\mathbb{R}^3$ passing through the origin:

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0,$$

where the a_i's are, obviously, real.

Notice that any plane

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = a_0,$$

which intersects the real sphere transversely will have its intersection diffeomorphic to a circle. Complexifying and taking the (Zariski) closure in $\mathcal{Q}^2$, these planes are given by the homoegeneous equations

$$a_1Z_1 + a_2Z_2 + a_3Z_3 = a_0Z_0,$$

which intersect S^2 in a circle C. Thus it divides the rational curve into 2 holomorphic disks, in which it intersects the divisor $\{z_0 = 0\}$ at infinity at the same point when a_0 varies. Any point q at $\mathcal{Q}^1$, the divisor of infinity will correspond to a point $[a_1 : a_2 : a_3]$ in $\mathbb{CP}^{2*}$ uniquely, as the point is given by the element representing the line that passes through both q and $\bar{q}$. (Here $q \neq \bar{q}$, thus $[a_1 : a_2 : a_3]$ is well-defined and it belongs to $\mathbb{RP}^{2*}$) Thus this shows that for any point $q \in \mathcal{Q}^1$, there is a 1-real-parameter family of holomorphic disks that passes through q and with its boundary lying on S^2.

One also notices that if a_0 changes such that the real plane intersects the real sphere at 1 point (non-transversally), after complexification the rational curve intersects the real sphere at 1 point only. In this case there will be no holomorphic disk with boundaries in S^2. Indeed, this is an example of "bubbling" in the boundary as the boundary shrinks to a point when a_0 approaches the value at which the 2 surfaces do not intersect in a circle on S^2.

Now fix $q \in \Delta$ and let $f(D)$ be a holomorphic disk with boundary on S^2. Notice that S^2 is the fixed point of the conjugation map in $\mathcal{Q}^2 \subset \mathbb{CP}^3$. Then $C := f(D) \cup \overline{f(D)}$ can be realized as a map from $\hat{\mathbb{C}}$ to $\mathcal{Q}^2$ such that it is holomorphic in $\hat{\mathbb{C}} - \mathbb{R}$. Thus by reflection principle, it is holomorphic in all of $\hat{\mathbb{C}}$. The construction gives C a rational curve in $\mathcal{Q}^2$ which intersects Δ at 2 distinct points. Thus C is a projective line on $\mathbb{CP}^3$ by Bézout's theorem. Since C is invariant under conjugation, C is the complexification of a real line, and therefore it is in the form of

$$a_1Z_1 + a_2Z_2 + a_3Z_3 = a_0Z_0,$$

with all $a_i \in \mathbb{R}$.

Since it intersects M in a circle, we conclude that we have its (Euclidean) distance from the origin less than 1, i.e. any holomorphic disk in $\mathcal{Q}^2$ with boundary on S^2 is given by

$$\{a_1 Z_1 + a_2 Z_2 + a_3 Z_3 = a_0 Z_0\} \cap \mathcal{Q}^2,$$

with the point of intersection with $\mathcal{Q}^1$ specified. Consider the set

$$\mathcal{M} := \left\{ \begin{array}{l} ([a_0 : a_1 : a_2 : a_3], [q_1 : q_2 : q_3]) \in \mathbb{RP}^{3*} \times \mathbb{CP}^2 \mid \\ a_0^2 < a_1^2 + a_2^2 + a_3^2;\ q_1^2 + q_2^2 + q_3^2 = 0;\ \text{and}\, a_1 q_1 + a_2 q_2 + a_3 q_3 = 0 \end{array} \right\}.$$

Here we can think of $\mathcal{M}$ as the "moduli space" of holomorphic disks. The map from $\mathcal{Q}^1 \times I \to \mathcal{M}$ given by

$$((q_1, q_2, q_3), t) \mapsto [t\sqrt{a_1^2 + a_2^2 + a_3^2} : a_1 : a_2 : a_3],$$

where

$$[a_1 : a_2 : a_3] = [\Im m\, q_2 \bar{q}_3 : \Im m\, q_3 \bar{q}_1 : \Im m\, q_1 \bar{q}_2]$$

is an diffeomorphism and the space of holomorphic disks (modulo reparametrizations) is isomorphic to $\mathcal{Q}^1 \times I$.

To investigate this phenomenon when the round S^2 is replaced by a docile $N \subset \mathcal{Q}^2 = \mathbb{CP}^1 \times \mathbb{CP}^1$, one needs some basic facts similar to those in [LM10].

Following Lemma 1 in [LM10], as the space of orientation preserving diffeomorphisms from $\mathbb{CP}^1$ to itself is connected, we can smoothly deform the map ψ to the identity and thus it gives a family of docile surfaces. As a result, any docile surface $N \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ is isotopic to the standard $S^2 \subset \mathcal{Q}^2$ through a family of other docile surfaces.

Following Lemma 2 in [LM10], we have that the homomorphism $H_2(\mathbb{CP}^1 \times \mathbb{CP}^1, N) \to \mathbb{Z}$ given by homological intersection with $[\Delta] \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1 - N)$ is an isomorphism.

Following Lemma 3 in [LM10], there is a Kähler metric h such that N is Lagrangian with respect to the Kähler form ω of h. The metric can be chosen such that ω represents $2\pi c_1(\mathcal{Q}^2)$ in the deRham cohomology. The metric can be constructed to vary smoothly with N, as N varies among docile surfaces.

Following with the definitions on embedded holomorphic disks in [LM10], one also defines

Definition V.2. Let D denote the closed unit disk in $\mathbb{C}$, and let Z be a complex manifold. A continuous map $f : D \to Z$ is called a *parametrized holomorphic disk* in Z if f is holomorphic in the open unit disk. If $f(\partial D) \subset W$ for a specified subset $W \subset Z$, we will say that f is a parametrized holomorphic disk in (Z, W).

By the proof of Proposition 1 in [LM10], given any docile surface $N \subset \mathcal{Q}^2$ and f a parametrized holmorphic disk in $(\mathcal{Q}^2, N)$ whose relative homology class $[f]$ generates $H_2(\mathcal{Q}^2, N) \cong \mathbb{Z}$, then f is a smooth embedding, $f(D)$ meets N only along $f(\partial D)$, and $f(D)$ meets $\mathcal{Q}$ transversely in a single point.

Let

$$\mathcal{M}^N = \left\{ f : (D, \partial D) \to (\mathcal{Q}^2, N) \,\middle|\, \begin{array}{c} f \text{ parametrized holomorphic disk,} \\ [f] \text{ generates } H_2(\mathcal{Q}^2, N) \end{array} \right\} / \sim,$$

where

$$f \sim g \Longleftrightarrow f = g \circ h \text{ for some automorphism } h \text{ of } D.$$

We will drop the superscript N if it is clear from the context that N is fixed. Here M^N is the moduli space of holomorphic disks where the boundaries lie on N.

Let $q \in \Delta \subset \mathcal{Q}^2$. Define $\mathcal{M}_q^N \subset \mathcal{M}$ to be the space of disks that pass through q. In this section we want to prove that

Proposition V.3. *Let $N \subset \mathcal{Q}^2$ be a docile surface and $q \in \Delta \subset \mathcal{Q}^2$. Then $\mathcal{M}_q^N$ is non-empty and* $\dim \mathcal{M}_q^N = 1$.

We first describe the Maslov index (c.f. [LM10]): Let $L \to D$ be a complex line bundle and $\ell \to \partial D$ is a real line sub-bundle of $L|_{\partial D}$, then the Maslov index $\mu(L,\ell)$ is obtained by trivializing L, viewing ℓ as a map $\partial D \to \mathbb{RP}^1$ and $\mu(L,\ell)$ is defined as the winding number of this map. Note that $\mu(L,\ell)$ is independent of the trivialization and amount to the first Chern class of the double of (L,ℓ). More generally, if $V \to D$ is a rank-r complex vector bundle, and if $v \to S^1$ is a rank-r real sub-bundle of $V|_{\partial D}$, the Maslov index $\mu(V,v)$ is defined to be the Maslov index of the associated line-bundle pair $(\bigwedge^r V, \bigwedge^r v)$.

If Z is a complex manifold, $W \subset Z$ is a maximal totally real submanifold, and f is a parameterized holomorphic disk in (Z,W), then the normal Maslov index of $f(D)$ is defined to be $\mu(N,\mathfrak{n})$, where N is the normal bundle of the disk and $\mathfrak{n} = TW/T(\partial f(D))$ is the relative normal bundle of its boundary.

Digression: Let X be a complex manifold and M be a maximal totally real submanifold. Assume that C is a rational curve in X such that it intersects M in an embedded circle. This would be the case for the compactification of the Grauert tubes of the CROSSes, where the intersection with M is a geodesic. In these cases, the rational curves are complexified compactified geodesics and each of these curves consists of two holomorphic disks with their boundaries conincide with the intersectoin circle with M. Since C is a rational curve, the normal bundle N of C can be written as

$$\mathcal{O}(k_1) \oplus \dots \mathcal{O}(k_{n-1}),$$

by Grothendieck's splitting principle. It is clear that N is the double of the normal bundle of $(N,\mathfrak{n})$ of each holomorphic disk. Thus $\mu(N,\mathfrak{n})$ is the first Chern class of

N of C and thus we have

$$\mu(N, \mathfrak{n}) = \sum_{i=1}^{n-1} k_i.$$

This is the end of the digression.

Now let $N \subset \mathcal{Q}^2$ be a docile surface and f be a parametrized holomorphic disk in $(\mathcal{Q}^2, N)$ whose relative homology class represents the generator of $H_2(\mathcal{Q}^2, N)$. Following the proof in Proposition 2 in [LM10], $f(D)$ has normal Maslov index 2. This means that the double of the normal bundle of $f(D)$, in the sense of [LeB06], is the $\mathcal{O}(2)$ line bundle of the double $\mathbb{CP}^1 = f(D) \cup \overline{f(D)}$, where $\overline{f(D)}$ is $f(D)$ equipped with the opposite complex structure. Since $H^1(\mathbb{CP}^1, \mathcal{O}(2)) = 0$, we have the Moduli space of nearby holomorphic disks in $(\mathcal{Q}^2, N)$ is smooth with an isomorphism

$$T_{f(D)}\mathcal{M} \cong H^0_{\mathbb{R}}(\mathbb{CP}^1, \mathcal{O}(2)),$$

where the right-hand side denotes the real-linear subspace of $H^0(\mathbb{CP}^1, \mathcal{O}(2))$ consisting of sections which are real along $\mathbb{RP}^1 \subset \mathbb{CP}^1$.

An element of $H^0_{\mathbb{R}}(\mathbb{CP}^1, \mathcal{O}(2))$ can be written as $\alpha Z_0^2 + \beta Z_0 Z_1 + \gamma Z_1^2$, where $[Z_0 : Z_1]$ are homogeneous coordinates of $\mathbb{CP}^1$. Here α, β, γ are real constants. By transforming the upper half plane into the unit disk, the element can be written as

$$(\alpha - \gamma - i\beta)\zeta^2 + 2(\alpha + \gamma)\zeta + (\alpha - \gamma + i\beta).$$

Let $a = \alpha - \gamma - i\beta \in \mathbb{C}$ and $b = 2(\alpha + \gamma) \in \mathbb{R}$, we can represent elements of $H^0_{\mathbb{R}}(\mathbb{CP}^1, \mathcal{O}(2))$, after a trivialization as

$$g(\zeta) = a\zeta^2 + b\zeta + \bar{a},$$

where $a \in \mathbb{C}$ and $b \in \mathbb{R}$.

Let f be a parametrized holomorphic disk in $\mathcal{M}$ such that $f(0) = q \in \Delta$. Then by the above trivialization, $\dim \mathcal{M}_q = 1$, because it corresponds to $a = 0$. So it suffices to prove the existence of holomorphic disks passing through q. This also suggests that for $f(D) \in \mathcal{M}_q$, we can represent elements of $T_{f(D)}\mathcal{M}_q$ as $g(\zeta) = b\zeta$ after the same trivialization. This in turn shows that for all disks close to $f(D)$, their boundaries won't intersect each other. Since $M \approx S^2$, $f(\partial D)$ divides M into two disks. This in turn shows that $\mathcal{M}_q$ cannot contain any loop.

Now assume that $f_{s_j} \in \mathcal{M}_q$ and this sequence of holomorphic disks converge to a possibly singular holomorphic curve X in the same relative homology class. By Gromov's compactness theorem, X is a union of holomorphic $\mathbb{CP}^1$'s and at most one holomorphic disk in $(\mathcal{Q}^2, N_\tau)$. With respect to the Kähler form described above, all of these disks $f_{s_j}(D)$ has an area of 4π. Thus we know that X has total area of 4π, by the Gromov's compactness theorem. If X contains a copy of $\mathbb{CP}^1$, then the area of the $\mathbb{CP}^1$ is at least 4π, which is given by the "coordinate lines". In the picture of $(z, w) \in \mathbb{CP}^1 \times \mathbb{CP}^1$, the coordinate lines coorespond to $\{z = \mathit{const}\}$ ("right line") or $\{w = \mathit{const}\}$ ("left line"). Thus, X either *only* contains a coordinate line, or X *only* contains a holomorphic disk with area 4π. Any coordinate lines intersect M at 1 point only, which means if X is a coordinate line, $f_{s_j}(\partial D)$ are circles on M converging to a point on M. If $f_{t_j}(D)$ converges to a holomorphic disk, then the limit $\lim_{s_j \to s} f_{s_j} \in \mathcal{M}_q$. This shows that for $f \in \mathcal{M}_q$, the connected component containing f in $\mathcal{M}_q$ is an open interval, say $f_s, s \in (-1, 1)$, while $\lim_{s \to -1} f_s$ and $\lim_{s \to 1} f_s$ are coordinate lines. Note that these two coordinate lines have to pass through q. But there are only 2 coordinate lines passing through a point in $\mathcal{Q}^2 = \mathbb{CP}^1 \times \mathbb{CP}^1$. And $\lim_{s \to 1} f_s$ and $\lim_{s \to -1} f_s$ definitely are different coordinate lines because they intersect M at different points. This shows that, without loss of generality, $\lim_{s \to 1} f_s$ and $\lim_{s \to -1} f_s$ correspond to the left line and the right line, respectively.

Similar to [LM10], we apply the continuity method. Let N_t be a smooth family of docile surfaces, $t \in [0,1]$ such that $N_1 = N$ and N_0 denotes the standard round $S^2 \subset \mathcal{Q}^2$. It is clear from the discussion above that the theorem holds for the standard round S^2. Let $E \subset [0,1]$ be the set of t for which such a disk exists. Thus $0 \in E$, so E is non-empty.

Lemma V.4. *E is open.*

Proof. By Theorem 4 of [LeB06], we have that a small deformation of N_t contains an $h^0(\mathbb{CP}^1, \mathcal{O}(2))$-dimensional family of holomorphic disks. Thus if $\tau \in E$, and for values of t in a small interval about τ, there is a family of holomorphic disks with boundary on N_t. □

Lemma V.5. *E is closed.*

Proof. Let t_j be a sequence of values in $[0,1]$ for which there exists a corresponding sequence of holomorphic disk $f_{t_j} \in \mathcal{M}^{N_{t_j}}$. Suppose $t_j \to \tau \in [0,1]$. we want to show that it converges to a holomorphic disk in $(\mathcal{Q}^2, N_\tau)$. Notice that by the proof in [LM10], and the existence of the Kähler form ω_t described earlier above, we can think of the sequence of disks $f_{t_j}(D)$ in $(\mathcal{Q}^2, S^2)$ which are holomorphic with respect to some fixed almost-complex structure. Again, since these Kähler forms correspond to the same holomogy class, we have each disk here would have an area of 4π. By the above discussion, this sequence of holomorphic disks either converges to a parametrized holomorphic disk in $(\mathcal{Q}^2, N_\tau)$, or a coordinate line.

Let q_1 (resp. q_2) be a point on M such that it corresponds to the intersection point of the left line (resp. right line) passing through $q \in \Delta$. For any $t \in E$, let $f_{t,s}$ be the family of disks in the connected component of $\mathcal{M}_q^{N_t}$ containing f_t with $f_{t,0} = f_t$. We know that $f_{t,s}$ converges to the left line (resp. right line) passing through q as $s \to 1$ (resp. -1). Then we have $f_{t,s}(\partial D) \to q_1$ as $s \to 1$ and $f_{t,s}(\partial D) \to q_2$ as $s \to -1$.

Pick small open sets U_1 (resp. (U_2) of M containing q_1 (resp. q_2). We can pick U_i such that for each t_j, there exists an s_j such that $f_{t_j,s_j}(\partial D) \subset M - U_1 - U_2$.

Now assume that $\tau \notin E$. This means that for *every* sequence $f_{t_j,s_j}(D)$ of holomorphic disks, the limit is a coordinate line, in which the image of the boundary converges to q_1 or q_2. This is impossible since we assume that $f_{t_j,s_j}(\partial D) \subset M - U_1 - U_2$ and thus it cannot converge to q_1 or q_2. This is a contradiction and thus $\tau \in E$. $\square$

Thus E is both open and closed in $[0,1]$. As E is non-empty, $E = [0,1]$. This shows that there exists a parametrized holomorphic disk passes through q and with boundary in N, and because of the existence, we know $\dim \mathcal{M}_q^N = 1$. Thus we have proved the existence of holomorphic disks for a general docile N.

5.2 Moduli space of holomorphic disks and case of higher dimensions

The above proof actually shows more:

Proposition V.6. *Let N is a docile surface. Then $\mathcal{M} = \mathcal{M}^N$ is diffeomorphic to $S^2 \times I$. In particular, $\mathcal{M}_q$ is connected and is diffeomorphic to an interval.*

Proof. The only thing we have to prove here is that $\mathcal{M}_q$ may contain more than one connected component. To prove this, we apply the continuity method backwards, similar to the proof in Proposition 1 in [LM10]. Since E is open and we know that there is a three-dimensional family of holomorphic disks near any fixed $f(D)$ when we perturb N, the topology of $\mathcal{M}_q^N$ won't change. This means that by the continuity argument, the topology of $\mathcal{M}_q^N$ is the same as $\mathcal{M}_q^{S^2} \approx I$. This shows that $\mathcal{M}_q^N$ is also connected and thus $\mathcal{M}$ is diffeomorphic to $S^2 \times I$. $\square$

Notice here the first factor S^2 is represented by the points in Δ, and $\dim I = \dim \mathcal{M}_q$.

If we look at the round S^n and its compactified Grauert tube $\mathcal{Q}^n$, if $q \in \mathcal{Q}^n$, one could see that $\mathcal{M}_q$ is an $(n-1)$-dimensional family of 2-planes that passes through q. Indeed $\mathcal{M}_q$ can be identified as an open set in the space of real projective lines in $\mathbb{CP}^{n+1}$ containing q. Since $q \in \mathcal{Q}^{n-1}$, which is not a real point, $q \neq \bar{q}$. The real projective lines have to contain both q and $\bar{q}$. Thus it will satisfy two equations and thus it is of dimension $n-1$.

To be specific, C will be given as the intersection of $n-1$ real projective lines with S^n. The lines can be written as

$$\begin{cases} a_{11}Z_1 & + & \ldots & + & a_{1s}Z_s & = & b_1 Z_0 \\ \vdots & + & \ddots & + & \vdots & = & \vdots \\ a_{r1}Z_1 & + & \ldots & + & a_{rs}Z_s & = & b_r Z_0 \end{cases},$$

or

$$\begin{pmatrix} a_{11} & \ldots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{r1} & \ldots & a_{rs} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_s \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}.$$

Here $r = n-1$ and $s = n+1$. Let

$$\mathbf{A} = \begin{pmatrix} a_{11} & \ldots & a_{1s} \\ \vdots & \ddots & \vdots \\ a_{r1} & \ldots & a_{rs} \end{pmatrix}; \quad \mathbf{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_s \end{pmatrix}; \text{ and } \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}.$$

The equation becomes

$$\mathbf{Az} = \mathbf{b}.$$

We want to minimize $\mathbf{z} \cdot \mathbf{z}$ subject to the constraints $\mathbf{Az} = \mathbf{b}$. Using the method of Lagrange multipliers, we have

$$\begin{cases} 2\mathbf{z} = \mathbf{A}^T \lambda \\ \mathbf{Az} = \mathbf{b} \end{cases}.$$

Let $\mathbf{B} = \mathbf{A}\mathbf{A}^T$. Since $\mathbf{A}$ has maximal rank, $\mathbf{B} = \mathbf{A}\mathbf{A}^T$ has maximal rank and thus $\mathbf{B}^{-1}$ exists. Then we have

$$\mathbf{z} = \mathbf{A}^T\mathbf{B}^{-1}\mathbf{b}$$

as the closest point of the line to the origin. We want the distance to be less than 1 so that it intersects S^n in a circle. This means

$$\mathbf{z}^T\mathbf{z} < 1,$$

or

$$\mathbf{b}^T\mathbf{B}^{-1}\mathbf{b} < 1.$$

Here $\mathbf{B}$ is a symmetric matrix with maximal rank. Therefore the set of $\mathbf{b}$ satisfying the above equation is an $(n-1)$-ball. To conclude, we show that $\mathcal{M}_q$ is diffeomorphic to the $(n-1)$-ball.

By looking at the Grauert tube picture of the CROSSes (other than $\mathbb{OP}^2$), we have the following observations.

Let (M, g) be a CROSS (other than $\mathbb{OP}^2$). Let C be a compactified complexified geodesic. Recall the normal Maslov index is defined to be the first chern class of the Maslov index of the normal bundle $N_{C|X}$ of C in X with respect to M. If the normal bundle is $\oplus_j \mathcal{O}(n_j)$, then $\mu = \sum_j n_j$'s. Let I be the index of a geodesic of M. Then as computed in [Bes78], we have

1. $M = \mathbb{RP}^n$, $I = 0$.
2. $M = S^n$, $I = n - 1$.
3. $M = \mathbb{CP}^n$, $I = 1$.
4. $M = \mathbb{HP}^n$, $I = 3$.

We observe that

Proposition V.7. *(Index formula) Let (M, g) be a CROSS of real dimension d other than the Cayley projective plane. Using the above notation, we have*

$$\mu = I + d - 1$$

Proof.

1. $M = \mathbb{RP}^n$.

   Since C intersects $\mathcal{Q}^{n-1}$ at two points, it follows that C has degree 1 and thus $N_{C|X} = \mathcal{O}(1)^{\oplus(n-1)}$. Thus $\mu = n - 1$ and the formula holds.

2. $M = S^n$.

   C intersects $\mathcal{Q}^{n-1}$ at two points. In $\mathbb{CP}^{n+1}$, $\mathcal{Q}^{n-1} = \mathcal{Q}^n \cap H$, where H is a hyperplane in $\mathbb{CP}^{n+1}$. This shows that C has degree 2 in $\mathbb{CP}^{n+1}$. By the relation between degrees and irreducible subvariety in projective space, we know that C must lie inside a linear $\mathbb{CP}^2$. So we get

$$\begin{aligned} N_{C|\mathbb{CP}^{n+1}} &= N_{C|\mathbb{CP}^2} \oplus N_{\mathbb{CP}_2|\mathbb{CP}^{n+1}}|_C \\ &= \mathcal{O}_C(4) \oplus \mathcal{O}_{\mathbb{CP}^2}(1)^{\oplus(n-1)}|_C \\ &= \mathcal{O}_C(4) \oplus \mathcal{O}_C(2)^{\oplus(n-1)}. \end{aligned}$$

   Since $N_{\mathcal{Q}^n|\mathbb{CP}^{n+1}}|_C = \mathcal{O}_{\mathbb{CP}^{n+1}}(2)|_C = \mathcal{O}_C(4)$ and

$$N_{C|\mathcal{Q}^n} \oplus N_{\mathcal{Q}^n|\mathbb{CP}^{n+1}}|_C = N_{C|\mathbb{CP}^{n+1}},$$

   we concluded that $N_{C|\mathcal{Q}^n} = \mathcal{O}_C(2)^{\oplus(n-1)}$ and thus $\mu = 2(n-1)$ and the formula holds.

3. $M = \mathbb{CP}^n$.

   Since C intersects the divisor at infinity $V = X - TM$ at two points, in which V has bidegree $(1, 1)$, and the fact that C is irreducible, we can conclude that C is of bidegree $(1, 1)$ in $\mathbb{CP}^n \times \mathbb{CP}^n$. Thus C must lie inside a $\mathbb{CP}^1 \times \mathbb{CP}^1$, each of the factor must be linearly embedded in $\mathbb{CP}^n \times \mathbb{CP}^n$. Then the normal bundle

$$\begin{aligned} N_{C|\mathbb{CP}^n\times\mathbb{CP}^n} &= N_{C|\mathbb{CP}^1\times\mathbb{CP}^1} \oplus N_{\mathbb{CP}^1\times\mathbb{CP}^1|\mathbb{CP}^n\times\mathbb{CP}^n}|_C \\ &= N_{C|\mathbb{CP}^1\times\mathbb{CP}^1} \oplus (\mathcal{O}_{\mathbb{CP}^1}(1) \oplus \mathcal{O}_{\mathbb{CP}^1}(1))|_C \\ &= \mathcal{O}_C(2) \oplus \mathcal{O}_C(1)^{\oplus 2(n-1)}. \end{aligned}$$

The relation $N_{C|\mathbb{CP}^1\times\mathbb{CP}^1}$ is from the case S^2 above the other one is from the fact that C has bidegree $(1,1)$. By this we know $\mu = 2 + 2(n-1) = 2n$, since $\dim_{\mathbb{R}} \mathbb{CP}^n = 2n$, the formula holds.

4. $M = \mathbb{HP}^n$ Without loss of generality, we assume that C lies in $Gr(2,4) \hookrightarrow Gr(2,2n+2)$. To calculate the normal bundle of C in $Gr(2,2n+2)$, we use that

$$N_{C|Gr(2n+2)} = N_{C|Gr(2,4)} \oplus N_{Gr(2,4)|Gr(2,2n+2)}|_C = N_0 \oplus N|_C.$$

The case for $n=1$ corresponds to the S^4 case above and thus $N_0 = \mathcal{O}_C(2)^{\oplus 3}$. To find $N|_C$, we first note that there is the sequence

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] & Gr(2,2n+2)\times\mathbb{C}^{2n+2} \arrow[r] & Q_{2n+2} \arrow[r] & 0 \\ 0 \arrow[r] & S \arrow[r] \arrow[u, equal] & Gr(2,4)\times\mathbb{C}^4 \arrow[r] \arrow[u, hook] & Q_4 \arrow[r] \arrow[u, hook] & 0 \end{tikzcd}$$

Here S is the tautological vector bundle of the Grassmannian. Notice that

$$\begin{aligned} Q_{2n+2}|_{Gr(2,4)} &= (Gr(2,2n+2)\times\mathbb{C}^{2n+2}/S)|_{Gr(2,4)} \\ &= (Gr(2,2n+2)\times(\mathbb{C}^4\oplus\mathbb{C}^{2n-2})/S)|_{Gr(2,4)} \\ &= Q_4 \oplus (Gr(2,2n+2)\times\mathbb{C}^{2n-2})|_{Gr(2,4)} \\ &= Q_4 \oplus Gr(2,4)\times\mathbb{C}^{2n-2}. \end{aligned}$$

Observe that the tangent bundle

$$\begin{aligned} TGr(2,2n+2)|_{Gr(2,4)} &= Hom(S, Q_{2n+2})|_{Gr(2,4)} \\ &= Hom(S, Q_{2n+2}|_{Gr(2,4)}) \\ &= Hom(S, Q_4 \oplus Gr(2,4) \times \mathbb{C}^{2n-2}) \\ &= Hom(S, Q_4) \oplus (S^*)^{\oplus(2n-2)} \\ &= TGr(2,4) \oplus (S^*)^{\oplus(2n-2)}. \end{aligned}$$

Thus we have

$$N|_C = N_{Gr(2,4)|Gr(2,2n+2)}|_C = (S^*)|_C^{\oplus(2n-2)}.$$

By the Plücker embedding of $i : Gr(2,4) \hookrightarrow \mathbb{CP}^5$, we have $\bigwedge^2 S^*$ correspond to $i^*\mathcal{O}(1)$. But the case of $\mathbb{HP}^1$ is the same as $\mathbb{S}^3$ and thus C corresponds to a degree 2 curve in $\mathbb{P}^5$. Thus we have $\wedge^2 S^*|_C = \mathcal{O}_C(2)$. By the splitting theorem again, $S^* = \mathcal{O}_C(a) + \mathcal{O}_C(b)$. Since $\wedge^2 S^*|_C = \mathcal{O}_C(a+b)$, it means that $a+b=2$. This means $\mu = 2+2+2+(2n-2)(a+b) = 6+4(n-1) = 4n+2$. Thus the formula holds since $\dim_{\mathbb{R}} \mathbb{HP}^n = 4n$.

□

To analyze this result, one notices that the tangent space of the moduli space $\mathcal{M}$ of parametrized holomorphic disks are given by

$$T_{f(D)}\mathcal{M} = H^0_{\mathbb{R}}(\mathbb{CP}^1, \mathcal{O}(N)).$$

If $N = \sum_j \mathcal{O}(n_j)$ and $n_j \geq 0$, then

$$H^0_{\mathbb{R}}(\mathbb{CP}^1, \mathcal{O}(N)) = \oplus H^0_{\mathbb{R}}(\mathbb{CP}^1, \mathcal{O}(n_j)).$$

Thus we have

$$\dim_{\mathbb{R}} \mathcal{M} = \sum_j h^0_{\mathbb{R}}(\mathbb{CP}^1, \mathcal{O}(n_j)) = \sum_j (1+n_j) = d - 1 + \mu.$$

Fix $q \in V$. We have

$$\dim_{\mathbb{R}} \mathcal{M} = \dim_{\mathbb{R}} \mathcal{M}_q + \dim_{\mathbb{R}} V.$$

But $\dim_{\mathbb{R}} V = 2d - 2$. Thus the formula reads

$$\dim_{\mathbb{R}} \mathcal{M}_q = \dim_{\mathbb{R}} \mathcal{M} - \dim_{\mathbb{R}} V = d - 1 + \mu - (2d-2) = \mu - (d-1) = I.$$

Since the index of a geodesic is the dimension of the negative eigenspace of the Hessian of the length function on M, it encodes the dimension of the deformation of the geodesic such that its length decreases. By looking at $\dim_{\mathbb{R}} \mathcal{M}_q$, at least in the S^n case, it corresponds to the dimension of sliding the projective line passing through q, in which the projective line intersects S^n at circles. Notice that there is only 1 projective line which corresponds to the compactified complexified geodesics, and all the others intersect S^n at circles whose length is less than the geodesics. This provides a geometrical interpretation of the index formula.

5.3 Further directions

Notice that even in the S^2 case, there is a one-dimensional family of holomorphic disks passing through $q \in \Delta$. We would like to produce a similar result as in [LM10],i.e. there is a foliation of $\mathcal{Q}^2 - S^2$ by a family of parametrized holomorphic disks on $(\mathcal{Q}^2, S^2)$. In the round S^2 case, it is clear that in this one-dimensional family of holomorphic disks passing through q, there is only one disk that corresponds to the complexified compactified geodesic. This suggests that in the general case, for each $q \in \Delta$, one should *choose* a holomorphic disk in $\mathcal{M}_q$ such that our choices of disks give a foliation of $X - S^2$ when q varies. We suggest to look at the Robin's constant:

Let w be the defining function of Δ such that $dw \neq 0$ at Δ. Let $q \in \Delta$ and $f : D \to X$ be a parametrized holomorphic disk with boundary in M, such that

$f(0) = q$. We define the Robin's constant

$$R(f) = \log |(w \circ f)'(0)|^2.$$

Notice that If $f(z)$ is replaced by $f(e^{i\theta} z)$, $R(f)$ is unchanged. If w is replaced by another defining function $\tilde{w}$, then $\tilde{w} = sw$ for some function $s \neq 0$. Then

$$\begin{aligned}
(\tilde{w} \circ f)'(0) &= (sw \circ f)'(0) \\
&= s(f(0))(w \circ f)'(0) + s'(f(0))(w \circ f(0)) \\
&= s(q)(w \circ f)'(0) + s'(q)w(q) \\
&= s(q)(w \circ f)'(0).
\end{aligned}$$

Thus $R(f)$ is replaced by $\log s(q) + R(f)$. If we fixed q, then $R(f)$ is shifted.

For each fixed q, we would like to choose a disk in $\mathcal{M}_q$ such that $R(f)$ is the smallest. In the round S^2, we compute the standard example and find that $R(f_a) = \log 4 - \log(1 - a^2)$, where f_a corresponds to the disk obtained by the intersection of $\{z_1 = 0\}$ and $\{z_1^2 + z_2^2 + z_3^2 = z_0^2\}$. Thus $a \in (-1, 1)$. Notice that $R(f_a) \to +\infty$ when $a \to \pm 1$, and $R(f_a)$ consists of only 1 critical point ($a = 0$), in which its second derivative is positive.

Let $\mathcal{M}_q = \{f_t | t \in (-1, 1)\}$. We pose the following:

Conjecture V.8. *Let $\mathcal{R}(t) = R(f_t)$.*

1. *$\mathcal{R}(t) \to +\infty$ when $t \to \pm 1$.*
2. *Let $t_0 \in (-1, 1)$ such that $\mathcal{R}'(t_0) = 0$. Then $\mathcal{R}''(t_0) \geq 0$.*

Both of this would suggest that there is only 1 minimum point and thus one can "choose" the correct disk.

CHAPTER VI

Further Research Directions

6.1 Docility on $\mathcal{Q}^2$

The Lebrun-Mason model on $\mathbb{CP}^2$ with the foliation of holomorphic disks are interesting in their own right. We would like to prove a theorem similar to that of Lebrun and Mason for the case when $\mathbb{CP}^2$ is replaced by $\mathcal{Q}^2$. As above, we prove the existence of a certain family of disks passing through each $q \in M$ in the case of $\mathcal{Q}^2$. We pose the following conjecture:

Conjecture VI.1. *For each $q \in M$, in the family of holomorphic disks through q, there exists one unique disk with minimal distortion (minimal Robin's constant). Varying q gives a foliation by disks of $\mathcal{Q}^2 - N$.*

We computed that this result is true for the round case (N being the anti-diagonal in $\mathbb{CP}^1 \times \mathbb{CP}^1$).

In the Grauert tube picture, recall that S^n can be embbeded into $\mathcal{Q}^n$, where the manifold of oriented geodesics N^+ can be realized as $\mathcal{Q}^{n-1}$, the intersection of $\mathcal{Q}^n$ with a generic hyperplane in $\mathbb{CP}^{n+1}$. Let M, diffeomorphic to S^n, be a perturbation of S^n. We also conjecture that

Conjecture VI.2. *For any points $q \in \mathcal{Q}^{n-1}$, there exists an $(n-1)$-dimensional family of embedded holomorphic disks passing through q such that their boundaries* *lie on M. Varying q we can find a foliation of $\mathcal{Q}^n - M$ by holomorphic disks by selecting the disk of minimal distortion through q.*

We suggest that the dimension of the family is $n-1$, which coincides with the index I of a geodesic in S^n, as we can see in Chapter V.

6.2 Other CROSSes

We showed that the only Zoll metric on S^n with an infinite Grauert tube must be the round one. The same technique could be applied to other CROSSes like $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$ or even $\mathbb{OP}^2$.

Question VI.3. *Let M be diffeomorphic to a CROSS. Does there exist any Zoll metric on M, other than the round ones, with an infinite Grauert tube?*

As the only Zoll metric on $\mathbb{RP}^n$ is the round one [Ber03], and we proved the case for S^n, we just have to extend our method to $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{OP}^2$. The question itself is not very strong since we know that any analytic Zoll perturbation (i.e. g_t is analytic in the t parameter) of the CROSSes for $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{OP}^2$ are isometric to the CROSSes [Tsu81].

Following our results in the round case of the spheres, we can just extend our theorem to all of the CROSSes and so their tangent bundle TM can be compactified to X by adding the points at infinity, in which $X - TM$ corresponds to the manifold of oriented geodesics N^+. We can even extend our proof of X being Kähler and $V = X - TM$ being a positive divisor. However, in the case of $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{OP}^2$, we don't have Audin's cohomological structure theorem for the associated cut as for the case of a Zoll g on S^n, and thus we could not find the relation between the canonical divisor K_X and the divisor V at inifinity, which was crucial for our arguments above. It would thus be interesting if we could generalize these topological results of polarized symplectic manifolds on $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{OP}^2$.

6.3 Finsler metric

In the Lebrun-Mason construction, there is a family of embedded circles on N that gives the boundary of holomorphic disks. We conjecture that

Conjecture VI.4. *There is a Finsler metric on N such that all the embedded circles are geodesics.*

In this case N is Finsler-Zoll. By looking at the moduli space of geodesics of N, it corresponds to the moduli space of holomorphic disks with boundaris on N. But this corresponds to points of M. This makes us wonder whether we could extend the Lebrun-Mason construction to a Finsler metric. For a Finsler-Zoll surface, we have to look at the unit tangent bundle and complexify it in a more or less canonical way. In this case we are looking for a complexification of the Finsler metric to each complexified tangent space.

It is worth noting that for a Finsler-Zoll metric on M, one could not look at the projectivized tangent bundle $\mathbb{P}TM$. One has to look at the sphere bundle $\mathbb{S}TM$ and, if the construction works, we could blow it down to the manifold of oriented geodesics N^+. This is like a double cover of N in the Lebrun-Mason picture and thus it looks like that this would be the picture on $\mathcal{Q}^2$, which we have discussed above. At this stage, we cannot figure out whether a Finsler-Zoll surface would give a Lebrun-Mason construction on $\mathcal{Q}^2$, nor whether the docility of a totally real surface N in $\mathcal{Q}^2$ actually corresponds to a Finsler-Zoll surface via some construction.

6.4 Duality

The main connection between a Zoll manifold M and its manifold of geodesics N is mysterious. Given a Zoll manifold M one can find its manifold of geodesics. Lebrun and Mason showed that if N is nice enough (docile in a certain embedding), it gives the Zoll manifold M back by looking at the moduli space of a family of closed curves. Here N is totally real and M is complex. In the view of symplectic cut, we have M being lagrangian and N being symplectic. Both pictures give this property but the roles of M and N are opposite, in the sense that one would relate symplectic objects with complex objects, and totally-real objects with lagrangian objects. In addition, there are foliations by curves (either disks or $\mathbb{CP}^1$ in both pictures) on the complement of the totally-real or lagrangian manifold. We would like to investigate more to see something deep and elegant behind the picture. For example, the interchange of the roles of complex and symplectic geometry seems reminiscent of mirror symmetry.

BIBLIOGRAPHY

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[Agu96] Raúl M. Aguilar, *Isotropic almost complex structures on tangent bundles*, Manuscripta Math. **90** (1996), no. 4, 429–436. MR 1403714 (97e:53049)

[Aud07] Michèle Audin, *Lagrangian skeletons, periodic geodesic flows and symplectic cuttings*, Manuscripta Math. **124** (2007), no. 4, 533–550. MR 2357797 (2008k:53192)

[Ber03] Marcel Berger, *A panoramic view of Riemannian geometry*, Springer-Verlag, Berlin, 2003. MR 2002701 (2004h:53001)

[Bes78] Arthur L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], vol. 93, Springer-Verlag, Berlin, 1978, With appendices by D. B. A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger and J. L. Kazdan. MR 496885 (80c:53044)

[Bur84] D. Burns, *Les équations de Monge-Ampère homogènes, et des exhaustions speciales de variétés affines*, Goulaouic-Meyer-Schwartz seminar, 1983–1984, École Polytech., Palaiseau, 1984, pp. Exp. No. 4, 10. MR 811647

[Dom62] Peter Dombrowski, *On the geometry of the tangent bundle*, J. Reine Angew. Math. **210** (1962), 73–88. MR 0141050 (25 #4463)

[Fun13] P. Funk, *Über Flächen mit lauter geschlossenen geodätischen Linien*, Math. Ann. **74** (1913), no. 2, 278–300. MR 1511763

[Gre63] L. W. Green, *Auf Wiedersehensflächen*, Ann. of Math. (2) **78** (1963), 289–299. MR 0155271 (27 #5206)

[GS91] Victor Guillemin and Matthew Stenzel, *Grauert tubes and the homogeneous Monge-Ampère equation*, J. Differential Geom. **34** (1991), no. 2, 561–570. MR 1131444 (93e:32018)

[GS92] ———, *Grauert tubes and the homogeneous Monge-Ampère equation. II*, J. Differential Geom. **35** (1992), no. 3, 627–641. MR 1163451 (94e:32032)

[Gui76] Victor Guillemin, *The Radon transform on Zoll surfaces*, Advances in Math. **22** (1976), no. 1, 85–119. MR 0426063 (54 #14009)

[KK00] Yasuyuki Kachi and János Kollár, *Characterizations of* $\mathbf{P}^n$ *in arbitrary characteristic*, Asian J. Math. **4** (2000), no. 1, 115–121, Kodaira's issue. MR 1802915 (2001k:14016)

[LeB06] Claude LeBrun, *Twistors, holomorphic disks, and Riemann surfaces with boundary*, Perspectives in Riemannian geometry, CRM Proc. Lecture Notes, vol. 40, Amer. Math. Soc., Providence, RI, 2006, pp. 209–221. MR 2237111 (2007g:53051)

[Lem93] László Lempert, *Complex structures on the tangent bundle of Riemannian manifolds*, Complex analysis and geometry, Univ. Ser. Math., Plenum, New York, 1993, pp. 235–251. MR 1211884 (94c:53083)

[LM02] Claude Lebrun and L. J. Mason, *Zoll manifolds and complex surfaces*, J. Differential Geom. **61** (2002), no. 3, 453–535. MR 1979367 (2004d:53043)

[LM10] Claude LeBrun and L. J. Mason, *Zoll metrics, branched covers, and holomorphic disks*, Comm. Anal. Geom. **18** (2010), no. 3, 475–502. MR 2747436 (2012c:53057)

[LS91] László Lempert and Róbert Szőke, *Global solutions of the homogeneous complex Monge-Ampère equation and complex structures on the tangent bundle of Riemannian manifolds*, Math. Ann. **290** (1991), no. 4, 689–712. MR 1119947 (92m:32022)

[PW91] Giorgio Patrizio and Pit-Mann Wong, *Stein manifolds with compact symmetric center*, Math. Ann. **289** (1991), no. 3, 355–382. MR 1096176 (92e:32009)

[Roc11] Frédéric Rochon, *On the uniqueness of certain families of holomorphic disks*, Trans. Amer. Math. Soc. **363** (2011), no. 2, 633–657. MR 2728581 (2011j:53079)

[Sto77] Wilhelm Stoll, *Variétés strictement paraboliques*, C. R. Acad. Sci. Paris Sér. A-B **285** (1977), no. 12, A757–A759. MR 0457793 (56 #15997)

[Sto80] ———, *The characterization of strictly parabolic manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **7** (1980), no. 1, 87–154. MR 577327 (81h:32028)

[Sző91] Róbert Szőke, *Complex structures on tangent bundles of Riemannian manifolds*, Math. Ann. **291** (1991), no. 3, 409–428. MR 1133340 (93c:53023)

[Tsu81] Chiaki Tsukamoto, *Infinitesimal Blaschke conjectures on projective spaces*, Ann. Sci. École Norm. Sup. (4) **14** (1981), no. 3, 339–356. MR 644522 (84k:58055)

[Wei74] Alan Weinstein, *On the volume of manifolds all of whose geodesics are closed*, J. Differential Geometry **9** (1974), 513–517. MR 0390968 (52 #11791)